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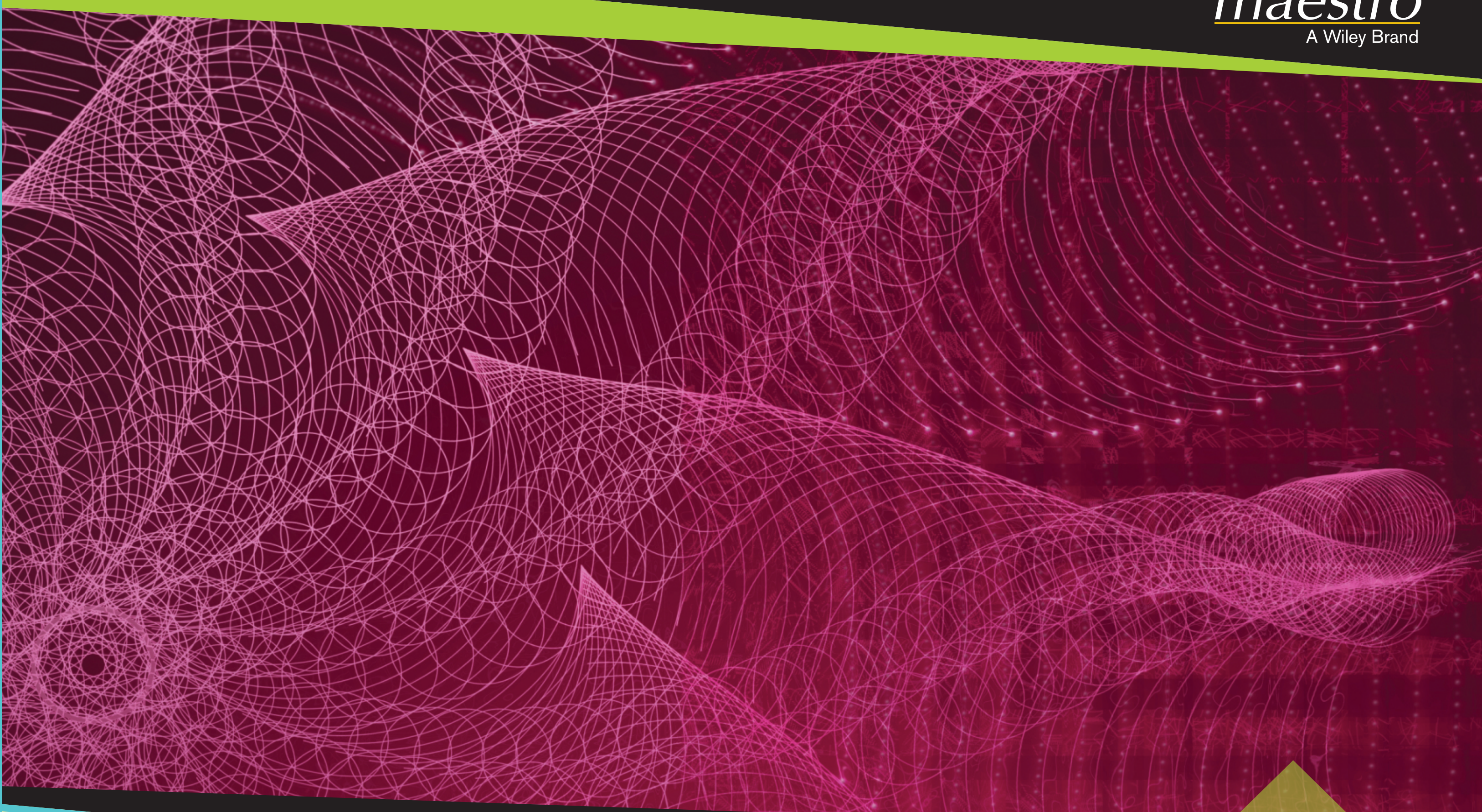

VOLUME – I

WILEY PROBLEMS IN MATHEMATICS FOR JEE

with Summarized Concepts

- Solved Examples
- Practice Exercises with Complete Solutions
- Covers Solved JEE 2018 (Main and Advanced) Mathematics Questions

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VOLUME –I

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Cover Image: Carlos_bcn/iStockphoto

Other Wiley Editorial Offices:

John Wiley & Sons, Inc. 111 River Street, Hoboken, NJ 07030, USA

Wiley-VCH Verlag GmbH, Pappellaee 3, D-69469 Weinheim, Germany

John Wiley & Sons Australia Ltd, 42 McDougall Street, Milton, Queensland 4064, Australia

John Wiley & Sons (Asia) Pte Ltd, 1 Fusionpolis Walk #07-01 Solaris, South Tower Singapore 138628

John Wiley & Sons Canada Ltd, 22 Worcester Road, Etobicoke, Ontario, Canada, M9W 1L1

First Edition: 2018

ISBN: 978-81-265-7629-6

ISBN: 978-81-265-8689-9 (ebk)

www.wileyindia.com

Printed at:

Note to the Student

Wiley Mathematics Problem Book is specifically designed to meet the needs of engineering (JEE) aspirants and give an edge to their preparation. The book offers complete coverage of the mathematics curriculum (of Class 11 syllabus) for JEE. It is enriched with unique elements and features that help students recapitulate the concepts, build problem-solving skills and apply them to solve all question-types asked in the engineering entrance examinations. The book is a valuable resource for both JEE (Main) and JEE (Advanced) aspirants. The chapter flow of the book is aligned with JEE Main syllabus and its coverage in the classroom. However, topics specific to JEE (Advanced) and advanced level questions are also covered both as solved examples and practice exercises.

We will now walk you through the target examinations and some key features of the book that enhance the learning experience.

TARGET EXAMINATION

Admission to Undergraduate Engineering Programs at IITs, NITs and other Center and State (participating) funded Technical Institutions use the Joint Entrance Examination Main (JEE Main) score as eligibility/merit criteria. The JEE (Main) is also an eligibility test for the Joint Entrance Examination Advanced [JEE (Advanced)], which is mandatory for the candidate if he/she is aspiring for admission to the undergraduate program offered by the IITs. The JEE (Advanced) scores are used as an eligibility criteria for admission into IITs.

An effective exam strategy for success in these examinations can be based on the detailed analysis of previous years question papers and planning your preparation accordingly. The Mathematics Question Paper of these examinations is a judicious mix of easy, moderate and tough questions. The analysis of question distribution over the units of mathematics syllabus for these examinations is given below.

EXAM ANALYSIS OF PAPERS

Mathematics question paper comes as an amalgamation of easy, moderate and tough questions. This section shows the unit-wise as well as chapter-wise analysis of previous 9 years (2010-2018) JEE Main and JEE Advanced papers.

JEE Main

Unit	Year								
	2010	2011	2012	2013	2014	2015	2016	2017	2018
Algebra	14	13	13	12	12	11	12	13	12
Calculus	8	10	9	8	9	8	7	10	8
Trigonometry	2	1	1	3	2	3	3	2	3
Analytical Geometry	6	6	7	7	7	8	8	5	7

JEE Advanced

Unit	Year								
	2010	2011	2012	2013	2014	2015	2016	2017	2018
Algebra	16	17	12	14	12	6	12	10	8
Trigonometry	5	1	2	4	3	1	2	1	1
Analytical Geometry	13	8	9	10	7	3	9	7	9
Differential Calculus	2	7	6	2	11	5	7	8	12
Integral Calculus	8	7	10	7	5	4	5	7	4
Vector	3	3	2	3	2	1	1	3	2

MATHEMATICS JEE MAIN PAPERS ANALYSIS (2010-2018)

Unit	Chapter	AIEEE 2010	AIEEE 2011	AIEEE 2012	JEE Main 2013	JEE Main 2014 (Offline)	JEE Main 2015 (Offline)	JEE Main 2016 (Offline)	JEE Main 2017 (Offline)	JEE Main 2018 (Offline)
Algebra	Complex Numbers and Quadratic Equations	2	2	2	3	3	2	2	1	1
	Permutations and Combinations	1	1	1	1		2	1	2	1
	Binomial Theorem		1	1	1	1		1		1
	Sequences and Series	2	1	2	1	2	2	2	3	2
	Statistics	1	1	1	1	1	1	1	1	1
	Mathematical Reasoning	1	1	1	1	1		1		1
	Matrices and Determinants	3	2	2	2	2	2	2	3	3
	Vector Algebra	2	2	2	1	1	1	1	1	1
	Probability	2	2	1	1	1	1	1	2	1
Calculus	Sets, Relations and Functions	1	2	1	1	1	2	1	2	2
	Limits, Continuity and Differentiability	3	3	2	2	3	3	2	2	1
	Application of Derivatives	1	1	3	1	2			2	1
	Integrals	1	2	2	3	2	2	1	2	2
	Application of Integrals	1	1	1	1	1		1	1	1
	Differential Equations	1	1				1	2	1	1
Trigonometry	Trignometric Functions	2	1	1	2	2	2	2	2	3
	Inverse Trignometric Functions				1		1	1		
Analytical Geometry	Conic Sections	2	2	3	3	3	4	5	3	5
	Three-Dimensional Geometry	4	4	4	4	4	4	3	2	2

MATHEMATICS JEE ADVANCED PAPERS ANALYSIS (2010-2018)

Unit	Chapter	IIT-JEE 2010						IIT-JEE 2011						IIT-JEE 2012						JEE Advanced 2013						JEE Advanced 2014						JEE Advanced 2015						JEE Advanced 2016						JEE Advanced 2017						JEE Advanced 2018					
		P	Q	R	S	T	U	P	Q	R	S	T	U	P	Q	R	S	T	U	P	Q	R	S	T	U	P	Q	R	S	T	U	P	Q	R	S	T	U	P	Q	R	S	T	U	P	Q	R	S	T	U	P	Q	R	S	T	U
Algebra	Complex Numbers		1		1						1		2	1						1	1	2							1								1						1		1						2				
	Quadratic Equations	1						2					1													1	1						1					1								2									1
	Permutations and Combinations	1																																				1											1				1		
	Sequence and Series	1					2						1	1		2					1				1						1						1	2											1						1
	Binomial Theorem	1												1												1											1						1												1
	Logarithms							1											1		1																																		
	Matrices and Determinants	1		3			2	2		3			1	2	1						2						2											1	1				1	1	1				1		1				1
	Probability	2							1	2				1	2					1		2			2	2		2			1			2				1		2				1	1						2				
Trigonometry	Properties and Solution of Triangles (Heights and Distances)		1				1							1							1						1												1																
	Trigonometric Equations						2						1													1																			1										
	Trigonometric Ratios and Identities						1								1						1																	1																	
	Inverse Trigonometric Function (Principal Values Only)																			1			1								1		1																		1				
Analytical Geometry	Rectangular Coordinate System																																																						
	Straight Lines and Pair of Lines	2			1		1	1					1	1						1	2																	1																	
	Conics		1	3			2	2	2				1	1	1				1			2	1		1			2			1		2				1		2	2					2	3					1		1		
	Circle			2				1								2					1					2	1				1								4										1		2	1			
	Three-Dimensional Geometry						1							2	1					1			1																					1							2				2
Differential Calculus	Sets and Relations							2																									1											1							1		1		2
	Limits							1	1					2							1							2			1						1		3				1		1										
	Functions	1					1							1	2												2		1					2			1													1	1		1		1
	Continuity and Differentiability								1				2	1						1						1	2		1		1								1					1	4				1	1	2				1
	Differentiation																																						2																

Unit	Chapter	IIT-JEE 2010						IIT-JEE 2011						IIT-JEE 2012						JEE Advanced 2013						JEE Advanced 2014						JEE Advanced 2015						JEE Advanced 2016						JEE Advanced 2017						JEE Advanced 2018					
		P	Q	R	S	T	U	P	Q	R	S	T	U	P	Q	R	S	T	U	P	Q	R	S	T	U	P	Q	R	S	T	U	P	Q	R	S	T	U	P	Q	R	S	T	U	P	Q	R	S	T	U	P	Q	R	S	T	U
Integral Calculus	Application of Derivatives	1							1		1		1						2	1	2	2											1					1							1	3									1
	Indefinite Integration													1																																									
	Definite Integration	1	1					2						1	1	2										1			1		1		1				1	1					1		1										1
	Area Under the Curve			3				1						1	1					1											1							1							1										
	Differential Equations		1				1						1		1					1						1							1						1					1											
Vector	Vectors	2					1	1	1				1	1					1	1			1		1		1				1						1		1					1		2									2

P: Single Correct Choice Type **Q:** One or More Than One Option Correct Type **R:** Paragraph Type

S: Matrix-Match Type **T:** Reasoning Type **U:** Integer Answer Type

FEATURES OF THE BOOK

A. Understand the Concepts

1. All the concepts as per the JEE curriculum are explained in simple steps to develop fundamental understanding of the subject.

> ### 1.1 Set Theory
> #### 1.1.1 Sets
> A set is a well-defined collection of objects or elements. Each element in a set is unique in its nature. Generally, but not necessarily, a set is denoted by a capital letter (e.g. A, B, U, V, etc.) and the elements are enclosed between brackets, '{ }', denoted by small letters $a, b, \ldots, x, y$, etc. For example, let us consider the following sets:
>
> A = Set of all small English alphabets = $\{a, b, c, \ldots, x, y, z\}$
>
> B = Set of all positive integers less than or equal to 10 = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
>
> R = Set of real numbers = $\{x: -\infty < x < \infty\}$

> **Key Points:**
> - If R is a relation from A to B and $(a, b) \notin R$, then we also write $a \not R b$ (read as a is not related to b).
> - In an identity relation on A, every element of A should be related to itself only.
> - aRb shows that a is the element of domain set and b is the element of range set.

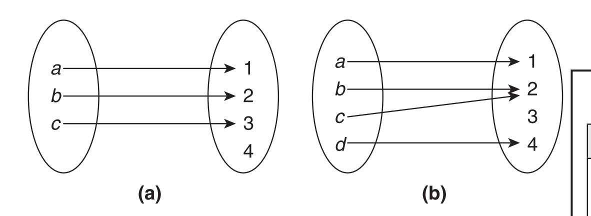
2. Important points to remember about concepts highlighted as Key Points.

B. Every Aspect of the Subject Covered

In form of formulas, figures, graphs and tables to enhance problem-solving skills.

1. $\sin 2A = 2\sin A\cos A = \dfrac{2\tan A}{1+\tan^2 A}$
2. $\cos 2A = 1 - 2\sin^2 A = 2\cos^2 A - 1 = \cos^2 A - \sin^2 A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$
3. $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$
4. $\sin 3A = 3\sin A - 4\sin^3 A = 4\sin(60^\circ - A)\sin A\sin(60^\circ + A)$
5. $\cos 3A = 4\cos^3 A - 3\cos A = 4\cos(60^\circ - A)\cos A\cos(60^\circ + A)$

$= \tan(60^\circ - A)\tan A\tan(60^\circ + A)$



Fig

Table 1.4 Functions and their respective periods

S. No.	$f(x)$	Period
1.	$\sin^n x$, $\cos^n x$, $\text{cosec}^n x$, $\sec^n x$	2π if n is an odd number; π if n is an even number
	$\tan^n x$, $\cot^n x$	$\pi\ \forall n \in N$
	$\lvert\sin x\rvert^n$, $\lvert\cos x\rvert^n$, $\lvert\tan x\rvert^n$, $\lvert\cot x\rvert^n$	$\pi\ \forall n \in N$
	$\{x\}$	1
	$f(x) = k$	Periodic function but it has no fundamental period.

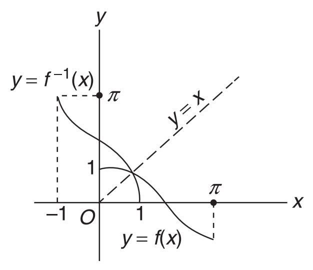


Figure 1.43

C. Reinforce Concepts

1. **Illustrations** pose a specific problem using concepts already presented and then work through the solution.

Illustration 3.4 Find the solution of equation $\sqrt{3}\cos\theta+\sin\theta=\sqrt{2}$.

Solution:

$$\sqrt{3}\cos\theta+\sin\theta=\sqrt{2}$$
$$\Rightarrow \frac{\sqrt{3}}{2}\cos\theta+\frac{1}{2}\sin\theta=\frac{1}{\sqrt{2}}$$
$$\Rightarrow \sin\frac{\pi}{3}\cos\theta+\cos\frac{\pi}{3}\sin\theta=\frac{1}{\sqrt{2}}$$
$$\Rightarrow \sin\left(\theta+\frac{\pi}{3}\right)=\sin\frac{\pi}{4}$$
$$\Rightarrow \theta=n\pi+(-1)^n\frac{\pi}{4}-\frac{\pi}{3}$$

2. **Your Turn** within each chapter is present to reinforce and check the understanding of the students.

Your Turn 2

1. Find the solution of the equation $\sqrt{3}\sin x+\cos x=4$.

Ans. No solution

2. Find the general solution of the equation $(\sqrt{3}-1)\sin\theta+(\sqrt{3}+1)\cos\theta=2$.

Ans. $\theta=2n\pi\pm\frac{\pi}{4}+\frac{\pi}{12}$

3. Find the solution of the equation $\sin x+\cos x-2\sqrt{2}\sin x\cos x=0$.

Ans. $x=n\pi+(-1)^n\frac{\pi}{6}-\left(\frac{\pi}{4}\right)$

3. **Additional Solved Examples** suitable for JEE exams are provided with in-depth solutions for the students to understand the logic behind and formula used.

Additional Solved Examples

1. Solve cot (sin x + 3) = 1.

Solution:

$$\sin x+3=n\pi\pm\frac{\pi}{4}\Rightarrow 2\le n\pi\pm\frac{\pi}{4}\le 4\Rightarrow n=1$$
$$\Rightarrow \sin x=\pi\pm\frac{\pi}{4}-3$$
$$\Rightarrow x=n\pi+(-1)^n\sin^{-1}\left(\frac{5\pi}{4}-3\right)\text{ or } n\pi+(-1)^n\sin^{-1}\left(\frac{3\pi}{4}-3\right)$$

D. Understanding the Exam Pattern

Through Previous Years' Solved JEE Main/AIEEE Questions and Previous Years' Solved JEE Advanced/IIT-JEE Questions.

Previous Years' Solved JEE Advanced/ IIT-JEE Questions

1. The number of solutions of the pair of equations $2\sin^2\theta$ … $= 0$ and $2\cos^2\theta - 3\sin\theta = 0$ in the interval $[0, 2\pi]$ is

(A) zero **(B)** one **(C)** two **(D)** …

[IIT-JE…

Solution: The first equation is

$$2\sin^2\theta - \cos 2\theta = 0$$

It can be written as

$$2\sin^2\theta - (1 - 2\sin^2\theta) = 0$$

$$\Rightarrow 4\sin^2\theta = 1$$

$$\Rightarrow \sin\theta = \frac{1}{2}, \frac{-1}{2}$$

Previous Years' Solved JEE Main/AIEEE Questions

1. A body weighing 13 kg is suspended by two strings 5-m and 12-m long, their other ends being fastened to the extremities of a rod 13-m long. If the rod be so held that the body hangs immediately below the middle point. The tensions in the strings are:

(A) 12 kg and 13 kg **(B)** 5 kg and 5 kg
(C) 5 kg and 12 kg **(D)** 5 kg and 13 kg

[AIEEE 2007]

Solution: See Fig. 2.20. Since, $13^2 = 5^2 + 12^2$, therefore,

$$\angle AOB = \frac{\pi}{2}$$

$\angle AOB$ is the angle in a semicircle with diameter AB and centre C.

E. Practice to Complete Your Learning

Through Practice Exercise 1 (JEE Main) and Practice Exercise 2 (JEE Advanced). All questions types as per JEE Main and Advanced covered.

Practice Exercise 1

1. If $f(x) = \cos(\log x)$, then $f(x)f(y) - \{(1/2)[f(x/y) + f(xy)]\}$ is

(A) -1 **(B)** $1/2$
(C) -2 **(D)** None of these

2. The values of b and c for which the identity $f(x+1) - f(x) = 8x + 3$ is satisfied, where $f(x) = bx^2 + cx + d$ are

(A) $b = 2, c = 1$ **(B)** $b = 4, c = -1$
(C) $b = -1, c = 4$ **(D)** $b = -1, c = 1$

3. If $f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$, then

(A) $f(\pi/4) = 2$ **(B)** $f(-\pi) = 2$
(C) $f(\pi) = 1$ **(D)** $f(\pi/2) = -1$

4. $f(x, y) = 1/(x+y)$ is a homogeneous function of degree

(A) 1 **(B)** -1
(C) 2 **(D)** -2

Practice Exercise 2

Single/Multiple Correct Choice Type Questions

1. If $y = f(x)$ be the concave upward function and $y = g(x)$ be a function such that $f'(x)\cdot g(x) - g'(x)\cdot f(x) = x^4 + 2x^2 + 10$, then

(A) $g(x)$ has at least one root between two consecutive roots of $f(x) = 0$
(B) $g(x)$ has at most one root between two consecutive roots of $f(x) = 0$
(C) if α and β are two consecutive roots of $f(x) = 0$, then $\alpha\beta < 0$
(D) when $f(x)$ increases $g(x)$ decreases

Matrix Match Type Questions

19. Match the following:

(A) The number of solution of $\frac{x}{2} + \frac{\sin x}{\cos x} = \frac{\pi}{4}\ln[-\pi, \pi]$	**(i)** 1
(B) The number of solution of equation $\sin^{-1}(\lvert x^2 - 1\rvert) + \cos^{-1}(\lvert 2x^2 - 5\rvert) = \frac{\pi}{2}$	**(ii)** 0
(C) The nu…	
(D) The nu…	

Comprehension Type Questions

Paragraph for Questions 16 and 17: A line $-\frac{x}{2} = -\frac{f(t)\cdot y}{t} = t^2 z = \lambda$ is the perpendicular to the line of the intersection of the planes $t\cdot f(t)x + f\left(\frac{1}{t^2}\right)z + f(-t) = 0$ and $ty + f(-t)z + f(t^2) = 0$, where $t \in R - \{0\}$.

16. $f(t)$ is

(A) even fu…
(B) odd fu…
(C) neither…
(D) both e…

Integer Type Questions

21. If $\theta_1, \theta_2, \theta_3$ are three values lying in $[0, 2\pi]$ for which $\tan\theta = \lambda$, then $\tan\frac{\theta_1}{3}\tan\frac{\theta_2}{3} + \tan\frac{\theta_2}{3}\tan\frac{\theta_3}{3} + \tan\frac{\theta_1}{3}\tan\frac{\theta_3}{3}$ is equal to __________.

22. The number of solutions that the equation $\sin[\cos(\sin x)] = \cos[\sin(\cos x)]$ has in $\left[0, \frac{\pi}{2}\right]$ is __________.

F. Check Your Performance and Problem-Solving Approach

Through Answer Key and Solution to practice exercises provided with explanation.

Answer Key

Practice Exercise 1

1. (D)	**2.** (B)	**3.** (A)	**4.** (D)	**5.** (A)	**6.** (C)
7. (B)	**8.** (A)	**9.** (D)	**10.** (B)	**11.** (C)	**12.** (A)
13. (B)	**14.** (A)	**15.** (A)	**16.** (C)	**17.** (D)	**18.** (C)
19. (C)	**20.** (A)	**21.** (C)	**22.** (A)	**23.** (B)	**24.** (B)
25. (A)					
31. (D)					

Solutions

Practice Exercise 1

1. Given that

$$f(x) = \cos(\log x) \Rightarrow f(y) = \cos(\log y)$$

Therefore,

$$f(x)f(y) - \frac{1}{2}\left[f\left(\frac{x}{y}\right) + f(xy)\right]$$

$$= \cos(\log x)\cos(\log y) - \frac{1}{2}\left[\cos\left(\log\frac{x}{y}\right) + \cos(\log xy)\right]$$

$$= \cos(\log x)\cos(\log y) - \frac{1}{2}[2\cos(\log x)\cos(\log y)] = 0$$

7. We have

$$f(x) = (x-1)(x-2)(x-3)$$

and $\quad f(1) = f(2) = f(3) = 0$

which implies that $f(x)$ is not one-to-one. For each $y \in R$, there exists $x \in R$ such that $f(x) = y$. Therefore, f is onto. Hence, $f: R \to R$ is onto but not one-to-one.

8. Here, $|x|$ is not one-one and

$$\lim_{x\to\infty} \frac{4x}{(\sqrt{x^2+8x+3} + \sqrt{x^2+4x+3}}$$

is not one-to-one. Also, x^2+1 is not one-to-one. However, $2x-5$ is one-to-one because

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Appendix: Chapterwise Solved JEE 2018 Questions A-1

1 Sets, Relations and Functions

1.1 Set Theory

1.1.1 Sets

A set is a well-defined collection of objects or elements. Each element in a set is unique in its nature. Generally, but not necessarily, a set is denoted by a capital letter (e.g. A, B, U, V, etc.) and the elements are enclosed between brackets, '{ }', denoted by small letters a, b, …, x, y, etc. For example, let us consider the following sets:

A = Set of all small English alphabets = $\{a, b, c, \ldots, x, y, z\}$

B = Set of all positive integers less than or equal to 10 = $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

R = Set of real numbers = $\{x: -\infty < x < \infty\}$

The elements of a set can be discrete (e.g. set of all English alphabets) or continuous (e.g. set of real numbers). The set may contain finite or infinite number of elements. A set may contain no elements and such a set is called 'void set' or 'null set' or 'empty set' and is denoted by ϕ. The number of elements of a set A is denoted by $n(A)$ and hence, $n(\phi) = 0$ as it contains no element.

1.1.2 Union of Sets

Let A and B be two sets. The union of set A and set B is the set of all elements which are in set A or in set B. We denote the union of set A and set B by $A \cup B$ which, usually, read as 'A union B' (Fig. 1.1).

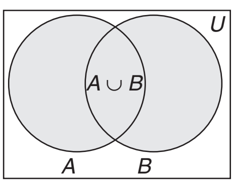


Figure 1.1

Symbolically, we represent 'A union B' as

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

For example, if $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 5, 6\}$ and $C = \{1, 2, 6, 8\}$, then $A \cup B \cup C = \{1, 2, 3, 4, 5, 6, 8\}$.

1.1.3 Intersection of Sets

Let A and B be two sets. The intersection of set A and set B is the set of all those elements that belong to both A and B (Fig. 1.2).

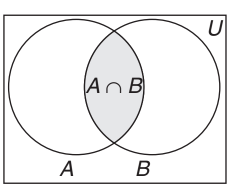


Figure 1.2

The intersection of A and B is denoted by $A \cap B$ (which is read as 'A intersection B'). Thus,

$$A \cap B = \{x: x \in A \text{ and } x \in B\}$$

For example, If $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 5, 6\}$ and $C = \{1, 2, 6, 8\}$, then $A \cap B \cap C = \{2\}$.

Note: Remember that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

1.1.4 Difference of Two Sets

The difference of set A to set B, which is denoted by $A - B$, is the set of those elements that exist in set A but it does not in set B:

$$A - B = \{x: x \in A \text{ and } x \notin B\}$$

In a similar manner,

$$B - A = \{x: x \in B \text{ and } x \notin A\}$$

In general,

$$A - B \neq B - A$$

For example, if $A = \{a, b, c, d\}$ and $B = \{b, c, e, f\}$, then $A - B = \{a, d\}$ and $B - A = \{e, f\}$.

1.1.5 Subset of a Set

A set A is said to be a subset of the set B if each element of the set A is also the element of the set B. The symbol used is '$\subseteq$'. That is,

$$A \subseteq B \Leftrightarrow (x \in A \Rightarrow x \in B)$$

Each set is a subset of its own set. Also a void set is a subset of any set. If there is at least one element in set B which does not belong to set A, then set A is a proper subset of set B and is denoted by $A \subset B$. If set B has n elements, then the total number of subsets of set B is 2^n. For example, if $A = \{a, b, c, d\}$ and $B = \{b, c, d\}$, then $B \subset A$ or equivalently $A \supset B$ (i.e. A is a superset of B).

1.1.6 Equality of Two Sets

Sets A and B are said to be equal if $A \subseteq B$ and $B \subseteq A$ which can be written as $A = B$.

1.1.7 Universal Set

As the name implies, universal set is a set with collection of all the elements and is denoted by U. For example, a set of real numbers R is a universal set whereas a set $A = \{x: x \leq 3\}$ is not a universal set as it does not contain the set of real numbers $x > 3$. Once the universal set is known, one can define the 'complementary set' of a set as the set of all the elements of the universal set which do not belong to that set. For example, if $A = \{x: x \leq 3\}$ then $\overline{A}$ (or A^c) = complimentary set of $A = \{x: x > 3\}$. Hence, we can say that $A \cup \overline{A} = U$, that is, a union of a set and its complimentary is always the universal set and $A \cap \bar{A} = \phi$, that is, the intersection of the set and its complimentary is always a void set. Some of the important properties of operations on sets are listed as follows:

1. $\bar{\bar{A}}$ [or $(A^c)^c$] $= A$, $A \cap A^c = \phi$ and $A \cup A^c = U$
2. $A \cup \phi = A$ and $A \cap \phi = \phi$
3. $A \cup U = U$ and $A \cap U = A$
4. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
5. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
6. $\overline{A \cup B} = \bar{A} \cap \bar{B}$
7. $\overline{A \cap B} = \bar{A} \cup \bar{B}$

1.1.8 Cartesian Product of Sets

The Cartesian product (also known as the cross product) of two sets A and B, denoted by $A \times B$ (in the same order) is the set of all ordered pairs (x, y) such that $x \in A$ and $y \in B$. What we mean by ordered pair is that the pair (a, b) is not the same pair as (b, a) unless $a = b$. It implies that $A \times B \neq B \times A$ in general. Also if set A contains m elements and set B contains n elements then $A \times B$ contains $m \times n$ elements. Similarly, we can define $A \times A = \{(x, y); x \in A$ and $y \in A\}$. We can also define Cartesian product of more than two sets. For example,

$$A_1 \times A_2 \times A_3 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n): a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}$$

Illustration 1.1 In a sports club, 65% of children play football, 70% play volleyball and 75% play basketball. What is the smallest percentage of children playing all these three games?

Solution: Out of 100 children, the total number of children who do not play

(i) football = 100 – 65 = 35
(ii) volleyball = 100 – 70 = 30
(iii) basketball = 100 – 75 = 25

So, the maximum number of children who do not play at least one game is

$$35 + 30 + 25 = 90$$

Thus, the minimum number of children who play all three games is

$$100 - 90 = 10$$

Hence, the smallest percentage of children playing all three games is 10%.

Note: The greatest percentage of children playing all three games = min(65%, 70%, 75%) = 65%.

Your Turn 1

1. In a sweet shop, normally, people buy either one cake or one box of chocolate. One day, the shop sold 57 cakes and 36 boxes of chocolates. How many customers were there that day if 12 people bought both a cake and a box of chocolates?

Ans. 81

2. A survey shows that 63% of the Americans like cheese whereas 76% of them like apples. If x% of the Americans like both cheese and apples, find the values of x.

Ans. $39 \le x \le 63$

1.2 Relation

Let A and B be two sets. A relation R from the set A to set B is a subset of the Cartesian product $A \times B$. Further, if $(x, y) \in R$, then we say that x is related to y and write this relation as $x R y$. Hence, $R\{(x, y); x \in A, y \in B, x R y\}$.

As an example, consider $A = \{1, 2, 3\}$ and $B = \{1, 8, 27\}$, so that $A \times B = \{(1,1), (1,8), (1,27), (2,1), (2,8), (2,27), (3,1), (3,8), (3,27)\}$.

Consider now a subset R of $A \times B$, as $R = \{(1, 1), (2, 8), (3, 27)\}$.

We notice that in every ordered pair of R, the second element is the cube of the first element, that is, the element of the ordered pairs of R has a common relationship which is "cube".

In case, we take $A = \{2, 4, 6\}$, $B = \{1, 5\}$, then

$$A \times B = \{(2, 1), (2, 5), (4, 1), (4, 5), (6, 1), (6, 5)\}$$

Consider now a subset R of $A \times B$ as

$$A \times B = \{(2, 1), (2, 5), (4, 1), (4, 5), (6, 1), (6, 5)\}$$

Here, the first element in each of the ordered pair is greater than the second element. Hence, the relationship is "greater than". Obviously, from the definition, $x R y$ and $y R x$ are not the same, since $R = \{(x, y): x \in A, y \in B, x R y\}$ and $R = \{(x, y): x \in B, y \in A, x R y\}$ are different.

1.2.1 Domain and Range of a Relation

Let R be a relation defined from a set A to a set B, i.e. $R \subseteq A \times B$. Then the set of all first elements of the ordered pairs in R is called the domain of R. The set of all second elements of the ordered pairs in R is called the range of R. That is, D = domain of $R = \{x : (x, y) \in R\}$ or $\{x : x \in A$ and $(x, y) \in R\}$, R^* = range of $R = \{y : (x, y) \in R\}$ or $\{y : y \in B$ and $(x, y) \in R\}$.

Clearly, $D \subseteq A$ and $R^* \subseteq B$.

For example, for R given in $\{(1, 1), (2, 8), (3, 27)\}$ above, domain of $R = \{1, 2, 3\}$, range of $R = \{1, 8, 27\}$.

Illustration 1.2 Let $A = \{1, 2, 3\}$ and $B = \{2, 4, 6, 8\}$. Let $R_1 = \{(1, 2), (2, 4), (3, 6)\}$ and $R_2 = \{(2, 4), (2, 6), (3, 8), (1, 6)\}$. Then find domains and range of relation R_1 and R_2.

Solution:

Domain:	$R_1 = \{1, 2, 3\}$
Range:	$R_1 = \{2, 4, 6\}$
Domain:	$R_2 = \{2, 3, 1\}$
Range:	$R_2 = \{4, 6, 8\}$

Your Turn 2

Find the domain and range of the following relations:

1. $\{(1, 2), (1, 4), (1, 6), (1, 8)\}$

Ans. Domain = {1}, Range = {2, 4, 6, 8}

2. $\{(x, x^3) : x$ is a prime number less than 10$\}$

Ans. Domain = {2, 3, 5, 7}, Range = {8, 27, 125, 343}

1.2.2 Types of Relation

1. **Binary relation:** If A is a non-empty set, then any subset of $A \times A$ is said to be binary relation on A or a relation on A.
2. **Reflexive relation:** A relation R on a set A is said to be a reflexive relation on A if

$$x R x, \text{ that is, } (x, x) \in R; \forall x \in A$$

3. **Symmetric relation:** A relation R on a set A is said to be a symmetric relation on A if

$$xRy \Rightarrow yRx$$

That is,

$$(x, y) \in R \Rightarrow (y, x) \in R \; \forall \; x, y \in A$$

4. **Anti-symmetric relation:** A relation R on a set A is said to be an anti-symmetric relation on A if

$$xRy \text{ and } yRx \Rightarrow x = y$$

That is,

$$(x, y) \in R \text{ and } (y, x) \in R$$
$$\Rightarrow x = y; \; \forall \; x, y \in A$$

5. **Transitive relation:** A relation R on a set A is said to be a transitive relation on A if

$$(x, y) \in R \text{ and } (y, z) \in R$$
$$\Rightarrow (x, z) \in R; \forall \; x, y, z \in A$$

That is,

$$xRy \text{ and } yRz \Rightarrow xRz$$

6. **Identity relation:** A relation R on a set A is said to be an identity relation on A if

$$R = \{(x, y): x \in A, y \in A, x = y\}$$

This is denoted by I_A. Therefore,

$$I_A = \{(x, x): x \in A\}$$

7. $A \times A$ is said to be the universal relation on A.
8. As $\phi \subset A \times A$, ϕ is a relation on A, called void relation on A.
   (a) Identity relation is always reflexive but a reflexive relation need not to be identity relation.
   (b) A relation which is not symmetric is not necessarily anti-symmetric.
9. **Inverse relation:** Let $R \subset A \times B$ be a relation from A to B. Then the inverse relation of R, denoted by R^{-1}, is a relation from B to A defined by

$$R^{-1} = \{(y, x): (x, y) \in R\}$$

Thus,

$$(x, y) \in R \Leftrightarrow (y, x) \in R^{-1}, \forall \; x \in A, y \in B$$

Clearly,

$$\text{Domain } R^{-1} = \text{Range } R$$
$$\text{Range } R^{-1} = \text{Domain } R$$

Also $(R^{-1})^{-1} = R$.

Let $A = \{1, 2, 4\}$, $B = \{3, 0\}$ and let $R = \{(1, 3), (4, 0), (2, 3)\}$ be a relation from A to B. Then $R^{-1} = \{(3, 1), (0, 4) (3, 2)\}$.

10. **Equivalence relation:** Let A be a non-empty set. Then a relation R on A is said to be equivalence relation if
    (i) R is reflexive **(ii)** R is symmetric **(iii)** R is transitive
11. **Partial order relation:** A relation R defined on a set A is said to be a "partial order relation" on A if it is simultaneously reflexive, transitive and antisymmetric on A.

Illustration 1.3 Let N be the set of all natural numbers. Let a relation R be defined on N by $R = \{(a, b): a, b \in N \text{ and } a \le b\}$. Show that R is a partial order relation.

Solution: R is reflexive because $a \le a \; \forall \; a \in N$.

R is transitive because $a \le b$ and $b \le c$, so $a \le c$, $\forall \; a, b, c \in N$.

R is anti-symmetric because $a \le b$ and $b \le a$, so $a = b \; \forall \; a, b \in N$.

Thus, R is a partial order relation.

12. **Total order relation:** A relation R on a set A is said to be a total order relation on A if R is a partial order relation on A such that given any $x, y \in A$, we must have either xRy or yRx.
13. **Composition of relations:** Let R and S be two relations from sets A to B and B to C respectively. Then we can define a relation SoR from A to C such that

$$(a, c) \in SoR \Leftrightarrow \exists \; b \in B \text{ such that } (a, b) \in R \text{ and } (b, c) \in S$$

This relation is called the composition of R and S.
In general, $RoS \neq SoR$. Also $(SoR)^{-1} = R^{-1}oS^{-1}$.

Your Turn 3

1. Let $A = (1, 2)$ and $B = \{3, 4\}$. Then find the number of relations from A to B.

Ans. 16

2. Let $R = \{(1, -1), (2, 0), (3, 1), (4, 2), (5, 3)\}$. Then (i) write R in set builder form (ii) represent R by arrow diagram.

Ans. (i) $R = \{(a, b): a \in N, 1 \le a \le 5, \; b = a - 2\}$

(ii)

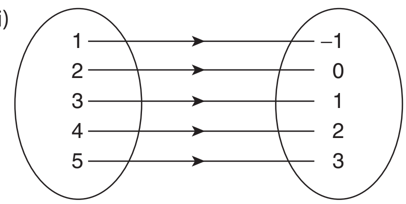


Illustration 1.4 Let T be the set of triangles in a plane and a relation r be defined by $xry \Leftrightarrow x$ is similar to y; $\forall \; x, y \in T$. Then show that r is an equivalence relation on T.

Solution:

1. Every triangle is similar to itself.
   Therefore, x is similar to x, $\forall \; x \in T$
   That is, xrx. So, r is reflexive on T.
2. $xry \Rightarrow x$ is similar to y
   $\Rightarrow y$ is similar to x
   $\Rightarrow yrx$
   Therefore, r is symmetric relation on T.
3. xry and $yrz \Rightarrow x$ is similar to y and y is similar to z
   $\Rightarrow x$ is similar to $z \Rightarrow xrz$
   Therefore, r is transitive relation. Thus, r is an equivalence relation on T.

Illustration 1.5 Let N be the set of all natural numbers. A relation R be defined on $N \times N$ by $(a, b) \, R \, (c, d) \Leftrightarrow a + d = b + c$. Show that R is an equivalence relation.

Solution:

1. $(a, b) \, R \, (a, b)$. For $a + b = b + a$
   Therefore, R is reflexive.
2. $(a, b) \, R \, (c, d) \Rightarrow a + d = b + c \Rightarrow c + b = d + a$
   $\Rightarrow (c, d) \, R \, (a, b)$
   Therefore, R is symmetric.
3. $(a, b) \, R \, (c, d)$ and $(c, d) \, R \, (e, f) \Rightarrow a + d = b + c$ and $c + f = d + e$
   $\Rightarrow a + d + c + f = b + c + d + e$
   $\Rightarrow a + f = b + e \Rightarrow (a, b) \, R \, (e, f)$

Therefore, R is transitive.

Thus, R is an equivalence relation on $N \times N$.

Illustration 1.6 If R is the relation 'is less than' from $A = \{1, 2, 3, 4, 5\}$ to $B = \{1, 4, 5\}$, write down the Cartesian product corresponding to R. Also find R^{-1} (aRb is a relation then $bR'a$ is relation inverse to R, i.e. $R' = R^{-1}$).

Solution: Clearly,

$$R = \{(a, b) \in A \times B : a < b\}$$

Therefore, $R = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$.

So, $R^{-1} = \{(4, 1), (5, 1), (4, 2), (5, 2), (4, 3), (5, 3), (5, 4)\}$.

Illustration 1.7 Let $A = \{3, 5\}$, $B = \{7, 11\}$ and $R = \{(a, b) : a \in A, b \in B, a - b \text{ is even}\}$. Show that R is a universal relation from A to B.

Solution: Given $A = \{3, 5\}$, $B = \{7, 11\}$.

Now,

$$R = \{(a, b) : a \in A, b \in B \text{ and } a - b \text{ is even}\} = \{(3, 7), (3, 11), (5, 7), (5, 11)\}$$

Also

$$A \times B = \{(3, 7), (3, 11), (5, 7), (5, 11)\}$$

Clearly, $R = A \times B$. Hence, R is a universal relation from A to B.

Key Points:

- If R is a relation from A to B and $(a, b) \notin R$, then we also write $a \not R b$ (read as a is not related to b).
- In an identity relation on A, every element of A should be related to itself only.
- aRb shows that a is the element of domain set and b is the element of range set.

1.3 Number Theory

1.3.1 Natural Numbers

The numbers 1, 2, 3, 4, … are called 'natural numbers' whose set is denoted by N. Thus,

$$N = \{1, 2, 3, 4, 5, \ldots\}$$

1.3.2 Integers

The numbers …, –3, –2, –1, 0, 1, 2, 3, … are called integers and the respective set is denoted by I or Z. Thus,

$$I \text{ (or } Z) = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

Remarks:

(i) Integers 1, 2, 3, … are called positive integers or natural numbers and they are denoted by I^+ or N.

(ii) Integers …, –3, –2, –1 are called negative integers which are denoted by I^-.

(iii) Integers 0, 1, 2, 3, … are called whole numbers or non-negative integers.

(iv) Integers …, –3, –2, –1, 0 are called non-positive integers.

1.3.3 Rational Numbers

The numbers which can be expressed in the form p/q, where p and q are integers, highest common factor (HCF) of p and q is 1 and $q \neq 0$, are called the 'rational numbers' and their set is denoted by Q. Thus,

$$Q = \left\{\frac{p}{q} : p,\ q \in I \text{ and } q \neq 0 \text{ and HCF of } p, q \text{ is } 1\right\}$$

It may be noted that every integer is a rational number since it can be written as $p/1$. It may also be noted that all recurring decimals are rational numbers, for example, $p = 0.\dot{3} = 0.33333\ldots$ Then,

$$10p - p = 3 \Rightarrow p = \frac{1}{3}$$

which is a rational number.

1.3.4 Irrational Numbers

There are numbers, which cannot be expressed in p/q form. These numbers are called irrational numbers and their set is denoted by Q^c (i.e. complementary set of Q). For example,

$$\sqrt{2},\ 1+\sqrt{3},\ \pi,\ \sqrt{3},\ e,\ \sqrt{5}, \ldots$$

Irrational numbers cannot be expressed as terminating decimals or recurring decimals.

Illustration 1.8 Prove that $\log_4 18$ is an irrational number.

Solution: Since we know that

$$\log_4 18 = \frac{1}{2} + \log_2 3$$

Let us assume the contrary that the number $\log_2 3$ is a rational number. Then

$$\log_2 3 = \frac{p}{q} \text{ (since } \log_2 3 > 0)$$

where both numbers p and q may be regarded as natural number.

$$\Rightarrow 2^p = 3^q$$

However, this is not possible for any natural number p and q. Hence, $\log_4 18$ is an irrational number.

1.3.5 Real Numbers

Real numbers are numbers that can be expressed as decimals, such as

$$\frac{4}{5} = 0.8000\ldots$$

$$\frac{1}{3} = 0.3333\ldots$$

$$\sqrt{2} = 1.4142\ldots$$

A real number can be represented geometrically as a point on number line called 'real line' (Fig. 1.3).

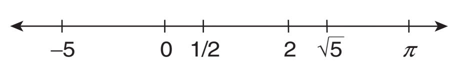


Figure 1.3

A set of real numbers consists of all rational and irrational numbers.

1.3.6 Number Chart

Figure 1.4 depicts the number chart.

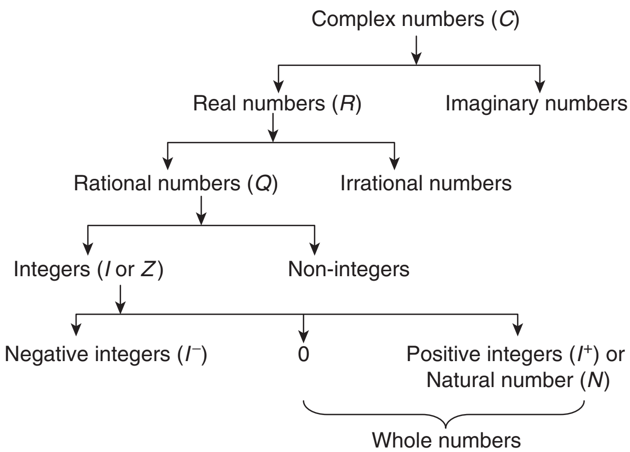


Figure 1.4

1.4 Intervals

A subset of real line is called an interval if it contains all the real numbers lying between every pair of its elements. Let $a, b \in R$. Then the set

(i) $\{x: x \in R, a \le x \le b\}$ is called a closed interval and is denoted by $[a, b]$.

(ii) $\{x: x \in R, a < x < b\}$ is called an open interval and is denoted by (a, b).

(iii) $\{x: x \in R, a \le x < b\}$ is called semi-closed or semi-open interval and is denoted by $[a, b)$. It is also called left-closed and right-open interval.

(iv) $\{x: x \in R, a < x \le b\}$ is called left-open and right-closed interval and is denoted by $(a, b]$.

Following are some of the examples:

1. The set of all real numbers x such that $4 \le x \le 7$ is the closed interval $[4, 7]$.
2. The set of all real numbers x such that $x < 4$ is the open interval $(-\infty, 4)$.

1.5 Basic Inequalities

The following are some important points to remember:

1. $a \le b \Rightarrow$ either $a < b$ or $a = b$.
2. $a < b$ and $b < c \Rightarrow a < c$.
3. $a < b \Rightarrow a + c < b + c \;\; \forall c \in R$.
4. $a < b$ and $c < d \Rightarrow a + c < b + d$ and $a - d < b - c$.
5. $a < b \Rightarrow ka < kb$ if $k > 0$ and $ka > kb$ if $k < 0$, that is, inequality sign reverses if both sides are multiplied by a negative number. In particular, $a < b \Rightarrow -a > -b$.
6. $0 < a < b \Rightarrow a^r < b^r$ if $r > 0$ and $a^r > b^r$ if $r < 0$.
7. $\left(f(x) + \dfrac{1}{f(x)}\right) \ge 2 \;\forall\; f(x) > 0$ and equality holds for $f(x) = 1$.
8. $\left(f(x) + \dfrac{1}{f(x)}\right) \le -2 \;\forall\; f(x) < 0$ and equality holds for $f(x) = -1$.

1.6 Logarithm

Following are some important points to remember:

1. The expression $\log_b x$ is valid for $x > 0$, $b > 0$ and $b \ne 1$
2. $b^{\log_b a} = a$
3. $\log_a b = \dfrac{\log_c b}{\log_c a}$
4. $\log_b a = \dfrac{1}{\log_a b}$ when both a and b are non-unity.
5. $\log_b a_1 \ge \log_b a_2 \Rightarrow \begin{cases} a_1 \ge a_2 > 0, \text{ if } b > 1 \\ 0 < a_1 \le a_2, \text{ if } 0 < b < 1 \end{cases}$

Illustration 1.9 Find values of x so that $\log_{|x|}|x-1| \ge 0$.

Solution: It is clear that

$$|x| > 0 \text{ and } |x| \ne 1 \Rightarrow x \ne 0, -1, 1$$

Also,

$$|x-1| > 0 \Rightarrow x \ne 1$$

Two case can be handled here:

Case 1: We have

$$0 < |x| < 1 \Rightarrow x \in (-1, 0) \cup (0, 1) \qquad (1)$$

Then,

$$\log_{|x|}|x-1| \ge 0 \Rightarrow \log_{|x|}|x-1| \ge \log_{|x|} 1$$
$$\Rightarrow 0 < |x-1| \le 1$$
$$\Rightarrow -1 \le x-1 \le 1 \text{ and } x \ne 1$$
$$\Rightarrow 0 \le x \le 2 \text{ and } x \ne 1$$
$$\Rightarrow x \in [0, 1) \cup (1, 2] \qquad (2)$$

From Eqs. (1) and (2), we have $x \in (0, 1)$.

Case 2: We have

$$|x| > 1 \Rightarrow x < -1 \text{ or } x > 1$$
$$x \in (-\infty, -1) \cup (1, \infty) \qquad (3)$$

Also,

$$\log_{|x|}|x-1| \ge 0$$
$$\Rightarrow |x-1| \ge 1 \Rightarrow x-1 \ge 1 \text{ or } x-1 \le -1$$
$$\Rightarrow x \ge 2 \text{ or } x \le 0$$

That is,

$$x \in (-\infty, 0] \cup [2, \infty) \qquad (4)$$

From Eqs. (3) and (4), we find that

$$x \in (-\infty, -1) \cup [2, \infty)$$

Hence,

$$x \in (0, 1) \cup (-\infty, -1) \cup [2, \infty)$$

or

$$x \in (-\infty, -1) \cup (0, 1) \cup [2, \infty)$$

Remarks: Often, one forgets to test for positive values of argument for which only log has some meaning. We need to be careful at this point.

Your Turn 4

1. Prove that $a^{\log_b c} = c^{\log_b a}$, where $a, b, c \in R^+$ and $b \neq 1$.
2. Solve for x, $\log_{1/\sqrt{2}}(x-1) > 2$. **Ans.** $x \in (1, 3/2)$

1.7 Wavy Curve Method

The method of intervals (or wavy curve) is used for solving inequalities of the form

$$f(x) = \frac{(x-a_1)^{n_1}(x-a_2)^{n_2}\ldots(x-a_k)^{n_k}}{(x-b_1)^{m_1}(x-b_2)^{m_2}\ldots(x-b_p)^{m_p}} > 0 \ (<0, \leq 0 \text{ or } \geq 0)$$

where $n_1, n_2, \ldots, n_k, m_1, m_2, \ldots, m_p$ are natural numbers and the numbers $a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_p$ are any real numbers such that $a_i \neq b_j$, for any $i = 1, 2, 3, \ldots, k$ and $j = 1, 2, 3, \ldots, p$. The following statements comprise these:

(i) All zeros of the function $f(x)$ contained on the left-hand side of the inequality should be marked on the number line with darkened black circles.

(ii) All points of discontinuities of the function $f(x)$ contained on the left-hand side of the inequality should be marked on the number line with empty white circles.

(iii) Check the value of $f(x)$ for any real number greater than the right most marked number on the number line.

(iv) From the right to left, starting above the number line [in the case of when the value of $f(x)$ is positive (in step iii), otherwise from below the number line], a wavy curve should be drawn which passes through all the marked points so that when passes through a simple point, the curve intersects the number line, and when passing through a double point, the curve remains located on one side of the number line.

(v) The appropriate intervals are chosen in accordance with the sign of inequality [the function $f(x)$ is positive wherever the curve is above the number line and it is negative if the curve is found below the number line]. Their union represents the solution of the inequality.

Remarks:

1. Points of discontinuity can never be included in the answer.
2. If you are asked to find the intervals where $f(x)$ is non-negative or non-positive, then make the intervals closed corresponding to the roots of the numerator and let it remain open corresponding to the roots of denominator.
3. The point where denominator is zero or function approaches infinity will never be included in the answer.

Illustration 1.10 Let us consider,

$$f(x) = \frac{(x-1)^3(x+2)^4(x-3)^5(x+6)}{x^2(x-7)^3}$$

Solve the following inequalities: (a) $f(x) > 0$, (b) $f(x) \geq 0$, (c) $f(x) < 0$ and (d) $f(x) \leq 0$.

Solution: On the number line, we mark zeros of the function, 1, –2, 3 and –6 (with black circles) and the points of discontinuity 0 and 7 (with white circles), isolate the double points: –2, and 0, and draw the curve of signs as shown in Fig. 1.5.

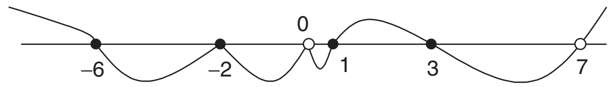


Figure 1.5

From the graph, we get the following:

(a) $x \in (-\infty, -6) \cup (1, 3) \cup (7, \infty)$

(b) $x \in (-\infty, -6] \cup \{-2\} \cup [1, 3] \cup (7, \infty)$

(c) $x \in (-6, -2) \cup (-2, 0) \cup (0, 1) \cup (3, 7)$

(d) $x \in [-6, 0) \cup (0, 1] \cup [3, 7)$

Illustration 1.11 Let us consider,

$$f(x) = \frac{\left(\sin x - \frac{1}{2}\right)(\ln x - 1)^2(x-2)(\tan x - \sqrt{3})}{(e^x - e^2)(x-3)^2}$$

Solve the following inequalities for $x \in (0, 2\pi)$: (a) $f(x) > 0$, (b) $f(x) \geq 0$, (c) $f(x) < 0$ and (d) $f(x) \leq 0$.

Solution: See Fig. 1.6. Clearly, $x \neq 2, 3, \frac{\pi}{2}, \frac{3\pi}{2}$ and $f(x) = 0$ for $x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{6}, e, \frac{4\pi}{3}$.

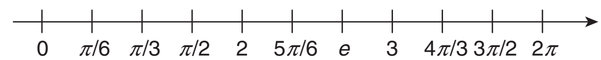


Figure 1.6

Now, sign of $f(x)$ does not change around $x = 2, e, 3$. Then, for $f(x) > 0$.

$$\left(\sin x - \frac{1}{2}\right)(\tan x - \sqrt{3}) > 0$$

$$\Rightarrow x \in \left(0, \frac{\pi}{6}\right) \cup \left(\frac{\pi}{3}, \frac{\pi}{2}\right) \cup \left(\frac{5\pi}{6}, \frac{4\pi}{3}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right) - \{e, 3\}$$

Hence,

(a) $x \in \left(0, \frac{\pi}{6}\right) \cup \left(\frac{\pi}{3}, \frac{\pi}{2}\right) \cup \left(\frac{5\pi}{6}, \frac{4\pi}{3}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right) - \{e, 3\}$

(b) $x \in \left(0, \frac{\pi}{6}\right] \cup \left[\frac{\pi}{3}, \frac{\pi}{2}\right) \cup \left[\frac{5\pi}{6}, \frac{4\pi}{3}\right] \cup \left(\frac{3\pi}{2}, 2\pi\right) - \{3\}$

(c) $x \in \left(\frac{\pi}{6}, \frac{\pi}{3}\right) \cup \left(\frac{\pi}{2}, \frac{5\pi}{6}\right) \cup \left(\frac{4\pi}{3}, \frac{3\pi}{2}\right) - \{2\}$

(d) $x \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right] \cup \left(\frac{\pi}{2}, \frac{5\pi}{6}\right] \cup \left[\frac{4\pi}{3}, \frac{3\pi}{2}\right) \cup \{e\} - \{2\}$

Your Turn 5

1. Solve the inequality, $2x^3 - 5x^2 + 2x \leq 0$.
   Ans. $x \in (-\infty, 0] \cup [1/2, 2]$
2. Solve the inequality, $\frac{x^2 - 3x - 18}{13x - x^2 - 42} \geq 0$.
   Ans. $x \in [-3, 6) \cup (6, 7)$
3. Solve the inequality, $\frac{(x-3)(x+2)}{x^2 - 1} < 1$.
   Ans. $x \in (-5, -1) \cup (1, +\infty)$
4. Solve the inequality, $\frac{3x+4}{x^2 - 3x + 5} < 0$.
   Ans. $x \in (-\infty, -4/3)$
5. Solve the inequality, $\frac{(x-1)(x-2)}{(2x-5)(x+4)} < 0$.
   Ans. $x \in (-4, 1) \cup (2, 5/2)$

1.8 Quadratic Expression

Consider a quadratic expression $f(x) = ax^2 + bx + c$, $a \neq 0$ and a, b, $c \in R$. Here, $f(x) = ax^2 + bx + c$ represents the equation of a parabola whose axis is parallel to y-axis. It is open upward (or concave) if $a > 0$ and open downward (or convex) if $a < 0$. It intersects with x-axis if $b^2 - 4ac > 0$ touches the x-axis if $b^2 - 4ac = 0$ and never intersects with x-axis if $b^2 - 4ac < 0$.

1.8.1 Concave (Open Upward) Parabola (i.e. when *a* > 0)

1. If $b^2 - 4ac < 0$, the parabola lies above x-axis (Fig. 1.7), that is, $f(x) > 0\ \forall\ x \in R$.

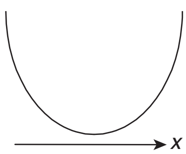


Figure 1.7

2. If $b^2 - 4ac = 0$, the parabola touches the x-axis at $x_1 = (-b/2a)$ and lies above the x-axis (Fig. 1.8) for the remaining values of x, that is, $f(x) \geq 0\ \forall\ x \in R$.

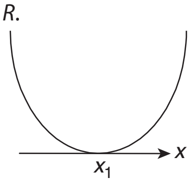


Figure 1.8

3. If $b^2 - 4ac > 0$, the parabola is cut by x-axis at two real points x_1 and x_2 (Fig. 1.9). If $x_1 < x_2$ (say), we can write
$$f(x) = ax^2 + bx + c = a(x - x_1)(x - x_2)$$
Here, $f(x) > 0\ \forall\ x \in (-\infty, x_1) \cup (x_2, \infty)$ and $f(x) < 0\ \forall\ x \in (x_1, x_2)$ and $f(x_1) = f(x_2) = 0$.

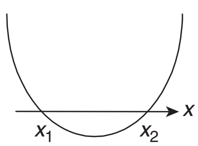


Figure 1.9

1.8.2 Convex (Open Downward) Parabola (i.e. when *a* < 0)

1. If $b^2 - 4ac < 0$, the parabola lies below the x-axis (Fig. 1.10), that is, $f(x) < 0\ \forall\ x \in R$.

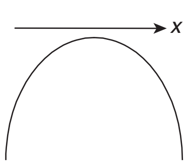


Figure 1.10

2. If $b^2 - 4ac = 0$, the parabola touches the x-axis at $x_1 = (-b/2a)$ and lies below the x-axis (Fig. 1.11) for the remaining values of x, that is, $f(x) \leq 0\ \forall\ x \in R$.

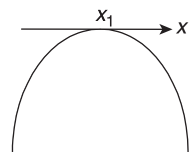


Figure 1.11

3. If $b^2 - 4ac > 0$, the parabola is cut by x-axis at two real points x_1 and x_2 ($x_1 < x_2$, say) (Fig. 1.12) and we can write
$$f(x) = ax^2 + bx + c = a(x - x_1)(x - x_2)$$
Here, $f(x) < 0\ \forall\ x \in (-\infty, x_1) \cup (x_2, \infty)$, $f(x) > 0\ \forall\ x \in (x_1, x_2)$ and $f(x_1) = f(x_2) = 0$.

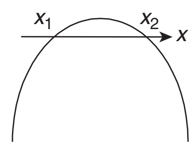


Figure 1.12

1.9 Absolute Value

Let $x \in R$, then the magnitude of x is called its absolute value and it is, in general, denoted by $|x|$. Thus, $|x|$ can be defined as
$$|x| = \begin{cases} -x, & x \leq 0 \\ x, & x > 0 \end{cases}$$
Note that $x = 0$ can be included either with positive values of x or with negative values of x. As we know, all real numbers can be plotted on the real number line, $|x|$, which, in fact, represents the distance of number x from the origin, measured along the number line. Thus, $|x| \geq 0$. Secondly, any point x lying on the real number line has its coordinates $(x, 0)$. Thus, its distance from the origin is $\sqrt{x^2}$. Hence, $|x| = \sqrt{x^2}$. Thus, we can define $|x|$ as $|x| = \sqrt{x^2}$, for example, if $x = -2.5$, then $|x| = 2.5$ if $x = 3.8$, then $|x| = 3.8$. There is another way to define $|x|$:
$$|x| = \max\{x, -x\}$$

1.9.1 Basic Properties of $|x|$

1. $||x|| = |x|$
2. Geometrical meaning of $|x - y|$: It is the distance between x and y.
3. $|x| > a \Rightarrow x > a$ or $x < -a$ if $a \in R^+$ and $x \in R^+$ if $a \in R^-$.
4. $|x| < a \Rightarrow -a < x < a$ if $a \in R^+$ and no solution if $a \in R^- \cup \{0\}$.
5. $|xy| = |x||y|$
6. $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|},\ y \neq 0$
7. $|x + y| \leq |x| + |y|$: This is an important and interesting basic property. Here, the equality sign holds if x and y either both are non-negative or non-positive (i.e. both $x, y \geq 0$, or both $x, y \leq 0$). In other words, in the case of $xy \geq 0$. This property is self-explanatory. Here, $|x| + |y|$ represents the sum of distances of

the numbers x and y from the origin and $|x+y|$ represents the distance of the number $x + y$ from the origin (or the distance between x and $-y$ measured along the number line).

8. $|x-y| \geq |x|-|y|$: This is a very useful and interesting property. Here, the equality sign holds if both x and y are non-negative or non-positive (i.e. both $x, y \geq 0$ or both $x, y \leq 0$). In other words, in the case of $xy \geq 0$. This property is self-explanatory. Here, $|x|-|y|$ represents the difference of the distances of the numbers x and y from the origin and $|x-y|$ represents the distance between x and y measured along the number line. The last two properties can be put in one compact form, that is, $|x|-|y| \leq |x \pm y| \leq |x|+|y|$.

Illustration 1.12 Solve the following inequalities for real values of x: (a) $|x-1| < 2$, (b) $|x-3| > 5$, (c) $0 < |x-1| \leq 3$, (d) $|x-1| + |2x-3| = |3x-4|$ and (e) $|(x-3)/(x^2-4)| \leq 1$.

Solution: Since both sides of the given inequality are non-negative for all x's, when squaring them, we get the inequality $(x-1)^2 < 4$ which is equivalent to the given inequality. Then, we have

$$x^2 - 2x - 3 < 0$$
$$\Rightarrow (x+1)(x-3) < 0$$
$$\Rightarrow x \in (-1, 3)$$

Alternate Method 1: Since we have

$$|x-1| = \begin{cases} x-1, & \text{if } x-1 \geq 0 \\ -(x-1), & \text{if } x-1 < 0 \end{cases}$$

$$\Rightarrow x-1 < 2, \text{ if } x-1 \geq 0$$

or $$-(x-1) < 2, \text{ if } x-1 < 0$$

$$\Rightarrow x \in [1, 3) \text{ or } (-1, 1)$$
$$\Rightarrow x \in (-1, 3)$$

Alternate Method 2:

(a) We may regard $|x - 1|$ as the distance on the number line between the points x and 1. Hence, we have to indicate on the number line all such points x which are at a distance less than 2 from the point having the coordinate 1 (Fig. 1.13). The desired solution is $(-1, 3)$.

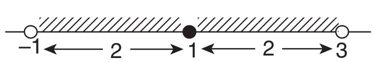


Figure 1.13

(b) We know that

$$|x-3| > 5$$
$$\Rightarrow x-3 < -5 \text{ or } x-3 > 5$$
$$\Rightarrow x < -2 \text{ or } x > 8$$
$$\Rightarrow x \in (-\infty, -2) \cup (8, \infty)$$

(c) We have

$$0 < |x-1| \leq 3$$

Here,

$$|x-1| > 0 \Rightarrow x \neq 1$$

Also,

$$|x-1| \leq 3$$
$$\Rightarrow -3 \leq x-1 \leq 3$$
$$\Rightarrow -2 \leq x \leq 4, x \neq 1$$
$$\Rightarrow x \in [-2, 1) \cup (1, 4]$$

(d) Since $3x - 4 = x - 1 + 2x - 3$, we get

$$|3x-4| = |x-1| + |2x-3|$$
$$\Rightarrow (x-1)(2x-3) \geq 0$$
$$\Rightarrow x \in (-\infty, 1] \cup [3/2, +\infty)$$

(e) We have

$$\left|\frac{x-3}{x^2-4}\right| \leq 1$$

It is clear that

$$x^2 - 4 \neq 0 \Rightarrow x \neq 2, -2$$

Now

$$\left|\frac{x-3}{x^2-4}\right| \leq 1 \Rightarrow -1 \leq \frac{x-3}{x^2-4} \leq 1$$

Consider

$$\frac{x-3}{x^2-4} \geq -1$$

$$\Rightarrow \frac{(x+2)(x-2)\{x-[(-1-\sqrt{29})/2]\}\{x-[(-1+\sqrt{29})/2]\}}{(x+2)^2(x-2)^2} \geq 0$$

Now, see Fig. 1.14.

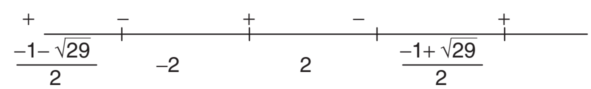


Figure 1.14

$$x \in \left(-\infty, \frac{-1-\sqrt{29}}{2}\right] \cup (-2, 2) \cup \left[\frac{-1+\sqrt{29}}{2}, \infty\right) \quad (1)$$

Now, consider

$$\frac{x-3}{x^2-4} \leq 1$$

$$\Rightarrow \frac{x-3-x^2+4}{(x^2-4)} \leq 0$$

$$\Rightarrow \frac{-x^2+x+1}{(x^2-4)} \leq 0$$

$$\Rightarrow \frac{x^2-x-1}{(x^2-4)} \geq 0$$

$$\Rightarrow \frac{(x+2)\ (x-2)\left(x-\frac{1+\sqrt{5}}{2}\right)\left(x-\frac{1-\sqrt{5}}{2}\right)}{(x+2)^2(x-2)^2} \geq 0$$

See Fig. 1.15.

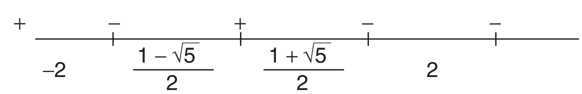


Figure 1.15

$$\Rightarrow x \in (-\infty, -2) \cup \left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right] \cup (2, \infty) \quad (2)$$

From Eqs. (1) and (2), we have

$$x \in \left(-\infty, \frac{-1-\sqrt{29}}{2}\right] \cup \left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right] \cup \left[\frac{-1+\sqrt{29}}{2}, +\infty\right)$$

Your Turn 6

1. Solve the following inequality: $|x-1|+2|x+1|+|x-2|\le 8$.

Ans. $x \in \left[-\frac{7}{4}, \frac{9}{4}\right]$

2. Solve the following inequality: $4|x^2-1|+|x^2-4|\ge 6$.

Ans. $x \in (-\infty, -\sqrt{2}] \cup \left[-\sqrt{\frac{2}{5}}, \sqrt{\frac{2}{5}}\right] \cup [\sqrt{2}, \infty)$

1.10 Greatest Integer

Let $x \in R$ be any real number. We can always infer that x to be lying between two consecutive integers, namely, I and $I + 1$, that is, $I \le x < (I + 1)$ (left hand equality would hold if x is an integer, otherwise $I < x < I + 1$). That means, we can always find an integer, namely, I so that the given real number x is always greater than or equals to I. This unique I is called the greatest integral value of x and is symbolically denoted by $[x]$, that is, $[x]$ stands for the greatest integer that is less than or equal to x. For example,

$$x = 2.69 \Rightarrow 2 < x < 3$$
$$\Rightarrow [x] = 2, x = -3.63 \Rightarrow -4 < x < -3$$
$$\Rightarrow [x] = -4, x = -3.99 \Rightarrow -4 < x < -3$$
$$\Rightarrow [x] = -4$$

In other words, if we list all the integers less than or equals to x, then the integer greatest among them is called the greatest integer of x. Greatest integer of x is also called integral part of x. It is obvious that if x is an integer, then $[x] = x$.

Points to Remember:

1. $I \le x < I + 1 \Leftrightarrow [x] = I$, for example, $[x] = 2 \Leftrightarrow 2 \le x < 3$.
2. $[x] > I \Rightarrow [x] \ge I + 1 \Rightarrow x \ge I + 1$.
3. $[x] < I \Rightarrow [x] \le I - 1 \Rightarrow x < I$.
4. $I_1 \le [x] \le I_2 \Rightarrow I_1 \le x < I_2 + 1 \Rightarrow x \in [I_1, I_2 + 1)$, for example, $-1 \le [x] \le 4 \Rightarrow -1 \le x < 5 \Rightarrow x \in [-1, 5)$.
5. $x - 1 < [x] \le x$

Explanation: We know that

$$I \le x < I + 1 \quad (1)$$
$$\Rightarrow [x] = I \Rightarrow [x] \le x$$

From Eq. (1), we get

$$I - 1 \le x - 1 < I \Rightarrow [x] > x - 1$$

1.11 Fractional Part

We have seen that $x \ge [x]$. The difference between the number x and its integral value $[x]$ is called the fractional part of x and is symbolically denoted by $\{x\}$. Thus,

$$\{x\} = x - [x]$$

For example, if $x = 4.92$, then $[x] = 4$ and $\{x\} = 0.92$. We know that

$$x - 1 < [x] \le x$$
$$\Rightarrow -x \le -[x] < 1 - x$$
$$\Rightarrow 0 \le x - [x] < 1$$
$$\Rightarrow 0 \le \{x\} < 1$$

We find that the fractional part of any number is always non-negative and less than one. If x is an integer, then

$$x = [x] \Rightarrow \{x\} = 0 \Rightarrow \{[x]\} = 0$$

1.12 Basic Properties of Greatest Integer and Fractional Part

1. $[[x]] = [x]$, $[\{x\}] = 0$, $\{[x]\} = 0$.
2. $x - 1 < [x] \le x$, $0 \le \{x\} < 1$.
3. $[n + x] = n + [x]$ where $n \in I$

   Explanation: Let $x = I_x + f_x$, where $I_x = [x]$ and $f_x = \{x\}$. Then

$$\Rightarrow x + n = I_x + n + f_x$$
$$\Rightarrow [x + n] = I_x + n = n + [x]$$
$$\Rightarrow [x + n] = n + [x]$$

4. Let $[x] + [-x] = \begin{cases} 0, & \text{if } x \in \text{integer} \\ -1, & \text{if } x \notin \text{integer} \end{cases}$

   Explanation: There can be two cases: Either x is an integer or x is not an integer.

   Case 1: If $x \in I$ (integer), then

$$[x] = I \text{ and } [-x] = -I$$
$$\Rightarrow [x] + [-x] = 0$$

   Case 2: If $I < x < I + 1$, then

$$-I - 1 < -x < -I$$
$$\Rightarrow [x] = I, [-x] = -I - 1 \Rightarrow [x] + [-x] = -1$$

5. We have

$$\{x\} + \{-x\} = \begin{cases} 0, & x \in \text{Integer} \\ 1, & x \notin \text{Integer} \end{cases}$$

   This property is the direct consequence of the above property. As we know

$$x = [x] + \{x\}, \forall x \in R \quad (1)$$
$$\Rightarrow -x = [-x] + \{-x\} \quad (2)$$

   From Eqs. (1) and (2), we get

$$[x] + [-x] + \{x\} + \{-x\} = 0$$

6. $[x + y] = \begin{cases} [x] + [y], & \text{if } \{x\} + \{y\} < 1 \\ [x] + [y] + 1, & \text{if } \{x\} + \{y\} \ge 1 \end{cases}$.

   Hence, $[x + y] \le [x] + [y] + 1$. Note that $[x] + [y]$ and $[x] + [y] + 1$ represent two consecutive integers.

   Explanation: Let $x = I_x + f_x$, $y = I_y + f_y$, where I_x and I_y represent the integral part of x and y, respectively, and f_x and f_y represent the fractional part of x and y, respectively. It is obvious that

$$[x] = I_x \text{ and } [y] = I_y$$

   and

$$0 \le f_x < 1, 0 \le f_y < 1$$
$$\Rightarrow 0 \le f_x + f_y < 2$$
$$\Rightarrow [f_x + f_y] = 0 \text{ or } 1$$

   Now,

$$x + y = I_x + I_y + f_x + f_y$$
$$\Rightarrow [x + y] = I_x + I_y + [f_x + f_y] = \begin{cases} I_x + I_y & \text{if } 0 \le f_x + f_y < 1 \\ I_x + I_y + 1 & \text{if } 1 \le f_x + f_y < 2 \end{cases}$$
$$\Rightarrow [x + y] \le [x] + [y] + 1$$

7. $\left[\frac{[x]}{n}\right]=\left[\frac{x}{n}\right],\ n\in N,\ x\in R.$

Explanation: Let $x = I_x + f_x$, where $I_x = [x]$ and $f_x = \{x\}$. An integer I_x can be written as

$$I_x = n\lambda + r$$

where λ is the quotient when I_x is divided by n and r the corresponding remainder, that is,

$$0 \le r \le n-1$$
$$\Rightarrow x = I_x + f_x = n\lambda + r + f_x$$
$$\Rightarrow \frac{x}{n} = \lambda + \frac{r+f_x}{n}$$

Now,

$$0 \le r \le n-1 \text{ and } 0 \le f_x < 1$$
$$\Rightarrow 0 \le r + f_x < n$$
$$\Rightarrow 0 \le \frac{r+f_x}{n} < 1$$
$$\Rightarrow \left[\frac{r+f_x}{n}\right] = 0$$

Also,

$$\left[\frac{x}{n}\right] = \lambda + \left[\frac{r+f_x}{n}\right]$$
$$\Rightarrow \left[\frac{x}{n}\right] = \lambda$$

Further,

$$[x] = I_x = n\lambda + r$$
$$\Rightarrow \frac{[x]}{n} = \lambda + \left(\frac{r}{n}\right) \Rightarrow \left[\frac{[x]}{n}\right] = \lambda \quad \left(\because 0 \le \frac{r}{n} < 1\right)$$
$$\Rightarrow \left[\frac{[x]}{n}\right] = \left[\frac{x}{n}\right]$$

Illustration 1.13 Prove that $[x] = \left[\frac{x}{2}\right]+\left[\frac{x+1}{2}\right]$, where [.] denote greatest integer function and n be any positive integer, then show that $\left[\frac{n+1}{2}\right]+\left[\frac{n+2}{4}\right]+\left[\frac{n+4}{8}\right]+\left[\frac{n+8}{16}\right]+\cdots = n$.

Solution: Let $x/2$ is an integer, say, n. Then

$$\left[\frac{x}{2}\right] = n \text{ and } \left[\frac{x+1}{2}\right] = \left[\frac{x}{2}+\frac{1}{2}\right] = n$$

Therefore,

$$\left[\frac{x}{2}\right]+\left[\frac{x+1}{2}\right] = 2n = x = [x]$$

Let $x/2$ is non-integer, say $x/2 = n + f$ (or $x = 2n + 2f$). Then

$$\left[\frac{x}{2}\right] = n$$
$$\left[\frac{x+1}{2}\right] = \left[n+f+\frac{1}{2}\right] \quad (1)$$

Since $0 \le f < 1$, we get

$$\frac{1}{2} \le f + \frac{1}{2} < \frac{3}{2}$$

We have the following two cases:

Case 1: If we have

$$\frac{1}{2} \le f + \frac{1}{2} < 1$$

then

$$\left[\frac{x+1}{2}\right] = n \quad \text{[from Eq. (1)]}$$

That is, if

$$0 \le f < \frac{1}{2}$$

then

$$\left[\frac{x}{2}\right]+\left[\frac{x+1}{2}\right] = 2n = [x] \quad (\because 0 \le 2f < 1)$$

Case 2: If we have

$$1 \le f + \frac{1}{2} < \frac{3}{2}$$

then

$$\left[\frac{x+1}{2}\right] = n+1$$

That is, if

$$\frac{1}{2} \le f < 1$$

Then

$$\left[\frac{x}{2}\right]+\left[\frac{x+1}{2}\right] = 2n+1 = [x] \quad (\because 1 \le 2f < 2)$$

Illustration 1.14 Solve the following equations: (a) $|2x-1| = 3[x] + 2\{x\}$ and (b) $x^2 - 4x + [x] + 3 = 0$.

Solution:

(a) It is given that

$$|2x-1| = 3[x] + 2\{x\}$$

Let $2x - 1 \le 0$, that is, $x \le 1/2$. The given equation yields

$$1 - 2x = 3[x] + 2\{x\}$$
$$\Rightarrow 1 - 2[x] - 2\{x\} = 3[x] + 2\{x\}$$
$$\Rightarrow 1 - 5[x] = 4\{x\} \Rightarrow \{x\} = \frac{1-5[x]}{4}$$
$$\Rightarrow 0 \le \frac{1-5[x]}{4} < 1$$
$$\Rightarrow 0 \le 1 - 5[x] < 4$$
$$\Rightarrow -\frac{3}{5} < [x] \le \frac{1}{5}$$

Now, $[x] = 0$ as zero is the only integer lying between $-3/5$ and $1/5$.

$$\{x\} = \frac{1}{4} \Rightarrow x = \frac{1}{4}$$

which is less than 1/2. Hence, 1/4 is one solution. Now, let $2x - 1 > 0$, that is, $x > 1/2$.

$$2x - 1 = 3[x] + 2\{x\}$$
$$\Rightarrow [x] = -1 \Rightarrow -1 \le x < 0$$

which is not a solution as $x > 1/2$. That is, $x = 1/4$ is the only solution.

(b) It is given that

$$x^2 - 4x + [x] + 3 = 0$$
$$\Rightarrow x^2 - 4x + x - \{x\} + 3 = 0$$
$$\Rightarrow x^2 - 3x + 3 = \{x\}$$
$$\Rightarrow 0 \le x^2 - 3x + 3 < 1$$

Now,

$$x^2 - 3x + 3 = x^2 - 3x + \frac{9}{4} + 3 - \frac{9}{4} = \left(x - \frac{3}{2}\right)^2 + \frac{3}{4} > 0$$

$$\Rightarrow x^2 - 3x + 3 > 0 \Rightarrow x \in R$$

Let us consider

$$x^2 - 3x + 3 < 1$$
$$\Rightarrow x^2 - 3x + 2 < 0$$
$$\Rightarrow (x-1)(x-2) < 0$$
$$\Rightarrow 1 < x < 2 \Rightarrow [x] = 1$$

Now, from the original equation, we have

$$x^2 - 4x + 4 = 0$$
$$\Rightarrow (x-2)^2 = 0$$
$$\Rightarrow x = 2$$

which does not satisfy $1 < x < 2$. Thus, the given equation does not have any solution.

Your Turn 7

1. Solve the inequality, $x[x] - x^2 - 3[x] + 3x > 0$, where [.] denotes the greatest integer function.
   Ans. $x \in (-\infty, 3) - I$
2. Solve $[x]^3 - 2[x] + 1 = 0$. **Ans.** $x \in [1, 2)$
3. Solve the inequality, $[x]^2 - 3[x] + 2 \le 0$.
   Ans. $x \in [1, 3)$
4. If $y = 3[x] + 1 = 2[x-3] + 5$, find the value of $[x + y]$.
   Ans. –7

1.13 Functions, Domain, Co-Domain, Range

Functions are the major tools for describing the real word in mathematical terms. The temperature at which water boils depends on the elevation above sea level (the boiling point drops as you ascend). The interest paid on a cash investment depends on the length of time the investment is held. In each case, the value of one variable quantity, which we denote by y, depends on the value of another variable quantity, which we denote by x. Since the value of y is completely determined by the value of x, we say that y is a function of x. Here, y is called dependent variable and x is called the independent variable.

Let X and Y be two non-empty sets. A function f of X into Y (or from X to Y), which is written as

$$f: X \to Y$$

is a rule or a correspondence which connect every member, say, x of X to exactly one member, say, y of Y. For example, when we study circles, if we take area as y and the radius as x, we have $y = \pi x^2$, we say that y is a function of x. The equation $y = \pi x^2$ is a rule (correspondence) that tells how to calculate a unique (single) output value of y for each possible input value of the radius x. Here, we say y is a function of x and represent it as $y = f(x)$, $y = g(x)$ or $y = h(x)$ normally.

The set of all possible input values of x for which $f(x)$ exists or is defined is called the 'domain' of the function. The set of all output values of the y is the 'range' of the function. Since in the case of radii, it cannot be negative, the domain is $[0, \infty)$ and so the range is also $[0, \infty)$.

X is the 'domain' of the function. $f(X)$ is the 'range' of the function and Y is 'co-domain' of the function. The range is always a subset of codomain.

$f(x)$ is also called the image of x under f or the f-image of x and x is called 'preimage' of y or $f(x)$.

Key Points:

1. $f: X \to Y$ is a function if each element x in X has a unique image $f(x)$ in Y.
2. $f: X \to Y$ is not a function if there is an element in X which does not have an f-image in Y.
3. $f: X \to Y$ is not a function if there is an element in X which has more than one f-image in Y.
4. Graphically, if a line parallel to y-axis (vertical line) cuts the graph of $y = f(x)$ at only one point, then $y = f(x)$ is called function in x.

Examples are listed as follows:

1. Let $X = R$, $Y = R$ and $y = f(x) = x^2$. Then, $f: X \to Y$ is a function since each element in X has exactly one f-image in Y. The range of $f = \{f(x): x \in X\} = \{x^2: x \in R\} = [0, \infty)$.
2. Let $X = R$, $Y = R$ and $y^2 = x$. Here, $f(x) = \pm\sqrt{x}$, that is, f is not a function of X into Y since $x > 0$ has two f-images in Y, and further, each $x < 0$ has no f-image in Y.

Illustration 1.15 If $f(x) = \ln\left(\frac{1-x}{1+x}\right)$, prove that $f\left(\frac{2x}{1+x^2}\right) = 2f(x)$.

Solution: We have

$$f(x) = \ln\left(\frac{1-x}{1+x}\right)$$
$$\Rightarrow f\left(\frac{2x}{1+x^2}\right) = \ln\left(\frac{1-[2x/(1+x^2)]}{1+[2x/(1+x^2)]}\right)$$
$$\Rightarrow f\left(\frac{2x}{1+x^2}\right) = \ln\left(\frac{1-x}{1+x}\right)^2 = 2\ln\left|\frac{1-x}{1+x}\right|$$
$$\Rightarrow f\left(\frac{2x}{1+x^2}\right) = 2\ln\left(\frac{1-x}{1+x}\right)$$
$$\Rightarrow f\left(\frac{2x}{1+x^2}\right) = 2f(x)$$

Illustration 1.16 If $f(x)$ satisfying the condition $f\left(x+\frac{1}{x}\right) = x^4 + \frac{1}{x^4}$. Then find the value of $f(5)$.

Solution: We have

$$f\left(x+\frac{1}{x}\right) = x^4 + \frac{1}{x^4}$$
$$\Rightarrow f\left(x+\frac{1}{x}\right) = \left(x^4 + \frac{1}{x^4} + 2\right) - 2 = \left(x^2 + \frac{1}{x^2}\right)^2 - 2$$
$$= \left(x^2 + \frac{1}{x^2+2-2}\right)^2 - 2$$
$$\Rightarrow f\left(x+\frac{1}{x}\right) = \left[\left(x+\frac{1}{x}\right)^2 - 2\right]^2 - 2$$

Let us consider

$$x + \frac{1}{x} = y$$
$$\Rightarrow f(y) = (y^2 - 2)^2 - 2$$
$$\Rightarrow f(5) = (25 - 2)^2 - 2$$
$$\Rightarrow f(5) = 527$$

Key Point:
For a function $f: X \to Y$, set X is called the domain of the function f. Set Y is called the codomain of the function f. Set of images of different elements of set X is called the range of the function f.

1.13.1 Some Important Functions

1. **Absolute Value Function:** $f: R \to R$ defined by

$$f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

which is called absolute value function (Fig. 1.16). Its domain is R and its range is $[0, \infty)$.

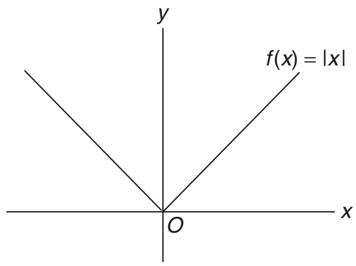


Figure 1.16

Properties of absolute value function:

(i) $|x + y| \leq |x| + |y|$ and equality holds if and only if $xy \geq 0$.

(ii) $|x - y| \geq ||x| - |y||$ and equality holds if and only if $xy \geq 0$.

(iii) $|xy| = |x||y|$

(iv) $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$

2. **Constant Function:** $f: R \to R$ defined by $f(x) = c$, $\forall\ x \in R$, where c is a constant, is called a constant function. Its domain is R and range is $\{c\}$. Graph of a constant function is a straight line parallel to x-axis (Fig. 1.17).

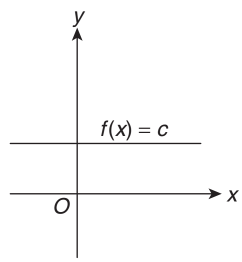


Figure 1.17

3. **Identity Function:** $f: R \to R$ defined by $f(x) = x$ is called the identity function (Fig. 1.18). Its domain is R and the range is also R.

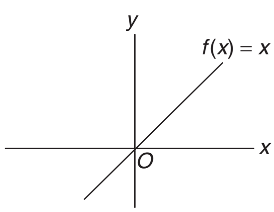


Figure 1.18

4. **Exponential Function:** Let $a \neq 1$ be a positive real number, then $f: R \to R$ defined by $f(x) = a^x$ is called exponential function. Its domain is R and range is $(0, \infty)$. The graph of $f(x) = a^x$, when $a > 1$ is shown in Fig. 1.19(a). The graph of $f(x) = a^x$, when $0 < a < 1$ is shown in Fig. 1.19(b).

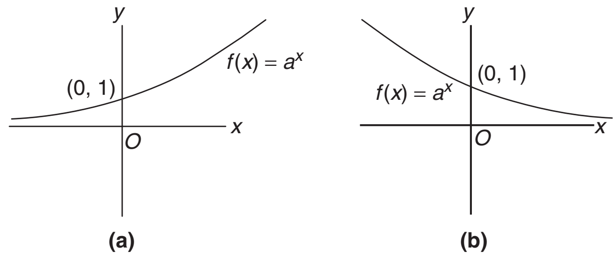


Figure 1.19

5. **Logarthmic Function:** Let $a \neq 1$ be a positive real number, then $f: (0, \infty) \to R$ defined by $f(x) = \log_a x$ is called logarthmic function. Its domain is $(0, \infty)$ and the range is R. The graph of $f(x) = \log_a x$, when $a > 1$, is shown in Fig. 1.20(a) and the graph of $f(x) = \log_a x$, when $0 < a < 1$, is shown in Fig. 1.20(b).

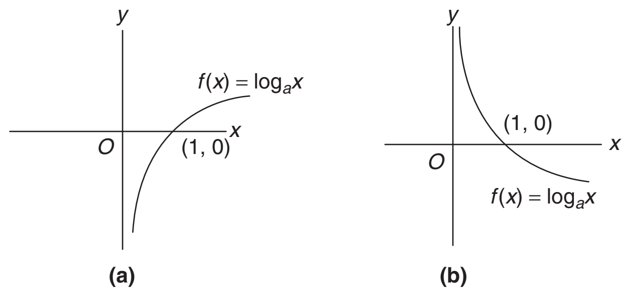


Figure 1.20

Properties of logarithmic function:

(i) $\log_a b = x \Leftrightarrow a^x = b$ (a is known as the base of the logarithm).

(ii) $\log_a b$ is real if $b > 0$, $a > 0$, $a \neq 1$.

(iii) $\log_a 1 = 0$.

(iv) $\log_a a = 1$.

(v) $\log_a (m \times n) = \log_a |m| + \log_a |n|$.

(vi) $\log_a \dfrac{m}{n} = \log_a |m| - \log_a |n|$.

(vii) $\log_a m^n = \begin{cases} n \log_a |m|, & \text{for } n \text{ even number} \\ n \log_a m, & \text{otherwise} \end{cases}$

(viii) $\log_{a^n} m = \begin{cases} \dfrac{1}{n} \log_{|a|} m, & \text{for } n \text{ even number} \\ \dfrac{1}{n} \log_a m, & \text{otherwise} \end{cases}$

(ix) $\log_a b = \dfrac{\log_x b}{\log_x a}$ for any positive $x \neq 1$.

(x) $\log_a b \times \log_b c = \log_a c$.

(xi) $\log_a b = \dfrac{1}{\log_b a}$.

(xii) $a^{\log_a x} = x$.

(xiii) $\log_a x = \log_a y \Leftrightarrow x = y$.

(xiv) $\log_a x > \log_a y \Rightarrow \begin{cases} x > y > 0 \text{ when } a > 1 \\ 0 < x < y \text{ when } 0 < a < 1 \end{cases}$

6. **The greatest integer function or Step function:** $f: R \to Z$ defined by $f(x) = [x]$, where $[x]$ denotes the greatest integer among all the integers less than or equal to x is called the greatest integer function, that is, $f(x) = n$, where $n \le x < n + 1$, $n \in I$ (set of integers) (Fig. 1.21). Its domain is R and the range is I. Following are some examples:

$$[2.5] = 2 \text{ since } 2 \le 2.5 < 3$$
$$[\pi^2] = 9 \text{ since } 9 \le \pi^2 < 10$$
$$[-4.5] = -5 \text{ since } -5 \le -4.5 < -4$$

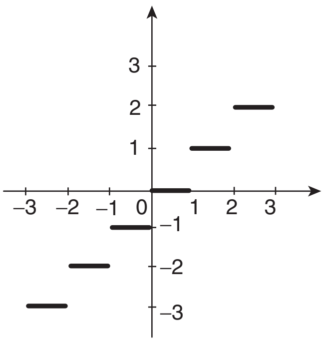


Figure 1.21

Properties of greatest integer function:

(i) $x - 1 < [x] \le x$.

(ii) $[-x] = \begin{cases} -[x] - 1 \text{ if } x \notin I \\ -[x] \quad \text{if } x \in I \end{cases}$

(iii) $[x + y] = [x] + [y]$ if $\{x\} + \{y\} < 1$. Here, $\{x\}$ denotes the fraction part of x.

(iv) $[x + y] = [x] + [y] + 1$ if $\{x\} + \{y\} \ge 1$. Here, $\{x\}$ denotes the fraction part of x.

7. **Fractional-part function:** $\{x\}$ denotes the fractional part of x which is equal to $x - [x]$. Following are some examples:

$$\{2.7\} = 0.7, \{3\} = 0, \{-3.2\} = 0.8$$

See Fig. 1.22. The domain is R and the range is $[0, 1)$.

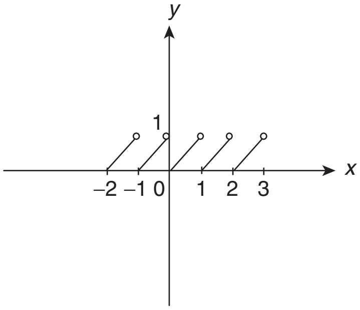


Figure 1.22

8. **Signum Function:** Signum function (Fig. 1.23) is given by

$$\text{sgn}(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -1 & \text{if} \quad x < 0 \end{cases}$$

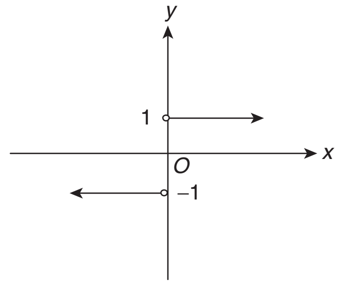


Figure 1.23

9. **Polynomial Function:** $f: R \to R$ defined by $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, where $a_0, a_1, a_2, \ldots, a_n \in R$, is called a polynomial function. If $a_n \ne 0$, the degree of $f(x)$ is n. Note that the range of $f(x)$ is R if and only if n is odd.

10. **Trigonometric Function:** The various trigonometric functions are listed in Table 1.1.

Table 1.1 Trigonometric functions

S. No.	Function	Domain	Range
1.	$y = \sin x$	R	$[-1, 1]$
2.	$y = \cos x$	R	$[-1, 1]$
3.	$y = \tan x$	$R - \{n\pi + \pi/2, n \in I\}$	R
4.	$y = \cot x$	$R - \{n\pi, n \in I\}$	R
5.	$y = \sec x$	$R - \{n\pi + \pi/2, n \in I\}$	$(-\infty, -1] \cup [1, \infty)$
6.	$y = \text{cosec} x$	$R - \{n\pi, n \in I\}$	$(-\infty, -1] \cup [1, \infty)$

11. **Inverse Circular Function:** The various inverse circular functions are listed in Table 1.2.

Table 1.2 Inverse circular functions

S. No.	Function	Domain	Range
1.	$y = \sin^{-1} x$	$-1 \le x \le 1$	$[-\pi/2, \pi/2]$
2.	$y = \cos^{-1} x$	$-1 \le x \le 1$	$[0, \pi]$
3.	$y = \tan^{-1} x$	$-\infty < x < \infty$	$(-\pi/2, \pi/2)$
4.	$y = \cot^{-1} x$	$-\infty < x < \infty$	$(0, \pi)$
5.	$y = \text{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[-\pi/2, 0) \cup (0, \pi/2]$
6.	$y = \sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$

1.13.2 Graphs of Trigonometric Functions

Graphs of some important trigonometric functions are shown in Fig. 1.24.

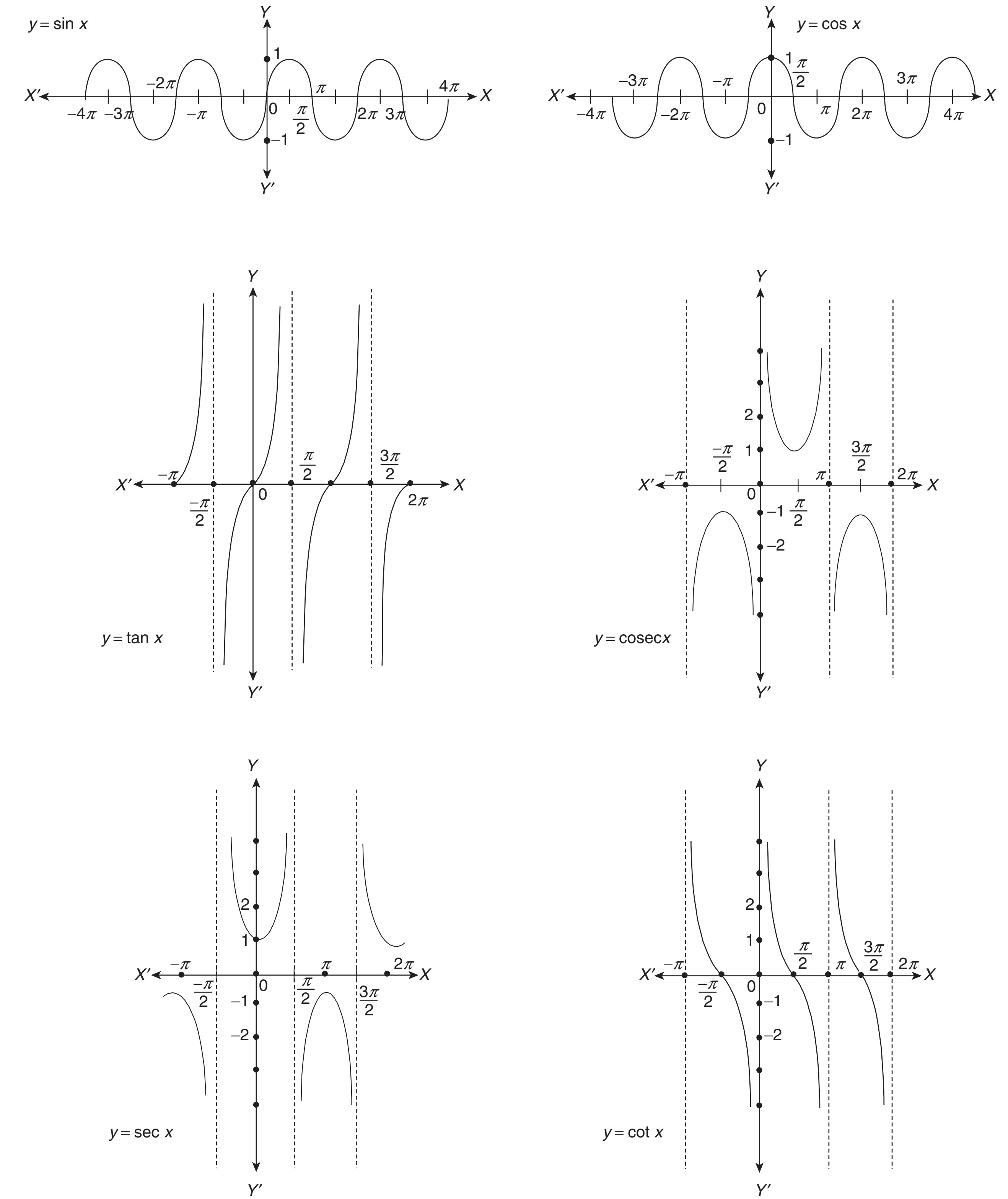


Figure 1.24

1.13.3 Graphs of Inverse Functions

Graphs of some important inverse functions are shown in Fig. 1.25.

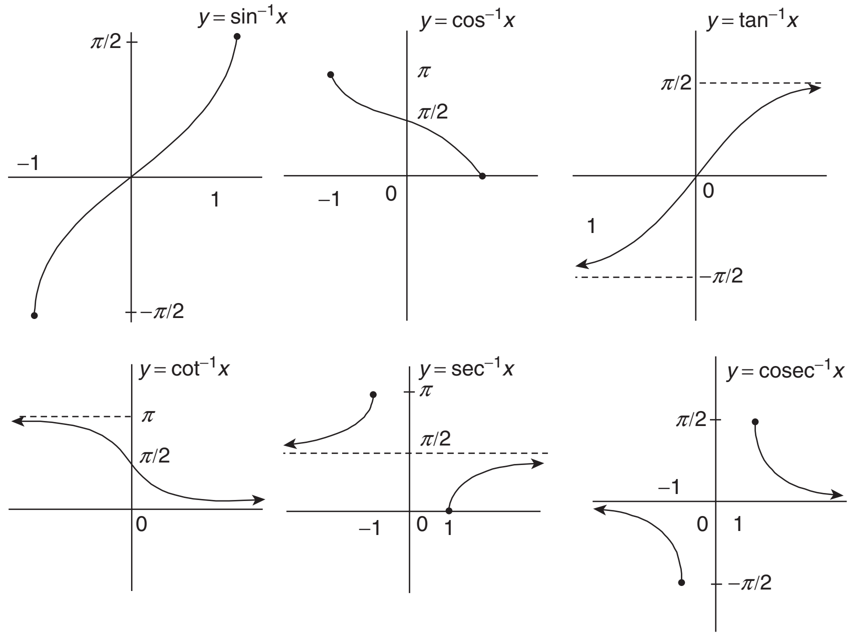


Figure 1.25

Illustration 1.17 Find the total number of positive real values of x such that x, $[x]$, $\{x\}$ are in HP where $[\cdot]$ denotes the greatest integer function and $\{\cdot\}$ denotes the fraction part.

Solution: We have

$$[x] = \frac{2\{x\}x}{\{x\} + x}$$

$\Rightarrow [x]\{x\} + x[x] = 2\{x\}x \Rightarrow [x]\{x\} + [x]([x] + \{x\}) = 2\{x\}([x] + \{x\})$

$$\Rightarrow [x]^2 = 2\{x\}^2 \Rightarrow \{x\}^2 = \frac{[x]^2}{2}$$

$$\Rightarrow 0 < \frac{[x]^2}{2} < 1 \Rightarrow 0 < [x]^2 < 2$$

$$\Rightarrow 0 < [x] < \sqrt{2} \Rightarrow [x] = 1$$

$$\Rightarrow \{x\} = \frac{1}{\sqrt{2}} \Rightarrow x = 1 + \frac{1}{\sqrt{2}} \Rightarrow x = \frac{\sqrt{2}+1}{\sqrt{2}}$$

which is the only possible value of x.

Illustration 1.18 Draw the graph of the following: (a) $f(x) = x|x|$, (b) $f(x) = |x - 1| + |x + 1|$, (c) $f(x) = 2x - \{x\}, x \in [-1, 2]$ and (d) $f(x) = x[x], x \in [-1, 2]$.

Solution:

(a) Since we have

$$|x| = \begin{cases} x & x \ge 0 \\ -x & x < 0 \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} x \cdot x & x \ge 0 \\ x(-x) & x < 0 \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} x^2 & x \ge 0 \\ -x^2 & x < 0 \end{cases}$$

Figure 1.26 shows the graph of $f(x) = x|x|$.

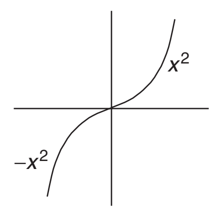


Figure 1.26

(b) We have

$$f(x) = |x - 1| + |x + 1|$$

Since

$$|x-1| = \begin{cases} x-1 & x-1 \ge 0 \Rightarrow x \ge +1 \\ -(x-1) & x-1 < 0 \Rightarrow x < +1 \end{cases}$$

and $|x+1| = \begin{cases} x+1 & x+1 \ge 0 \Rightarrow x \ge -1 \\ -(x+1) & x+1 < 0 \Rightarrow x < -1 \end{cases}$

we get

$$f(x) = \begin{cases} -2x & x \in (-\infty - 1) \\ 2 & x \in [-1, 1) \\ 2x & x \in [1, \infty) \end{cases}$$

Figure 1.27 shows the graph of $f(x) = |x - 1| + |x + 1|$.

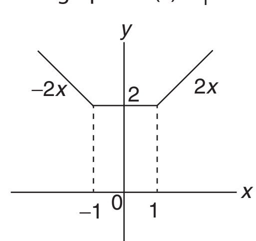


Figure 1.27

(c) We have

$$f(x) = 2x - \{x\} x \in [-1, 2]$$

Since

$$\{x\} = \begin{cases} \vdots \\ x+2 & -2 \le x < -1 \\ x+1 & -1 \le x < 0 \\ x & 0 \le x < 1 \\ x-1 & 1 \le x < 2 \\ x-2 & 2 \le x < 3 \\ \vdots \end{cases}$$

we get

$$f(x) = \begin{cases} 2x-(x+1) & -1 \le x < 0 \\ 2x-(x) & 0 \le x < 1 \\ 2x-(x-1) & 1 \le x < 2 \\ 2x-(0) & x = 2 \end{cases}$$

$$f(x) = \begin{cases} x-1 & -1 \le x < 0 \\ x & 0 \le x < 1 \\ x+1 & 1 \le x < 2 \\ 4 & x = 2 \end{cases}$$

Figure 1.28 shows the graph of $f(x) = 2x - \{x\}, x \in [-1, 2]$.

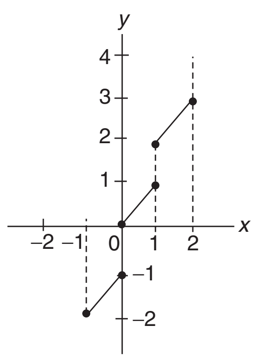


Figure 1.28

(d) We have

$$f(x) = x[x], x \in [-1, 2]$$

Since

$$[x] = \begin{cases} -1 & -1 \le x < 0 \\ 0 & 0 \le x < 1 \\ 1 & 1 \le x < 2 \\ 2 & x = 2 \end{cases}$$

we have

$$f(x) = \begin{cases} x(-1) & -1 \le x < 0 \\ x(0) & 0 \le x < 1 \\ x(1) & 1 \le x < 2 \\ x(2) & x = 2 \end{cases}$$

That is,

$$f(x) = \begin{cases} -x & -1 \le x < 0 \\ 0 & 0 \le x < 1 \\ x & 1 \le x < 2 \\ 4 & x = 2 \end{cases}$$

Figure 1.29 shows the graph of $f(x) = x[x], x \in [-1, 2]$.

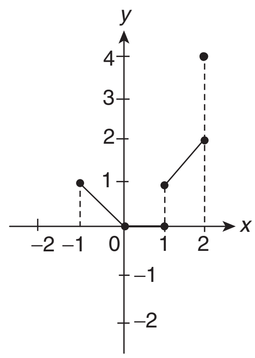


Figure 1.29

Illustration 1.19 Solve the following inequalities: (a) $\frac{1}{|x|-3} < \frac{1}{2}$ and (b) $|2 - |[x] - 1|| \le 2$ and (c) $|x| + |x - 3| > 3$.

Solution:

(a) We have

$$\frac{1}{|x|-3} < \frac{1}{2}$$

$$\Rightarrow \frac{1}{|x|-3} - \frac{1}{2} < 0 \Rightarrow \frac{2-|x|+3}{2(|x|-3)} < 0$$

$$\Rightarrow \frac{5-|x|}{|x|-3} < 0 \Rightarrow \frac{|x|-5}{|x|-3} > 0$$

$$\Rightarrow |x| > 5 \text{ or } |x| < 3 \Rightarrow x \in (-\infty, -5) \cup (5, \infty) \text{ or } x \in (-3, 3)$$

$$\Rightarrow x \in (-\infty, -5) \cup (-3, 3) \cup (5, \infty)$$

(b) We have

$$|2 - |[x] - 1|| \le 2$$

$$\Rightarrow ||[x] - 1| - 2| \le 2 \Rightarrow -2 \le |[x] - 1| - 2 \le 2$$

$$\Rightarrow 0 \le |[x] - 1| \le 4 \Rightarrow -4 \le [x] - 1 \le 4$$

$$\Rightarrow -3 \le [x] \le 5 \Rightarrow x \in [-3, 6)$$

(c) We have

$$|x| + |x - 3| > 3$$

If $x < 0, -x - (x - 3) > 3$, then

$$x < 0, -2x + 3 > 3$$

$$\Rightarrow x < 0, x < 0$$

$$\Rightarrow x < 0$$

or if $0 \le x < 3, x - (x - 3) > 3$, then

$$0 \le x < 3, 3 > 3 \Rightarrow x \in \phi$$

or if $x \ge 3, x + |x - 3| > 3$, then

$$x \ge 3, x > 3 \Rightarrow x > 3$$

Hence, the solution of $|x| + |x - 3| > 3$ is $x \in (-\infty, 0) \cup (3, \infty)$.

Illustration 1.20 Solve the inequality, $\log_{3x+5}(9x^2+8x+8)>2$.

Solution: For the logarithm to be defined, we need to have

$$3x+5>0,\ 3x+5\neq 1 \text{ and } 9x^2+8x+8>0$$

If $3x+5>1$, then the inequality gives

$$\log_{3x+5}(9x^2+8x+8)>\log_{3x+5}(3x+5)^2$$
$$\Rightarrow 9x^2+8x+8>(3x+5)^2$$
$$\Rightarrow 9x^2+8x+8>9x^2+30x+25$$
$$\Rightarrow 22x+17<0 \Rightarrow x<\frac{-17}{22}$$

However,

$$3x+5>1 \Rightarrow x>\frac{-4}{3}$$

and $\quad 9x^2+8x+8>0\ \forall x\in R$ ($D = 8^2 - 4.9.8 < 0, a = 9 > 0$)

Thus,

$$\frac{-4}{3}<x<\frac{-17}{22}$$

If $0<3x+5<1$, then the inequality gives

$$\log_{3x+5}(9x^2+8x+8)>\log_{3x+5}(3x+5)^2$$
$$\Rightarrow 9x^2+8x+8<(3x+5)^2$$
$$\Rightarrow x>-\frac{17}{22}$$

However,

$$0<3x+5<1 \Rightarrow -5<3x<-4 \Rightarrow \frac{-5}{3}<x<\frac{-4}{3}$$

There is no value of x satisfying

$$x>\frac{-17}{22} \text{ and } \frac{-5}{3}<x<\frac{-4}{3}$$

Hence, the solution set is

$$\left(\frac{-4}{3},\frac{-17}{22}\right)$$

Your Turn 8

1. Find all values of the parameter $a\in R$ for which the following inequality is valid $\forall\, x\in R$: $1+\log_5(x^2+1)\geq \log_5(ax^2+4x+a)$.
   Ans. (2, 3]
2. Solve $2[x]=x+\{x\}$, where [·] and {·} denote the greatest integer function and fractional part, respectively.
   Ans. $x=0$ and $x=\frac{3}{2}$
3. If $f(x)=\begin{cases}[x] & 0\leq\{x\}<\frac{1}{2}\\ [x]+1 & \frac{1}{2}<\{x\}<1\end{cases}$, then prove that $f(x)=-f(-x)$ (where [·] and {·} denote the greatest integer function and fractional part.
4. Solve the following inequalities:
   (a) $\log_{x^2}\left(\frac{x}{|x|}-x\right)\geq 0$
   (b) $\log_{1/2}(x^2-5x+7)>0$
   Ans. (a) $x\in(-\infty,-2]\cup(0,1)$; (b) $x\in(2,3)$
5. Solve the following equations: (a) $[x]^2-5[x]+6=0$, where [·] denotes the greatest integer function and (b) $|x-1|+|2x-3|=|3x-4|$.
   Ans. (a) $x\in[2,4)$; (b) $x\in(-\infty,1]\cup\left[\frac{3}{2},\infty\right)$

1.14 Algebra of Functions

Let us consider two functions, f: $D_1\to R$ and g: $D_2\to R$. We describe the functions $f+g$, $f-g$, $f\cdot g$ and f/g as follows:

1. $f+g$: $D\to R$ is a function defined by $(f+g)x=f(x)+g(x)$, where $D=D_1\cap D_2$.
2. $f-g$: $D\to R$ is a function defined by $(f-g)x=f(x)-g(x)$, where $D=D_1\cap D_2$
3. $f\cdot g$: $D\to R$ is a function defined by $(f\cdot g)x=f(x)\cdot g(x)$, where $D=D_1\cap D_2$.
4. f/g: $D\to R$ is a function defined by $(f/g)x=\frac{f(x)}{g(x)}$, where $D=\{x: x\in D_1\cap D_2, g(x)\neq 0\}$.

Key Points:
1. If f and g are two functions, then the sum of the functions $f+g$ is defined as $(f+g)(x)=f(x)+g(x)$, $\forall x\in$ (Domain f) $\cap$ (Domain g).
2. Similarly, we can define product fg and quotient f/g, respectively, as follows:
   (i) $\forall x\in$ (Domain f) $\cap$ (Domain g), $(fg)\cdot(x)=f(x)\cdot g(x)$.
   (ii) $\forall x\in$ (Domain f) $\cap$ (Domain g) $-\{x: g(x)=0\}$, $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$.
3. If k is any real number and f is a function, then kf is defined as $\forall x\in$ (Domain f) by $(kf)x=kf(x)$.

Illustration 1.21 Let $f(x)=\sqrt{6-x}$, $g(x)=\sqrt{x-2}$. Then find $f+g$, $f-g$, $f\cdot g$ and f/g.

Solution:

$$(f+g)x=\sqrt{6-x}+\sqrt{x-2},\ 2\leq x\leq 6$$
$$(f-g)x=\sqrt{6-x}-\sqrt{x-2},\ 2\leq x\leq 6$$
$$(f\cdot g)x=\sqrt{6-x}\cdot\sqrt{x-2}=\sqrt{(6-x)\cdot x-2)},\ 2\leq x\leq 6$$
$$(f/g)x=\frac{\sqrt{6-x}}{\sqrt{x-2}}=\sqrt{\frac{6-x}{x-2}},\ 2<x\leq 6$$

Illustration 1.22 Find the domain of following functions: (a) $f(x)=\sqrt{(x^2-5x+6)/(x+4)}$, (b) $f(x)=\sqrt{(3-|x|)/(|x|-5)}$ and (c) $f(x)=1/\sqrt{x-[x]}$.

Solution:

(a) We know that $f(x)=\sqrt{(x^2-5x+6)/(x+4)}$ is real and defined if and only if

$$\frac{x^2-5x+6}{x+4}\geq 0$$

and

$$x+4\neq 0$$
$$\Rightarrow \frac{(x-2)(x-3)}{(x+4)}\geq 0$$

and

$$x\neq -4$$
$$\Rightarrow x\in(-4,2]\cup[3,\infty)$$

(b) We know that $f(x)=\sqrt{(3-|x|)/(|x|-5)}$ is real and defined if and only if

$$\frac{3-|x|}{|x|-5}\geq 0$$

and

$$|x|-5\neq 0$$

$$\Rightarrow \frac{|x|-3}{|x|-5}\leq 0$$

and

$$|x|\neq +5$$

$$\Rightarrow |x|\in[3,5)$$

$$\Rightarrow x\in(-5,-3]\cup[3,5)$$

(c) We know that $f(x)=1/\sqrt{x-[x]}$ is real and defined if and only if $x-[x]>0$. Thus, $\{x\}>0$.

$$x\in R-I$$

because $\{x\}$ is always positive and is equal to zero if x is of integer.

Illustration 1.23 Find the domain of following functions: (a) $f(x) = \cos^{-1}(\log_3 x)$ and (b) $f(x) = \log_{x+1}(x^2-5x+6)$.

Solution:

(a) We have

$$f(x)=\cos^{-1}(\log_3 x)$$

Since $\log_3 x$ is defined if and only if $x>0$ and $\cos^{-1}(\log_3 x)$ is defined if and only if $-1\leq \log_3 x\leq 1$, for the domain, we have

$$x>0 \text{ and } -1\leq \log_3 x\leq 1$$

$$\Rightarrow x>0 \text{ and } \log_3\frac{1}{3}\leq x\leq \log_3 3$$

$$\Rightarrow x>0 \text{ and } \frac{1}{3}\leq x\leq 3$$

$$\Rightarrow \frac{1}{3}\leq x\leq 3$$

$$\Rightarrow x\in\left[\frac{1}{3},3\right]$$

(b) We have

$$f(x)=\log_{x+1}(x^2-5x+6)$$

Since $\log_{x+1}(x^2-5x+6)$ is defined if and only if $x^2-5x+6>0$ and $x+1>0$, $x+1\neq 1$, for the domain, we have

$$(x-2)(x-3)>0$$

and

$$x>-1, x\neq 0$$

$$\Rightarrow x\in(-\infty,2)\cup(3,\infty)$$

and

$$x\in(-1,\infty)$$

$$\Rightarrow x\in(-1,2)\cup(3,\infty)$$

and

$$x\neq 0$$

$$\Rightarrow x\in(-1,0)\cup(0,2)\cup(3,\infty)$$

Illustration 1.24 Find the domain of the following functions: (a) $f(x)=\dfrac{\sec^{-1}(x-5)}{x-[x]}$ and (b) $f(x)=\cos^{-1}\left(\dfrac{2-|x|}{4}\right)+\log_{10}(3-x)$.

Solution:

(a) We have

$$f(x)=\frac{\sec^{-1}(x-5)}{x-[x]}$$

Since $\sec^{-1}(x-5)$ is defined if and only if $|x-5|\geq 1$ and $x-[x]\neq 0$, for the domain, we get

$$|x-5|\geq 1$$

and

$$x-[x]\neq 0$$

$$\Rightarrow x-5\leq -1$$

or

$$-5\geq 1 \text{ and } \{x\}\neq 0$$

$$\Rightarrow x\leq 4$$

or

$$x\geq 6 \text{ and } x\notin I$$

$$\Rightarrow x\in(-\infty,4]\cup[6,\infty)$$

and

$$x\notin I$$

$$\Rightarrow x\in(-\infty,4]\cup[6,\infty)-I$$

(b) We have

$$f(x)=\cos^{-1}\left(\frac{2-|x|}{4}\right)+\log_{10}(3-x)$$

Let D_1 be the domain of $\cos^{-1}[(2-|x|)/4]$, D_2 be the domain of $\log_{10}(3-x)$ and D be the domain of $f(x)$. Now,

$$D=D_1\cap D_2$$

For D_1, we have

$$-1\leq\frac{2-|x|}{4}\leq 1$$

$$\Rightarrow -4\leq 2-|x|\leq 4\Rightarrow -6\leq -|x|\leq 2$$

$$\Rightarrow 6\geq |x|\geq -2\Rightarrow -2\leq |x|\leq 6$$

$$\Rightarrow 0\leq |x|\leq 6$$

$$\Rightarrow x\in[-6,6]$$

For D_2, we have $3-x>0$, $x<3$ and $x\in(-\infty,3)$. Hence,

$$D\equiv x\in[-6,6] \text{ and } x\in(-\infty,3)$$

That is,

$$D\equiv x\in[-6,3]$$

Illustration 1.25 Find the domain of following: (a) $f(x)=x^{\cos^{-1}x}$, $x>0$ and (b) $f(x)={}^{3x-1}C_{20-4x}$.

Solution:

(a) Since $x>0$ and $\cos^{-1}x\in R$, we have

$$x>0 \text{ and } x\in[-1,1]$$

$$\Rightarrow x\in(0,1]$$

(b) We have $f(x)={}^{3x-1}C_{20-4x}$. That is,

$$3x-1\geq 20-4x$$

$$\Rightarrow 7x\geq 21\Rightarrow x\geq 3$$

Also, we know that

$$3x-1\geq 0\Rightarrow x\geq\frac{1}{3}$$

and

$$20-4x\geq 0\Rightarrow x\leq 5$$

$$\Rightarrow x\in[3,5]$$

However,

$$3x-1\in N \text{ and } 20-4x\in N_0$$

where $N_0=N\cup\{0\}$). Hence, the domain of $f(x)=\{3,4,5\}$.

Your Turn 9

1. Find the domain of each of the following functions:

   (a) $f(x) = 1/(x^2+5x+6)$, (b) $f(x) = \sqrt{x^2+5x+6}$, (c) $f(x) = \log_x 2$, (d) $f(x) = \sqrt{\log_{0.5} x}$ and (e) $f(x) = \log\ [(x+1)(x+2)/(x+3)]$.

   Ans. (a) $R - \{-2, -3\}$; (b) $(-\infty, -3] \cup [-2, \infty)$; (c) $(0, 1) \cup (1, \infty)$; (d) $(0, 1]$; (e) $(-3, -2) \cup (-1, \infty)$

2. Find the domain of each of the following functions:

   (a) $f(x) = 1/\sqrt{2x^2 - 7x - 4}$

   (b) $f(x) = \log\ [(x^2 - 5x + 6)/(x^2 + 4x + 6)]$

   (c) $f(x) = \sin^{-1}[(1 + x^2)/2x] + \sqrt{x^2 - 16}$

   (d) $f(x) = \log_x 5 + \sqrt{\cos(\sin x)}$

   Ans. (a) $\left(-\infty, -\frac{1}{2}\right) \cup (4, \infty)$; (b) $(-\infty, 2) \cup (3, \infty)$; (c) ϕ; (d) $(0, 1) \cup (1, \infty)$

3. $f(x)$ is defined over [0, 1]. Find the domain of the following functions: (a) $f(2x + 3)$, (b) $f(\sin x)$ and (c) $f(\cos x)$.

   Ans. (a) $\left[\frac{-3}{2}, -1\right]$, (b) $\bigcup_{n \in Z} [2n\pi, (2n+1)\pi]$

   (c) $\bigcup_{n \in Z} \left[\left(2n - \frac{1}{2}\right)\pi, \left(2n + \frac{1}{2}\right)\pi\right]$

1.15 Methods to Determine Range

1. If $y = f(x)$, it should be expressed as $x = g(y)$, then the domain of $g(y)$ [under the condition $x \in$ domain of $f(x)$] represents the range of $f(x)$.
2. The following methods can also be followed:
   - **(i)** If $f(x) = a\sin x + b\cos x$, the range of $f(x)$ is $\left[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}\right]$.
   - **(ii)** If $f(x) = \sin^{2n}x + \cos^{2n}x$, $n \in N$, the range of $f(x)$ is $\left[\frac{1}{2^{n-1}}, 1\right]$.
   - **(iii)** If $f(x) = \sin^{2n+1}x + \cos^{2n+1}x$, $n \in N$, the range of $f(x)$ is $[-1, 1]$.
   - **(iv)** $f(x) = \sin x$ and $g(x) = \cos x$ are bounded function, that is, $-1 \le \sin x \le 1$ and $-1 \le \cos x \le 1$.
3. Range of $f(x)$ depends on its domain.
4. Range can also be found by finding absolute maxima and absolute minima of $f(x)$.
5. If $f(x)$ defined on $[a, b]$ and $f(x)$ is increasing function, then the range of $f(x)$ is $[f(a), f(b)]$. If $f(x)$ defined on $[a, b]$ and $f(a)$ is decreasing function, then the range of $f(x)$ is $[f(b), f(a)]$.

Note: If $f(x)$ is increasing function, then $f'(x) \ge 0$; if $f(x)$ is decreasing function, then $f'(x) \le 0$. For example, $\log_a x (a < 1)$, e^{-x}, $\sin^{-1}x$, $\cos^{-1}x$, $\cot^{-1}x$, $\text{cosec}^{-1}x$ are decreasing function.

6. The set of y-coordinates of the graph of a function is the range.

Illustration 1.26 Find the range of the following functions: (a) $f(x) = (x^2 - 1)/(x^2 + x + 1)$, (b) $f(x) = 2e^x/(3e^x + 5)$ and (c) $f(x) = \sqrt{16 - x^2}$.

Solution:

(a) We have

$$f(x) = \frac{x^2 - 1}{x^2 + x + 1}$$

The domain of $f(x) \equiv R$. Now,

$$y = f(x)$$

$$\Rightarrow y = \frac{x^2 - 1}{x^2 + x + 1}$$

$$\Rightarrow yx^2 + yx + y - x^2 + 1 = 0$$

$$\Rightarrow x^2(y - 1) + x(y) + y + 1 = 0$$

$$\Rightarrow x = \frac{-y \pm \sqrt{y^2 - 4(y+1)(y-1)}}{2(y-1)}$$

Since x is defined if and only if

$$y^2 - 4(y + 1)(y - 1) \ge 0 \text{ and } y - 1 \ne 0$$

$$\Rightarrow y^2 - 4(y^2 - 1) \ge 0 \text{ and } y \ne 1$$

$$\Rightarrow -3y^2 + 4 \ge 0 \text{ and } y \ne 1$$

$$\Rightarrow y^2 - \left(\frac{2}{\sqrt{3}}\right)^2 \le 0 \text{ and } y \ne 1$$

$$\Rightarrow \left(y + \frac{2}{\sqrt{3}}\right)\left(y - \frac{2}{\sqrt{3}}\right) \le 0 \text{ and } y \ne 1$$

$$\Rightarrow y \in \left[\frac{-2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right] \text{ and } y \ne 1$$

$$\Rightarrow y \in \left[\frac{-2}{\sqrt{3}}, 1\right) \cup \left(1, \frac{2}{\sqrt{3}}\right]$$

Hence, the range of $f(x)$ is

$$\left[\frac{-2}{\sqrt{3}}, 1\right) \cup \left(1, \frac{2}{\sqrt{3}}\right]$$

(b) We have

$$f(x) = \frac{2e^x}{3e^x + 5}$$

Domain of $f(x)$ is R.

$$y = \frac{2e^x}{3e^x + 5}$$

$$\Rightarrow e^x = \frac{5y}{2 - 3y}$$

$$\Rightarrow x = \log_e\left(\frac{5y}{2 - 3y}\right)$$

$$\Rightarrow \frac{5y}{2 - 3y} > 0 \text{ and } 2 - 3y \ne 0$$

$$\Rightarrow \frac{5y}{3y - 2} < 0 \text{ and } y \ne 2/3$$

$$\Rightarrow y \in \left(0, \frac{2}{3}\right)$$

Hence, the range of $f(x)$ is $\left(0, \frac{2}{3}\right)$.

(c) We have

$$f(x) = \sqrt{16 - x^2}$$

The domain of $f(x)$ is $x \in [-4, 4]$.

$$y = \sqrt{16 - x^2}$$

$$\Rightarrow y^2 = 16 - x^2 \Rightarrow x^2 = \sqrt{16-y^2}$$

$$\Rightarrow 16 - y^2 \geq 0 \Rightarrow y \in [-4, 4]$$

However,

$$x \in [-4, 4] \Rightarrow y \geq 0$$

$$\Rightarrow y \in [0, 4]$$

Hence, the range of $f(x)$ is [0, 4].

Illustration 1.27 Find the range of following functions: (a) $f(x) = \sin^4 x + \cos^4 x$, (b) $f(x) = 3\sin^2 x + \sin 2x + 3$ and (c) $f(x) = \sin^2 x + 3\sin x + 5$.

Solution:

(a) We have

$$f(x) = \sin^4 x + \cos^4 x$$

Since the range of $\phi(x)$ is

$$\sin^{2n}x + \cos^{2n}x \equiv \left[\frac{1}{2^{n-1}}, 1\right]$$

for $f(x)$, $x = 2$, the range of $f(x)$ is

$$\left[\frac{1}{2^{2-1}}, 1\right] \equiv \left[\frac{1}{2}, 1\right]$$

(b) We have

$$f(x) = 3\sin^2 x + \sin 2x + 3$$

$$= 3\left(\frac{1-\cos 2x}{3}\right) + \sin 2x + 3$$

That is,

$$f(x) = \sin 2x - \frac{3}{2}\cos 2x + \frac{9}{2}$$

The range of $f(x)$ is

$$\left[-\sqrt{1^2 + \frac{3^2}{2^2}} + \frac{9}{2}, \sqrt{1^2 + \frac{3^2}{2^2}} + \frac{9}{2}\right]$$

That is,

$$\left[\frac{9-\sqrt{13}}{2}, \frac{9+\sqrt{13}}{2}\right]$$

(c) We have

$$f(x) = \sin^2 x + 3\sin x + 5$$

Here, $\sin x$ is a bounded function. Thus,

$$f(x) = \left(\sin^2 x + \frac{3}{2}\right)^2 + 5 - \frac{9}{4}$$

$$f(x) = \left(\sin x + \frac{3}{2}\right)^2 + \frac{11}{4}$$

Since $-1 \leq \sin x \leq 1$, we get

$$+\frac{1}{2} \leq \sin x + \frac{3}{2} \leq \frac{5}{2}$$

$$\Rightarrow \frac{1}{4} \leq \left(\sin x + \frac{3}{2}\right)^2 \leq \frac{25}{4}$$

$$\Rightarrow 3 \leq \left(\sin x + \frac{3}{2}\right)^2 + \frac{11}{4} \leq 9$$

$$\Rightarrow 3 \leq f(x) \leq 9$$

Hence, the range of $f(x)$ is [3, 9].

Illustration 1.28 Find the range of the following functions: (a) $f(x) = \sin^{-1}(\log[x]) + \log(\sin^{-1}[x])$, where [·] is the greatest integer function and (b) $f(x) = \sin^{-1}x + \tan^{-1}x + \cos^{-1}x$.

Solution:

(a) Since $\sin^{-1}(\log[x])$ is defined if and only if $-1 \leq \log[x] \leq 1$, we have

$$e^{-1} \leq [x] \leq e \Rightarrow [x] = 1, 2$$

Now, $\log(\sin^{-1}[x])$ is defined if and only if $\sin^{-1}[x] > 0$ and $-1 \leq [x] < 1$. So,

$$[x] = 1$$

Hence, $f(x)$ is defined if $[x] = 1$ only.

$$f(x) = \sin^{-1}\log 1 + \log \sin^{-1} 1 = \sin^{-1}(0) + \log\left(\frac{\pi}{2}\right)$$

$$\Rightarrow f(x) = \log\frac{\pi}{2}$$

The range of $f(x)$ is $\left\{\log\frac{\pi}{2}\right\}$.

(b) We have

$$f(x) = \sin^{-1}x + \tan^{-1}x + \cos^{-1}x$$

The domain of $f(x)$ is [–1, 1].

$$f(x) = \frac{\pi}{2} + \tan^{-1}x \qquad (\text{As } \sin^{-1}x + \cos^{-1}x = \pi/2)$$

Now, $-1 \leq x \leq 1$.

$$\frac{-\pi}{4} \leq \tan^{-1}x \leq \frac{\pi}{4}$$

$$\frac{\pi}{4} \leq \tan^{-1}x + \frac{\pi}{2} \leq \frac{3\pi}{4}$$

$$\frac{\pi}{4} \leq f(x) \leq \frac{3\pi}{4}$$

Hence, the range of $f(x)$ is

$$\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$$

Illustration 1.29 Find the range of $f(x) = \sqrt{x-1} + \sqrt{7-x}$.

Solution: We have

$$f(x) = \sqrt{x-1} + \sqrt{7-x}$$

The domain of $f(x)$ is [1, 7].

$$f'(x) = \frac{1}{2\sqrt{x-1}} - \frac{1}{2\sqrt{7-x}}$$

$$f'(x) = 0$$

$$\frac{1}{2\sqrt{x-1}} - \frac{1}{2\sqrt{7-x}} = 0$$

$$\Rightarrow x - 1 = 7 - x$$

$$\Rightarrow x = 4$$

Now, $f(1) = \sqrt{6}$, $f(4) = 2\sqrt{3}$ and $f(7) = \sqrt{6}$. Hence, the range of $f(x)$ is $[\sqrt{6}, 2\sqrt{3}]$.

Illustration 1.30 Find the range of following functions: (a) $f(x) = \sec^{-1}(x^2 + 3x + 1)$ and (b) $f(x) = \log_e(\{x\}^2 + 3\{x\} + 2)$, where {·} is the fractional part.

Solution:

(a) We have

$$f(x) = \sec^{-1}(x^2 + 3x + 1)$$

Let $t = x^2 + 3x + 1$ for $x \in R$. Then

$$t \in \left[\frac{-5}{4}, \infty\right)$$

However,

$$y = \sec^{-1}(t) \Rightarrow t \in \left[\frac{-5}{4}, -1\right] \cup [1, \infty)$$

$$y \in \left[\sec^{-1}\frac{-5}{4}, \sec^{-1}(-1)\right] \cup [\sec^{-1}1, \sec^{-1}\infty)$$

$$y \in \left[\sec^{-1}\frac{-5}{4}, \pi\right] \cup \left[0, \frac{\pi}{2}\right)$$

Hence, the range of $f(x)$ is

$$\left[\sec^{-1}\frac{-5}{4}, \pi\right] \cup \left[0, \frac{\pi}{2}\right)$$

(b) We have

$$f(x) = \log(\{x\}^2 + 3\{x\} + 2)$$

Let $t = \{x\}^2 + 3\{x\} + 2$. Then

$$t = \left(\{x\} + \frac{3}{2}\right)^2 + 2 - \frac{9}{4} \Rightarrow t = \left(\{x\} + \frac{3}{2}\right)^2 - \frac{1}{4}$$

Since $0 \le \{x\} < 1$, we have

$$t \in [2, 6)$$

$$\Rightarrow \log t \in [\log 2, \log 6)$$

Since $y = \log t$, we have

$$y \in [\log 2, \log 6)$$

Hence, the range of $f(x)$ is $[\log 2, \log 6)$.

Illustration 1.31 Find the range of function $f(x) = |x - 1| + |x - 2|$.

Solution:

$$f(x) = \begin{cases} 3 - 2x, & x < 1 \\ 1, & 1 \le x < 2 \\ 2x - 3, & 2 \le x \end{cases}$$

The graph of $f(x)$ is shown in Fig. 1.30.

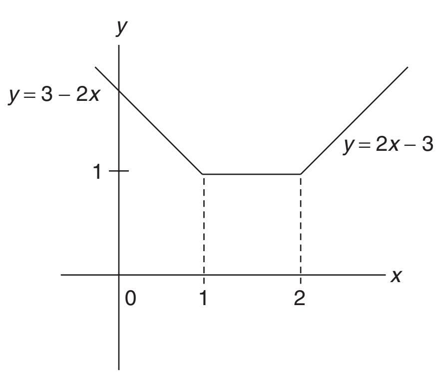


Figure 1.30

From the graph, we see that the set of y-coordinate of $f(x)$ is $[1, \infty)$. Hence, the range of $f(x)$ is $[1, \infty)$.

Your Turn 10

1. Find the range of each of the following functions:

(a) $f(x) = x^2 + 5x + 6$; (b) $f(x) = \sqrt{x^2 - 5x + 6}$; (c) $f(x) = x/(x^2 + 1)$; (d) $f(x) = (x + 2)/(x + 4)$; (e) $f(x) = \sqrt{\log x}$; (f) $f(x) = 1 - \sqrt{x}$ and (g) $f(x) = 1/(1 + \sqrt{x})$.

Ans. (a) $\left[-\frac{1}{4}, \infty\right)$; (b) $[0, \infty)$; (c) $\left[-\frac{1}{2}, \frac{1}{2}\right]$; (d) $(-\infty, 1) \cup (1, \infty)$; (e) $[0, \infty)$; (f) $(-\infty, 1]$; (g) $(0, 1]$

2. Find the domain and range of $f + g$ and $f \cdot g$ for each of the following functions:
(a) $f(x) = x$, $g(x) = \sqrt{x - 1}$ and (b) $f(x) = \sqrt{x + 1}$, $g(x) = \sqrt{x - 1}$.

Ans. (a) $D_{f+g} = D_{f \cdot g} = [1, \infty)$, R_{f+g} $[1, \infty)$, $R_{f \cdot g}$ $[0, \infty)$; (b) $D_{f+g} = D_{f \cdot g} = [1, \infty)$, $R_{+g} = [\sqrt{2}, \infty)$, $R_{f \cdot g} = [0, \infty)$

3. In each of the following, find the domain and range of f/g and g/f: (a) $f(x) = 2$, $g(x) = x^2 + 1$ and (b) $f(x) = 1$, $g(x) = 1 + \sqrt{x}$.

Ans. (a) $D_{f/g} = D_{g/f} = R$, $R_{f/g} = (0, 2]$, $R_{g/f} = \left[\frac{1}{2}, \infty\right)$; (b) $D_{f/g} = D_{g/f} = [0, \infty)$, $R_{f/g} = (0, 1]$, $R_{g/f} = [1, \infty)$

4. Find the domain and range of the function f, defined by $f(x) = (x^2 + 1)/\ln(x^2 + 1)$.

Ans. Domain of f is $R - \{0\}$; Range of f is $[e, \infty)$

5. Find the range of each of the following functions:

(a) $f(x) = 2 + 3\sin x - 4\cos x$; (b) $f(x) = 2 - 3x - 5x^2$; (c) $f(x) = \sqrt{x - x^2}$; (d) $f(x) = 4^x - 2^x + 1$; (e) $f(x) = \log(\sin^{-1}x)$; and (f) $f(x) = 4^x + 2^x + 1$

Ans. (a) $[-3, 7]$; (b) $\left(-\infty, \frac{49}{20}\right]$; (c) $\left[0, \frac{1}{2}\right]$; (d) $\left[\frac{3}{4}, \infty\right)$; (e) $\left(-\infty, \log\frac{\pi}{2}\right]$; (f) $(1, \infty)$

1.16 Composition of Functions

Let $f: X \to Y_1$ and $g: Y_2 \to Z$ be two functions and the set $D = \{x \in X: f(x) \in Y_2\}$. Then the function h defined, on D, by $h(x) = g[f(x)]$ is called the composite function of g and f and is denoted by gof. It is also called function of a function (Fig. 1.31).

Note: (1) Domain of gof is D which is a subset of Y_2 (the domain of g); (2) Range of gof is a subset of the range of g.

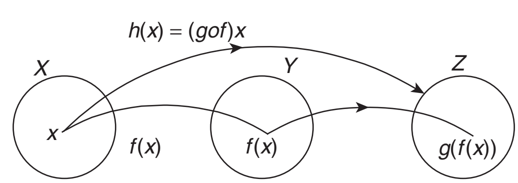


Figure 1.31

Key Points:

1. Let us consider the two functions, $f: X \to Y_1$ and $g: Y_1 \to Y$. We define the function $h: X \to Y$ such that $h(x) = g[f(x)]$ (Fig. 1.32). To obtain $h(x)$, first, we take the f-image of an element $x \in X$ so that $f(x) \in Y_1$, which is the domain of $g(x)$. Then take g-image of $f(x)$, that is, $g[f(x)]$ which would be an element of set Y.
2. The function h is called the composition of f and g and is denoted by gof. Thus,
$$(gof)x = g[f(x)]$$
and the domain is
$$(gof) = \{x: x \in \text{Domain}(f), f(x) \in \text{Domain}(g)\}$$

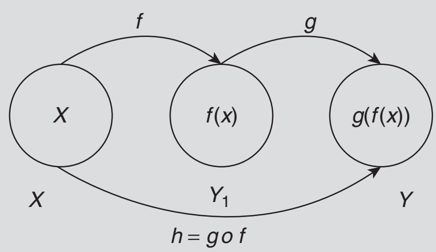


Figure 1.32

3. Similarly, we can define $(fog)x = f[g(x)]$ and Domain(fog) = $\{x: x \in \text{Domain}(g), g(x) \in \text{Domain}(f)\}$. In general, $fog \neq gof$.

Illustration 1.32 Wherever is possible, describe fog and gof for the following functions: (a) $f(x) = \sqrt{x+3}, g(x) = 1 + x^2$ and (b) $f(x) = \sqrt{x}, g(x) = x^2 - 1$.

Solution:

(a) Domain of the function f is $[-3, \infty)$; the range of the function f is $[0, \infty)$; the domain of the function g is R; the range of the function g is $[1, \infty)$.

(i) For $gof(x)$: Since the range of f is a subset of the domain of g, the domain of gof is $[-3, \infty)$, which is equal to the domain of f.
$$gof(x) = g[f(x)] = g(\sqrt{x+3}) = 1 + (x+3) = x + 4$$
The range of gof is $[1, \infty)$.

(ii) For $fog(x)$: Since the range of g is the subset of the domain of f, the domain of fog is R, which is equal to the domain of g.
$$fog(x) = f[g(x)] = f(1 + x^2) = \sqrt{x^2 + 4}$$
The range of fog is $[2, \infty)$.

(b) We have
$$f(x) = \sqrt{x}, g(x) = x^2 - 1$$
The domain of f is $[0, \infty)$ the range of f is $[0, \infty)$; the domain of g is R; the range of g is $[-1, \infty)$.

(i) For $gof(x)$: Since the range of f is a subset of the domain of g, the domain of gof is $[0, \infty)$ and
$$g[f(x)] = g(\sqrt{x}) = x - 1$$
The range of gof is $[-1, \infty)$.

(ii) For $fog(x)$: Since the range of g is not a subset of the domain of f, that is
$$[-1, \infty) \not\subset [0, \infty)$$
Therefore, fog is not defined on whole of the domain of g. The domain of fog is $\{x \in R$; the domain of g: $g(x) \in [0, \infty)$, which is equal to the domain of f. Thus, the domain of fog is
$$D = \{x \in R: x \le -1 \text{ or } x \ge 1\} = (-\infty, -1] \cup [1, \infty).$$
Now,
$$fog(x) = f[g(x)] = f(x^2 - 1) = \sqrt{x^2 - 1}$$
And its range is $[0, \infty)$.

Illustration 1.33 Two functions are defined as follows:
$$f(x) = \begin{cases} x+1, & x \le 1 \\ 2x+1, & 1 < x \le 2 \end{cases}, \quad g(x) \begin{cases} x^2, & -1 \le x < 2 \\ x+2, & 2 \le x \le 3 \end{cases}$$
Find fog and gof.

Solution: We have
$$(fog)(x) = f[g(x)] = \begin{cases} g(x)+1, & g(x) \le 1 \\ 2g(x)+1, & 1 < g(x) \le 2 \end{cases}$$
Let us consider $g(x) \le 1$. Then
$$x^2 \le 1, -1 \le x < 2 \Rightarrow -1 \le x \le 1, -1 \le x < 2 \Rightarrow -1 \le x \le 1$$
$$x + 2 \le 1, 2 \le x \le 3 \Rightarrow x \le -1, 2 \le x \le 3 \Rightarrow x = \phi$$
Let us consider $1 < g(x) \le 2$. Then
$$1 < x^2 \le 2, -1 \le x < 2$$
$$\Rightarrow x \in [-\sqrt{2}, -1) \cup (1, \sqrt{2}], -1 \le x < 2 \Rightarrow 1 < x \le \sqrt{2}$$
$$1 < x + 2 \le 2, 2 \le x \le 3 \Rightarrow -1 < x \le 0, 2 \le x \le 3, x = \phi$$
Thus,
$$f[g(x)] = \begin{cases} x^2+1, & -1 \le x \le 1 \\ 2x^2+1, & 1 < x \le \sqrt{2} \end{cases}$$
Now, let us consider gof. Then
$$gof = g[f(x)] = \begin{cases} f^2(x), & -1 \le f(x) < 2 \\ f(x)+2, & 2 \le f(x) \le 3 \end{cases}$$
Let us consider $-1 \le f(x) < 2$. Then
$$-1 \le x + 1 < 2, x \le 1 \Rightarrow -2 \le x < 1, x \le 1 \Rightarrow -2x \le x < 1$$
$$-1 \le 2x + 1 < 2, 1 < x \le 2 \Rightarrow -1 \le x < 1/2, 1 < x \le 2 \Rightarrow x = \phi$$
Let us consider $2 \le f(x) \le 3$. Then
$$2 \le x + 1 \le 3, x \le 1 \Rightarrow 1 \le x \le 2, x \le 1 \Rightarrow x = 1$$
$$2 \le 2x + 1 \le 3, 1 < x \le 2 \Rightarrow 1 \le 2x \le 2, 1 < x \le 2$$
$$\Rightarrow 1/2 \le x \le 1, 1 < x \le 2 \Rightarrow x = \phi$$
Thus,
$$g(f(x)) = \begin{cases} (x+1)^2, & -2 \le x < 1 \\ x+3, & x = 1 \end{cases}$$
If we like, we can also write
$$g[f(x)] = (x+1)^2, -2 \le x \le 1$$

Your Turn 11

1. If $u(x) = 4x - 5$, $v(x) = x^2$ and $f(x) = 1/x$, then find the formula for each of the following: (a) $u\{u[f(x)]\}$; (b) $u\{f[v(x)]\}$ and (c) $f\{v[u(x)]\}$.

Ans. (a) $(16/x) - 25$; (b) $(4/x^2) - 5$; (c) $1/(4x-5)^2$

2. If $f(x) = \sqrt{x}$, $g(x) = x/4$ and $h(x) = 4x - 8$, then find the formula for each of the following: (a) $h\{g[f(x)]\}$, (b) $h\{f[g(x)]\}$, (c) $g\{h[f(x)]\}$ and (d) $g\{f[h(x)]\}$.

Ans. (a) $\sqrt{x} - 8$; (b) $2\sqrt{x} - 8$; (c) $\sqrt{x} - 2$; (d) $\sqrt{x-2}/2$

3. Let f: $R \to R$ and g: $R \to R$ be the functions defined as the following:

$$f(x) = \begin{cases} 2x & x < 1 \\ 2x^2 - 1 & x \geq 1 \end{cases}, \quad g(x) = \begin{cases} x+2, & x < 0 \\ 2x, & x \geq 0 \end{cases}$$

Find $(f+g)(x)$, $(fg)(x)$ and $f[g(x)]$.

Ans. $$(f+g)(x) = \begin{cases} 3x+2 & x < 0 \\ 4x & 0 \leq x < 1; \\ 2x^2+2x-1 & x \geq 1 \end{cases}$$

$$(fg)(x) = \begin{cases} 2x^2+4x & x < 0 \\ 4x^2 & 0 \leq x < 1; \\ 4x^3 - 2x & x \geq 1 \end{cases} f[g(x)] = \begin{cases} 2x+4 & x < -1 \\ 2x^2+8x+7 & -1 \leq x < 0 \\ 4x & 0 \leq x < \frac{1}{2} \\ 8x^2 - 1 & x \geq \frac{1}{2} \end{cases}$$

4. (a) If $f(x) = \ln(3x)$, find $g(x)$ such that $f[g(x)] = x$. (b) Given that $f(x) = 1/(1-x)$, find $f\{f[f(x)]\}$.

Ans. (a) $e^x/3$; (b) x

1.17 Types of Functions

1.17.1 One-to-One Function or Injective Function

A function $f(x)$ is said to be one-to-one function on a domain D if $f(x_1) \neq f(x_2)$ when $x_1 \neq x_2$. That is, when $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$ (Fig. 1.33).

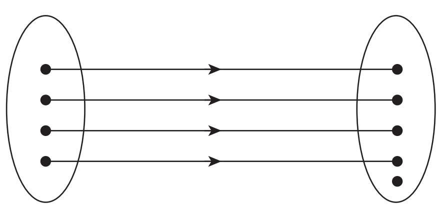


Figure 1.33

Two different values of x do not have the same values of y. An example is as follows: $f(x) = \sqrt{x}$ is one-to-one on any domain of non-negative numbers because $\sqrt{x_1} \neq \sqrt{x_2}$, whenever $x_1 \neq x_2$.

1.17.2 Many-to-One Function

If there exist at least two distinct elements in a domain, whose f-images are the same, then f is called many-to-one function (Fig. 1.34). Here, $f(x_1) = f(x_2)$ where $x_1 \neq x_2$. An example is as follows: $g(x) = \sin x$ is not one-to-one (hence many-to-one) because for many different values of x, we get the same values of $g(x)$.

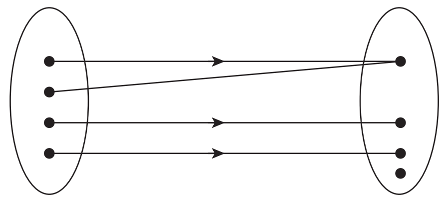


Figure 1.34

1.17.3 Methods to Identify if a Function is One-to-One or Many-to-One

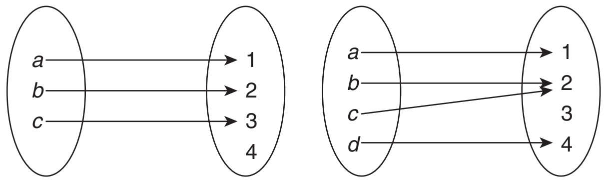
1. Let $x_1, x_2 \in$ Domain of f and if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ for every x_1, x_2 in the domain, then f is one-to-one function or else it is many-to-one function.
2. Conversely, if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for every x_1, x_2 in the domain, then f is one-to-one function or else it is many-to-one function.
3. If the function is entirely increasing or decreasing in the domain, then f is one-to-one function or else many-to-one function.
4. Any continuous function $f(x)$ which has at least one local maxima or local minima is many-to-one function.
5. All even functions are many-to-one function.
6. All polynomials of even degree defined in the domain R have at least one local maxima or minima and hence, they are many-to-one functions in the domain R. Polynomials of odd degree may be either one-to-one functions or many-to-one functions.
7. If f is a rational function, then $f(x_1) = f(x_2)$ is satisfied when $x_1 = x_2$ in the domain. Hence, we can write $f(x_1) - f(x_2) = (x_1 - x_2)g(x_1, x_2)$ where $g(x_1, x_2)$ is some function in x_1 and x_2. Now, if $g(x_1, x_2) = 0$ gives some solution which is different from $x_1 = x_2$ and which lies in the domain, then f is many-to-one function or else one-to-one function.
8. Draw the graph of $y = f(x)$ and determine whether $f(x)$ is one-to-one function or many-to-one function.

Some Important Results:

1. Let f: $X \to Y$ and g: $Y \to Z$.
   - **(i)** If both f and g are one-to-one, then so is gof.
   - **(ii)** If gof is one-to-one, then f is one-to-one function; however, g may not be one-to-one function.
2. One-to-one is the property of the domain of a function; for example, f: $R \to R$ defined by $f(x) = x^2$ is not one-to-one function but f: $R^+ \to R$ defined by $f(x) = x^2$ is one-to-one function.
3. Any function will be either one-to-one function or many-to-one function.

Key Points:
A function f is said to be one-to-one function if it does not take the same values at two distinct points in its domain. Thus if $x_1, x_2 \in$ Domain f, then $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$. Alternatively, f is one-to-one function if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Note: (1) A real function is one-to-one in its domain if $f'(x) > 0\ \forall x \in$ Domain f or $f'(x) < 0\ \forall x \in$ Domain f where f' is derivative of f. Here, $f'(x)$ can vanish at some points but such points must not form intervals and (2) a function f is many-to-one function [Fig. 1.35(b)] if it is not one-to-one function [1.35(a)].



(a) (b)

Figure 1.35

Illustration 1.34 Let $f: R \to R$ be defined as $f(x) = (x^2 + 4x + 7)/(x^2 + x + 1)$. Determine if $f(x)$ is one-to-one or many-to-one.

Solution: For the given function $f(x_1) = f(x_2)$, we get

$$\frac{x_1^2 + 4x_1 + 7}{x_1^2 + x_1 + 1} = \frac{x_2^2 + 4x_2 + 7}{x_2^2 + x_2 + 1}$$

$$\Rightarrow (x_1 - x_2)(2x_1 + 2x_2 + 1 + x_1x_2) = 0$$

Here, one of the solutions of $2x_1 + 2x_2 + 1 + x_1x_2 = 0$ is $x_1 = 0$ and the other one is $x_2 = -1/2$. Hence, $f(0) = f(-1/2) = 7$ so that $f(x)$ is many-to-one function.

Note: Any function is either one-to-one function or many-to-one function.

Illustration 1.35 Let $f: R \to R$ defined by $f(x) = x^3 + px^2 + 3x + 2010$. Then find the range of p for which f is a one-to-one function.

Solution: We have

$$f(x) = x^3 + px^2 + 3x + 2010$$
$$\Rightarrow f'(x) = 3x^2 + 2px + 3$$

For $f(x)$ to be one-to-one function, we need to have $f'(x) \geq 0$ or ≤ 0. Here, $f'(x)$ is the quadratic expression and coefficient of $x^2 > 0$ so that $f'(x) \geq 0$.

$$D \leq 0$$
$$\Rightarrow 4p^2 - 36 \leq 0$$
$$\Rightarrow p^2 \leq 9 \Rightarrow -3 \leq p \leq 3$$

Illustration 1.36 Check $f: R \to R$ defined by $f(x) = (x - 2)(x - 3)(x - 4)$ is one-to-one function or many-to-one function.

Solution: Obviously, the horizontal line cuts the graph of $f(x)$ at more than one point (Fig. 1.36). Hence, $f(x)$ is many-to-one function.

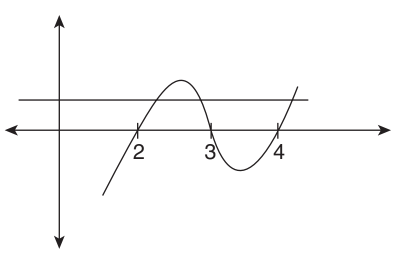
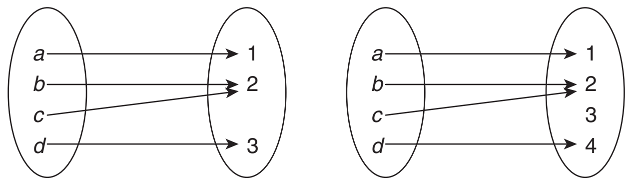


Figure 1.36

1.17.4 Onto Function (or Surjective Function) and Into Function

If each element in a co-domain has at least one pre-image in the domain, that is, if the range is equal to the co-domain, then the function is an onto function. If there exists at least one element in the co-domain of the function which does not have its pre-image, then the function is an into function. In other words, a function is called into function when its range is a proper subset of co-domain. Following are some examples:

1. The function $f: R \to R$ which is defined by $f(x) = x^2$ is not an onto function [Fig. 1.37(b)] (hence it is an into function [Fig. 1.37(b)]) as the negative real numbers have no pre-images. In fact, codomain is R and the range is $[0, \infty) \neq R$.
2. The function $f: R \to [-1, 1]$ which is defined by $f(x) = \sin x$ is an onto function [Fig. 1.37(a)] (hence it is not an into function [Fig. 1.37(b)]) as for $x \in R$, $\sin x$ takes all real values in $[-1, 1]$. In fact, the codomain is $[-1, 1]$, which is also its range.



onto (a) into (b)

Figure 1.37

Some Important Results:

Let $f: X \to Y$ and $g: Y \to Z$. Then

(i) If both f and g are onto functions, then gof is also an onto function.

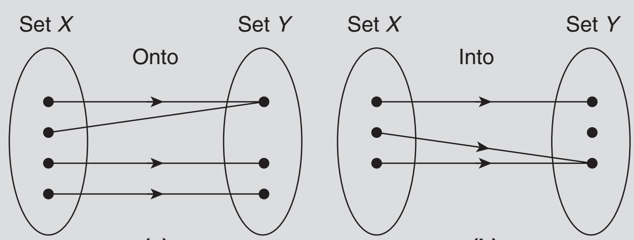
(ii) If gof is an onto function, then g is also an onto function; however, f may not be an onto function.

(iii) If $f: R \to R$ and $f(x)$ is the odd-degree polynomial, then $f(x)$ is an onto function. If $f(x)$ is an even-degree polynomial, then $f(x)$ is also an into function.

(iv) If onto is the property of co-domain of $f(x)$, that is, $f: R \to R$ which is defined by $f(x) = x^2$ is not onto. However, $f: R \to R^+ \cup \{0\}$ which is defined by $f(x) = x^2$ is an onto function.

Note: If $f(x)$ is an one-to-one as well as onto function, then $f(x)$ is said to be a bijective function.

Key Points:
A function $f: X \to Y$ is said to be an onto function if each element in Y is the image of at least one element in X (Fig. 1.38). Thus, for an onto function f, range of f is codomain Y. A function f is into function if it is not onto function, that is, range $\subset Y$.



(a) (b)

Figure 1.38

Illustration 1.37 Prove that $f: R - \{5\} \to R - \{1\}$ defined by $f(x) = (x - 3)/(x - 5)$ is bijective.

Solution: To prove $f(x)$ is one-to-one, we proceed as follows: We have

$$f(x)=\frac{x-3}{x-5}$$

$$x_1, x_2 \in R-\{5\}$$

Therefore,

$$f(x_1)=f(x_2) \Rightarrow \frac{x_1-3}{x_1-5}=\frac{x_2-3}{x_2-5}$$

$$\Rightarrow x_1x_2-5x_1-3x_2+15=x_1x_2-3x_1-5x_2+15$$

$$\Rightarrow(-5x_1+3x_1)+(5x_2-3x_2)=0$$

$$\Rightarrow 2(x_1-x_2)=0$$

$$\Rightarrow x_1-x_2=0$$

Now,

$$f(x_1)=f(x_2) \Rightarrow x_1=x_2 \text{ (unique)}$$

Hence, $f(x)$ is one-to-one function. To prove $f(x)$ is an onto function, we proceed as follows:

$$f(x)=\frac{x-3}{x-5}$$

$$y=\frac{x-3}{x-5}$$

$$\Rightarrow(x-5)y=x-3 \Rightarrow x(y-1)=-3+5y$$

$$\Rightarrow x=\frac{5y-3}{y-1}$$

Hence, the range of $f(x)$ is $R-\{1\}$; therefore, $f(x)$ is an onto function. Hence, $f(x)$ is bijective function.

Your Turn 12

1. Identify the type(s) of function(s) of each of the following functions (one-to-one/many-to-one and into/onto) if defined as $f: D \to R$, where D is its domain: (a) $f(x) = |x|$; (b) $f(x) = \sin x$; (c) $f(x) = x + 5$; (d) $f(x) = x^2 + 2x + 3$; (e) $f(x) = 2x + \sin x + 5$; (f) $f(x) = x^3$; (g) $f(x) = x^5$; (h) $f(x) = 2x^4 + 5$; (i) $f(x) = x^3 + 1$; (j) $f(x) = \log x$; (k) $f(x) = \log_e |x|$; (l) $f(x) = (x+1)/(x+5)$; (m) $f(x) = 1/x^3$; (n) $f(x) = 1/x^2$ and (o) $f(x) = e^x$.

   Ans. (a) Many-to-one/into; (b) Many-to-one/into; (c) One-to-one/onto; (d) Many-to-one/into; (e) One-to-one/onto; (f) One-to-one/onto; (g) One-to-one/onto; (h) Many-to-one/into; (i) One-to-one/onto; (j) One-to-one/onto; (k) Many-to-one/onto; (l) One-to-one/into; (m) One-to-one/into; (n) Many-to-one/into; (o) One-to-one/into

2. By choosing suitable domain and co-domain, make each of the following functions invertible (one-to-one and onto):
   (a) $f(x) = x^2 + x + 1$, (b) $f(x) = x^2$, (c) $f(x) = 2^{x(x+1)}$, (d) $f(x) = \sin x$, (e) $f(x) = \cos x$, (f) $f(x) = |\log x|$ and (g) $f(x) = |x^2 + 5x + 6|$.

   Ans. (a) $\left[-\frac{1}{2},\infty\right) \to \left[\frac{3}{4},\infty\right)$; (b) $[0, \infty) \to [0, \infty)$; (c) $\left[-\frac{1}{2},\infty\right) \to [2^{-1/4}, \infty)$; (d) $\left[\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right] \to [-1, 1]$; (e) $[0, \pi] \to [-1, 1]$; (f) $[1, \infty) \to (0, \infty)$; (g) $[-2, \infty) \to [0, \infty)$

3. Show that the function $f: R \to R$ which is defined by $f(x) = (x^2+2x+5)/(x^2+x+1)$ is many one and into.

4. Prove that $f: (-1, 1) \to R$ which is defined by

$$f(x)=\begin{cases}\dfrac{x}{1+x}, & -1<x\le 0\\ \dfrac{x}{1-x}, & 0<x<1\end{cases}$$

   is a bijective function.

5. Find all linear functions from $[-2, 4]$ to $[10, 15]$ which are onto functions.

   Ans. $f(x)=\frac{5}{6}x+\frac{35}{3}$ or $-\frac{5}{6}x+\frac{40}{3}$.

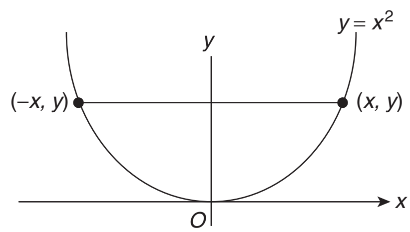
1.17.5 Even Function and Odd Function

A function $y = f(x)$ is even if $f(-x) = f(x)$ for every number x in the domain of f; for example, $f(x) = x^2$ is even because

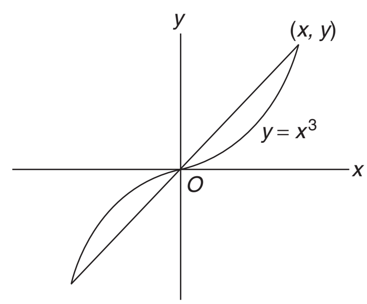
$$f(-x) = (-x)^2 = x^2 = f(x)$$

The graph of an even function $y = f(x)$ is symmetric about y-axis [Fig. 1.39(a)], that is, (x, y) lies on the graph $\Leftrightarrow$ $(-x, y)$ lies on the graph.

A function $y = f(x)$ is an odd function if $f(-x) = -f(x)$ for every x in the domain of f. That is, $f(x) = x^3$ is odd because $f(-x) = -x^3 = -f(x)$. The graph of an odd function $y = f(x)$ is symmetric about origin, that is, if (x, y) lies on the graph $\Leftrightarrow$ $(-x, -y)$ lies on the graph [Fig. 1.39(b)].



(a)



(b)

Figure 1.39

Note:

1. For domain R, even functions are not one-to-one functions.
2. Every function can be written as a sum of an even function and an odd function, that is,

$$f(x)=\frac{f(x)+f(-x)}{2}+\frac{f(x)-f(-x)}{2}$$

   where the domain lies on both sides of origin.

3. The function $f(x) = 0$ is the only function which is both even function and odd function.
4. Every odd, continuous function passes through the origin.

Table 1.3 lists comparison of different properties of the two functions $f(x)$ and $g(x)$ whether they are even or odd functions.

Table 1.3 Properties of functions $f(x)$ and $g(x)$

f(x)	g(x)	f(x) + g(x)	f(x) – g(x)	f(x) · g(x)	f(x)/g(x)	(fog)x
Even	Even	Even	Even	Even	Even	Even
Even	Odd	Neither Even nor Odd	Neither Even nor Odd	Odd	Odd	Even
Odd	Even	Neither Even nor Odd	Neither Even nor Odd	Odd	Odd	Even
Odd	Odd	Odd	Odd	Even	Even	Odd

Illustration 1.38 State whether the following functions are even or odd: (a) $f(x)=\log(x+\sqrt{1+x^2})$, (b) $f(x)=\dfrac{x}{a^x-1}+\dfrac{x}{2}+1$; (c) $f(x)=\begin{cases} x|x| & x\le -1 \\ [x+1][1-x] & -1<x<1 \\ -x|x| & x\ge 1\end{cases}$, where [·] represents the greatest integer function and (d) $f(x+y)=f(x)+f(y)\ \forall x, y\in R$.

Solution:

(a) We have

$$f(x)=\log(x+\sqrt{1+x^2})$$

That is,

$$f(-x)=\log(-x+\sqrt{1+x^2})$$

$$f(-x)=\log\frac{(-x+\sqrt{1+x^2})(x+\sqrt{1+x^2})}{(x+\sqrt{1+x^2})}$$

$$f(-x)=\log\left(\frac{1+x^2-x^2}{x+\sqrt{1+x^2}}\right)$$

Therefore,

$$f(-x)=\log\left(\frac{1}{x+\sqrt{1+x^2}}\right)$$

Now,

$$f(-x)=-\log(x+\sqrt{1+x^2})=-f(x)$$

Hence, $f(x)$ is an odd function.

(b) We have

$$f(x)=\frac{x}{a^x-1}+\frac{x}{2}+1$$

That is,

$$f(-x)=\frac{-x}{a^{-x}-1}-\frac{x}{2}+1$$

$$f(-x)=\frac{-x\,.\,a^x}{1-a^x}-\frac{x}{2}+1=\frac{x(a^x-1+1)}{(a^x-1)}-\frac{x}{2}+1$$

$$f(-x)=x+\frac{x}{a^x-1}-\frac{x}{2}+1=\frac{x}{a^x-1}+\frac{x}{2}+1=f(x)$$

Hence, $f(x)$ is an even function.

(c) We have

$$f(x)=\begin{cases} x|x| & x\le -1 \\ [x+1]+[1-x] & -1<x<1 \\ -x|x| & x\ge 1\end{cases}$$

That is,

$$f(-x)=\begin{cases} (-x)|-x| & -x\le -1 \\ [-x+1]+[1+x] & -1<-x<1 \\ -(-x)|-x| & -x\ge 1\end{cases}$$

$$f(-x)=\begin{cases} -x|x| & x\ge 1 \\ [1+x]+[1-x] & -1<x<1=f(x) \\ x|x| & x\le 1\end{cases}$$

Hence, $f(x)$ is an even function.

(d) We have

$$f(x+y)=f(x)+f(y)\ \forall x\in R$$

Replacing x and y by zero, we get

$$f(0)=2f(0)$$

$$\Rightarrow f(0)=0$$

Replacing y by $-x$, we get

$$f(0)=f(x)+f(-x)$$

$$\Rightarrow 0=f(x)+f(-x)$$

$$\Rightarrow f(-x)=-f(x)$$

Hence, $f(x)$ is an odd function.

Key Points:

1. If $f: X\to Y$ is a real-valued function such that $\forall x\in D\Rightarrow -x\in D$ (where D is the domain of f). If $f(-x) = f(x)$ for every $x\in D$, then f is said to be an even function and if $f(-x) = -f(x)$, then f is said to be an odd function. The graphs of even functions are symmetric about y-axis [That is, if (x, y) lies on the curve, then $(-x, y)$ also lies on the curve and those of the odd functions are symmetric about the origin [That is, if (x, y) lies on the curve, then $(-x, -y)$ also lies on the curve].
2. For any function f, $f(x) + f(-x)$ is an even function and $f(x) - f(-x)$ is an odd function. Thus, any function $y = f(x)$ can be expressed uniquely as the sum of an even function and an odd function as follows:

$$y=f(x)=\left\{\frac{f(x)+f(-x)}{2}\right\}+\left\{\frac{f(x)-f(-x)}{2}\right\}$$

Note: For two real functions f and g, the following features hold:

(i) $f+g$ is an even function if both f and g are even functions; $f+g$ is an odd function if both f and g are odd functions; $f+g$ is neither even function nor odd function if one of them is an odd function and the other is an even function.

(ii) fg is an even function if both f and g are even functions or both are odd functions; fg is an odd function if one of them is an even function and the other is an odd function.

(iii) fog is an even function if at least one of the functions (f and g) is an even function; fog is odd if both f and g are odd.

(iv) f is an even function $\Rightarrow f'$ is an odd function;
f is an odd function $\Rightarrow f'$ is an even function;

f is an odd function $\Rightarrow \int_a^x f(t)dt$ is an even function for any $a \in R$;

f is an even function $\Rightarrow \int_0^x f(t)dt$ is an odd function.

(v) If f is an even function, then it cannot be one-to-one function. An odd function may or may not be an one-to-one function.

Extension of Domain:

Let a function be defined on certain domain which is entirely non-negative (or non-positive). Then the domain of $f(x)$ can be extended to the set X = {$-x$: $x \in$ domain of $f(x)$} in two ways:

1. ***Even Extension*:** The even extension is obtained by defining a new function $f(-x)$ for $x \in$ X, such that $f(-x) = f(x)$.
2. ***Odd Extension*:** The odd extension is obtained by defining a new function $f(-x)$ for $x \in$ X such that $f(-x) = -f(x)$.

Illustration 1.39 Find the even and odd extension of the function

$$f(x)=\begin{cases} x^5+x^4, & 0 \le x \le 2 \\ x^3+3, & 2 < x \le 4 \end{cases}$$

Solution: The even function of $f(x)$ is given by

$$g(x)=\begin{cases} -x^3+3, & -4 \le x < 2 \\ -x^3+x^4, & 2 \le x \le 0 \end{cases}$$

The odd extension of $f(x)$ is given by

$$h(x)=\begin{cases} x^3-3, & -4 \le x < -2 \\ x^3-x^4, & -2 \le x \le 0 \end{cases}$$

1.17.6 Identical Function

Two functions $y = f(x)$ and $y = g(x)$ are said to be identical if
(1) the domain and the range of both functions are equal and
(2) both functions should be equal $\forall x \in$ domain, that is
$f(x) = g(x)\ \forall x \in$ domain.

Example: Consider $f(x) = \log x - \log(x + 1)$ and $g(x) = \log[x/(x+1)]$.

$$f(x) = \log x - \log(x+1) = \log\left(\frac{x}{x-1}\right)$$

Now, the domain of $f(x)$ is

$$x > 0 \text{ and } x + 1 > 0$$

$$\Rightarrow x > 0 \text{ and } x > -1$$

$$\Rightarrow x \in (0, \infty)$$

The domain of $g(x)$ is

$$\frac{x}{x+1} > 0$$

$$\Rightarrow x \in (-\infty, -1) \cup (0, \infty)$$

Hence, the domain of $f(x) \neq$ domain of $g(x)$. Therefore, $f(x)$ and $g(x)$ are not identical functions.

Note: The two functions $f(x)$ and $g(x)$ become identical if the domain of $g(x)$ is defined as $x \in (0, \infty)$.

1.17.7 Periodic Function

A function $f(x)$ is said to be periodic if there is a positive number p such that $f(x + p) = f(x)$ for all $x \in$ D. The smallest value of such p is called the principal or fundamental period of f.

If we draw the graph of a periodic function $f(x)$, we find that the graph gets repeated after each interval of length p; for example, $y = \sin x$ is periodic with period 2π as $\sin(x + 2\pi) = \sin x$. Graphical representation of $y = \sin x$ is shown in Fig. 1.40.

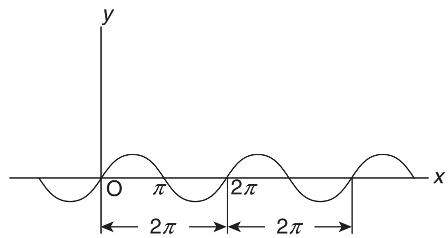


Figure 1.40

Rules to Find Period of Periodic Function:

1. If $f(x)$ is periodic with period p, then $af(x) + b$, where $a, b \in R$ ($a \neq 0$) is also a periodic function with period p.
2. If $f(x)$ is periodic with period p, then $f(ax + b)$, where $a \in R-\{0\}$ and $b \in R$, is also periodic with period $p/|a|$.
3. If $f(x)$ is periodic with p as the period and $g(x)$ is periodic with q as the period ($p \neq q$) and LCM of p and q exists, then $f(x) + g(x)$ is periodic with period equal to LCM of p and q where $f(x)$ and $g(x)$ cannot be interchanged by adding a positive number in x which is less than LCM of p and q [in which case, this number becomes the period of $f(x) + g(x)$]. Note that LCM of p and q exists if and only if p/q is a rational number.
4. If $f(x)$ is periodic with period p, then $1/f(x)$ is also periodic with the same period p.
5. If $f(x)$ is periodic with period p, then $\sqrt{f(x)}$ is also periodic with the same period p.
6. If $f(x)$ is a periodic function with period p and $g(x)$ is a strictly monotonic function, then $g[f(x)]$ will also be periodic with period p.
7. If $f(x)$ is a periodic function with period p and $g(x)$ is any other function, then $g\,o\,f(x)$ is periodic (period may be less than p) but $f\,o\,g(x)$ may or may not be periodic. For example, $\sin x^2$ is not periodic but $\sin(x + \sin x)$ is periodic with period 2π.
8. Constant function is periodic with no fundamental period.

Note:

1. LCM of $\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}\right) = \frac{\text{LCM of}(a_1, a_2, a_3)}{\text{HCF of}(b_1, b_2, b_3)}$.
2. LCM of a rational number and an irrational number does not exist. In general, LCM of irrational numbers does not exist. If irrational numbers are the multiples of same irrational number, then existence of LCM is possible. For example,

$$\text{LCM of } 2\pi \text{ and } 3\pi = (\text{LCM of 2 and 3})\pi = 6\pi$$

Functions and Their Periods:

Table 1.4 lists some functions and their respective periods.

Table 1.4 Functions and their respective periods

S. No.	$f(x)$	Period
1.	$\sin^n x$, $\cos^n x$, $\text{cosec}^n x$, $\sec^n x$	2π if n is an odd number; π if n is an even number
2.	$\tan^n x$, $\cot^n x$	$\pi\ \forall n \in N$
3.	$\lvert\sin x\rvert^n$, $\lvert\cos x\rvert^n$, $\lvert\tan x\rvert^n$, $\lvert\cot x\rvert^n$	$\pi\ \forall n \in N$
4.	$\{x\}$	1
5.	$f(x) = k$	Periodic function but it has no fundamental period.

Definition: A function $f: X \to Y$ is said to be a periodic function if there exists a positive real number p such that $f(x + p) = f(x)$, $\forall x \in X$. The least of all such positive numbers p is called the 'principal period' or simply 'period' of f. All periodic functions can be analysed over an interval of one period within the domain as the same pattern shall be repetitive over the entire domain.

Remarks:

1. If $f(x)$ is periodic with period p, then $af(x) + b$ where $a, b \in R$ $(a \neq 0)$ is also periodic with period p.
2. If $f(x)$ is periodic with period p, then $f(ax + b)$ where $a, b \in R$ $(a \neq 0)$ is also periodic with period $p/|a|$.
3. Let $f(x)$ has period $p = m/n$ ($m, n \in N$ and coprime), $g(x)$ has period $q = r/s$ ($r, s \in N$ and coprime) and let t be the LCM of p and q, that is,
$$t = \frac{\text{LCM of }(m, r)}{\text{HCF of }(n, s)}$$
Then t shall be the period of $f + g$ in which there does not exist a positive number k ($<t$) for which $f(k + x) = g(x)$ and $g(k + x) = f(x)$, else k will be the period. The same rule is applicable for any other algebraic combination of $f(x)$ and $g(x)$.
4. If f is periodic and g is non-periodic, then $f[g(x)]$ is non-periodic and $g[f(x)]$ is periodic.
5. If f is periodic with period T, then f' is periodic with the same period.
6. A periodic function cannot be one-to-one function.

Illustration 1.40 Find the period of the following functions: (a) $f(x) = \cos 4x + \sin \pi x$, (b) $f(x) = \sin 6\pi\{x\}$ and (c) $f(x) = e^{\sin^2 x + \sin^2[x + (\pi/3)] + \cos x[\cos(x + \pi/3)]}$.

Solution:

(a) We have
$$f(x) = \cos 4x + \sin \pi x$$
The period of $\cos 4x$ is
$$\frac{2\pi}{4} = \frac{\pi}{2}$$
The period of $\sin \pi x$ is
$$\frac{2\pi}{\pi} = 2$$
The period of $f(x)$ is
$$\text{LCM}\left\{\frac{\pi}{2}, 2\right\}$$
Since $\text{LCM}\left\{\frac{\pi}{2}, 2\right\}$ does not exist, $f(x)$ is not periodic.

(b) We have
$$f(x) = \sin 6\pi\{x\}$$
$$= \sin 6\pi(x - [x])$$
$$= -\sin(6\pi[x] - 6\pi x)$$
Since $[x]$ is the integer and $6\pi\,[x]$ is the multiple of 2π and $\sin(2n\pi - \theta) = -\sin\theta$, so
$$f(x) = \sin 6\pi x$$
So, the period of $f(x)$ is
$$\frac{2\pi}{6\pi} = \frac{1}{3}$$

(c) We have
$$f(x) = e^{\sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + \cos x \cos\left(x + \frac{\pi}{3}\right)}$$
Let us consider
$$\phi(x) = \sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + \cos x \cos\left(x + \frac{\pi}{3}\right)$$
$$\phi(x) = \frac{1 - \cos 2x}{2} + \frac{1 - \cos\left(2x + \frac{2\pi}{3}\right)}{2} + \frac{\cos\left(2x + \frac{\pi}{3}\right) + \cos\frac{\pi}{3}}{2}$$
which is obtained by using
$$2\cos A \cos B = \cos(A + B) + \cos(A - B)$$
$$\phi(x) = \frac{1}{2}\left\{\frac{5}{2} - \left[\cos 2x + \cos\left(2x + \frac{2\pi}{5}\right) + \cos\left(2x + \frac{\pi}{3}\right)\right]\right\}$$
$$= \frac{1}{2}\left\{\frac{5}{2} - 2\cos\left(2x + \frac{\pi}{3}\right)\cos\frac{\pi}{3} + \cos\left(2x + \frac{\pi}{3}\right)\right\}$$
which is obtained by using
$$\cos A + \cos B = 2\cos\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)$$
Therefore,
$$\phi(x) = \frac{1}{2}\left\{\frac{5}{2} + \cos\left(2x + \frac{\pi}{3}\right)\left(1 - 2\cos\frac{\pi}{3}\right)\right\} = \frac{5}{4}$$
Here, $f(x) = e^{5/4}$ is a constant and hence, $f(x)$ is periodic function and it has no fundamental period.

1.17.8 Inverse of a Function

Let $f: X \to Y$ be a function defined by $y = f(x)$ such that f is both one-to-one and onto functions (i.e. bijective function). Then there exists a unique function $g: Y \to X$ such that for each $y \in Y$,
$$g(y) = x \Leftrightarrow y = f(x)$$
The function g so defined is called the inverse of the function f (Fig. 1.41). Note that
$$f[g(x)] = g[f(x)] = x$$
and hence f and g are inverses of each other. The function f is called invertible if inverse of f exists.

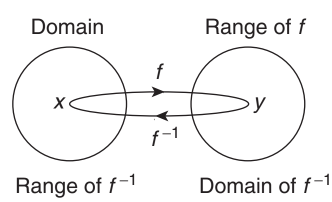


Figure 1.41

1. **How to find f^{-1}**
   Step 1: Solve the equation $y = f(x)$ for x in terms of y.
   Step 2: Interchange x and y. The resulting formula is $y = f^{-1}(x)$.

Example: Inverse of $y = (1/2)x + 1$
   Step 1: $x = 2y - 2$.
   Step 2: $y = f^{-1}(x) = 2x - 2$ which is the inverse of $f(x) = (1/2)x + 1$.

2. **How to draw graph of f^{-1}**

Reflect the graph of $g(x)$ about the line $y = x$ (Fig. 1.42).
Example: $f(x) = x^2 + 1, x \ge 0$.

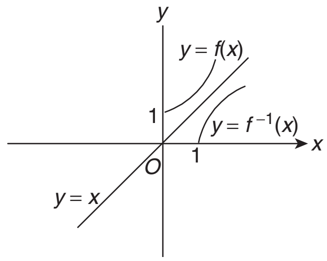


Figure 1.42

Example: $f(x) = \cos x, x \in [0, \pi]$ (Fig. 1.43).

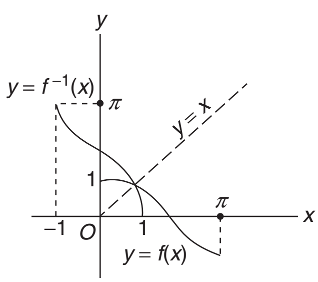


Figure 1.43

Note: Given an invertible function f, it is not always possible to find an explicit formula for f^{-1}. For example, if $f(x) = x + \sin x$, then f is invertible, but f^{-1} cannot be found. However, the graph of $y = f^{-1}(x)$ can be drawn.

Definition: A function whose domain and range are the subsets of the set of real numbers, R, is called a 'real-valued function' or a 'real function'. For a real function, f, the set of values of x for which $f(x)$ is a real number is defined as "domain of definition" and corresponding set of values of $f(x)$ is defined as the 'range of definition'.

Note: To find the domain of complicated functions, we can use the following relations:

1. Domain $(f + g)$ = (Domain f) $\cap$ (Domain g).
2. Domain (fg) = (Domain f) $\cap$ (Domain g).
3. Domain (f/g) = (Domain f) $\cap$ (Domain g) $- \{x: g(x) = 0\}$.
4. Domain $[\sqrt{f(x)}] = \{x : f(x) \ge 0\}$.
5. Domain $[\log_a f(x)] = \{x: f(x) > 0\}$, when $a > 0, a \ne 1$.
6. Domain $[a^{f(x)}]$ = Domain f, when $a > 0$.

Illustration 1.41 Let $f: R \to R$ defined by $f(x) = (e^x - e^{-x})/2$. Is $f(x)$ invertible? If so, find its inverse.

Solution: To check for invertibility of $f(x)$:

(i) For one-to-one function: Let $x_1 x_2 \in R$ and $x_1 < x_2$. Then

$$e^{x_1} < e^{x_2} \quad \text{(As } e > 1) \quad (1)$$

Also,

$$x_1 < x_2 \Rightarrow -x_2 < -x_z$$
$$\Rightarrow e^{-x_2} < e^{-x_1} \quad (2)$$

On adding Eqs. (1) and (2), we get

$$\frac{1}{2}(e^{x_1} - e^{-x_1}) < \frac{1}{2}(e^{x_2} - e^{-x_2})$$
$$\Rightarrow f(x_1) < f(x_2)$$

where f is an increasing function. Hence, $f(x)$ is a one-to-one function.

(ii) For onto function: Since $x \to \infty$, $f(x) \to \infty$. Similarly, at $x \to -\infty$, $f(x) \to -\infty$. That is,

$$-\infty < f(x) < \infty, x \in (-\infty, \infty)$$

Hence, the range of f is the same as the set R. Therefore, $f(x)$ is an onto function. Since $f(x)$ is both one-one and onto functions, $f(x)$ is invertible.

(iii) To find f^{-1}: We have

$$y = f(x)$$
$$y = \frac{e^x - e^{-x}}{2}$$
$$\Rightarrow e^x - e^{-x} = 2y$$
$$\Rightarrow e^{2x} - 2ye^x - 1 = 0$$
$$\Rightarrow e^x = \frac{2y \pm \sqrt{4y^2 + 4}}{2}$$
$$\Rightarrow e^x = y \pm \sqrt{y^2 + 1}$$
$$\Rightarrow e^x = y + \sqrt{y^2 + 1} \quad (e^x \text{ is positive})$$

Now,

$$x = \log_e (y + \sqrt{y^2 + 1})$$
$$\Rightarrow f^{-1}(x) = \log_e (x + \sqrt{x^2 + 1}).$$

Your Turn 13

1. Define a function $f(x)$ suitably in the interval $[0, \infty)$ so that $f(x)$ may be an (a) even real function and (b) odd real function whose definition is as follows: $f(x) = \begin{cases} 1, & x < -1 \\ -x, & -1 \le x \le 0 \end{cases}$

   Ans. (a) $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$; (b) $f(x) = \begin{cases} -x & 0 \le x \le 1 \\ -1 & x > 1 \end{cases}$

2. Find the period of each of the following functions (where [•] denotes the greatest integer function): (a) $f(x) = \sin x + \tan x$, (b) $f(x) = |\sin x| + |\cos x|$, (c) $f(x) = 2 + x - [x]$, (d) $f(x) = \sin x + x - [x]$ and (e) $f(x) = \cos(\cos x) + \cos(\sin x)$.

   Ans. (a) 2π; (b) $\pi/2$; (c) 1; (d) The function is not periodic; (e) $\pi/2$

3. Find the period of each of the following functions (if exists): (a) $f(x) = \cos(6 - 3x)$, (b) $f(x) = |\sin x| - |\cos x|$, (c) $f(x) = nx - [nx]$, $n \in N$, (d) $f(x) = \cos 2x + \tan 2x + \cos 2^2 x + \tan 2^2 x + \cdots + \cos 2^n x + \tan 2^n x$ and (e) $f(x) = \cos\sqrt{x}$.

   Ans. (a) $2\pi/3$; (b) π; (c) $1/n$; (d) π; (e) Non-periodic

4. (a) Let f: $[-\sqrt{2}+1, \sqrt{2}+1] \to \left[\dfrac{-\sqrt{2}+1}{2}, \dfrac{\sqrt{2}+1}{2}\right]$ be a function which is defined by $f(x) = \dfrac{1-x}{1+x^2}$. Show that f is invertible and find its inverse. (b) Show that $f: \left(-\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right) \to R$ which is defined by $f(x) = \dfrac{3x - x^3}{1-3x^2}$ is invertible and f^{-1} is an increasing function. (c) Prove that the function $f(x) = x + \sin x$ is not a periodic function.

5. Are each of the following functions identical? (a) $f(x) = x/x^2$ and $\phi(x) = 1/x$; (b) $f(x) = x^2/x$ and $\phi(x) = x$; (c) $f(x) = x$ and $\phi(x) = \sqrt{x^2}$ and (d) $f(x) = \log x^2$ and $\phi(x) = 2\log x$.

Ans. (a) Yes; (b) No; (c) No; (d) No

1.17.9 Basic Transformation on Graph

1. Drawing Graph of $y = f(x) + b$, $b \in R$ from Known Graph of $y = f(x)$: It is obvious that the domain of $f(x)$ and $f(x) + b$ are the same. Let us take any point x_0 in the domain of $f(x)$.

$$y\big|_{x=x_0} = f(x_0)$$

The corresponding point on $f(x) + b$ is $f(x_0) + b$. Now,

$$b > 0 \Rightarrow f(x_0) + b > f(x_0)$$

which means that the corresponding point on $f(x) + b$ is lying at a distance b units above the point on $f(x)$. Now,

$$b < 0 \Rightarrow f(x_0) + b < f(x_0)$$

which means that the corresponding point on $f(x) + b$ is lying at a distance b units below the point on $f(x)$. Accordingly, the graph of $f(x) + b$ can be obtained by translating the graph of $f(x)$ either in the positive direction on y-axis (if $b > 0$) or in the negative direction on y-axis (if $b < 0$), through a distance $|b|$ units (Fig. 1.44).

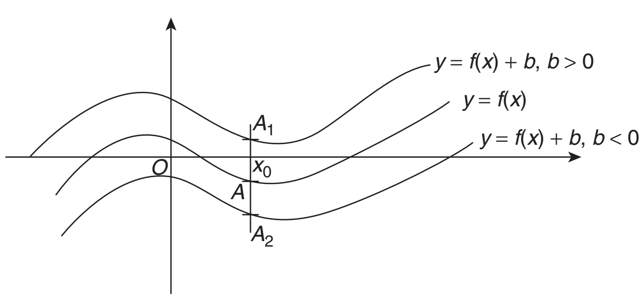


Figure 1.44

2. Drawing Graph of $[y] = f(x)$ from Known Graph of $y = f(x)$: It is clear that $[y] = f(x)$ makes sense only when $f(x)$ is an integer. If $f(x)$ is not an integer, the graph of $[y] = f(x)$ would not exist. As such, first of all, we locate those points which make $f(x)$ as an integer. To do this, we draw lines parallel to x-axis passing through different integral points lying on y-axis till the entire graph of $f(x)$ is covered up. The procedure is depicted in Fig. 1.45.

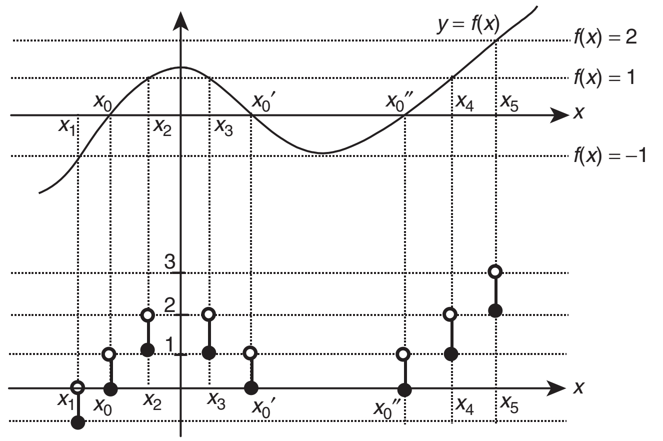


Figure 1.45

3. Drawing Graph of $y = [f(x)]$ from Known Graph of $y = f(x)$: It is clear that if $n \le f(x) < n + 1$, $n \in I$, then $[f(x)] = n$. Thus, we would draw the lines parallel to x-axis passing through different integral points. Hence, the values of x can be obtained so that $f(x)$ lies between two successive integers. This procedure is depicted in Fig. 1.46.

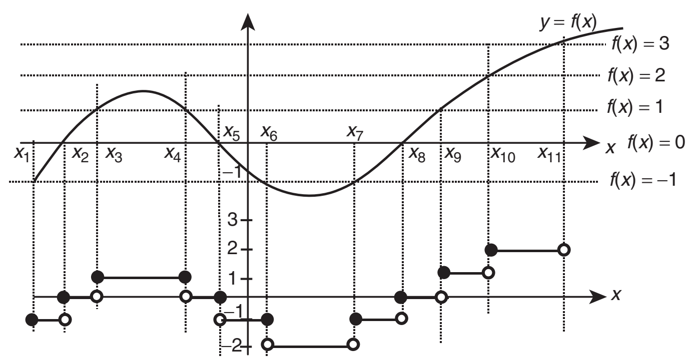


Figure 1.46

4. Drawing Graph of $y = f([x])$ from Known Graph of $y = f(x)$: We have

$$x \in [0, 1) \Rightarrow [x] = 0 \Rightarrow f([x]) = f(0) \;\; \forall x \in [0, 1)$$

Similarly,

$$f([x]) = f(1) \;\forall x \in [1, 2)$$

In general,

$$f([x]) = f(n) \;\forall x \in [n, n+1),\, n \in I$$

which implies that we just draw lines parallel to y-axis passing through the different integral points lying on x-axis. Fig. 1.47 depicts the procedure.

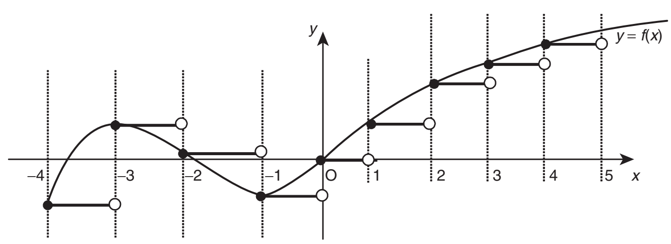


Figure 1.47

5. **Drawing Graph of $y = |f(x)|$ from Known Graph of $y = f(x)$:** We have

$$|f(x)| = f(x) \text{ if } f(x) \geq 0 \text{ and } |f(x)| = -f(x) \text{ if } f(x) < 0$$

which means that the graph of $f(x)$ and $|f(x)|$ would coincide if $f(x) \geq 0$ and the sections, where $f(x) < 0$, get inverted in the upwards direction. Figure 1.48 depicts the procedure.

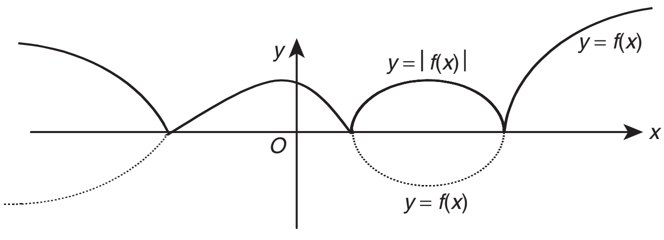


Figure 1.48

6. **Drawing Graph of $y = f(|x|)$ from Known Graph of $y = f(x)$:** It is clear that

$$f(|x|) = \begin{cases} f(x), & x \geq 0 \\ f(-x), & x < 0 \end{cases}$$

Thus, $f(|x|)$ is an even function. Graphs of $f(|x|)$ and $f(x)$ would be identical in the first and the fourth quadrants (since $x \geq 0$) and the graph of $f(|x|)$ is symmetrical about y-axis (since $(|x|)$ is even). The procedure is depicted in Fig. 1.49.

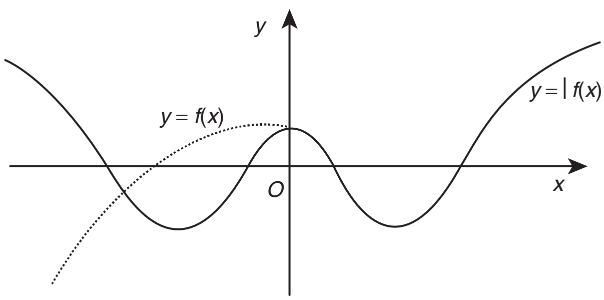


Figure 1.49

7. **Drawing Graph of $|y| = f(x)$ from Known Graph of $y = f(x)$:** We have $|y| \geq 0$. If $f(x) < 0$, the graph of $|y| = f(x)$ does not exist and if $f(x) \geq 0$, $|y| = f(x)$ would give $y = \pm f(x)$. Hence, the graph of $|y| = f(x)$ exists only in the regions where $f(x)$ is non-negative and it is reflected about x-axis only on those regions. The regions where $f(x) < 0$ are neglected. The procedure is depicted in Fig. 1.50. The dotted lines show the graph of $y = f(x)$ and the normal lines depict the corresponding graph of $|y| = f(x)$.

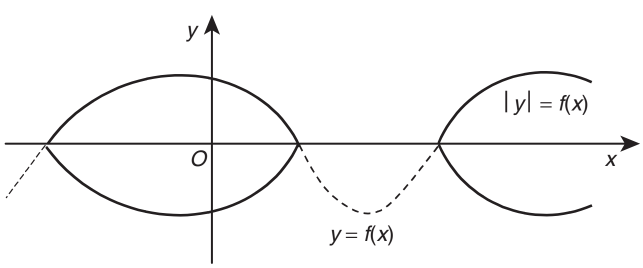


Figure 1.50

8. **Drawing Graph of $y = f(x + a)$, $a \in R$ from Known Graph of $y = f(x)$:** Let us take any point $x_0 \in$ domain of $f(x)$, and set $x + a = x_0$ or $x = x_0 - a$. Now,

$$a > 0 \Rightarrow x < x_0 \text{ and } a < 0 \Rightarrow x > x_0$$

which tells that the mean x_0 and $x_0 - a$ give us the same abscissa for $f(x)$ and $f(x + a)$, respectively. As such for $a > 0$, the graph of $f(x + a)$ can be obtained simply by translating the graph of $f(x)$ in the negative direction on x-axis through a distance a units. If $a < 0$, the graph of $f(x + a)$ can be obtained by translating the graph of $f(x)$ in the positive direction on x-axis through a distance a units. The procedure is depicted in Fig. 1.51.

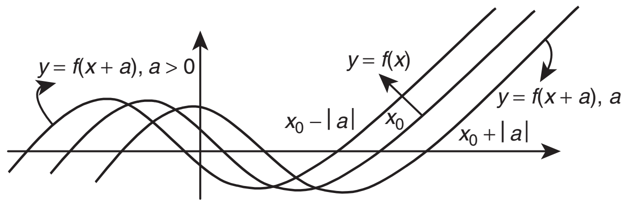


Figure 1.51

9. **Drawing Graph of $y = af(x)$ from Known Graph of $y = f(x)$:** We know that the corresponding points (points with the same x-coordinates) have their ordinates in the ratio of $1 : a$ (where $a > 0$). Figure 1.52 depicts the procedure.

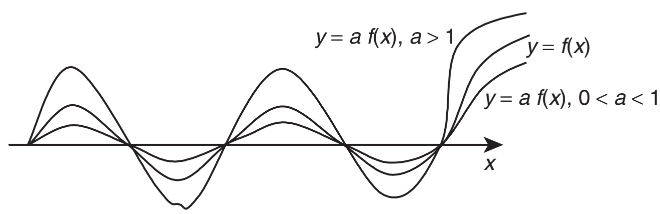


Figure 1.52

10. **Drawing Graph of $y = f(ax)$ from Known Graph of $y = f(x)$:** Let us consider any point $x_0 \in$ domain of $f(x)$. Let

$$ax = x_0 \text{ or } x = \frac{x_0}{a}$$

Now, if $0 < a < 1$, then $x > x_0$ and $f(x)$ will stretch by a units against y-axis, and if $a > 1$, $x < x_0$, then $f(x)$ compresses by a units against y-axis. Note that the points of maxima and minima are on the line parallel to x-axis for both curves. Figure 1.53 depicts the procedure.

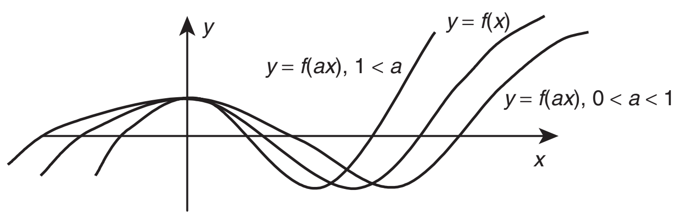


Figure 1.53

11. **Drawing Graph of $y = f(x)\sin x$ from Known Graph of $y = f(x)$:** We have

$$-f(x) \leq f(x)\sin x \leq f(x)$$

Hence, the graph of $f(x)\sin x$ lies in between the graphs of $y = f(x)$ and $y = -f(x)$. It leads to just drawing the graph of $\sin x$ in between the graphs of $y = \pm f(x)$, for example, the graph of $y = x\sin x$ is shown in Fig. 1.54.

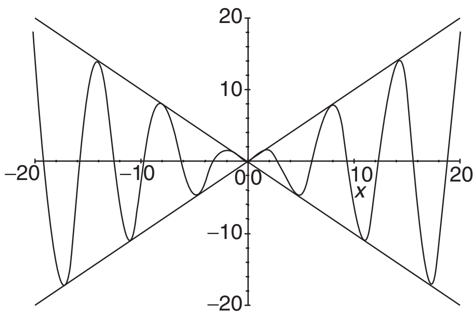


Figure 1.54

Note: If $f(x)$ is even/odd function, $f(x)\sin x$ becomes odd/even function and we need to pay attention to the symmetry of $f(x)\sin x$. Similar treatment can be given to $y = f(x)\cos x$.

12. Drawing Graph of $y = f^{-1}(x)$ from Known Graph of $y = f(x)$: For drawing the graph of $y = f^{-1}(x)$, first, we need to find the interval in which the function is bijective (i.e. invertible). Then, take the reflection of $y = f(x)$ (within the invertible region) about the line $y = x$. The reflected region gives the graph of $y = f^{-1}(x)$. For example, let us draw the graph of $y = \sin^{-1}x$ (Fig. 1.55). We know that $y = f(x) = \sin x$ is invertible if

$$f: [-\pi/2, \pi/2] \to [-1, 1]$$

which implies that the inverse mapping is

$$f^{-1}: [-1, 1] \to [-\pi/2, \pi/2]$$

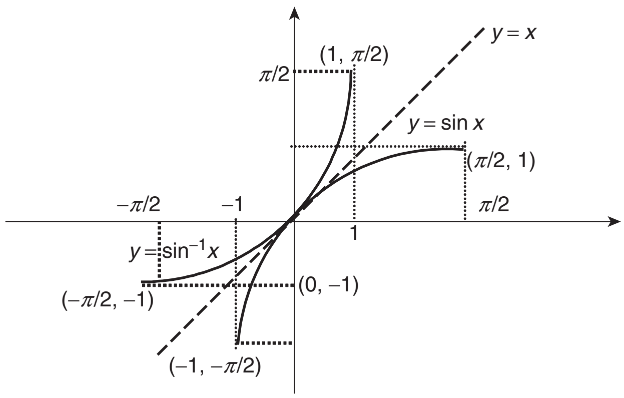


Figure 1.55

Note: It is clear that at least one root of the equation $f(x) - f^{-1}(x) = 0$ would lie on the line $y = x$ when the root exists.

Illustration 1.42 Draw the graphs of the following: (a) $y = [\sin x]$, (b) $|y| = \sin x$, (c) $[y] = \cos x$ and (d) $y = \cos[x]$, where $x \in [0, 2\pi]$.

Solution:

(a) The graph of $y = [\sin x]$ is shown in Fig. 1.56.

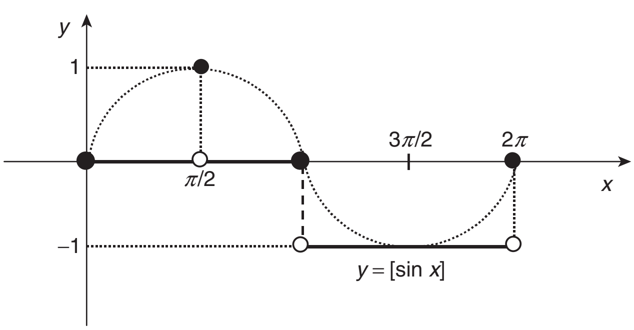


Figure 1.56

(b) The graph of $|y| = \sin x$ is shown in Fig. 1.57.

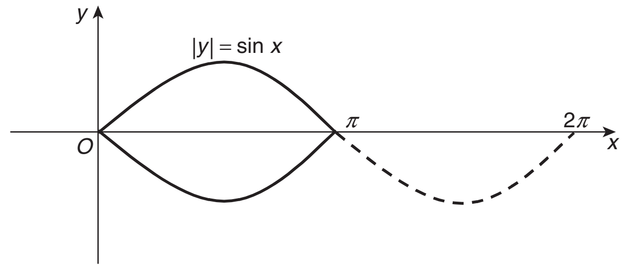


Figure 1.57

(c) The graph of $[y] = \cos x$ is shown in Fig. 1.58.

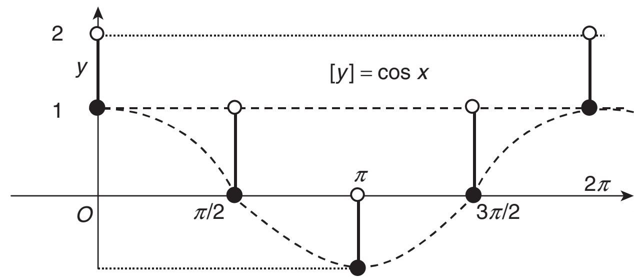


Figure 1.58

(d) The graph of $y = \cos[x]$ is shown in Fig. 1.59.

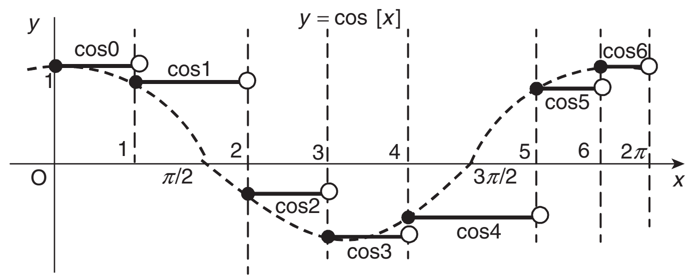
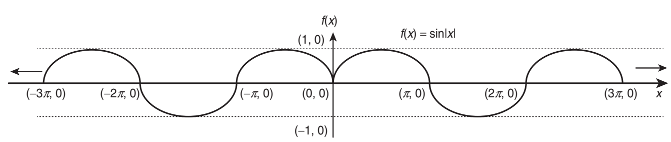


Figure 1.59

Your Turn 14

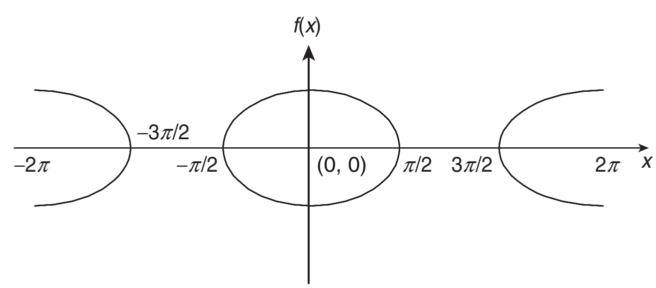
1. Draw the graph of $f(x) = \sin|x|$, $x \in [-2\pi, 2\pi]$.

Ans.



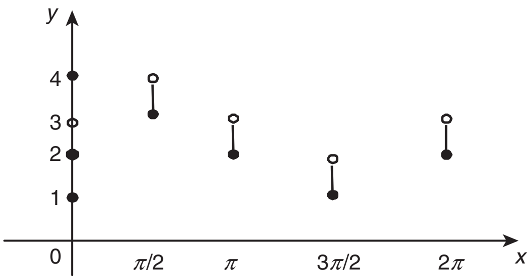
2. Draw the graph of $|f(x)| = \cos x$, $x \in [-2\pi, 2\pi]$.

Ans.



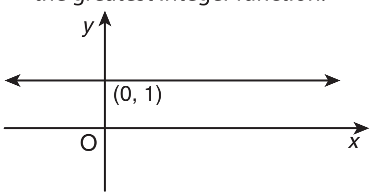
3. Draw the graph of $[f(x)] = 2 + \sin x$, $x \in [0, 2\pi]$ where [·] denotes the greatest integer function.

Ans.



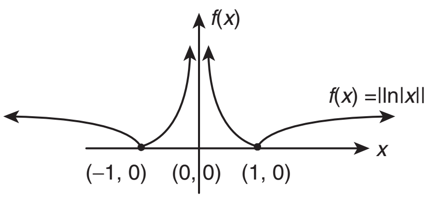
4. Draw the graph of $f(x) = [|\sin x| + |\cos x|]$, $x \in R$ where [·] denotes the greatest integer function.

Ans.



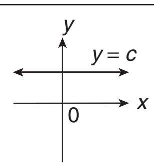
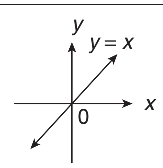
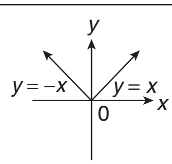
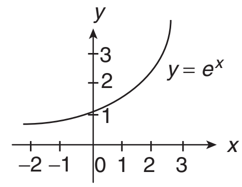
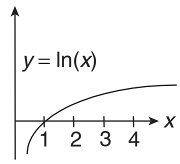
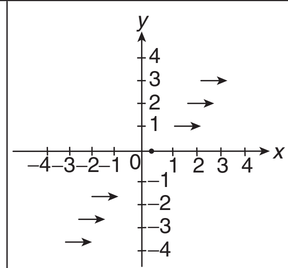
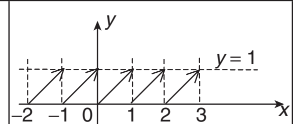
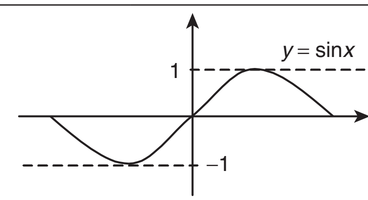
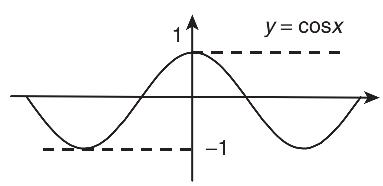
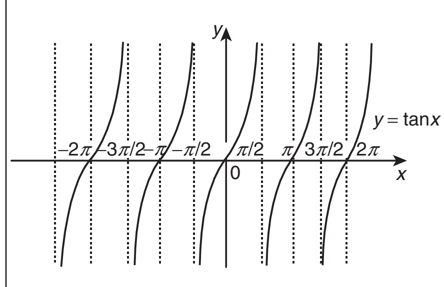
5. Draw the graph of $f(x) = |\ln|x||$, $x \in R \sim \{0\}$.

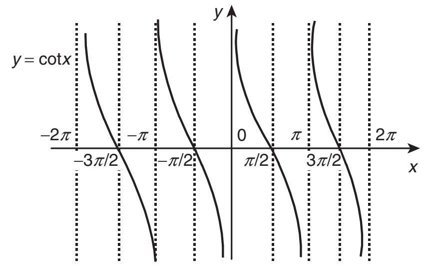
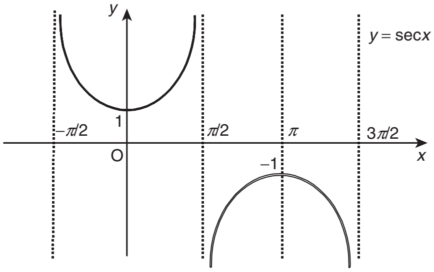
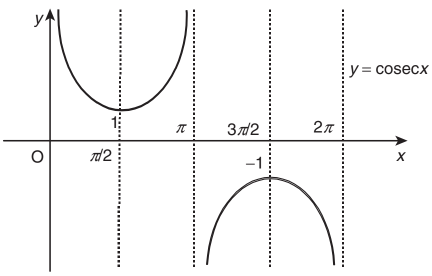
Ans.



Domain and Range of Some Important Real Functions and Their Graphs: A convenient and useful method for studying a function is to study it through its graph. To draw the graph of a function $f: X \to Y$, we choose a system of coordinate axes in the plane such that to each $x \in X$, there corresponds the ordered pair $[x, f(x)]$ which determines a point in the plane. The set of all points $\{(x, f(x)): x \in X\}$ is the graph of f. We discuss some examples of functions and their graphs in Table 1.5.

Table 1.5 Functions and their graphs

S. No.	Function	Domain and Range of Definition	Graph
1.	A constant function	$f: R \to \{c\}$ defined by $f(x) = c$	
2.	The identity function	$f: R \to R$ defined by $f(x) = x$	
3.	The absolute value function	$f: R \to [0, \infty)$ defined by $f(x) = \|x\|$	
4.	The exponential function	$f: R \to (0, \infty)$ defined by $f(x) = e^x$	
5.	The natural logarithmic function	$f: (0, \infty) \to R$ defined by $f(x) = \ln x$	
6.	The greatest integer function	$f: R \to Z$ defined by $f(x) = [x]$ the greatest integer $\le x$	
7.	The fractional part of x	$f: R \to R$ defined by $f(x) = \{x\}$	
8.	Polynomial functions	$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ where $a_0, a_1, \ldots, a_n$ are real numbers, $a_0 \ne 0$.	
9.	Rational functions	$f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials in x. Domain is $R - \{x: q(x) = 0\}$.	
10.	Trigonometric or circular functions	  	

Definitions:

1. **Bijective Function:** If a function f is both one-to-one function and onto function, then f is said to be a bijective function.
2. **Inverse of a Function:** If $f: X \to Y$ be a function which is defined by $y = f(x)$ such that f is both one-to-one function and onto function, then there exists a unique function $g: Y \to X$ such that for each $y \in Y$, $g(y) = x$ if and only if $y = f(x)$. The function g so defined is called the 'inverse of f' which is denoted by f^{-1}. Also if g is the inverse of f, then f is the inverse of g and the two functions f and g are said to be inverses of each other.

Note: Let $f: X \to Y$ be a bijective function, then

(i) Domain $f = X =$ Range f^{-1}; Range $f = Y =$ Domain f^{-1}.
(ii) $f^{-1}: Y \to X$ is also bijective.
(iii) $f^{-1} o f(x) = x\ \forall x \in$ Domain f and $f o f^{-1}(x) = x\ \forall x \in$ Domain f^{-1}.
(iv) The graphs of $y = f(x)$ and $y = f^{-1}(x)$ are symmetric about the line $y = x$ for a real function f.

Additional Solved Examples

1. If $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$, then does (a) $\{(1, a), (2, b), (2, c), (3, c)\}$ and (b) $\{(2, b), (3, b)\}$ represent a function $A \to B$.

Solution:

(a) Since two-ordered pairs $(2, b)$, $(2, c)$ have the same first coordinates, $\{(1, a), (2, b), (2, c), (3, c)\}$ does not represent a function, $A \to B$.

(b) Since the element 1 of A is not associated with some element of B, that is, 1 is not the first coordinate of any ordered pair so $\{(2, b), (3, b)\}$ does not represent a function $A \to B$. (See Fig. 1.60.)

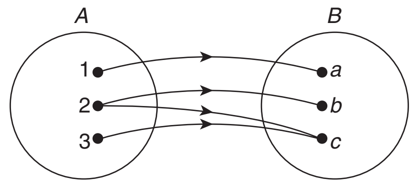


Figure 1.60

2. If the functions f and g are given by $f = \{(1, 2), (3, 5), (4, 6)\}$ and $g = \{(2, 3), (5, 1), (6, 3)\}$, then find fog and gof.

Solution: It is given that

$$f(1) = 2;\ f(3) = 5;\ f(4) = 6$$

and

$$g(2) = 3;\ g(5) = 1;\ g(6) = 3$$

$$fog(2) = f\{g(2)\} = f(3) = 5$$
$$fog(5) = f\{g(5)\} = f(1) = 2$$
$$fog(6) = f\{g(6)\} = f(3) = 5$$
$$\Rightarrow fog = \{(2, 5), (5, 2), (6, 5)\}$$

Similarly,

$$gof = \{(1, 3), (3, 1), (4, 3)\}$$

3. Given the function $f(x) = \dfrac{a^x + a^{-x}}{2}$ $(a > 0)$. Show that $f(x + y) + f(x - y) = 2f(x)f(y)$.

Solution: Given that

$$f(x) = \frac{a^x + a^{-x}}{2} \quad (1)$$

Therefore,

$$f(y) = \frac{a^y + a^{-y}}{2}$$

$$f(x+y) = \frac{a^{x+y} + a^{-(x+y)}}{2};\ f(x-y) = \frac{a^{x-y} + a^{-(x-y)}}{2}$$

$$f(x+y) + f(x-y) = \frac{a^{x+y} + a^{-(x+y)} + a^{x-y} + a^{-(x-y)}}{2}$$

$$= \frac{a^x a^y + a^{-x}a^{-y} + a^x a^{-y} + a^{-x}a^y}{2}$$

$$= \frac{a^y(a^x + a^{-x}) + a^{-y}(a^x + a^{-x})}{2}$$

$$= 2\,\frac{(a^x + a^{-x})}{2}\cdot\frac{(a^y + a^{-y})}{2} = 2f(x)f(y)$$

4. Let $f: R \to R$ be defined by $f(x) = \cos(5x + 2)$. Is f invertible? Justify your answer.

Solution: We know that any function $f: A \to B$ is invertible if and only if it is bijective. Now, $f: R \to R$, which is given by $f(x) = \cos(5x + 2)$ is neither injective nor surjective. For $-1 \le \cos(5x + 2) \le 1$, the range of $f \ne R$. Therefore, f is not surjective. Also

$$f\left(x + \frac{2\pi}{5}\right) = \cos\left\{5\left(x + \frac{2\pi}{5}\right) + 2\right\} = \cos(2\pi + 5x + 2) = \cos(5x + 2) = f(x)$$

Therefore, f is not injective. Thus, f is not invertible. For the existence of inverse of a function, the given function must be one-to-one and onto.

Note: All periodic functions are many-to-one. Hence, they are not invertible when they are defined on the whole of R. Each of these can be made invertible on a restricted domain. For example, if

$y = \sin x$ is restricted to the domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then it is invertible and its inverse is given by $y = \sin^{-1}x$, where $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

5. If $f(x)$ is a periodic function with principal period T then prove that the function $f(ax + b)$ is periodic with principal period $T/|a|$.

Solution: We have

$$f\left[a\left(x+\frac{T}{|a|}\right)+b\right] = f\left(ax+b+\frac{aT}{|a|}\right) = f(ax+b)$$

[$\therefore$ T is a period of $f(x)$]

Thus,

$$T/|a| \text{ is a period of } f(ax+b) \quad (1)$$

Further, let a real number $t > 0$ be a period of $f(ax + b)$. Then

$$f(ax+b) = f[a(x+t)+b],\ \forall x = f(ax+b+at),\ \forall x$$

$$\Rightarrow |a|t \text{ is a period of } f(x)$$

$$\Rightarrow |a|t \geq T \quad (\therefore T \text{ is the principal of } f(x))$$

$$\Rightarrow t \geq \frac{T}{|a|} \quad (2)$$

From Eqs. (1) and (2), it follows that $\frac{T}{|a|}$ is the principal period of $f(ax + b)$.

6. Let $f(x)$ be a polynomial function of degree n satisfying the condition $f(x) + f(1/x) = f(x)f(1/x)$ $\forall x \in R - \{0\}$. Then, prove that $f(x) = 1 \pm x^n$.

Solution: Let us consider that $f(x) = a_0 + a_1x + \cdots + a_nx^n$. Thus,

$$(a_0 + a_1x + \cdots + a_nx^n) + \left(a_0 + \frac{a_1}{x} + \cdots + \frac{a_n}{x^n}\right)$$

$$= (a_0 + a_1x + \cdots + a_nx^n)\left(a_0 + \frac{a_1}{x} + \cdots + \frac{a_n}{x^n}\right)$$

Multiplying on both sides with x^n, we get

$$(a_0x^n + a_1x^{n+1} + \cdots + a_nx^{2n}) + (a_0x^n + a_1x^{n-1} + - + a_n)$$
$$= (a_0 + a_1x + \cdots + a_nx^n)(a_0x^n + a_1x^{n-1} + \cdots + a_n)$$

Equating the coefficients of $x^{2n}, x^{2n-1}, \ldots, x^{n+1}$ on both sides, we get

$$a_n = a_0a_n \Rightarrow a_0 = 1 \quad [\text{since } a_n \neq 0]$$

That is,

$$a_{n-1} = a_{n-1}a_0 + a_na_1 \Rightarrow a_na_1 = 0 \Rightarrow a_1 = 0$$

Similarly, we get

$$a_2 = a_3 = \cdots = a_{n-1} = 0$$

Now, equate the coefficient of x^n on both sides, we get

$$2a_0 = a_0^2 + a_1^2 + \cdots + a_{n-1}^2 + a_n^2$$
$$\Rightarrow a_n^2 = 1 \Rightarrow a_n = \pm 1$$

Hence, $f(x) = 1 \pm x^n$.

7. Let

$$f(x) = \begin{cases} 1 + x^3, & x < 0 \\ x^2 - 1, & x \geq 0 \end{cases}$$

$$g(x) = \begin{cases} (x-1)^{1/3}, & x < 0 \\ (x+1)^{1/2}, & x \geq 0 \end{cases}$$

Then, evaluate $g[f(x)]$.

Solution: When $x < 0$, $f(x) = 1 + x^3$. For $x < 0$, $1 + x^3 \in (-\infty, 1)$. Now, $(-\infty, 0) = (-\infty, -1) \cup [-1, 0)$. Similarly, when $x > 0$,

$$f(x) = x^2 - 1$$

For $x \geq 0$, $x^2 - 1 \in [-1, \infty)$. Now,

$$[0, \infty) = [0, 1) \cup [1, \infty) \text{ and } [-1, \infty) = [-1, 0) \cup [0, \infty)$$

$$G[f(x)] = \begin{cases} x & x < -1 \\ (x^3+2)^{\frac{1}{2}} & -1 \leq x < 0 \\ (x^2-2)^{\frac{1}{3}} & 0 \leq x < 1 \\ x & x \geq 1 \end{cases}$$

8. Find the domain and range of the following functions: (a) $f(x) = (x^2 - 2x + 9)/(x^2 + 2x + 9)$ and (b) $f(x) = 1/\sqrt{|x| - x}$.

Solution:

(a) The domain of f is the set of all real values of x for which $f(x)$ is real. Since $x^2 + 2x + 9 > 0$ $\forall x \in R$, therefore, the domain of f is the whole set R. The range of f is the set of all real values of y for which x is real and a member of domain of f. Now,

$$y = \frac{x^2 - 2x - 9}{x^2 + 2x + 9} \Leftrightarrow yx^2 + 2yx + 9y = x^2 - 2x + 9$$

That is,

$$(y-1)x^2 + (2y+2)x + 9(y-1) = 0$$

Now, if $y = 1$, the above equation reduces to $x = 0$, that is, for $x = 0$, $y = 1$. Thus, $1 \in$ Range. Further if $y \neq 1$, then $(y - 1)x^2 + (2y + 2)x + 9(y - 1) = 0$ is a quadratic equation in x and has real roots if

$$(2y+2)^2 - 36(y-1)^2 \geq 0$$

That is,

$$2y^2 - 5y + 2 \leq 0$$

That is, if $-1/2 \leq y \leq 2$ which gives that $[-1/2, 2] - \{1\}$ is another part of the range. Hence, the range is $[-1/2, 2]$.

(b) We have

$$y = \frac{1}{\sqrt{|x| - x}}$$

The domain of f is the set of all real values of x for which y is real. Here, the fact that y is real implies that

$$|x| - x > 0 \Leftrightarrow x < 0$$

Therefore, the domain is $(-\infty, 0)$. The range is the set of all real values of y for which x is real and $x \in (-\infty, 0)$. Therefore, clearly we understand that

$$y > 0 \quad (1)$$

$$y = \frac{1}{\sqrt{|x| - x}}$$

$$\Rightarrow \sqrt{|x| - x} = \frac{1}{y} \Rightarrow |x| - x = \frac{1}{y^2} \Rightarrow -2x = \frac{1}{y^2}$$

Clearly,

$$x \text{ is real if } y \neq 0 \quad (2)$$

From Eqs. (1) and (2), x is real if $y > 0$. Hence, the range is $(0, \infty)$.

9. Find if $f(x) = (3x - 4)/(x^2 + 1)$ is one-to-one.

Solution: The domain of $f(x)$ is whole of set R.

$$y = \frac{3x - 4}{x^2 + 1}$$
$$\Rightarrow yx^2 - 3x + y + 4 = 0$$

which is quadratic in x if $y \neq 0$. Thus, this gives two real values of x if

$$9 - 4y(y + 4) > 0$$

and if $y \in (-9/2, 1/2)$. Therefore, f is not one-to-one function.

10. Does the inverse of $f(x) = \{2 + (x - 3)^3\}^{1/3}$ exist? If so, find it.

Solution: We have

$$f(x) = \{2 + (x - 3)^3\}^{1/3}$$

Since $2 + (x - 3)^3$ is a polynomial function, it is continuous. Its domain and range both equal to R. Hence, it is onto (surjective). Thus, $f(x)$ being a positive rational power of the continuous function is also continuous.

$$f(x) = \frac{1}{3}\{2 + (x - 3)^3\}^{-2/3}[3(x - 3)^2] = \frac{(x-3)^2}{[2+(x-3)^3]} > 0\ \forall x$$

except at $x = 3$ and $x = 3 - 2^{1/3}$. Therefore, $f(x)$ is monotonically strictly increasing and so invertible. Let

$$y = \{2 + (x - 3)^3\}^{1/3}$$

Then,

$$y^3 = 2 + (x - 3)^3$$

or

$$x = 3 + (y^3 - 2)^{1/3}$$

Thus,

$$f^{-1}(x) = 3 + (x^3 - 2)^{1/3}$$

11. If $(x^2 - 3x - 3)^{x+1} > 1$, find $x \in (-1, \infty)$.

Solution: Clearly, $x^2 - 3x - 3 > 0$ because it is the base. Therefore, by sign-scheme,

$$x < \frac{3-\sqrt{21}}{2} \text{ or } x > \frac{3+\sqrt{21}}{2}$$

Therefore, taking logarithm, we get

$$\log_{10}(x^2 - 3x - 3)^{x+1} > \log_{10} 1 = 0$$

or

$$(x + 1)\log_{10}(x^2 - 3x - 3) > 0$$

Following two cases arise:

Case 1: $x + 1 > 0$, $\log_{10}(x^2 - 3x - 3) > 0$
Case 2: $x + 1 < 0$, $\log_{10}(x^2 - 3x - 3) < 0$

Now,

$$x > -1,\ x^2 - 3x - 3 > 1$$

$$x^2 - 3x - 3 > 1 \Rightarrow x^2 - 3x - 4 > 0$$

The corresponding equation is

$$x^2 - 3x - 4 = 0$$

or

$$(x + 1)(x - 4) = 0$$

Therefore, $x = -1, 4$. Therefore, the sign-scheme of $x^2 - 3x - 4$, $x \in R$ is shown in Fig. 1.61.

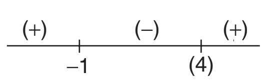


Figure 1.61

Therefore,

$$x^2 - 3x - 4 > 0 \Rightarrow x < -1 \text{ or } x > 4$$

However, $x > -1$. So $x > 4$. Similarly,

$$x < -1 \text{ and } -1 < x < 4$$

There is no such x exists. Therefore, the solution set is $(4, \infty)$.

12. If $f(x) = [\sin(\sin nx)]/[\tan(x/n)]$ has period 6π where $n \in N$. Find the minimum value of n.

Solution: The period of $\sin nx$ is $2\pi/n$ and the period of $\tan(x/n)$ is $\pi/(1/n) = n\pi$. Since, $f(x)$ can repeat only when $\sin(\sin nx)$ and $\tan(x/n)$ repeats at the same time. So,

$$\text{Period of } f(x) = \text{LCM}\left(\frac{2\pi}{n}, n\pi\right)$$

$$= \text{LCM}\left(\frac{2n\pi}{n^2}, \frac{2n\pi}{2}\right) = \frac{\text{LCM}(2n\pi, 2n\pi)}{\text{HCF}(n^2, 2)}$$

Case 1: Here, n^2 is even which implies that $n^2 = 2k$. The period of $f(x)$ is

$$\frac{2n\pi}{\text{HCF}(2k, 2)} = \frac{2n\pi}{2} = n\pi$$

$$\Rightarrow n\pi = 6\pi \Rightarrow n = 6$$

Case 2: Here, n^2 is odd which implies that $n^2 = 2k + 1$. The period of $f(x)$ is

$$\frac{2n\pi}{\text{HCF}(2k+1, 2)} = \frac{2n\pi}{1}$$

$$\Rightarrow 2n\pi = 6\pi \Rightarrow n = 3$$

13. Find the domain of the function

$$f(x) = \frac{1}{\sqrt{|\sin x| + \sin x}}$$

Solution: For $f(x)$ to take real value,

$$|\sin x| + \sin x > 0$$
$$\Rightarrow |\sin x| > -\sin x$$

Draw the graph of $y = |\sin x|$ and $y = -\sin x$ in $x \in [0, 2\pi]$. From graph in Fig. 1.62, $|\sin x| > -\sin x\ \forall x \in (0, \pi)$. Generalise the answer to get

$$\text{Domain of } f(x) \equiv [2n\pi, (2n + 1)\pi]$$

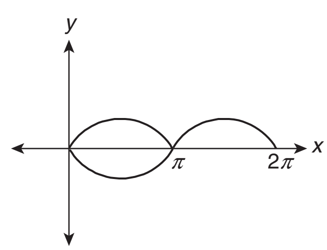


Figure 1.62

14. If $f(x) = \frac{\sin x}{\sqrt{1+\tan^2 x}} - \frac{\cos x}{\sqrt{1+\cot^2 x}}$. Find the range of $f(x)$.

Solution: We know that

$$f(x) = \frac{\sin x}{|\sec x|} - \frac{\cos x}{|\operatorname{cosec} x|}$$

Clearly, the domain of $f(x)$ is

$$R \sim \left\{n\pi, (2n+1)\frac{\pi}{2} : n \in I\right\}$$

and period of $f(x)$ is 2π.

$$f(x) = \begin{cases} 0 & x \in \left(0, \frac{\pi}{2}\right) \\ -\sin 2x & x \in \left(\frac{\pi}{2}, \pi\right) \\ 0 & x \in \left(\pi, \frac{3\pi}{2}\right) \\ \sin 2x & x \in \left(\frac{3\pi}{2}, 2\pi\right) \end{cases}$$

Thus, the range of $f(x)$ is $(-1, 1)$.

15. Let $f(x, y)$ be a periodic function which satisfies the condition $f(x, y) = f(2x + 2y, 2y - 2x)$ $\forall x, y \in R$ and let $g(x)$ the function defined as $g(x) = f(2^x, 0)$. Prove that $g(x)$ is periodic and find its period.

Solution: We have

$$f(x, y) = f(2x + 2y, 2y - 2x)$$
$$= f[2(2x + 2y) + 2(2y - 2x), 2(2y - 2x) - 2(2x - 2y)]$$

That is,

$$f(x, y) = f(8y, -8x)$$
$$= f[8(-8x), -8(8y)]$$
$$= f(-64x, -64y)$$
$$= f[64(64x), 64(64)y]$$
$$= f(x, y) = f(2^{12}x, 2^{12}y)$$

That is,

$$f(x, 0) = f(2^{12}x, 0)$$

Therefore,

$$f(2^y, 0) = f(2^{12}2^y, 0)$$
$$f(2^y, 0) = f(2^{12+y}, 0)$$
$$\Rightarrow g(y) = g(y + 12)$$

Hence, $g(x)$ is periodic and its period is 12.

Previous Years' Solved JEE Main/AIEEE Questions

1. The largest interval lying in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for which the function $\left[f(x) = 4^{-x^2} + \cos^{-1}\left(\frac{x}{2} - 1\right) + \log(\cos x)\right]$ is defined is

(A) $[0, \pi]$ **(B)** $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(C) $\left[-\frac{\pi}{4}, \frac{\pi}{2}\right)$ **(D)** $\left[0, \frac{\pi}{2}\right)$

[AIEEE 2007]

Solution:

$$4^{-x^2} = \frac{1}{4^{x^2}} \text{ defined } \forall x \quad (1)$$

$$0 \le \cos^{-1}\left(\frac{x}{2} - 1\right) \le \pi$$

$$\Rightarrow \cos^{-1} \le \cos^{-1}\left(\frac{x}{2} - 1\right) \le \cos^{-1}(-1)$$

$$\Rightarrow -1 \le \frac{x}{2} - 1 \le 1 \Rightarrow 1 \ge \frac{x}{2} \ge 2 \quad (2)$$

$$\Rightarrow 0 \le x \le 4$$

$\log \cos x$ is defined when $\cos x > 0$, that is, in $\left[0, \frac{\pi}{2}\right)$ (3)

(A part of domain where $\cos x > 0$)

Therefore, from Eqs. (1), (2) and (3), largest interval is

$$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left[0, \frac{\pi}{2}\right)$$

Hence, the correct answers are options (B) and (D).

2. Let $f : R \rightarrow R$ be a function defined by $f(x) = \text{Min}\{x+1, |x|+1\}$. Then which of the following is true?

(A) $f(x) \ge 1$ for all $x \in R$

(B) $f(x)$ is not differentiable at $x = 1$

(C) $f(x)$ is differentiable everywhere

(D) $f(x)$ is not differentiable at $x = 0$ **[AIEEE 2007]**

Solution: It is given that $f(x) = \text{Min}\{x+1, |x|+1\}$.

From Fig. 1.63, we have $f(x) = x + 1 \; \forall x \in R$

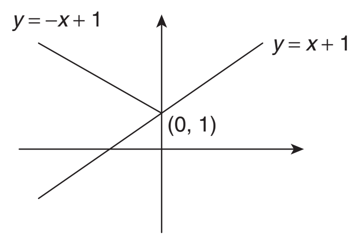


Figure 1.63

When $x < 0$, $(x + 1)$ is minimum.

When $x \ge 0$, $(x + 1)$ is minimum.

Therefore, overall $f(x) = x + 1$ is differentiable in R.

Hence, the correct answer is option (C).

3. Let $f : N \rightarrow Y$ be a function defined as $f(x) = 4x + 3$, where $Y = \{y \in N : y = 4x + 3 \text{ for some } x \in N\}$. Show that f is invertible and its inverse is

(A) $g(y) = \frac{3y+4}{3}$ **(B)** $g(y) = 4 + \frac{y+3}{4}$

(C) $g(y) = \frac{y+3}{4}$ **(D)** $g(y) = \frac{y-3}{4}$

[AIEEE 2008]

Solution: Let

$$f(x_1) = f(x_2), x_1, x_2 \in N \Rightarrow 4x_1 + 3 = 4x_2 + 3 \Rightarrow x_1 = x_2$$

Thus

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

Hence, the function is one-one.

Let $y \in Y$ be a number of the form $y = 4k + 3$ for some $k \in N$. Then

$$y = f(x)$$
$$\Rightarrow 4k + 3 = 4x + 3 \Rightarrow x = k \in N$$

Thus, the function is onto.

The function, being both one-one and onto is invertible.

$$y = 4x + 3 \Rightarrow x = \frac{y-3}{4}$$

Therefore,

$$f^{-1}(x) = \frac{x-3}{4} \Rightarrow g(y) = \frac{y-3}{4}$$

$g(y)$ is the inverse of the function.

Hence, the correct answer is option (D).

4. Let R be the real line. Consider the following subsets of the plane $R \times R$.
$S = \{(x, y) : y = x + 1 \text{ and } 0 < x < 2\}$, $T = \{(x, y) : x - y \text{ is an integer}\}$.
Which one of the following is true?

(A) neither S nor T is an equivalence relation on R
(B) both S and T are equivalence relations on R
(C) S is an equivalence relation on R but T is not
(D) T is an equivalence relation on R but S is not

[AIEEE 2008]

Solution: We have,

$$T = \{(x, y) : x - y \in I\}$$

As $0 \in I$, T is a reflexive relation. If

$$x - y \in I \Rightarrow y - x \in I$$

Then T is symmetrical as well.

If $x - y = I_1$ and $y - z = I_2$, then $x - z = (x - y) + (x - z) = I_1 + I_2 \in I$; therefore, T is transitive as well.

Hence, T is an equivalence relation. Clearly,

$$x \neq x + 1 \Rightarrow (x, x) \notin S$$

Therefore, S is not reflexive.

Hence, the correct answer is option (D).

5. If A, B and C are three sets such that $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then

(A) $A = B$ **(B)** $A = C$
(C) $B = C$ **(D)** $A \cap B = \phi$

[AIEEE 2009]

Solution:

$$A \cup B = A \cup C$$
$$\Rightarrow (A \cup B) \cap C = (A \cup C) \cap C$$
$$\Rightarrow (A \cap C) \cup (B \cap C) = C$$
$$\Rightarrow (A \cap B) \cup (B \cap C) = C \quad (1)$$

Again,

$$A \cup B = A \cup C$$
$$\Rightarrow (A \cup B) \cap B = (A \cup C) \cap B$$
$$\Rightarrow B = (A \cap B) \cup (C \cap B)$$
$$\Rightarrow B = (A \cap B) \cup (B \cap C) \quad (2)$$

From Eqs. (1) and (2),

$$B = C$$

Hence, the correct answer is option (C).

6. $\int_0^{\pi} [\cot x]dx$, $[\cdot]$. denotes the greatest integer function, is equal to

(A) $\frac{\pi}{2}$ **(B)** 1
(C) -1 **(D)** $-\frac{\pi}{2}$ **[AIEEE 2009]**

Solution: We have,

$$I = \int_0^{\pi} [\cot x]dx \quad (1)$$

That is,

$$\int_0^{\pi} [\cot(\pi - x)]dx = \int_0^{\pi} [-\cot x]dx \quad (2)$$

Adding Eqs. (1) and (2), we get

$$2I = \int_0^{\pi} [\cot x]dx + \int_0^{\pi} [-\cot x]dx = \int_0^{\pi} (-1)dx = [-x]_0^{\pi} = -\pi$$

$$\left[\begin{array}{r} \because [x] + [-x] = -1 \text{ if } x \notin Z \\ = 0 \text{ if } x \in Z \end{array}\right]$$

Therefore,

$$I = -\frac{\pi}{2}$$

Hence, the correct answer is option (D).

7. For real x, let $f(x) = x^3 + 5x + 1$, then

(A) f is one-one but not onto R
(B) f is onto R but not one-one
(C) f is one-one and onto R
(D) f is neither one-one nor onto R

[AIEEE 2009]

Solution: See Fig. 1.64. We have,

$$f(x) = x^3 + 5x + 1$$

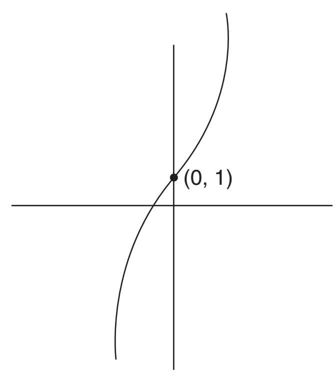


Figure 1.64

Now, $f'(x) = 3x^3 + 5 > 0, \forall x \in R$. Therefore, $f(x)$ is strictly increasing function and hence it is one-one function. Clearly, $f(x)$ is a continuous function and also increasing on R.

$$\lim_{x \to -\infty} f(x) = -\infty \text{ and } \lim_{x \to -\infty} f(x) = \infty$$

Therefore, $f(x)$ takes every value between $-\infty$ and ∞. Thus, $f(x)$ is onto function.

Hence, the correct answer is option (C).

8. Let $f(x) = (x+1)^2 - 1, x \geq -1$

Statement-1: The set $\{x : f(x) = f^{-1}(x)\} = \{0, -1\}$

Statement-2: f is a bijection.

(A) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(B) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(C) Statement-1 is true, Statement-2 is false
(D) Statement-1 is false, Statement-2 is true

[AIEEE 2009]

Solution:

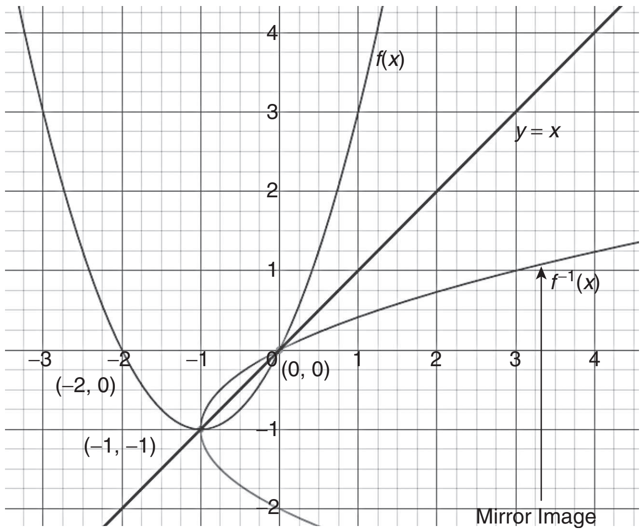


Figure 1.65

From Fig. 1.65, f is one-one in $[-1, \infty)$. There is no information about co-domain and therefore $f(x)$ is not necessarily onto function. Therefore, S_1 is true but S_2 is false.

Hence, the correct answer is option (C).

9. Let S be a non-empty subset of R. Consider the following statement: P: There is a rational number $x \in S$ such that $x > 0$. Which of the following statements is the negation of the statement P?
(A) There is no rational number $x \in S$ such that $x \le 0$
(B) Every rational number $x \in S$ satisfies $x \le 0$
(C) $x \in S$ and $x \le 0 \Rightarrow x$ is not rational
(D) There is a rational number $x \in S$ such that $x \le 0$

[AIEEE 2010]

Solution: Given that S is a non-empty subset of R.

- P: There is a rational number $x \in S$ such that $x > 0$
  Now we need to find the negation of P.
  Clearly, P is equivalent to saying that "There is a positive rational number in S.
  So its negation, ~P is "There is no positive rational number in S".
  ~P: There exists no positive rational number in S.
- $\Leftrightarrow$ ~P: Every rational number $x \in S$ satisfies $x \le 0$.

Hence, the correct answer is option (B).

10. Consider the following relations:

$R = \{(x, y) | x, y$ are real numbers and $x = wy$ for some rational number $w\}$;

$$S = \left\{ \left(\frac{m}{n}, \frac{p}{q} \right) \middle| \begin{array}{l} m, n, p \text{ and } q \text{ are integers such} \\ \text{that } n, q \neq 0 \text{ and } qm = pn \end{array} \right\}.$$

Then
(A) neither R nor S is an equivalence relation
(B) S is an equivalence relation but R is not an equivalence relation
(C) R and S both are equivalence relations
(D) R is an equivalence relation but S is not an equivalence relation.

[AIEEE 2010]

Solution: Let us consider the relation R:

(i) R is reflexive:
Since for any $x \in R, x = 1 \cdot x$ where 1 is rational, so $(x, x) \in R\ \forall x$.

(ii) R is not symmetric
Since $(\sqrt{2}, 0) \not R$ and $\sqrt{2} = \omega \cdot 0$ is not true for any ω rational, so R is not an equivalence relation.

Now let us consider the relation S:

(i) S is reflexive: $\frac{m}{n} s \frac{p}{q} \Leftrightarrow qm = pn$

Therefore, $\frac{m}{n} s \frac{m}{n}$ is reflexive.

(ii) S is symmetric: since

$$\frac{m}{n} s \frac{p}{q} \Rightarrow \frac{p}{q} s \frac{m}{n}$$

Therefore, S is symmetric.

Also

$$\frac{m}{n} s \frac{p}{q}, \frac{p}{q} s \frac{r}{s} \Rightarrow qm = pn,\ ps = rq \Rightarrow ms = rn.$$

Thus, S is transitive.

Therefore, S is an equivalence relation.

Hence, the correct answer is option (B).

11. Let f: $R \to R$ be defined by $f(x) = \begin{cases} k - 2x, \text{ if } x \le -1 \\ 2x + 3, \text{ if } x > -1 \end{cases}$. If f has a local minimum at $x = -1$, then a possible value of k is

(A) 0 **(B)** $-\frac{1}{2}$
(C) -1 **(D)** 1

[AIEEE 2010]

Solution: At $x = -1$, f is continuous if,

$$\lim_{x \to -1-} f(x) = \lim_{x \to -1+} f(x) = f(-1)$$

That is,

$$k + 2 = 2(-1) + 3 = k + 2 \Rightarrow k = -1$$

For, $k = -1$, f is continuous at $x = -1$; $f'(-1)$ does not exist. And,

$$f'(x) < 0 \text{ for } x < -1;$$

$$f'(x) > 0 \text{ for } x < -1.$$

Therefore, f has a local minimum at $x = -1$.

Hence, the correct answer is option (C).

12. Let R be the set of real numbers
Statement-1: $A = \{(x, y) \in R \times R: y - x$ is an integer$\}$ is an equivalence relation on R.
Statement-2: $B = \{(x, y) \in R \times R : x = \alpha y$ for some rational number $\alpha\}$ is an equivalence relation on R.

(A) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(B) Statement-1 is true, Statement-2 is false.
(C) Statement-1 is false, Statement-2 is true.
(D) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

[AIEEE 2010, 2011]

Solution:

- $x - y$ is an integer
- $x - x = 0$ is an integer $\Rightarrow A$ is reflexive
- $x - y$ is an integer $\Rightarrow y - x$ is an integer $\Rightarrow A$ is symmetric
- $x - y, y - z$ are integers
  As sum of two integers is an integer. Therefore, $(x-y)+(y-z) = x-z$ is an integer, which implies that A is transitive. Hence, Statement-1 is true.
  Also,
- $\frac{x}{x} = 1$ is a rational number $\Rightarrow B$ is reflexive
- $\frac{x}{y} = \alpha$ is rational $\Rightarrow \frac{y}{x}$ need not be rational, that is, $\frac{0}{1}$ is rational $\Rightarrow \frac{1}{0}$ is not rational

Hence, B is not symmetric, that is, B is not an equivalence relation.

Hence, the correct answer is option (B).

13. The domain of the function $f(x) = \frac{1}{\sqrt{|x| - x}}$ is

(A) $(0, \infty)$ **(B)** $(-\infty, 0)$
(C) $(-\infty, \infty) - \{0\}$ **(D)** $(-\infty, \infty)$

[AIEEE 2010, 2011]

Solution:

$$\frac{1}{\sqrt{|x| - x}} \Rightarrow |x| - x > 0 \Rightarrow |x| > x \Rightarrow x \text{ is negative.}$$

Therefore, $x \in (-\infty, 0)$.

Hence, the correct answer is option (B).

14. Let $X = \{1, 2, 3, 4, 5\}$. The number of different ordered pairs (Y, Z) that can be formed such that $Y \subseteq X, Z \subseteq X$ and $Y \cap Z$ is empty, is

(A) 5^2 **(B)** 3^5
(C) 2^5 **(D)** 5^3

[AIEEE 2012]

Solution: It is given that $Y \subseteq X, Z \subseteq X$. Let $a \in X$. Then we have following chances that

(i) $a \in Y, \quad a \in Z$

(ii) $a \notin Y, \quad a \in Z$

(iii) $a \in Y, \quad a \notin Z$

(iv) $a \notin Y, \quad a \notin Z$

It is required that $Y \cap Z = \phi$. Hence, the items (ii), (iii), (iv) above are chances for 'a' to satisfy $Y \cap Z = \phi$. Therefore, $Y \cap Z = \phi$ has 3 chances for a. Thus, for five elements of X, the number of required chances is $3 \times 3 \times 3 \times 3 \times 3 = 3^5$.

Hence, the correct answer is option (B).

15. If $a \in R$ and the equation, $-3(x - [x])^2 + 2(x - [x]) + a^2 = 0$ (where $[x]$ denotes the greatest integer $\leq x$) has no integral solution, then all possible values of a lie in the interval

(A) $(-2, -1)$ **(B)** $(-\infty, -2) \cup (2, \infty)$
(C) $(-1, 0) \cup (0, 1)$ **(D)** $(1, 2)$

[JEE MAIN 2014 (OFFLINE)]

Solution:

$$-3(x - [x])^2 + 2(x - [x]) + a^2 = 0 \Rightarrow -3\{x\}^2 + 2\{x\} + a^2 = 0$$

$$\Rightarrow a^2 = 3\{x\}^2 - 2\{x\} = 3\left[\{x\}^2 - \frac{2}{3}\{x\} + \left(\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right)^2\right] = 3\left[\{x\} - \frac{1}{3}\right] - \frac{1}{3}$$

Now we know that $0 \leq \{x\} < 1$. Therefore,

$$-\frac{1}{3} \leq \{x\} - \frac{1}{3} < \frac{2}{3} \Rightarrow 0 \leq \left[\{x\} - \frac{1}{3}\right]^2 < \frac{4}{9} \Rightarrow 0 \leq 3\left[\{x\} - \frac{1}{3}\right]^2 < \frac{4}{3}$$

$$\Rightarrow -\frac{1}{3} \leq 3\left[\{x\} - \frac{1}{3}\right]^2 - \frac{1}{3} < 1$$

$$\Rightarrow -1 \leq a^2 < 1$$

Only possibility for non-integral solution is $0 < a^2 < 1$.
Thus, $a^2 \geq 0$, but when $a = 0$, there is integral solution for $\{x\} = 0$
Therefore, $(-1, 0) \cup (0, 1)$

Hence, the correct answer is option (C).

16. Let P be the relation defined on the set of all real numbers such that

$$P = \{(a, b): \sec^2 a - \tan^2 b = l\}. \text{ Then } P \text{ is}$$

(A) reflexive and symmetric but not transitive.
(B) reflexive and transitive but not symmetric.
(C) symmetric and transitive but not reflexive.
(D) an equivalence relation.

[JEE MAIN 2014 (ONLINE SET – 1)]

Solution:

$$P = \{(a, b): \sec^2 a - \tan^2 b = 1\}$$

Since, $\sec^2 a - \tan^2 a = 1$ true $\Rightarrow a\,R\,a$ i.e. reflexive

$$aRb \Rightarrow \sec^2 a - \tan^2 b = 1 \Rightarrow 1 + \tan^2 a - \sec^2 b + \cancel{1} = \cancel{1}$$

$$\Rightarrow \sec^2 b = 1 + \tan^2 a \Rightarrow b\,R\,a$$

Therefore, P is symmetric.

$$a\,R\,b \text{ and } b\,R\,c \Rightarrow \sec^2 a - \tan^2 b = 1 \text{ and } \sec^2 b - \tan^2 c = 1$$

$$\Rightarrow \sec^2 a - (\sec^2 b - 1) = 1 \Rightarrow \sec^2 a - \tan^2 c = 1 \Rightarrow a\,R\,c$$

Therefore, P is Transitive.

Hence, the correct answer is option (D).

17. A relation on the set $A = \{x : |x| < 3, \in xZ\}$, where Z is the set of integers is defined by $R = \{(x, y) : y = |x|, x \neq -1\}$. Then the number of elements in the power set of R is

(A) 32 **(B)** 16
(C) 8 **(D)** 64

[JEE MAIN 2014 (ONLINE SET – 3)]

Solution:

$$A = \{x: |x| < 3, x \in Z\}, = \{-2, -1, 0, 1, 2\}$$

$$R = \{(x, y): y = |x|, x \neq -1\}$$

Therefore, number of elements in power set $= 2^4 = 16$

Hence, the correct answer is option (B).

18. Let $f: R \to R$ be defined by $f(x) = \dfrac{|x|-1}{|x|+1}$ then f is

(A) both one-one and onto
(B) one-one but not onto
(C) onto but not one-one
(D) neither one-one nor onto

[JEE MAIN 2014 (ONLINE SET – 4)]

Solution: Checking one-one

$$\frac{|x_1|-1}{|x_1+1|} = \frac{|x_2|-1}{|x_2|+1} \Rightarrow |x_1||x_2| + |x_1| - |x_2| - 1 = |x_1||x_2| - |x_1| + |x_2| - 1$$

$\Rightarrow 2|x_1| = 2|x_2| \Rightarrow |x_1| = |x_2|$

Therefore,

$$x_1 = \pm x_2 \Rightarrow \text{Not one-one}$$

Checking onto

Let

$$\frac{|x|-1}{|x|+1} = 1 \Rightarrow |x| - 1 = |x| + 1$$

Therefore, $f(x)$ does not take value 1. For any x, f is not onto.

Hence, the correct answer is option (D).

19. Let $f(n) = \left[\dfrac{1}{3} + \dfrac{3n}{100}\right]n$, where $[n]$ denotes the greatest integer less than or equal to n. Then $\sum_{n-1}^{56} f(n)$ is equal to

(A) 56 **(B)** 689
(C) 1287 **(D)** 1399

[JEE MAIN 2014 (ONLINE SET – 4)]

Solution:

$$f(1) = \left[\frac{1}{3} + \frac{3}{100}\right]1 = 0;$$

$$f(2) = \left[\frac{1}{3} + \frac{6}{100}\right]2 = 0 \ldots\ldots\ f(22) = \left[\frac{1}{3} + \frac{66}{100}\right]22 = \left[\frac{100+198}{300}\right]22 = 0$$

$$f(23) = \left[\frac{1}{3} + \frac{69}{100}\right]23 = \left[\frac{100+207}{300}\right]23 = 23 \ldots\ldots$$

$$f(55) = \left[\frac{1}{3} + \frac{165}{100}\right]55 = \left[\frac{100+495}{300}\right]55 = 55$$

$$f(56) = \left[\frac{1}{3} + \frac{168}{100}\right]56 = \left[\frac{100+504}{300}\right]56 = 2 \times 56 = 112$$

Therefore,

$$\sum_{n-1}^{56} f(n) = 0 + (23 + 24 + \cdots 55) + 112 = \frac{33}{2}[46 + (33-1)1] + 112$$

$$= \frac{33}{2}[46 + 33 - 1] + 112$$

$$= \frac{33}{2}[78] + 112 = 33 \times 39 \times 112 = 1399$$

Hence, the correct answer is option (D).

20. The function $f(x) = |\sin 4x| + |\cos 2x|$, is a periodic function with period

(A) 2π **(B)** π
(C) $\dfrac{\pi}{2}$ **(D)** $\dfrac{\pi}{4}$

[JEE MAIN 2014 (ONLINE SET – 4)]

Solution:

Period of $|\sin 4x|$ is $\dfrac{\pi}{4}$, so period of $|\sin\theta|$ is π.

Period of $|\cos 2x|$ is $\dfrac{\pi}{2}$, so period of $|\cos\theta|$ is π.

Therefore, period of $f(x)$ is LCM of periods$= \dfrac{\text{LCM of } \pi \text{ and } \pi}{\text{GCD of 4 and 2}} = \dfrac{\pi}{2}$

Thus, sin 4x and cos 2x are not complimentary.

Hence, the correct answer is option (C).

21. If the function $g(x) = \begin{cases} k\sqrt{x+1}, & 0 \le x \le 3 \\ mx + 2, & 3 < x \le 5 \end{cases}$ is differentiable, then the value of $k + m$ is

(A) $\dfrac{16}{5}$ **(B)** $\dfrac{10}{3}$
(C) 4 **(D)** 2

[JEE MAIN 2015 (OFFLINE)]

Solution: We have

$$g(x) = \begin{cases} k\sqrt{x+1}, & 0 \le x \le 3 \\ mx + 2, & 3 < x \le 5 \end{cases}$$

$$g(3^-) = 2k;\ g(3^+) = 3m + 2;\ g(3) = 2k$$

$$\Rightarrow 2k = 3m + 2 \qquad (1)$$

Also,

$$g'(x) = \begin{cases} \dfrac{k}{2\sqrt{x+1}}; & 0 < x < 3 \\ m; & 3 < x < 5 \end{cases}$$

$$\Rightarrow g'(3^-) = \frac{k}{4};\ g'(3^+) = m$$

$$\Rightarrow \frac{k}{4} = m \Rightarrow k = 4m$$

Therefore, from Eq. (1),

$$m = \frac{2}{5};\ k = \frac{8}{5} \Rightarrow k + m = 2$$

Hence, the correct answer is option (D).

22. The largest value of r for which the region represented by the set $\{\omega \in C: |\omega - 4 - i| \le r\}$ is contained in the region represented by the set $\{z \in C / |z-1| \le |z+i|\}$, is equal to

(A) $\sqrt{17}$ **(B)** $2\sqrt{2}$
(C) $\dfrac{3}{2}\sqrt{2}$ **(D)** $\dfrac{5}{2}\sqrt{2}$

[JEE MAIN 2015 (ONLINE SET – 1)]

Solution: See Fig. 1.66.

$$R_1 = \{w \in C : |\omega - (4+i)| \le r\};\ R_2 = \{z \in C : |z-1| \le |z+i|\}$$

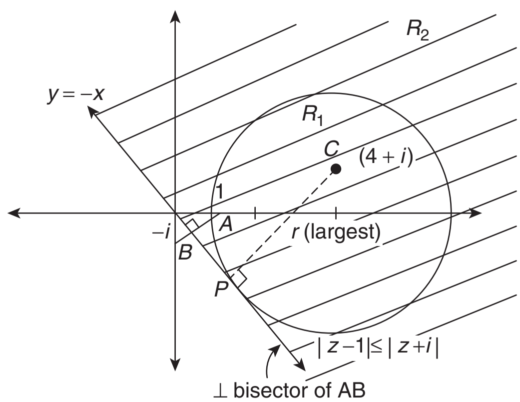


Figure 1.66

Therefore, largest $'r' = CP = \dfrac{|4+1|}{\sqrt{(1)^2+(1)^2}} = \dfrac{5}{2} = \dfrac{5\sqrt{2}}{2}$

Hence, the correct answer is option (D).

23. If Rolle's theorem holds for the function $f(x) = 2x^3 + bx^2 + cx$, $x \in [-1,1]$, at the point $x = \frac{1}{2}$, then $2b + c$ equals

(A) 1 **(B)** -1

(C) 2 **(D)** -3

[JEE MAIN 2015 (ONLINE SET – 1)]

Solution:

$$f(-1) = f(1) \text{ and } f'\left(\frac{1}{2}\right) = 0 \Rightarrow -2 + b - c = 2 + b + c \Rightarrow c = -2$$

and

$$f'(x) = 6x^2 + 2bx + c \Rightarrow f'\left(\frac{1}{2}\right) = \frac{3}{2} + b - 2 = 0 \Rightarrow b = \frac{1}{2}$$

$$\Rightarrow 2b + c = -1$$

Hence, the correct answer is option (B).

24. Let $A = \{x_1, x_2, \ldots, x_7\}$ and $B = \{y_1, y_2, y_3\}$ be two sets containing seven and three distinct elements respectively. Then the total number of functions $f: A \to B$ that are onto, if there exists exactly three elements x in A such that $f(x) = y_2$, is equal to

(A) $14\cdot{}^7C_2$ **(B)** $16\cdot{}^7C_3$

(C) $12\cdot{}^7C_3$ **(D)** $14\cdot{}^7C_3$

[JEE MAIN 2015 (ONLINE SET – 2)]

Solution: We have $A = \{x_1, x_2, x_3, \ldots x_7\}$ and $B = \{y_1, y_2, y_3\}$.

3 elements in A having image y_2 can be chosen in ${}^7C_3$ ways. Now we are left with 4 elements in A which are to be associated with y_1 or y_3 i.e. each of 4 elements in A has 2 choices y_1 or y_3 i.e. in $(2)^4$ ways. But there are 2 ways when one element of B will remain associated i.e. when all 4 are associated with y_1 or y_3.

Therefore, required number of functions $= {}^7C_3((2)^4 - 2)$

$$= 14\cdot({}^7C_3)$$

Hence, the correct answer is option (D).

25. Let k be a non-zero real number. If

$$f(x) = \begin{cases} \dfrac{(e^x-1)^2}{\sin\left(\dfrac{x}{k}\right)\log\left(1+\dfrac{x}{4}\right)}, & x \neq 0 \\ 12, & x = 0 \end{cases}$$ is a continuous function,

then the value of k is

(A) 1 **(B)** 2

(C) 3 **(D)** 4

[JEE MAIN 2015 (ONLINE SET – 2)]

Solution: For continuity at $x = 0$

$$\lim_{x\to 0}\left\{\frac{(e^x-1)^2}{\sin\left(\frac{x}{k}\right).\ln\left(1+\frac{x}{4}\right)}\right\} = 12 \Rightarrow \lim_{x\to 0}\left[\frac{\left(\frac{e^x-1}{x}\right)^2}{\frac{\sin\left(\frac{x}{k}\right)}{k\left(\frac{x}{k}\right)}\cdot\frac{\ln\left(1+\frac{x}{4}\right)}{4.\left(\frac{x}{4}\right)}}\right]$$

$$\Rightarrow \left\{\frac{(1)^2}{\left(\frac{1}{k}\right)}\cdot\frac{1}{\frac{1}{4}(1)}\right\} = 12 \Rightarrow 4k = 12 \Rightarrow k = 3$$

Hence, the correct answer is option (C).

26. Let $f: R \to R$ be a function such that $f(2 - x) = f(2 + x)$ and $f(4 - x) = f(4 + x)$, for all $x \in R$ and $\int_0^2 f(x)dx = 5$. Then the value of $\int_{10}^{50} f(x)dx$ is

(A) 80 **(B)** 100

(C) 125 **(D)** 200

[JEE MAIN 2015 (ONLINE SET – 2)]

Solution: Given, $f: R \to R$

$$f(2-x) = f(2+x) \forall x \in 1R \quad (1)$$
$$f(4-x) = f(4+x) \forall x \in 1R \quad (2)$$
$$\int_0^2 f(x)dx = 5, \int_{10}^{50} f(x)dx = ?$$

Replacing x by $(x - 2)$ in Eq. (1), we get

$f(2 - (x - 2)) = f(2 + x - 2) \Rightarrow f(x) = f(4 - x) = f(4 + x)$ [from Eq. (2)]

Therefore, $f(x) = f(4 + x)$.

So, $f(x)$ is periodic function, with period 4.

Also $f(2 - x) = f(2 + x)$ so, $f(x)$ is symmetric about the line $x = 2$.

$$\int_0^2 f(x)dx = \int_2^4 f(x)dx = 5$$

$$\Rightarrow \int_0^4 f(x)dx = 10$$

Therefore, $\displaystyle\int_{10}^{50} f(x)dx = 10\int_0^4 f(x)dx = 100$

Hence, the correct answer is option (B).

27. Let $f:(-1,1)\to R$ be a continuous function. If $\int_0^{\sin x} f(t)dt = \frac{\sqrt{3}}{2}x$, then $f\left(\frac{\sqrt{3}}{2}\right)$ is equal to

(A) $\frac{\sqrt{3}}{2}$ **(B)** $\sqrt{3}$

(C) $\sqrt{\frac{3}{2}}$ **(D)** $\frac{1}{2}$

[JEE MAIN 2015 (ONLINE SET – 2)]

Solution: $f:(-1,1)\to R$ is continuous and

$$\int_0^{\sin x} f(t)dt = \frac{\sqrt{3}}{2}x;\ f\left(\frac{\sqrt{3}}{2}\right) = ?$$

Differentiating both sides w.r.t. x, gives

$$f(\sin x)\frac{d}{dx}(\sin x) = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \cos x f(\sin x) = \sqrt{3}/2 \therefore \frac{1}{2}f\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2};\left(\text{at } x = \frac{\pi}{3}\right) \Rightarrow f\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

Hence, the correct answer is option (B).

28. If $f(x)+2f\left(\frac{1}{x}\right) = 3x, x\neq 0$ and $S=\{x\in R: f(x)=f(-x)\}$, then S

(A) contains more than two elements.
(B) is an empty set.
(C) contains exactly one element.
(D) contains exactly two elements.

[JEE MAIN 2016 (OFFLINE)]

Solution: We have

$$f(x)+2f\left(\frac{1}{x}\right) = 3x, x\neq 0$$

Replacing x by 1/2, we get

$$f\left(\frac{1}{x}\right)+2f(x) = 3/x$$

$$f\left(\frac{1}{x}\right) = \frac{3}{x}-2f(x)$$

$$f(x)+\frac{6}{x}-4f(x) = 3x$$

$$\Rightarrow \frac{6}{x}-3x = 3f(x)$$

$$\Rightarrow f(x) = \frac{2}{x}-x$$

Now,

$$f(x) = f(-x)$$

$$\frac{2}{x}-x = -\frac{2}{x}+x$$

$$\frac{4}{x} = 2x$$

That is,

$$\frac{2}{x} = x \Rightarrow x^2 = 2 \Rightarrow x = -\sqrt{2}, \sqrt{2}$$

Therefore, S contains only two elements.

Hence, the correct answer is option (D).

29. The number of $x\in[0,2\pi]$ for which $\left|\sqrt{2\sin^4+18\cos^2 x} - \sqrt{2\cos^4+18\sin^2 x}\right| = 1$ is

(A) 2 **(B)** 4

(C) 6 **(D)** 8

[JEE MAIN 2016 (ONLINE SET – 1)]

Solution: We have

$$\left|\sqrt{2\sin^4 x+18\cos^2 x} - \sqrt{2\cos^4 x+18\sin^2 x}\right| = 1$$

That is,

$$f(x) = \left|\sqrt{2\sin^4 x+8\cos^2 x} - \sqrt{2\cos^4 x+18\sin^2 x}\right|$$

Now, $f(x) = f\left(\frac{\pi}{2}-x\right)$, so $f(x)$ is symmetric about $x = \frac{\pi}{4}$.

If $f(x)$ has solution in $(0, \pi/4)$, then in $(0, 2\pi)$, there are eight solutions exist.

Hence, the correct answer is option (D).

30. For $x\in R$, $x\neq 0, x\neq 1$, let $f_0(x) = \frac{1}{1-x}$ and $f_{n+1}(x) = f_0(f_n(x))$, n = 0, 1, 2, …. Then, the value of $f_{100}(3)+f_1\left(\frac{2}{3}\right)+f_2\left(\frac{3}{2}\right)$ is equal to

(A) $\frac{8}{3}$ **(B)** $\frac{4}{3}$

(C) $\frac{5}{3}$ **(D)** $\frac{1}{3}$

[JEE MAIN 2016 (ONLINE SET – 1)]

Solution: We have

$$f_0(x) = \frac{1}{1-x}$$

$$f_1(x) = f_0(f_0(x)) = \frac{1}{1-f_0(x)} = \frac{1}{1-[1/(1-x)]} = \frac{1-x}{1-x-1} = \frac{x-1}{x}$$

$$f_1\left(\frac{2}{3}\right) = \frac{(2/3)-1}{2/3} = -1/2$$

$$f_2(x) = f_0(f_1(x)) = \frac{1}{1-f_1(x)} = \frac{1}{1-[(x-1)/x]} = \frac{x}{x-x+1} = x$$

$$f_2\left(\frac{3}{2}\right) = \frac{3}{2}$$

$$f_3(x) = f_0(f_2(x)) = \frac{1}{1-x}$$

$$f_{100}(x) = \frac{x-1}{x} \Rightarrow f_{100}(3) = \frac{3-1}{3} = \frac{2}{3}$$

$$f_{100}(3)+f_1\left(\frac{2}{3}\right)+f_2\left(\frac{3}{2}\right) = \frac{3}{2}-\frac{1}{2}+\frac{2}{3} = 1+\frac{2}{3} = \frac{5}{3}$$

Hence, the correct answer is option (C).

31. Let $P=\{\theta : \sin\theta-\cos\theta = \sqrt{2}\cos\theta\}$ and $Q=\{\theta : \sin\theta+\cos\theta = \sqrt{2}\sin\theta\}$ be two sets. Then

(A) $P\subset Q$ and $Q-P\neq\phi$ **(B)** $Q\not\subset P$

(C) $P = Q$ **(D)** $P \not\subset Q$

[JEE MAIN 2016 (ONLINE SET – 2)]

Solution: We have

$$P = \{\theta : \sin\theta - \cos\theta = \sqrt{2}\cos\theta\}$$
$$Q = \{\theta : \sin\theta + \cos\theta = \sqrt{2}\sin\theta\}$$

Therefore,

$$\sin\theta - \cos\theta = \sqrt{2}\cos\theta$$
$$\sin^2\theta + \cos^2\theta = 2\sin\theta\cos\theta = 2\cos^2\theta$$
$$\sin^2\theta - \cos^2\theta = 2\sin\theta\cos\theta$$
$$(\sin\theta + \cos\theta)\sqrt{2}\cos\theta = 2\sin\theta\cos\theta$$
$$\sin\theta + \cos\theta = \sqrt{2}\sin\theta$$
$$P = Q$$

Hence, the correct answer is option (C).

Previous Years' Solved JEE Advanced/ IIT-JEE Questions

1. Let $F(x)$ be an indefinite integral of $\sin^2 x$.
Statement-1: The function $F(x)$ satisfies $F(x + \pi) = F(x)$ for all real x.
because
Statement-2 : $\sin^2(x + \pi) = \sin^2 x$ for all real x.
(A) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(B) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(C) Statement-1 is True, Statement-2 is False
(D) Statement-1 is False, Statement-2 is True

[IIT-JEE 2007]

Solution: We have

$$f(x) = \int \sin^2 x dx$$
$$= \int \frac{1-\cos 2x}{2} dx$$
$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right] + c$$

Clearly,

$$f(x+\pi) \neq f(x)$$

Hence, Statement-1 is false and Statement-2 is true because $\sin^2 x$ is periodic function with period π.

Hence, the correct answer is option (D).

2. Let $f(x) = \frac{x}{(1+x)^{1/n}}$ for $n \geq 2$ and $g(x) = \underbrace{(f \, o f \, o \cdots o f)}_{f \text{ occurs } n \text{ times}}(x)$. Then $\int x^{n-1} g(x) dx$ equals

(A) $\frac{1}{n(n-1)}(1-nx^n)^{1-\frac{1}{n}} + K$ **(B)** $\frac{1}{n-1}(1-nx^n)^{1-\frac{1}{n}} + K$

(C) $\frac{1}{n(n-1)}(1+nx^n)^{1-\frac{1}{n}} + K$ **(D)** $\frac{1}{n+1}(1+nx^n)^{1+\frac{1}{n}} + K$

[IIT-JEE 2007]

Solution: We have

$$f(x) = \frac{x}{(1+x)^{1/n}}$$

Therefore,

$$f \, o f(x) = \frac{f(x)}{(1+[f(x)]^n)^{1/n}}$$
$$= \frac{x}{(1+2x^n)^{1/n}}$$

Now, we have

$$f \, o f \, o f(x) = f \, o f[f(x)]$$
$$= \frac{f(x)}{(1+2[f(x)]^n)^{1/n}}$$
$$= \frac{x}{(1+3x^n)^{1/n}}$$

$$\Rightarrow g(x) = (f \, o f \, o f \, o \cdots o f)(x) \text{ (where } f \text{ occurs } n \text{ times)}$$
$$= \frac{x}{(1+nx^n)^{1/n}}$$

Therefore,

$$I = \int x^{n-2}.g(x)dx$$
$$= \int \frac{x^{n-1}dx}{(1+nx^n)^{1/n}}$$

Substituting $1 + nx^n = t$, we get

$$n^2 x^{n-1} dx = dt$$
$$\Rightarrow I = \frac{1}{n^2}\int \frac{dt}{t^{1/n}}$$
$$= \frac{1}{n^2}\frac{t^{1-(1/n)}}{1-(1/n)} + K$$
$$= \frac{(1+nx^n)^{1-(1/n)}}{n(n-1)} + K$$

Hence, the correct answer is option (A).

Paragraph for Questions 3 to 5: Let A, B, C be three sets of complex numbers as defined below

$A = \{z : \text{Im}\, z \geq 1\}$

$B = \{z : |z - 2 - i| = 3\}$

$C = \{z : \text{Re}((1-i)z) = \sqrt{2}\}$

3. The number of elements in the set $A \cap B \cap C$ is
(A) 0 **(B)** 1
(C) 2 **(D)** ∞

[IIT-JEE 2008]

Solution: A = set of points on and above the line $y = 1$ in the argand plane.

B = set of points on the circle $(x - 2)^2 + (y - 1)^2 = 3^2$

C = set of point lies on the line $x + y = \sqrt{2}$

Hence $A \cap B \cap C$ = has only one point of intersection.

Hence, the correct answer is option (B).

4. Let z be any point in $A \cap B \cap C$. Then, $|z+1-i|^2 + |z-5-i|^2$ lies between
(A) 25 and 29 **(B)** 30 and 34
(C) 35 and 39 **(D)** 40 and 44

[IIT-JEE 2008]

Solution: The points (–1, 1) and (5, 1) are the extremities of a diameter of the given circle. Hence

$$|z+1-i|^2+|z-5-i|^2=36$$

Hence, the correct answer is option (C).

5. Let z be any point in $A\cap B\cap C$ and let w be any point satisfying $|w-2-i|<3$. Then $|z|-|w|+3$ lies between

(A) –6 and 3 **(B)** –3 and 6
(C) –6 and 6 **(D)** –3 and 9

[IIT-JEE 2008]

Solution:

$$||z|-|w||<|z-w|$$

and $|z-w|$ = Distance between z and w, where z is fixed. Hence, distance between z and w would be maximum for diametrically opposite points.

$$|z-w|<6$$
$$\Rightarrow -6<|z|-|w|<6$$
$$-3<|z|-|w|+3<9$$

Hence, the correct answer is option (D).

6. For any real number x, let $|x|$ denote the largest integer less than or equal to x. Let f be a real valued function defined on the interval [–10, 10] by

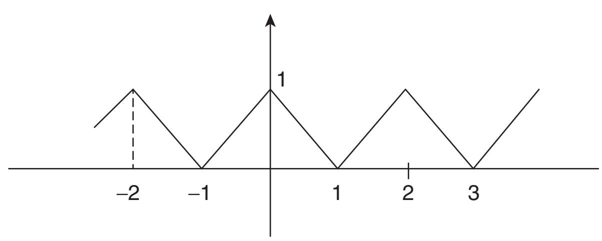
$$f(x)=\begin{cases} x-[x] & \text{if } [x] \text{ is odd}\\ 1+[x]-x & \text{if } [x] \text{ is even}\end{cases}$$

Then the value of $\dfrac{\pi^2}{10}\displaystyle\int_{-10}^{10} f(x)\cos\pi x\,dx$ is

[IIT-JEE 2010]

Solution:

$$f(x)=\begin{cases} x-1, & 1\le x<2\\ 1-x, & 0\le x<1\end{cases}$$



Now, $f(x)$ is periodic with period 2. Therefore,

$$I=\int_{-10}^{10} f(x)\cos\pi x\,dx$$

$$2\int_0^{10} f(x)\cos\pi x\,dx=2\times 5\int_0^2 f(x)\cos\pi x\,dx$$

$$=10\left[\int_0^1(1-x)\cos\pi x\,dx+\int_1^2(x-1)\cos\pi x\,dx\right]=10(I_1+I_2)$$

$$\Rightarrow I_2=\int_1^2(x-1)\cos\pi x\,dx \quad \text{put } x-1=t$$

$$I_2=-\int_0^1 t\cos\pi t\,dt$$

$$I_1=\int_0^1(1-x)\cos\pi x\,dx=-\int_0^1 x\cos(\pi x)\,dx$$

Therefore,

$$I=10\left[-2\int_0^1 x\cos\pi x\,dx\right]$$

$$=-20\left[x\frac{\sin\pi x}{\pi}+\frac{\cos\pi x}{\pi^2}\right]_0^1$$

$$=-20\left[-\frac{1}{\pi^2}-\frac{1}{\pi^2}\right]=\frac{40}{\pi^2}\quad\therefore \frac{\pi^2}{10}I=4$$

Hence, the correct answer is (4).

7. Let f be a real-valued function defined on the interval (–1, 1) such that $e^{-x}f(x)=2+\displaystyle\int_0^x\sqrt{t^4+1}\,dt$, for all $x\in(-1,1)$ and let f^{-1} be the inverse function of f. Then $(f^{-1})'(2)$ is equal to

(A) 1 **(B)** 1/3
(C) 1/2 **(D)** $1/e$

[IIT-JEE 2010]

Solution:

$$e^{-x}f(x)=2+\int_0^x\sqrt{t^4+1}\,dt \qquad (1)$$

$$f(f^{-1}(x))=x$$
$$\Rightarrow f'(f^{-1}(x))(f^{-1}(x))'=1$$
$$\Rightarrow (f^{-1}(2))'=\frac{1}{f'(f^{-1}(2))}$$
$$\Rightarrow f(0)=2\Rightarrow f^{-1}(2)=0$$
$$(f^{-1}(2))'=\frac{1}{f'(0)}$$
$$e^{-x}(f'(x)-f(x))=\sqrt{x^4+1}$$

Put

$$x=0\Rightarrow f'(0)-2=1\Rightarrow f'(0)=3$$
$$(f^{-1}(2))'=1/3$$

Hence, the correct answer is option (B).

8. Let S = {1, 2, 3, 4}. The total number of unordered pairs of disjoint subsets of S is equal to

(A) 25 **(B)** 34
(C) 42 **(D)** 41

[IIT-JEE 2010]

Solution: Total number of unordered pairs of disjoint subsets is

$$\frac{3^4+1}{2}=41$$

Hence, the correct answer is option (D).

9. Let $P=\{\theta:\sin\theta-\cos\theta=\sqrt{2}\cos\theta\}$ and $Q=\{\theta:\sin\theta+\cos\theta=\sqrt{2}\sin\theta\}$ be two sets. Then

(A) $P\subset Q$ and $Q-P\neq\varnothing$ **(B)** $Q\not\subset P$
(C) $P\not\subset Q$ **(D)** $P=Q$

[IIT-JEE 2011]

Solution: In set P,

$$\sin\theta = (\sqrt{2}+1)\cos\theta \Rightarrow \tan\theta = \sqrt{2}+1$$

In set Q,

$$(\sqrt{2}-1)\sin\theta = \cos\theta \Rightarrow \tan\theta = \frac{1}{\sqrt{2}-1} = \sqrt{2}+1 \Rightarrow P = Q$$

Hence, the correct answer is option (D).

10. Let $f(x) = x^2$ and $g(x) = \sin x$ for all $x \in R$. Then the set of all x satisfying $(f\,o\,g\,o\,g\,o\,f)(x) = (g\,o\,g\,o\,f)(x)$, where $(fog)(x) = f(g(x))$, is

(A) $\pm\sqrt{n\pi}, n \in \{0, 1, 2, ...\}$

(B) $\pm\sqrt{n\pi}, n \in \{1, 2, ...\}$

(C) $\frac{\pi}{2} + 2n\pi, n \in \{..., -2, -1, 0, 1, 2...\}$

(D) $2n\pi, n \in \{..., -2, -1, 0, 1, 2, ...\}$

[IIT-JEE 2011]

Solution:

$$(f\,o\,g\,o\,g\,o\,f)(x) = \sin^2(\sin x^2)$$

$$(g\,o\,g\,o\,f)(x) = \sin^2(\sin x^2)$$

Therefore,

$$\sin^2(\sin x^2) = \sin(\sin x^2)$$

$$\Rightarrow \sin(\sin x^2)[\sin(\sin x^2) - 1] = 0$$

$$\Rightarrow \sin(\sin x^2) = 0 \text{ or } 1$$

$$\Rightarrow \sin x^2 = n\pi \text{ or } 2m\pi + \pi/2, \text{ where } m, n \in I$$

$$\Rightarrow \sin x^2 = 0$$

$$\Rightarrow x^2 = n\pi \Rightarrow x = \pm\sqrt{n\pi}, n \in \{0, 1, 2, ...\}.$$

Hence, the correct answer is option (A).

11. Match the statements given in Column I with the intervals/union of intervals given in Column II:

Column I	Column II
(A) The set $\left\{\text{Re}\left(\frac{2iz}{1-z^2}\right) : z \text{ is a complex number}, \lvert z\rvert = 1, z \neq \pm 1\right\}$ is	**(p)** $(-\infty, -1) \cup (1, \infty)$
(B) The domain of the function $f(x) = \sin^{-1}\left(\frac{8(3)^{x-2}}{1-3^{2(x-1)}}\right)$ is	**(q)** $(-\infty, 0) \cup (0, \infty)$
(C) If $f(\theta) = \begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$, then the set $\left\{f(\theta) : 0 \le \theta < \frac{\pi}{2}\right\}$ is	**(r)** $(2, \infty)$
(D) If $f(x) = x^{3/2}(3x - 10), x \ge 0$, then $f(x)$ is increasing in	**(s)** $(-\infty, -1) \cup (1, \infty)$
	(t) $(-\infty, 0) \cup (2, \infty)$

[IIT-JEE 2011]

Solution:

(A) $$z = \frac{2i(x+iy)}{1-(x+iy)^2} = \frac{2i(x+iy)}{1-(x^2-y^2+2ixy)}$$

Using $1 - x^2 = y^2$

$$z = \frac{2ix - 2y}{2y^2 - 2ixy} = -\frac{1}{y}$$

Since $-1 \le y \le 1 \Rightarrow -\frac{1}{y} \le -1$ or $-\frac{1}{y} \ge 1$

(B) For domain

$$-1 \le \frac{8 \cdot 3^{x-2}}{1-3^{2(x-1)}} \le 1$$

$$\Rightarrow -1 \le \frac{3^x - 3^{x-2}}{1-3^{2x-2}} \le 1$$

Case 1: $\frac{3^x - 3^{x-2}}{1-3^{2x-2}} - 1 \le 0$

$$\Rightarrow \frac{(3^x - 1)(3^{x-2} - 1)}{(3^{2x-2} - 1} \ge 0$$

$$\Rightarrow x \in (-\infty, 0] \cup (1, \infty)$$

Case 2: $\frac{3^x - 3^{x-2}}{1-3^{2x} - 2} + 1 \ge 0$

$$\Rightarrow \frac{(x^{x-2} - 1)(3^x + 1)}{(3^x \cdot 3^{x-2} - 1)} \ge 0$$

$$\Rightarrow x \in (-\infty, 1) \cup [2, \infty)$$

So, $x \in (-\infty, 0) \cup [2, \infty)$.

(C) $R_1 \to R_1 + R_3$

$$f(\theta) = \begin{vmatrix} 0 & 0 & 2 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$$

$$= 2(\tan^2\theta + 1) = 2\sec^2\theta.$$

(D) $$f'(x) = \frac{3}{2}(x)^{1/2}(3x - 10) + (x)^{3/2} \times 3 = \frac{15}{2}(x)^{1/2}(x - 2)$$

Therefore, $f(x)$ is increasing when $x \ge 2$.

Hence, the correct matches are (A) → (s), (B) → (t), (C) → (r), (D) → (r)

12. The function $f : [0, 3] \to [1, 29]$, defined by $f(x) = 2x^3 - 15x^2 + 36x + 1$, is

(A) one-one and onto

(B) onto but not one-one

(C) one-one but not onto

(D) neither one-one nor onto

[IIT-JEE 2012]

Solution:

$$f(x) = 2x^3 - 15x^2 + 36x + 1$$

$$f'(x) = 6x^2 - 30x + 36$$

$$= 6(x^2 - 5x + 6)$$

$$= 6(x-2)(x-3)$$

$f(x)$ is increasing in [0, 2] and decreasing in [2, 3]. $f(x)$ is many one.

$$f(0) = 1$$
$$f(2) = 29$$
$$f(3) = 28$$

Range is [1, 29].
Hence, $f(x)$ is many-one-onto.

Hence, the correct answer is option (B).

13. Let $f:(-1,1) \to R$ be such that $f(\cos 4\theta) = \dfrac{2}{2-\sec^2\theta}$ for $\theta \in \left(0, \dfrac{\pi}{4}\right) \cup \left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$. Then the value(s) of $f\left(\dfrac{1}{3}\right)$ is (are)

(A) $1-\sqrt{\dfrac{3}{2}}$ **(B)** $1+\sqrt{\dfrac{3}{2}}$

(C) $1-\sqrt{\dfrac{2}{3}}$ **(D)** $1+\sqrt{\dfrac{2}{3}}$

[IIT-JEE 2012]

Solution: For

$$\theta \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$$

Let $\cos 4\theta = 1/3$. Then

$$\cos 2\theta = \pm\sqrt{\frac{1+\cos 4\theta}{2}} = \pm\sqrt{\frac{2}{3}}$$

$$f\left(\frac{1}{3}\right) = \frac{2}{2-\sec^2\theta} = \frac{2\cos^2\theta}{2\cos^2\theta - 1} = 1 + \frac{1}{\cos 2\theta}$$

$$f\left(\frac{1}{3}\right) = 1-\sqrt{\frac{3}{2}} \text{ or } 1+\sqrt{\frac{3}{2}}$$

Hence, the correct answers are options (A) and (B).

14. Let $f(x) = x\sin \pi x, x > 0$. Then for all natural numbers n, $f'(x)$ vanishes at

(A) A unique point in the interval $\left(n, n+\dfrac{1}{2}\right)$

(B) A unique point in the interval $\left(n+\dfrac{1}{2}, n+1\right)$

(C) A unique point in the interval $(n, n+1)$

(D) Two points in the interval $(n, n+1)$

[JEE ADVANCED 2013]

Solution: We have

$$f(x) = x\sin \pi x$$

$$f'(x) = \sin \pi x + \pi x \cos \pi x = 0$$

$$\Rightarrow -\tan \pi x = \pi x$$

It is clear from Fig. 1.67 that $f'(x)$ has one root in $\left(n+\dfrac{1}{2}, n+1\right)$ and $f'(x)$ also has one root in $(n, n+1)$.

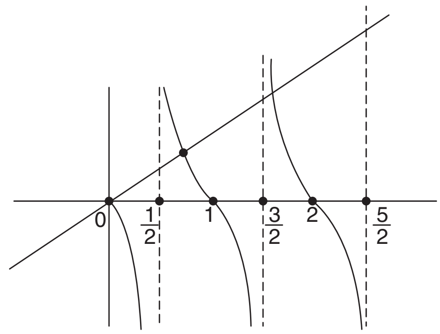


Figure 1.67

Hence, the correct answers are options (B) and (C).

15. Let $\omega = \dfrac{\sqrt{3}+i}{2}$ and $P = \{\omega^n : n = 1,2,3,....\}$. Further $H_1 = \left\{z \in C : \text{Re}\, z > \dfrac{1}{2}\right\}$ and $H_2 = \left\{z \in C : \text{Re}\, z > \dfrac{-1}{2}\right\}$, where C is the set of all complex numbers. If $z_1 \in P \cap H_1, z_2 \in P \cap H_2$ and O represents the origin, then $\angle z_1 o z_2 = ?$

(A) $\dfrac{\pi}{2}$ **(B)** $\dfrac{\pi}{6}$ (C) $\dfrac{2\pi}{3}$ **(D)** $\dfrac{5\pi}{6}$

[JEE ADVANCED 2013]

Solution: We note that $|\omega| = 1$. We also note that α_i are possible values of z_1 and β_i are possible values of z_2, where i = 1, 2, 3. Therefore,

$$\omega = \frac{\sqrt{3}}{2} + \frac{i}{2};$$

$$\omega = e^{i\frac{\pi}{6}}$$

$$\omega^2 = e^{i\frac{\pi}{3}}; \omega^3 = e^{i\frac{\pi}{2}}; \omega^4 = e^{2i\frac{\pi}{3}}; \omega^5 = e^{i\frac{5\pi}{6}}$$

Thus, $\angle z_1 o z_2$ can take the values $\dfrac{2\pi}{3}, \dfrac{5\pi}{6}$. (See Fig. 1.68.)

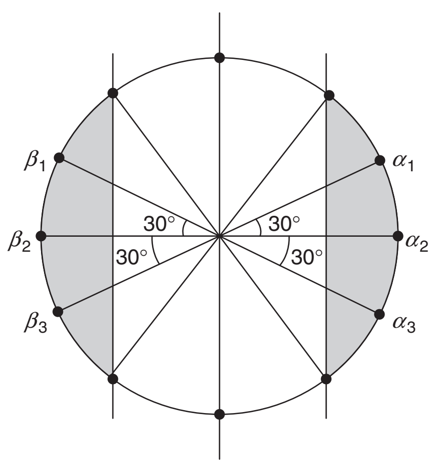


Figure 1.68

Hence, the correct answers are options (C) and (D).

Paragraph for Questions 16 and 17: Let $S = S_1 \cap S_2 \cap S_3$, where

$$S_1 = \{z \in C : |z| < 4\}, S_2 = \left\{z \in C : \text{Im}\left[\frac{z-1+\sqrt{3}i}{1-\sqrt{3}i}\right] > 0\right\} \text{ and}$$

$$S_3 = \{z \in C : \text{Re}\, z > 0\}.$$

[JEE ADVANCED 2013]

16. Area of S = ?

(A) $\frac{10\pi}{3}$ **(B)** $\frac{20\pi}{3}$

(C) $\frac{16\pi}{3}$ **(D)** $\frac{32\pi}{3}$

Solution: As we see, S_1 represents circle with centre (0, 0) and radius 4

$$S_1 : |z| < 4 \Rightarrow x^2 + y^2 < 16$$

Therefore,

$$S_1 : \text{Im}\left[\frac{z-1+\sqrt{3}i}{1-\sqrt{3}i}\right] > 0$$

$$\text{Im}\left[\frac{[(x-1)+(y+\sqrt{3}i)][1+\sqrt{3}i]}{2}\right] > 0$$

Also

$$S_2 \equiv y + \sqrt{3}x > 0$$
$$S_3\ \text{Re}(z) > 0, (x > 0)$$
$$S = S_1 \cap S_2 \cap S_3$$

The area of the shaded region (see Fig. 1.69) is

$$OAB + OBC = \frac{\pi(4)^2}{4} + \frac{60}{360} \times \pi(4)^2$$
$$= 4\pi + \frac{16\pi}{6}$$
$$= 4\pi + \frac{8\pi}{3}$$
$$= \frac{20\pi}{3}$$

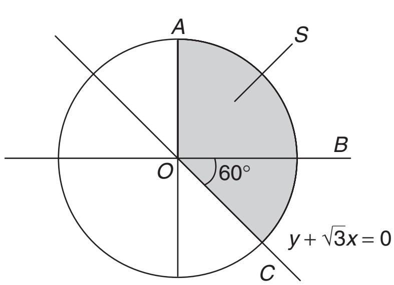


Figure 1.69

Hence, the correct answer is option (B).

17. $\min_{z \in S} |1 - 3i - z| =$

(A) $\frac{2-\sqrt{3}}{2}$ **(B)** $\frac{2+\sqrt{3}}{2}$

(C) $\frac{3-\sqrt{3}}{2}$ **(D)** $\frac{3+\sqrt{3}}{2}$

Solution: We have $\min|1-3i-z|$. The minimum distance of z from (1, –3) from $y + \sqrt{3}x = 0$ is

$$\left|\frac{-3+\sqrt{3}}{2}\right| = \frac{3-\sqrt{3}}{2}$$

Hence, the correct answer is option (C).

18. Let $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R$ be given by $f(x) = (\log(\sec x + \tan x))^3$. Then

(A) $f(x)$ is an odd function
(B) $f(x)$ is a one-one function
(C) $f(x)$ is an onto function
(D) $f(x)$ is an even function

[JEE ADVANCED 2014]

Solution:

$$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R$$

$$f(x) = \{\log(\sec x + \tan x)\}^3$$

$$f(-x) = \{\log(\sec x - \tan x)\}^3 = \left\{\log\left(\frac{1}{\sec x + \tan x}\right)\right\}^3$$

$$\{\because \sec^2 x - \tan^2 x = 1\}$$

Now,

$$f(x) + f(-x) = \{\log(\sec x + \tan x)\}^3 + \{-\log(\sec x + \tan x)\}^3 = 0$$

Therefore,

$$f(-x) = -f(x) \Rightarrow f(x) \text{ is odd} \quad (1)$$

Also,

$$f'(x) = 3\{\log(\sec x + \tan x)\}^2 \frac{1}{\sec x + \tan x} \times (\sec x \tan x + \sec^2 x)$$

$$= 3\{\log(\sec x + \tan x)\}^2 \sec x > 0$$

As in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\sec x$ is +ve.

Note: $\sec x + \tan x = 1 \Rightarrow x = 0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

But at $x = 0$,

$$\log(\sec x + \tan x) = -\infty$$

$$\Rightarrow \{\log(\sec x + \tan x)\}^2 > 0$$

Therefore, $f(x)$ is strictly increasing in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. (2)

Now when $x \to \frac{\pi^-}{2}, f(x) \to \infty$ and $\pi \to \frac{\pi^+}{2}, f(x) \to -\infty$.

Therefore, $f(x)$ being continuous in its domain, it covers whole co-domain, i.e. R. (See Fig. 1.70.)

Hence, it is onto. (3)

Note: $\lim\limits_{x \to \frac{\pi}{2}-} (\sec x + \tan x)$

$$= \lim_{h \to 0}\left(\sec\left(\frac{+\pi}{2} - h\right) + \tan\left(\frac{\pi}{2} - h\right)\right)$$

$$= \lim_{h \to 0}(\operatorname{cosec} h + \cot h) = \frac{1 + \cos h}{\sin h} = +\infty$$

$\lim\limits_{x \to \frac{-\pi}{2}+} (\sec x + \tan x)$

$$= \lim_{h \to 0}\left\{\sec\left(\frac{-\pi}{2} + h\right) + \tan\left(\frac{-\pi}{2} + h\right)\right\}$$

$$= \lim_{h \to 0}\left(\operatorname{cosec}\left(\frac{\pi}{2} - h\right) - \tan\left(\frac{\pi}{2} - h\right)\right)$$

$$= \lim_{h \to 0}(\sec h + \cot h)$$

$$= \lim_{h \to 0}\left(\frac{1 - \cos h}{\sin h}\right) = \frac{2 \sin^2 \frac{h}{2}}{2 \sin \frac{h}{2} \cos \frac{h}{2}}$$

$$= \lim_{h \to 0} \tan\frac{h}{2} = 0$$

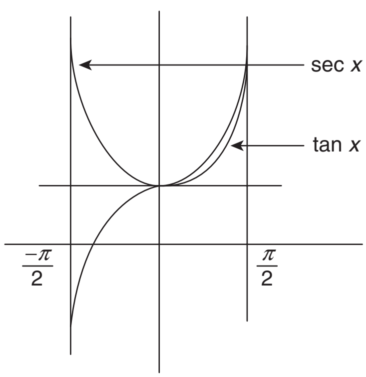


Figure 1.70

Therefore from Eqs. (1), (2) and (3), we can conclude that the correct options are (A), (B) and (C).

Hence, the correct answers are options (A), (B) and (C).

19. Let $f : [0, 4\pi] \to [0, \pi]$ be defined by $f(x) = \cos^{-1}(\cos x)$. The number of points $x \in [0, 4\pi]$ satisfying the equation $f(x) = \frac{10 - x}{10}$ is _______.

[JEE ADVANCED 2014]

Solution: We need to find the point of intersection of the curves

$$f_1(x) = \cos^{-1}(\cos x) \text{ and } f_2(x) = \frac{10 - x}{10}$$

in the domain $[0, 4\pi]$.

$f_1(x)$ is a period function with period 2π and $f_2(x)$ is a straight line plotting both graphs (see Fig. 1.71).

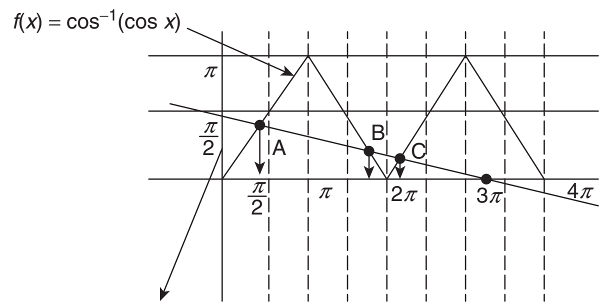


Figure 1.71

Therefore, A, B and C are the points of intersection of both curves which obviously satisfy the given equations, hence there are three such points.

Hence, the correct answer is (3).

20. Let $f_1: R \to R$, $f_2: [0, \infty) \to R$, $f_3: R \to R$ and $f_4: R \to [0, \infty)$ be defined by

$$f_1(x) = \begin{cases} |x| & \text{if } x < 0. \\ e^x & \text{if } x \geq 0 \end{cases}$$

$$f_2(x) = x^2$$

$$f_3(x) = \begin{cases} \sin x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

and

$$f_4(x) = \begin{cases} f_2(f_1(x)) & \text{if } x < 0 \\ f_2(f_1(x)) - 1 & \text{if } x \geq 0 \end{cases}$$

List I	List II
P. f_4 is	**1.** onto but not one-one
Q. f_3 is	**2.** neither continuous nor one-one
R. $f_2 o f_1$ is	**3.** differentiable but not one-one
S. f_2 is	**4.** continuous and one-one

	P	Q	R	S
(A)	3	1	4	2
(B)	1	3	4	2
(C)	3	1	2	4
(D)	1	3	2	4

[JEE ADVANCED 2014]

Solution: See Fig. 1.72.

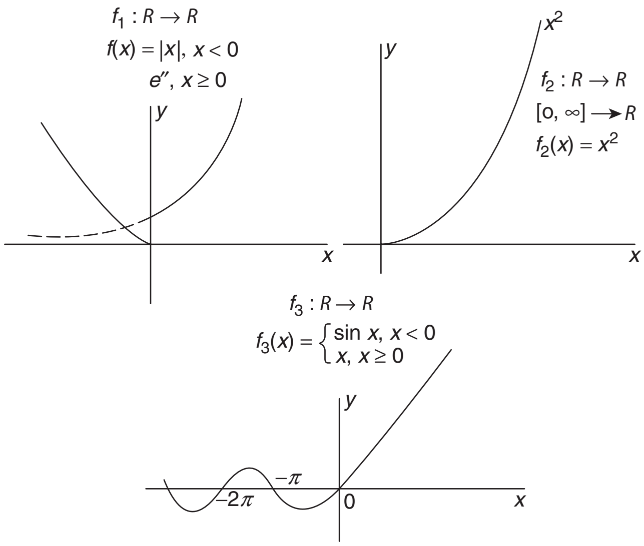


Figure 1.72

$$f_2(f_1(x) = (f_1(x))^2 = \begin{cases} \to |x|^2, \pi < 0 \text{ and } |x| \text{ valid} \\ \text{for } f_1(x) \in R \\ \to (e^x)^2, \pi \geq 0 \ \& \ e^x \text{ valid} \end{cases}$$

Range $f_1 \subset$ Dom f_2

$$\therefore f_2(f_1(x)) = x^2, \quad \pi < 0$$
$$= e^{2x}, \quad \pi \geq 0$$

See Fig. 1.73.

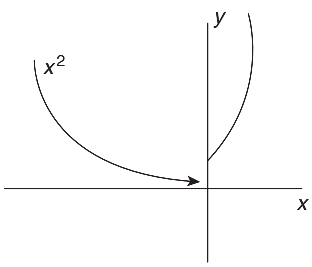


Figure 1.73

Therefore,

$$f_4(x) = \begin{cases} x^2, & x < 0 \\ e^{2x} - 1, & x \geq 0 \end{cases}$$

See Fig. 1.74.

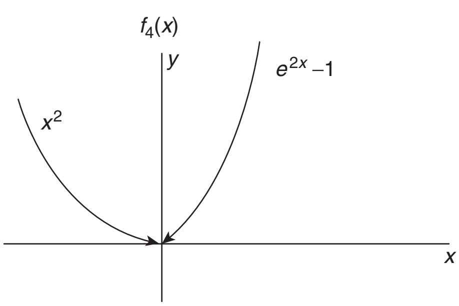


Figure 1.74

For (P) in List I:

f_4 Dom. R
Range $[0, \infty]$
Codomain = $[0\ \infty]$, therefore onto.

Now,

$$\text{LHL at } 0 = \lim_{x \to 0-} x^2 = 0$$
$$\text{RHL at } 0 = \lim_{x \to 0+} e^{2x} - 1 = 1 - 1 = 0$$
$$f_4(0) = 0$$

Therefore, continuous at 0.

Now

$$\text{LHD at } 0 = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{h^2 - 0}{-h} = \lim_{h \to 0} -h = 0$$
$$\text{RHD at } 0 = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{e^{2h} - 1 - 0}{h}$$
$$= \lim_{h \to 0} \frac{e^{2h} - 1}{2h} \cdot 2 = 1 \cdot 2 = 2$$

Therefore, not derivable at 0.
It is not one-one (obvious from graph). Therefore,

$$(P) \to (1)$$

For (Q) in List I:
f_3 in neither one-one nor onto. It is derivable at 0.

$$\text{LHD} = \lim_{h \to 0} \frac{\sin(0-h) - h}{-h} = \lim_{h \to 0} \frac{-\sin h}{-h} = \pm 1$$
$$\text{RHD} = \lim_{h \to 0} \frac{0 + h - 0}{h} = 1$$

Therefore, it is derivable at 0. Hence,
$(Q) \to (3)$
$(R) \to (2)$ (obvious from graph).
$(S) \to (4)$ (obvious from graph.)

Hence, the correct answer is option (D).

21. Let $f(x) = \sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)$ for all $x \in R$ and $g(x) = \frac{\pi}{2}\sin x$ for all $x \in R$. Let $(fog)(x)$ denote $f(g(x))$ and $(gof)(x)$ denote $g(f(x))$. Then which of the following is (are) true?

(A) Range of f is $\left[-\frac{1}{2}, \frac{1}{2}\right]$

(B) Range of fog is $\left[-\frac{1}{2}, \frac{1}{2}\right]$

(C) $\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{\pi}{6}$

(D) There is an $x \in R$ such that $(gof)(x) = 1$

[JEE ADVANCED 2015]

Solution:

$$f(x) = \sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right) \ \forall x \in R$$

and

$$g(x) = \frac{\pi}{2}\sin x \ \ \forall x \in R$$

Now,

$$fog(x) = f(g(x)) = \sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin\left(\frac{\pi}{2}\sin x\right)\right)\right)$$

Therefore,

$$\sin x \in [-1, 1]$$

$$\Rightarrow \frac{\pi}{2}\sin x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\Rightarrow \sin\left(\frac{\pi}{2}\sin x\right) \in [-1, 1]$$

$$\Rightarrow \frac{\pi}{2}\sin\left(\frac{\pi}{2}\sin x\right) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\Rightarrow \sin\left(\frac{\pi}{2}\sin\left(\frac{\pi}{2}\sin x\right)\right) \in [-1, 1]$$

$$\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin\left(\frac{\pi}{2}\sin x\right)\right) \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$$

$$\Rightarrow \sin\left[\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin\left(\frac{\pi}{2}\sin x\right)\right)\right] \in \left[\frac{-1}{2}, \frac{1}{2}\right]$$

$$\Rightarrow \text{Range of } fog \text{ is } \left[\frac{-1}{2}, \frac{1}{2}\right].$$

and

$$gof(x) = g(f(x)) = \frac{\pi}{6}\sin(f(x))$$

$$= \frac{\pi}{2}\sin\underbrace{\left[\underbrace{\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)}_{\text{belongs to }\left[\frac{-\pi}{6},\frac{\pi}{6}\right]}\right]}_{\text{belongs to }\left[\frac{-1}{2},\frac{1}{2}\right]}$$

$$\Rightarrow gof(x) \in \left[\frac{-\pi}{2}\sin\frac{1}{2}, \frac{\pi}{2}\sin\frac{1}{2}\right]$$

Let $\frac{\pi}{2}\sin\frac{1}{2} > 1$. Then

$$\sin\frac{1}{2} > \frac{2}{\pi} > \frac{2}{4} = \frac{1}{2}$$

which is false as $\frac{1}{2} < \frac{\pi}{6}$ rad, so

$$\sin\frac{1}{2} < \frac{1}{2}$$

$$\Rightarrow gof\,(x) \neq 1 \text{ for any } x \in R.$$

Also, $f(x) = \sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)$ belongs to $\left[\frac{-\pi}{6}, \frac{\pi}{6}\right]$. so,

$$f(x) \in \left[\frac{-1}{2}, \frac{1}{2}\right]$$

$$\Rightarrow \text{Range of } f = \left[\frac{-1}{2}, \frac{1}{2}\right]$$

and

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)}{\frac{\pi}{2}\sin x}$$

$$= \lim_{x\to 0}\frac{\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)}{\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)} \times \frac{\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)}{\left(\frac{\pi}{2}\sin x\right)} = \frac{\pi}{6}$$

Hence, the correct answers are options (A), (B) and (C).

22. Let $S = \left\{x \in (-\pi, \pi) : x \neq 0, \pm\frac{\pi}{2}\right\}$. The sum of all distinct solution of the equation $\sqrt{3}\sec x + \operatorname{cosec} x + 2(\tan x - \cot x) = 0$ in the set S is equal to

(A) $-\frac{7\pi}{9}$ **(B)** $-\frac{2\pi}{9}$

(C) 0 **(D)** $\frac{5\pi}{9}$

[JEE ADVANCED 2016]

Solution: Let us consider

$$S = \left\{x \in (-\pi, \pi), x \neq 0, \pm\frac{\pi}{2}\right\}$$

The given equation is

$$\sqrt{3}\sec x + \operatorname{cosec} x + 2(\tan x - \cot x) = 0$$

$$\Rightarrow \frac{\sqrt{3}}{\cos x} + \frac{1}{\sin x} + 2\left(\frac{\sin x}{\cos x} - \frac{\cos x}{\sin x}\right) = 0$$

$$\Rightarrow \sqrt{3}\sin x + \cos x + 2(\sin^2 x - \cos^2 x) = 0$$

$$\Rightarrow \sqrt{3}\sin x + \cos x = 2\cos 2x$$

$$\Rightarrow \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x = \cos 2x$$

$$\Rightarrow \cos x\cos\frac{\pi}{3} + \sin x\sin\frac{\pi}{3} = \cos 2x$$

$$\Rightarrow \cos 2x = \cos\left(x - \frac{\pi}{3}\right)$$

$$\Rightarrow 2x = 2n\pi \pm \left(x - \frac{\pi}{3}\right)$$

$$(x \in I)$$

- **Case 1:** When $2x = 2n\pi + x - \frac{\pi}{3}$, we have $x = 2n\pi - \frac{\pi}{3}$.

  If $n = 0$, we get $x = -\frac{\pi}{3}$.

  If $n = 1$, we get $x = 2\pi - \frac{\pi}{3}$.

  If $n = -1$, we get $x = -2\pi - \frac{\pi}{3}$.

- **Case 2:** When $2x = 2n\pi - x + \frac{\pi}{3}$, we get $x = \frac{2n\pi}{3} + \frac{\pi}{9}$.

If $n = 0$, we get $x = \frac{\pi}{9}$.

If $n = 1$, we get $x = \frac{2\pi}{3} + \frac{\pi}{9}$.

If $n = 2$, we get $n = 2x = \frac{4\pi}{3} + \frac{\pi}{9}$.

If $n = -1$, we get $x = \frac{-2\pi}{3} + \frac{\pi}{9}$.

Therefore, the sum of all distinct solutions of the given equation is

$$\frac{-\pi}{3} + \frac{\pi}{9} + \frac{2\pi}{3} + \frac{\pi}{9} - \frac{2\pi}{3} + \frac{\pi}{9} = 0$$

Hence, the correct answer is option (C).

23. Let $f : R \to R$, $g : R \to R$ and $h : R \to R$ be differentiable functions such that $f(x) = x^2 + 3x + 2$, $g(f(x)) = x$ and $h(g(g(x))) = x$ for all $x \in R$. Then

(A) $g'(2) = \frac{1}{15}$ **(B)** $h'(1) = 666$

(C) $h(0) = 16$ **(D)** $h(g(3)) = 36$

[JEE ADVANCED 2016]

Solution:

(A) It is given that $f: R \to R$, $g: R \to R$ and $h: R \to R$ are differentiable functions.

Now, $f(x) = x^3 + 3x + 2$

Differentiating w.r.t. to x, we get

$$f'(x) = 3x^2 + 3$$

Also,

$$g(f(x)) = x$$

Now,

$$g'(f(x)) \cdot f'(x) = 1$$

$$f(x) = 2 \Rightarrow x^3 + 3x + 2 = 2$$

$$\Rightarrow x^3 + 3x = 0 \Rightarrow x(x^2 + 3) = 0 \Rightarrow x = 0$$

Now,

$$g'(f(0)) = \frac{1}{f'(0)} \Rightarrow g'(2) = \frac{1}{3}$$

Hence, option (A) is incorrect.

(B) For all $x \in R$:

$$h(g(g(f(x))) = x$$

$$h(g(g(x))) = x$$

Now,

$$x \to f(x) \Rightarrow h(g(g(f(x))) = f(x)$$

$$\Rightarrow h(g(x)) = f(x)$$

$$\Rightarrow h'(g(x)) \cdot g'(x) = f'(x) = 3x^2 + 3 \quad (1)$$

For all $x \in R$:

$$g(f(x)) = x$$

Now,

$$x = 1 \Rightarrow g(f(1)) = 1 \Rightarrow g(6) = 1 \quad (\because f(1) = 6)$$

Substituting $x = 6$ in Eq. (1), we get

$$h'(g(6)) \cdot g'(6) = 3(6^2) + 3 = 111$$

Therefore,

$$h'(1) = \frac{111}{g'(6)} \left(g'(6) = \frac{1}{f'(1)} \right)$$

That is,

$$h'(1) = 111 \cdot f'(x) = 111 \times (3 + 3) = 666$$

Hence, option (B) is correct.

(C) $h(g(g(x))) = x$

For $g(g(x)) = 0$, we have

$$g(x) = g^{-1}(0) = 2$$

$$\Rightarrow x = g^{-1}(2) = f(2) = 16$$

$$\Rightarrow h(0) = 16$$

Hence, option (C) is correct.

(D) Here, $g(g(x)) = g(3)$ which implies that

$$g(x) = 3 \Rightarrow x = g^{-1}(3) = f(3) = 38$$

Hence, option (D) is incorrect.

Hence, the correct answers are options (B) and (C).

Practice Exercise 1

1. If $f(x) = \cos(\log x)$, then $f(x)f(y) - \{(1/2)[f(x/y) + f(xy)]\}$ is

(A) −1 **(B)** 1/2

(C) −2 **(D)** None of these

2. The values of b and c for which the identity $f(x+1) - f(x) = 8x + 3$ is satisfied, where $f(x) = bx^2 + cx + d$ are

(A) $b = 2, c = 1$ **(B)** $b = 4, c = -1$

(C) $b = -1, c = 4$ **(D)** $b = -1, c = 1$

3. If $f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$, then

(A) $f(\pi/4) = 2$ **(B)** $f(-\pi) = 2$

(C) $f(\pi) = 1$ **(D)** $f(\pi/2) = -1$

4. $f(x, y) = 1/(x + y)$ is a homogeneous function of degree

(A) 1 **(B)** −1

(C) 2 **(D)** −2

5. Let x be a non-zero rational number and y be an irrational number. Then xy is

(A) Rational **(B)** Irrational

(C) Non-zero **(D)** None of these

6. Numerical value of the expression $\left| \frac{3x^3 + 1}{2x^2 + 2} \right|$ for $x = -3$ is

(A) 4 **(B)** 2

(C) 3 **(D)** 0

7. The function $f : R \to R$ defined by $f(x) = (x-1)(x-2)(x-3)$ is

(A) One-to-one but not onto

(B) Onto but not one-to-one

(C) Both one-to-one and onto

(D) Neither one-to-one nor onto

8. Which one of the following is a bijective function on the set of real numbers?

(A) $2x-5$ **(B)** $|x|$

(C) x^2 **(D)** x^2+1

9. Let $f(x)=\dfrac{x^2-4}{x^2+4}$ for $|x|>2$. Then the function $f:(-\infty,-2]\cup[2,\infty)\to(-1,1)$ is

(A) One-to-one and into **(B)** One-to-one and onto

(C) Many-to-one and into **(D)** Many-to-one and onto

10. Let the function $f: R\to R$ be defined by $f(x)=2x+\sin x, x\in R$. Then f is

(A) One-to-one and onto **(B)** One-to-one but not onto

(C) Onto but not one-to-one **(D)** Neither one-to-one nor onto

11. If $f:[0,\infty)\to[0,\infty)$ and $f(x)=\dfrac{x}{1+x}$, then f is

(A) One-to-one and onto

(B) One-to-one but not onto

(C) Onto but not one-to-one

(D) Neither one-to-one nor onto

12. If $f:R\to S$ defined by $f(x)=\sin x-\sqrt{3}\cos x+1$ is onto, then the interval of S is

(A) [–1, 3] **(B)** [1, 1]

(C) [0, 1] **(D)** [0, –1]

13. $f(x)=x+\sqrt{x^2}$ is a function from $R\to R$, then $f(x)$ is

(A) Injective **(B)** Surjective

(C) Bijective **(D)** None of these

14. If $(x,y)\in R$ and $x,y\neq 0$; $f(x,y)\to(x/y)$, then this function is a/an

(A) Surjection **(B)** Bijection

(C) One-to-one **(D)** None of these

15. The domain of the function $f(x)=\dfrac{\sin^{-1}(3-x)}{\ln(|x|-2)}$ is

(A) [2, 4] **(B)** (2, 3) ∪ (3, 4]

(C) [2, ∞) **(D)** $(-\infty,-3)\cup[2,\infty)$

16. The domain of $f(x)=\dfrac{\log_2(x+3)}{x^2+3x+2}$ is

(A) $R-\{-1,-2\}$ **(B)** $(-2,+\infty)$

(C) $R-\{-1,-2,-3\}$ **(D)** $(-3,+\infty)-\{-1,-2\}$

17. The domain of the derivative of the function

$$f(x)=\begin{cases}\tan^{-1}x, & |x|\le 1\\ \frac{1}{2}(|x|-1), & |x|>1\end{cases}\text{ is}$$

(A) $R-\{0\}$ **(B)** $R-\{1\}$

(C) $R-\{-1\}$ **(D)** $R-\{-1,1\}$

18. The domain of the function $f(x)=\log_{3+x}(x^2-1)$ is

(A) $(-3,-1)\cup(1,\infty)$ **(B)** $[-3,-1)\cup[1,\infty)$

(C) $(-3,-2)\cup(-2,-1)\cup(1,\infty)$ **(D)** $[-3,-2)\cup(-2,-1)\cup[1,\infty)$

19. The domain of the function $f(x)=\sqrt{2-2x-x^2}$ is

(A) $-\sqrt{3}\le x\le\sqrt{3}$ **(B)** $-1-\sqrt{3}\le x\le -1+\sqrt{3}$

(C) $-2\le x\le 2$ **(D)** $-2+\sqrt{3}\le x\le -2-\sqrt{3}$

20. The domain of the function $f(x)=(x-3)/(x-1)\sqrt{x^2-4}$ is

(A) (1, 2) **(B)** $(-\infty,-2)\cup(2,\infty)$

(C) $(-\infty,-2)\cup(1,\infty)$ **(D)** $(-\infty,\infty)-\{1,\pm 2\}$

21. The domain of the function $\sqrt{\log(x^2-6x+6)}$ is

(A) $(-\infty,\infty)$ **(B)** $(-\infty,3-\sqrt{3})\cup(3+\sqrt{3},\infty)$

(C) $(-\infty,1]\cup[5,\infty)$ **(D)** $[0,\infty)$

22. The domain of the function $f(x)=\sin^{-1}(1+3x+2x^2)$ is

(A) $(-\infty,\infty)$ **(B)** (–1, 1)

(C) $\left[-\dfrac{3}{2},0\right]$ **(D)** $\left(-\infty,\dfrac{-1}{2}\right)\cup(2,\infty)$

23. The domain of $f(x)=(x^2-1)^{-1/2}$ is

(A) $(-\infty,-1)\cup(1,\infty)$ **(B)** $(-\infty,-1]\cup(1,\infty)$

(C) $(-\infty,-1]\cup[1,\infty)$ **(D)** None of these

24. The domain of the function $y=1/\sqrt{|x|-x}$ is

(A) $(-\infty,0)$ **(B)** $(-\infty,0]$

(C) $(-\infty,-1)$ **(D)** $(-\infty,\infty)$

25. The range of $f(x)=\sec\left(\dfrac{\pi}{4}\cos^2 x\right), -\infty<x<\infty$ is

(A) $[1,\sqrt{2}]$ **(B)** $[1,\infty)$

(C) $[-\sqrt{2},-1]\cup[1,\sqrt{2}]$ **(D)** $(-\infty,-1]\cup[1,\infty)$

26. The range of the function $f(x)=\dfrac{x^2+x+2}{x^2+x+1}; x\in R$ is

(A) $(1,\infty)$ **(B)** (1,11/7]

(C) (1,7/3] **(D)** (1,7/5]

27. The range of $f(x)=\cos 2x-\sin 2x$ contains the set

(A) [2, 4] **(B)** [–1, 1]

(C) [–2, 2] **(D)** [–4, 4]

28. The interval for which $\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}=\dfrac{\pi}{2}$ holds

(A) $[0,\infty)$ **(B)** [0,3]

(C) [0, 1] **(D)** [0, 2]

29. For $\theta>\dfrac{\pi}{3}$, the value of $f(\theta)=\sec^2\theta+\cos^2\theta$ always lies in the interval

(A) (0, 2) **(B)** [0, 1]

(C) (1, 2) **(D)** $[2,\infty)$

30. The function $f(x)=\sin[\log(x+\sqrt{x^2+1})]$ is

(A) Even function **(B)** Odd function

(C) Neither even nor odd **(D)** Periodic function

31. If $y=f(x)=(x+2)/(x-1)$, then x is

(A) $f(y)$ **(B)** $2f(y)$

(C) $1/f(y)$ **(D)** None of these

32. If the function $f:[1,\infty)\to[1,\infty)$ is defined by $f(x)=2^{x(x-1)}$, then $f^{-1}(x)$ is

(A) $\left(\frac{1}{2}\right)^{x(x-1)}$ **(B)** $\frac{1}{2}(1+\sqrt{1+4\log_2 f(x)}$

(C) $\frac{1}{2}(1-\sqrt{1+4\log_2 f(x)}$ **(D)** Not defined

33. If $f(x)=3x-5$, then $f^{-1}(x)$

(A) Is given by $1/(3x-5)$

(B) Is given by $(x+5)/3$

(C) Does not exist because f is not one-to-one

(D) Does not exist because f is not onto

34. Let $f(\theta)=\sin\theta(\sin\theta+\sin 3\theta)$, then

(A) $f(\theta)\geq 0$ only when $\theta\geq 0$ **(B)** $f(\theta)\leq 0$ for all real θ

(C) $f(\theta)\geq 0$ for all real θ **(D)** $f(\theta)\leq 0$ only when $\theta\leq 0$

35. Let $f(x)=\sin x+\cos x, g(x)=x^2-1$. Thus, $g[f(x)]$ is invertible for $x\in R$

(A) $\left[-\frac{\pi}{2},0\right]$ **(B)** $\left[-\frac{\pi}{2},\pi\right]$

(C) $\left[-\frac{\pi}{4},\frac{\pi}{4}\right]$ **(D)** $\left[0,\frac{\pi}{2}\right]$

36. Let $g(x) = 1 + x - [x]$ and $f(x)=\begin{cases}-1, x<0\\ 0, x=0.\\ 1, x>0\end{cases}$ Then $\forall\ x, f[g(x)]$ is equal to

(A) x **(B)** 1

(C) $f(x)$ **(D)** $g(x)$

37. If $f(x)=\frac{ax}{x+1}, x\neq -1$, then for what value of a is $f[f(x)]=x$?

(A) $\sqrt{2}$ **(B)** $-\sqrt{2}$

(C) 1 **(D)** -1

38. If $f(x)=(a-x^n)^{1/n}$ where $a>0$ and n is a positive integer, then $f[f(x)]$

(A) x^3 **(B)** x^2

(C) x **(D)** None of these

39. If X and Y are two non-empty sets, where $f:X\to Y$ is the function, is defined such that $f(c)=\{f(x):x\in C\}$ for $C\subseteq X$ and $f^{-1}(D)=\{x:f(x)\in D\}$ for $D\subseteq Y$ for any $A\subseteq X$ and $B\subseteq Y$, then

(A) $f^{-1}[f(A)]=A$

(B) $f^{-1}[f(A)]=A$ only if $f(x)=Y$

(C) $f[f^{-1}(B)]=B$ only if $B\subseteq f(x)$

(D) $f[f^{-1}(B)]=B$

40. If $f(x)=\begin{cases}x, \text{when } x \text{ is rational}\\ 0, \text{when } x \text{ is irrational}\end{cases}$; $g(x)=\begin{cases}0, \text{when } x \text{ is rational}\\ x, \text{when } x \text{ is irrational}\end{cases}$, then $(f-g)$ is

(A) One-to-one and onto

(B) One-to-one, but not onto

(C) Not one-to-one, but onto

(D) Neither one-to-one nor onto

41. If $f(x)$ and $g(x)$ be two given functions with all real numbers as their domain, then $h(x) = [f(x) + f(-x)]\ [g(x) - g(-x)]$ is

(A) An odd function

(B) An odd function when both f and g are odd

(C) An odd function when f is even and g is odd

(D) None of these

42. If $f(x)=\sqrt{4-x^2}+(1/\sqrt{|\sin x|-\sin x})$, then the domain of $f(x)$ is

(A) $[-2, 0]$ **(B)** $(0, 2]$

(C) $[-2, 2\}$ **(D)** $[-2, 0)$

43. If $f(x) = x^3 + 3x^2 + 12x - 2\sin x$, where $f: R\to R$, then

(A) $f(x)$ is many-to-one and onto

(B) $f(x)$ is one-to-one and onto

(C) $f(x)$ is one-to-one and into

(D) $f(x)$ is many-to-one and into

44. If $f(x)=\ln\left(\frac{x^2+e}{x^2+1}\right)$, then the range of $f(x)$ is

(A) $(0, 1)$ **(B)** $[0, 1]$

(C) $[0, 1)$ **(D)** $(0, 1]$

45. A function $f(x)$ is defined for all real x and satisfied $f(x + y) = f(xy)\ \forall x, y$. If $f(1) = -1$, then $f(2006)$ equals

(A) -2006 **(B)** 2006

(C) -1 **(D)** None of these

46. Let $y = f(x)$ be a real-valued function with domain as all real numbers. If the graph of the function is symmetrical about the line $x = 1$, then $\forall\alpha\in R$, which one is correct?

(A) $f(\alpha) = f(\alpha + 1)$ **(B)** $f(\alpha - 1) = f(\alpha)$

(C) $f(\alpha - 1) = f(\alpha + 1)$ **(D)** $f(1 - \alpha) = f(1 + \alpha)$

47. The range of the function $f(x)=\sin^{-1}\left[x^2+\frac{1}{2}\right]+\cos^{-1}\left[x^2-\frac{1}{2}\right]$, where $[\cdot]$ is the greatest integer function, is

(A) $\left\{\frac{\pi}{2},\pi\right\}$ **(B)** $\left\{0,\frac{\pi}{2}\right\}$

(C) $\{\pi\}$ **(D)** $\left(0,\frac{\pi}{2}\right)$

48. If $f o g = |\sin x|$ and $g o f = \sin^2\sqrt{x}$, then $f(x)$ and $g(x)$ are

(A) $f(x)=\sqrt{\sin x}, g(x)=x^2$ **(B)** $f(x) = |x|, g(x) = \sin x$

(C) $f(x)=\sqrt{x}, g(x)=\sin^2 x$ **(D)** $f(x)=\sin\sqrt{x}, g(x)=x^2$

49. If $f(x)$ is a function that is odd and even simultaneously, then $f(3) - f(2)$ is equal to

(A) 1 **(B)** -1

(C) 0 **(D)** None of these

50. The domain of $f(x) = \sqrt{\log_{1/4}\left(\frac{5x-x^2}{4}\right)} + {}^{10}C_x$ is

(A) $(0, 1] \cup [4, 5)$ **(B)** $(0, 5)$
(C) $\{1, 4\}$ **(D)** None of these

51. If $f: R \to R$, where $f(x) = ax + \cos x$. If $f(x)$ is bijective, then

(A) $a \in R$ **(B)** $a \in R^+$
(C) $a \in R^-$ **(D)** $a \in R - (-1, 1)$

52. If f is a function such that $f(0) = 2$, $f(1) = 3$ and $f(x + 2) = 2f(x) - f(x + 1)\ \forall\ x \in R$, then $f(5)$ is

(A) 7 **(B)** 13
(C) 1 **(D)** None of these

53. The number of real roots of $3^x + 4^x + 5^x - 6^x = 0$ is/are

(A) Two **(B)** More than two
(C) One **(D)** Equation does not have any real root

54. The range of $f(x) = \sin[\sin^{-1}\{x\}]$, where $\{.\}$ denotes the function part of x, is

(A) $[0, 1)$ **(B)** $[0, 1]$
(C) $(-1, 1)$ **(D)** None of these

55. Let $f(x) = \frac{3}{x-2} + \frac{4}{x-3} + \frac{5}{x-4}$. Then $f(x) = 0$ has

(A) Exactly one real root $\in (2, 3)$
(B) At least one real root $\in (3, 4)$
(C) At least one real root $\in (2, 3)$
(D) None of these

56. If $f(x) = \sin\sqrt{[a]}\,x$, (where [·] denotes the greatest integer function), has π as its fundamental period, then

(A) $a = 1$ **(B)** $a \in [1, 2)$
(C) $a = 9$ **(D)** $a \in [4, 5)$

57. If $f(x) = \begin{cases} x, & \text{when } x \text{ is rational} \\ 1-x, & \text{when } x \text{ is irrational} \end{cases}$, then $f o f(x)$ is given as

(A) 1 **(B)** x
(C) $1 + x$ **(D)** None of these

58. If $f(x)$ is defined on domain [0, 1], then $f(2\sin x)$ is defined on

(A) $\bigcup_{n \in I}\left\{\left[2n\pi, 2n\pi + \frac{\pi}{6}\right] \bigcup \left[2n\pi + \frac{5\pi}{6}, (2n+1)\pi\right]\right\}$

(B) $\bigcup_{n \in I}\left[2n\pi, 2n\pi + \frac{\pi}{6}\right]$

(C) $\bigcup_{n \in I}\left[2n\pi + \frac{5\pi}{6}, (2n+1)\pi\right]$

(D) None of these

59. Let $f: R \to R$ and $g: R \to R$ be two one-to-one and onto functions such that they are the mirror images of each other about the line $y = a$. If $h(x) = f(x) + g(x)$, then $h(x)$ is

(A) One-to-one and onto **(B)** One-to-one and into
(C) Many-to-one and onto **(D)** Many-to-one and into

60. If $f(x) = \cos|x| + \left[\left|\frac{\sin x}{2}\right|\right]$, (where [·] denotes the greatest integer function), then

(A) $f(x)$ is periodic **(B)** $f(x)$ is odd
(C) $f(x)$ is non-periodic **(D)** None of these

61. The range of the function $f(x) = \sqrt{{}^{x^2+4x}C_{2x^2+3}}$ is

(A) $\{1, 2\sqrt{3}\}$ **(B)** $\{1, 2\sqrt{3}, 3\sqrt{5}\}$
(C) $\{1, 2, 3\}$ **(D)** $\{1, 2\}$

62. The number of solutions of $\log_{\sin x} 2^{\tan x} > 0$ in the interval $\left(0, \frac{\pi}{2}\right)$ is

(A) 0 **(B)** 1
(C) 2 **(D)** 3

63. If $5^x + (2\sqrt{3})^{2x} \geq 13^x$, then the solution set for x is

(A) $[2, \infty)$ **(B)** $\{2\}$
(C) $(-\infty, 2]$ **(D)** $[0, 2]$

64. If domain of $f(x)$ is $[-1, 2]$, then the domain of $f([x] - x^2 + 4)$, where [·] denotes the greatest integer function, is

(A) $[-1, \sqrt{7}]$ **(B)** $[-\sqrt{3}, -1] \cup [-\sqrt{3}, \sqrt{7}]$
(C) $(-1, \sqrt{7}]$ **(D)** $[-\sqrt{3}, -1] \cup (\sqrt{3}, \sqrt{7})$

65. The period of the function $f(x) = [5x + 7] + \cos\pi x - 5x$, where [·] denotes the greatest integer function, is

(A) 3 **(B)** 2π
(C) 2 **(D)** None of these

66. The period of the function $f(x) = [x] + [2x] + [3x] + \cdots + [nx] - \frac{n(n+1)}{2}x$, when $x \in N$ is

(A) n **(B)** 1
(C) $1/n$ **(D)** None of these

67. Let $S_n = \sum_{r=1}^{n} r!\ (n > 6)$, then $S_n - 7\left[\frac{S_n}{7}\right]$ (where [·] denotes the greatest integer function) is equal to

(A) $\left[\frac{n}{7}\right]$ **(B)** $n! - 7\left[\frac{n!}{7}\right]$
(C) 5 **(D)** 3

68. The period of the function $f(x) = \frac{1}{3}(\sin 3x + |\sin 3x| + [\sin 3x])$ where [·] denotes the greatest integer function

(A) $\pi/3$ **(B)** $2\pi/3$
(C) $4\pi/3$ **(D)** π

69. If $f(x) = 1/[|\sin x| + |\cos x|]$ (where [·] denotes the greatest integer function), then

(A) $f(x)$ is an even function
(B) $f(x)$ is an odd function
(C) The range of $f(x)$ contains two elements
(D) None of these

70. If $f(x) = \sqrt{\sec^{-1}[(2-|x|)/4]}$, then the domain of $f(x)$ is

(A) $[-2, 2]$ **(B)** $[-6, 6]$
(C) $(-\infty, -6] \cup [6, \infty)$ **(D)** $[-6, -2] \cup [2, 6]$

71. The number of solution(s) of the equation $x^2 - 2 - 2[x] = 0$ ([.] denotes the greatest integer function) is(are)

(A) One (B) Two
(C) Zero (D) Infinity

72. The domain of the function $f(x) = \sin^{-1}[1 + \cos x] + \sqrt{16 - x^2}$ ([·] denotes the greatest integer function) is
(A) [–4, 4] (B) (–4, 4)
(C) [0, 2π] (D) None of these

73. If the function $f: [2, \infty) \to [1, \infty)$ is defined by $f(x) = 3^{x(x-2)}$, then what is $f^{-1}(x)$?
(A) $1 + \sqrt{1 + \log_3 x}$ (B) $1 - \sqrt{1 + \log_3 x}$
(C) $1 + \sqrt{1 - \log_3 x}$ (D) Does not exist

74. $f: [-4, 4] \sim \{-\pi, 0, \pi\} \to R$, when $f(x) = \cot(\sin x) + [x^2/a]$, when [·] denotes the greatest integer function. If f be an odd function, then the set of values of a is
(A) $(-16, 16) \sim \{0\}$ (B) $(-\infty, -16) \cup (16, \infty)$
(C) $[-16, -16] \sim \{0\}$ (D) $(-\infty, -16] \cup [16, \infty)$

75. Let A ≡ {1, 2, 3, 4}, B = {a, b, c}, then the number of function from A→ B, which are not onto is
(A) 81 (B) 24
(C) 8 (D) 45

76. The number of solutions of $\log_x 3 = 2x - 3$ is
(A) 3 (B) 1
(C) 2 (D) 0

77. The domain of $f(x) = \sqrt{(x-1)/(x - 2\{x\})}$, where $\{x\}$ denotes the fractional part of x, is
(A) $(-\infty, 0) \cup (0, 2]$ (B) [1, 0)
(C) $(-\infty, \infty) \sim (0, 2]$ (D) $(-\infty, 0) \cup (0, 1] \cup [2, \infty)$

78. Let $f(x) = \sqrt{[\sin 2x] - [\cos 2x]}$, (where [·] denotes the greatest integer function). Then the range of $f(x)$ is
(A) {0} (B) {1}
(C) {0, 1} (D) $\{0, 1, \sqrt{2}\}$

79. The number of points (x, y), where the curves $|y| = \ln|x|$ and $(x-1)^2 + y^2 - 4 = 0$ cut each other, is
(A) 2 (B) 3
(C) 1 (D) 6

80. If $f(x + y, x - y) = xy$, then the arithmetic mean of $f(x, y)$ and $f(y, x)$ is
(A) x (B) y
(C) 0 (D) xy

81. If $a, b \in R$, then the period of $f(x) = x - [x + a] - b$, where [·] denotes the greatest integer function, is
(A) 1 (B) $|a - b|$
(C) $|a + b|$ (D) None of these

82. The domain of the function $f(x) = \sqrt{x^{12} - x^3 + x^4 - x + 1}$ is
(A) (–1, 1) (B) (–∞, –1)
(C) (1, ∞) (D) (–∞, ∞)

83. The domain of definition of the function $f(x) = \sqrt{4^x + (64)^{(x-2)/3} - [(1/2)(72 + 2^{2x})]}$ is
(A) (– ∞, ∞) (B) (–∞, –3]
(C) [3, ∞) (D) $\left[\frac{1}{9}, 1\right)$

84. Let $f(x) = \int_0^x \log\frac{1 - \tan t}{1 + \tan t}\,dt$. Then $f(x)$ is
(A) An odd function
(B) An even function
(C) Both even as well as odd
(D) Neither even nor odd

85. If $f(x + y) = f(x) + f(y) - xy - 1\ \forall x, y \in R$ and $f(1) = 1$, then the number of solutions of the equation $f(n) = n, n \in N$ is
(A) 0 (B) 1
(C) 2 (D) n

86. If $f(x)$ is an even function and satisfies the relation $x^2 f(x) - 2f(1/x) = g(x)$, $x \neq 0$, where $g(x)$ is an odd function, then the value of $f(2)$ is
(A) 1/2 (B) 2
(C) 4/5 (D) 0

87. X and Y are two sets and $f: X \to Y$. If $\{f(c) = y; c \subset X, y \subset Y\}$ and $\{f^{-1}(d) = x; d \subset Y, x \subset X\}$, then the true statement is
(A) $f[f^{-1}(b)] = b$ (B) $f^{-1}[f(a)] = a$
(C) $f[f^{-1}(b)] = b, b \subset y$ (D) $f^{-1}[f(a)] = a, a \subset x$

88. If $f(x) = x/\sqrt{1 + x^2}$, then $f \circ f \circ f(x)$ equals to
(A) $x/\sqrt{1 + 3x^2}$ (B) $3x/\sqrt{1 + x^2}$
(C) $x/(1 + x^2)^{1/6}$ (D) x

89. The function $f(x) = (x^2 + 2x + c)/(x^2 + 4x + 3c)$ has the range $(-\infty, \infty)$ for the allowed values of $x \in R$ if
(A) $0 < c < 1$ (B) $0 \leq c \leq 1$
(C) $0 < c \leq 1$ (D) $0 \leq c < 1$

90. Let $A = \{1, 2, 3\}$. Then the total number of distinct relations that can be defined over A is
(A) 2^9 (B) 6
(C) 8 (D) None of these

91. Given two finite sets A and B such that $n(A) = 2$, $n(B) = 3$. Then total number of relations from A to B is
(A) 4 (B) 8
(C) 64 (D) None of these

92. The relation R defined on the set of natural numbers as {(a, b) : a differs from b by 3} is given by
(A) {(1, 4, (2, 5), (3, 6), ...}
(B) {(4, 1), (5, 2), (6, 3), ...}
(C) {(1, 3), (2, 6), (3, 9), …}
(D) None of these

93. The relation R is defined on the set of natural numbers as $\{(a, b) : a = 2b\}$. Then R^{-1} is given by
(A) {(2, 1), (4, 2), (6, 3), ...} (B) {(1, 2), (2, 4), (3, 6), ...}
(C) R^{-1} is not defined (D) None of these

94. The relation $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$ on set $A = \{1, 2, 3\}$ is
(A) Reflexive but not symmetric
(B) Reflexive but not transitive
(C) Symmetric and transitive
(D) Neither symmetric nor transitive

95. The relation "less than" in the set of natural numbers is
(A) Only symmetric (B) Only transitive
(C) Only reflexive (D) Equivalence relation

96. Let $P = \{(x, y) \mid x^2 + y^2 = 1, x, y \in R\}$. Then P is
(A) Reflexive (B) Symmetric
(C) Transitive (D) Anti-symmetric

97. Let R be an equivalence relation on a finite set A having n elements. Then the number of ordered pairs in R is
(A) Less than n (B) Greater than or equal to n

(C) Less than or equal to n **(D)** None of these

98. For real numbers x and y, we write $xRy \Leftrightarrow x - y + \sqrt{2}$ is an irrational number. Then the relation R is
(A) Reflexive **(B)** Symmetric
(C) Transitive **(D)** None of these

99. Let X be a family of sets and R be a relation on X defined by 'A is disjoint from B'. Then R is
(A) Reflexive **(B)** Symmetric
(C) Anti-symmetric **(D)** Transitive

100. If R be a relation < from $A = \{1, 2, 3, 4\}$ to $B = \{1, 3, 5\}$, that is, $(a,b) \in R \Leftrightarrow a < b$, then RoR^{-1} is
(A) $\{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)\}$
(B) $\{(3, 1) (5, 1), (3, 2), (5, 2), (5, 3), (5, 4)\}$
(C) $\{(3, 3), (3, 5), (5, 3), (5, 5)\}$
(D) $\{(3, 3) (3, 4), (4, 5)\}$

101. A relation from P to Q is
(A) A universal set of $P \times Q$ **(B)** $P \times Q$
(C) An equivalent set of $P \times Q$ **(D)** A subset of $P \times Q$

102. Let $A = \{a, b, c\}$ and $B = \{1, 2\}$. Consider a relation R defined from set A to set B. Then R is equal to set
(A) A **(B)** B
(C) $A \times B$ **(D)** $B \times A$

103. Let $n(A) = n$. Then the number of all relations on A is
(A) 2^n **(B)** $2^{(n)!}$
(C) 2^{n^2} **(D)** None of these

104. Let R be a reflexive relation on a finite set A having n-elements, and let there be m-ordered pairs in R. Then
(A) $m \geq n$ **(B)** $m \leq n$
(C) $m = n$ **(D)** None of these

105. The relation R defined on the set $A = \{1, 2, 3, 4, 5\}$ by $R = \{(x, y) : |x^2 - y^2| < 16\}$ is given by
(A) $\{(1, 1), (2, 1), (3, 1), (4, 1), (2, 3)\}$
(B) $\{(2, 2), (3, 2), (4, 2), (2, 4)\}$
(C) $\{(3, 3), (3, 4), (5, 4), (4, 3), (3, 1)\}$
(D) None of these

106. A relation R is defined from $\{2, 3, 4, 5\}$ to $\{3, 6, 7, 10\}$ by $xRy \Leftrightarrow x$ is relatively prime to y. Then domain of R is
(A) $\{2, 3, 5\}$ **(B)** $\{3, 5\}$
(C) $\{2, 3, 4\}$ **(D)** $\{2, 3, 4, 5\}$

107. Let R be a relation on N defined by $x + 2y = 8$. Then the domain of R is
(A) $\{2, 4, 8\}$ **(B)** $\{2, 4, 6, 8\}$
(C) $\{2, 4, 6\}$ **(D)** $\{1, 2, 3, 4\}$

108. If $R = \{(x, y) | x, y \in Z, x^2 + y^2 \leq 4\}$ is a relation in Z, then domain of R is
(A) $\{0, 1, 2\}$ **(B)** $\{0, -1, -2\}$
(C) $\{-2, -1, 0, 1, 2\}$ **(D)** None of these

109. R is a relation from $\{11, 12, 13\}$ to $\{8, 10, 12\}$ defined by $y = x - 3$. Then R^{-1} is
(A) $\{(8, 11), (10, 13)\}$ **(B)** $\{(11, 18), (13, 10)\}$
(C) $\{(10, 13), (8, 11)\}$ **(D)** None of these

Practice Exercise 2

Single/Multiple Correct Choice Type Questions

1. If $y = f(x)$ be the concave upward function and $y = g(x)$ be a function such that $f'(x) \cdot g(x) - g'(x) \cdot f(x) = x^4 + 2x^2 + 10$, then
(A) $g(x)$ has at least one root between two consecutive roots of $f(x) = 0$
(B) $g(x)$ has at most one root between two consecutive roots of $f(x) = 0$
(C) if α and β are two consecutive roots of $f(x) = 0$, then $\alpha\beta < 0$
(D) when $f(x)$ increases $g(x)$ decreases

2. If $\log_2\left(\log_{\frac{1}{2}}(\log_2(x))\right) = \log_3\left(\log_{\frac{1}{3}}(\log_3(y))\right) = \log_5\left(\log_{\frac{1}{5}}(\log_5(z))\right) = 0$ for positive x, y and z, then which of the following is/are NOT true?
(A) $z < x < y$ **(B)** $x < y < z$
(C) $y < z < x$ **(D)** $z < y < x$

3. If a function satisfies $(x - y)f(x + y) - (x + y)f(x - y) = 2(x^2y - y^3)$ $\forall$ $x, y \in R$ and $f(1) = 2$, then
(A) $f(x)$ must be polynomial function
(B) $f(3) = 12$
(C) $f(0) = 0$
(D) $f(x)$ may not be differentiable

4. If $f(x) \cdot g(y) = g'(y) - f'(x) \cdot g(y), \forall x, y \in R$ and $g'(0) = 1$, $g(0) = 1$, $f'(0) = -5$, then
(A) $f(0) = 6$ **(B)** $f(1) = e$
(C) $g(1) = e$ **(D)** $g(-1) = \frac{1}{e}$

5. If $13[x] + 25\{x\} = 271$ where $[\cdot]$ denotes the integral part of x and $\{x\}$ denotes the fractional part of x, then value of $[x]$ is/are
(A) 18 **(B)** 19
(C) 20 **(D)** 21

6. The equation $\sin x = [1 + \sin x] + [1 - \cos x]$ has (where $[\cdot]$ represents the greatest integer function)
(A) no solution in $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ **(B)** no solution in $\left[\frac{\pi}{2}, \pi\right]$
(C) no solution in $\left[\pi, \frac{3\pi}{2}\right]$ **(D)** no solution for $\forall x \in R$

7. Let $O(0, 0)$ and $C(\pi, 0)$ be the given points. If least and the greatest value of x satisfies the equation $\cos 2x = |\sin x|$ in $\left(-\frac{\pi}{2}, \pi\right)$ represents the points A and B, respectively, then
(A) $OABC$ is a rhombus
(B) $OABC$ is a parallelogram

(C) $OB = AC$
(D) area of a quadrilateral $OABC$ is $\frac{\pi}{2}$ sq. units

8. For $x > 0$, let $f(x) = x^{2/3}(6-x)^{1/3}$ and $g(x) = x \cdot \ln x$ which of the following is/are true?
(A) Exactly 1 real solution exist for the equation $f(x) = g(x)$
(B) $f(x) \geq g(x), \forall x \in [1, 4]$
(C) For $y = f(x)$, (6, 0) is a point of the inflexion
(D) For $x \in (0, 4)$ the equation $f(x) = f^{-1}(x)$ has only one real solution

9. If $f(x)$ be a periodic function with the period T so that $f(x+13) + f(x+630) = 0$ and $\sum_{p=1}^{777} \int_p^{p+T} f(x)dx = \lambda \int_0^T f(x)dx$, then
(A) $T = 1234$ **(B)** $[\lambda+1]+[T+1] = 2013$
(C) λ and T both are prime **(D)** only λ is prime

10. $f(x) = \sin\ (2(\sqrt{[a]})\, x)$, where [·] denote the greatest integer function, has the fundamental period π for
(A) $a = \frac{3}{2}$ **(B)** $a = \frac{5}{4}$
(C) $a = \frac{2}{3}$ **(D)** $a = \frac{4}{5}$

11. Let $f(x)$ be a real-valued function defined on: $R \to R$ such that $f(x) = [x]^2 + [x+1] - 3$, where $[x]$ = the greatest integer $\leq x$. Then
(A) $f(x)$ is a many-one and into function
(B) $f(x) = 0$ for infinite number of values of x
(C) $f(x) = 0$ for only two real values
(D) none of these

12. If $f: R \to R$, $f(x) = e^{-|x|} - e^x$ is a given function, then which of the following are correct?
(A) f is many-one into function
(B) f is many-one onto function
(C) range of f is $[0, \infty]$
(D) range of f is $(-\infty, 0]$

13. Which of the following pair(s) of functions are identical?
(A) $f(x) = \cos(2\tan^{-1}x)$, $g(x) = \frac{1-x^2}{1+x^2}$
(B) $f(x) = \frac{2x}{1+x^2}$, $g(x) = \sin(2\cot^{-1}x)$
(C) $f(x) = \tan x + \cot x$, $g(x) = 2\,\text{cosec}\, 2x$
(D) $f(x) = e^{\ln(\text{sgn}\cot^{-1}x)}$, $g(x) = e^{\ln[1+\{x\}]}$,
where sgn(·), [·], {·} denotes signum, greatest integer and fractional part functions, respectively.

14. If $f(x) = \sin$ for $x \in \left(0, \frac{\pi}{4}\right)$ is invertible, where {·} and [·] represent the fractional part and the greatest integer functions, respectively, then $f^{-1}(x)$ is
(A) $\sin^{-1}x$ **(B)** $\frac{\pi}{2} - \cos^{-1}x$
(C) $\sin^{-1}\{x\}$ **(D)** $\cos^{-1}\{x\}$

15. Range of $f(x) = \log_{\sqrt[3]{10}}(\sqrt{5}(2\sin x + \cos x) + 5)$ is
(A) [0, 1] **(B)** [0, 3]
(C) $\left(-\infty, \frac{1}{3}\right)$ **(D)** None of these

Comprehension Type Questions

Paragraph for Questions 16 and 17: A line $-\frac{x}{2} = -\frac{f(t)\cdot y}{t} = t^2 z = \lambda$ is the perpendicular to the line of the intersection of the planes $t\cdot f(t)x + f\left(\frac{1}{t^2}\right)z + f(-t) = 0$ and $ty + f(-t)z + f(t^2) = 0$, where $t \in R - \{0\}$.

16. $f(t)$ is
(A) even function
(B) odd function
(C) neither even nor odd function
(D) both even and odd function

17. If $t = \tan\theta$, where $\theta \in R - \left\{(2n+1)\frac{\pi}{2}, n\pi\right\}$; $n \in I$, then
(A) $f(\tan\theta) = -\tan(2\theta)\cdot f(\cot^2\theta)$
(B) $f(\tan\theta) = -\sin(2\theta)\cdot f(\cot^2\theta)$
(C) $f(\tan\theta) = -\sin(2\theta)\cdot f(\sec^2\theta)e$
(D) $f(\tan 2\theta) = -\tan\theta \cdot f(\cot\theta)$

Paragraph for Questions 18 and 19: Consider $f(x) = \frac{(x-\alpha)(x-\beta)}{(x-\gamma)(x-\delta)}$; where $0 < \alpha < \beta < \gamma < \delta$.

18. Number of extremas for $f(x)$ will be
(A) 1 **(B)** 0
(C) 2 **(D)** cannot be determined

19. Function $f: D \to R$, where D is the domain and R is the set of real number, will be
(A) one to one **(B)** many to one
(C) bijective **(D)** none of these

Paragraph for Questions 20 and 21: A cubic function $f(x) = -x^3 + ax^2 + bx + c$. If $f(x)$ is an odd function and $f(x) = 0$ at $x = -1$. Now the domain of function is reduced, so as to make $f(x)$ invertible such that $f^{-1}(x)$ remains in 2nd and 4th quadrant. Then

20. $|f^{-1}(x)| + f^{-1}(|x|) = 0$ has
(A) no solution **(B)** exactly one solution
(C) infinite solutions **(D)** exactly three solutions

21. Range of $f^{-1}|x|$ is
(A) $(-\infty, 0)$ **(B)** $(-\infty, -1)$
(C) $(0, \infty)$ **(D)** $(1, \infty)$

Matrix Match Type Questions

22. Match the following:

Column I	Column II
(A) The number of the possible values of k if fundamental period of $\sin^{-1}(\sin kx)$ is $\frac{\pi}{2}$, is	**(p)** 1
(B) Numbers of the elements in the domain of $f(x) = \tan^{-1}x + \sin^{-1}x + \sec^{-1}x$ is	**(q)** 2

Column I	Column II
(C) Period of the function $f(x) = \sin\left(\frac{\pi x}{2}\right) \cdot \text{cosec}\left(\frac{\pi x}{2}\right)$ is	**(r)** 3
(D) If the range of the function $f(x) = \cos^{-1}[5x]$ is $\{a, b, c\}$ and $a+b+c = \frac{\lambda\pi}{2}$, then λ is equal to (where [·] denotes greatest integer)	**(s)** 4
	(t) 0

23. Match the following:

Column I	Column II
(A) Function $f:\left[0, \frac{\pi}{3}\right] \to [0, 1]$ defined by $f(x) = \sqrt{\sin x}$ is	**(p)** one to one function
(B) Function $f:(1, \infty) \to (1, \infty)$ defined by $f(x) = \frac{x+3}{x-1}$ is	**(q)** many-one function
(C) Function $f:\left[-\frac{\pi}{2}, \frac{4\pi}{3}\right] \to [-1, 1]$ defined by $f(x) = \sin x$ is	**(r)** into function
(D) Function $f:(2, \infty) \to [8, \infty)$ defined by $f(x) = \frac{x^2}{x-2}$ is	**(s)** onto function

24. Match the following:

Column I	Column II
(A) If the smallest positive integral value of x for which $x^2 - x - \sin^{-1}(\sin 2) < 0$ is λ, then $3 + \lambda$ is equal to	**(p)** 4
(B) Number of solution(s) of $2[x] = x + 2\{x\}$ is (where [·], {·} are the greatest integer and least integer functions, respectively)	**(q)** 1
(C) If $x^2 + y^2 = 1$ and maximum value of $x + y$ is $\frac{\sqrt{2}\,\lambda}{3}$, then λ is equal to	**(r)** 2
(D) $f\left(x+\frac{1}{2}\right) + f\left(x-\frac{1}{2}\right) = f(x)$ for all $x \in R$, then period of $f(x)$ is	**(s)** 0
	(t) 3

Integer Type Questions

25. Consider the two polynomials $f(x)$ and $g(x)$ as $g(x) = \sum_{r=0}^{200} \alpha_r x^r$ and $f(x) = \sum_{r=0}^{200} \beta_r x^r$. Given (i) $\beta_r = 1 \;\forall r \geq 100$ (ii) $f(x+1) = g(x)$. Let $A = \sum_{r=100}^{200} \alpha_r$. Find the remainder when A is divided by 15.

26. Let $p(x) = x^5 + x^2 + 1$ have roots x_1, x_2, x_3, x_4 and x_5, $g(x) = x^2 - 2$, then find the value of $g(x_1)g(x_2)g(x_3)g(x_4)g(x_5) - 30g(x_1x_2x_3x_4x_5)$.

27. $f: R \to R$ is $f(x) = \ln(x + \sqrt{x^2+1})$, then find the number of solutions to the equation $|f^{-1}(x)| = e^{-|x|}$.

28. Let $f(x) = 30 - 2x - x^3$, then find the number of positive integral values of x which satisfies $f(f(f(x))) > f(f(-x))$.

29. Let $f(x) = (x+1)(x+2)(x+3)(x+4) + 5$, where $x \in [-6, 6]$. If the range of the function is $[a, b]$ where $a, b \in N$, then find the value of $\frac{a+b}{1683}$.

Answer Key

Practice Exercise 1

1. (D)	**2.** (B)	**3.** (D)	**4.** (B)	**5.** (B)	**6.** (A)
7. (B)	**8.** (A)	**9.** (C)	**10.** (A)	**11.** (B)	**12.** (A)
13. (D)	**14.** (A)	**15.** (B)	**16.** (D)	**17.** (C)	**18.** (C)
19. (B)	**20.** (B)	**21.** (C)	**22.** (C)	**23.** (A)	**24.** (A)
25. (A)	**26.** (C)	**27.** (B)	**28.** (C)	**29.** (D)	**30.** (B)
31. (A)	**32.** (B)	**33.** (B)	**34.** (C)	**35.** (C)	**36.** (B)
37. (D)	**38.** (C)	**39.** (C)	**40.** (A)	**41.** (A)	**42.** (D)
43. (B)	**44.** (D)	**45.** (C)	**46.** (D)	**47.** (C)	**48.** (C)
49. (C)	**50.** (C)	**51.** (D)	**52.** (B)	**53.** (C)	**54.** (A)
55. (A)	**56.** (D)	**57.** (B)	**58.** (A)	**59.** (D)	**60.** (A)
61. (A)	**62.** (A)	**63.** (C)	**64.** (D)	**65.** (C)	**66.** (B)
67. (C)	**68.** (B)	**69.** (A)	**70.** (C)	**71.** (A)	**72.** (D)
73. (A)	**74.** (B)	**75.** (D)	**76.** (C)	**77.** (D)	**78.** (C)
79. (B)	**80.** (C)	**81.** (A)	**82.** (D)	**83.** (C)	**84.** (B)
85. (B)	**86.** (D)	**87.** (D)	**88.** (A)	**89.** (C)	**90.** (A)

91. (C) **92.** (B) **93.** (B) **94.** (A) **95.** (B) **96.** (B)
97. (B) **98.** (A) **99.** (B) **100.** (C) **101.** (D) **102.** (C)
103. (C) **104.** (A) **105.** (D) **106.** (D) **107.** (C) **108.** (C)
109. (A)

Practice Exercise 2

1. (A, C) **2.** (B, C, D) **3.** (A, B, C) **4.** (A, C, D) **5.** (B, C) **6.** (A, B, C, D)
7. (B, D) **8.** (A, C, D) **9.** (A, B) **10.** (A, B) **11.** (A, B) **12.** (A, D)
13. (A, B, C, D) **14.** (A, B, C) **15.** (D) **16.** (B) **17.** (A) **18.** (C)
19. (B) **20.** (C) **21.** (B) **22.** (A) → (q), (B) → (q), (C) → (q), (D) → (r) **23.** (A) → (p, r), (B) → (p, s), (C) → (q, s), (D) → (q, s) **24.** (A) → (p), (B) → (t), (C) → (t), (D) → (t) **25.** (1) **26.** (7)
27. (2) **28.** (2) **29.** (3)

Solutions

Practice Exercise 1

1. Given that

$$f(x)=\cos(\log x) \Rightarrow f(y)=\cos(\log y)$$

Therefore,

$$f(x)f(y)-\frac{1}{2}\left[f\left(\frac{x}{y}\right)+f(xy)\right]$$

$$=\cos(\log x)\cos(\log y)-\frac{1}{2}\left[\cos\left(\log\frac{x}{y}\right)+\cos(\log xy)\right]$$

$$=\cos(\log x)\cos(\log y)-\frac{1}{2}[2\cos(\log x)\cos(\log y)]=0$$

2. We have

$$f(x+1)-f(x)=8x+3$$

$$\Rightarrow [b(x+1)^2+c(x+1)+d]-(bx^2+cx+d)=8x+3$$

$$\Rightarrow (2b)x+(b+c)=8x+3$$

$$\Rightarrow 2b=8,\ b+c=3 \Rightarrow b=4,\ c=-1$$

3. We have

$$f(x)=\cos[\pi^2]x+\cos[-\pi^2]x$$

$$f(x)=\cos(9x)+\cos(-10x)=\cos(9x)+\cos(10x)$$

$$=2\cos\left(\frac{19x}{2}\right)\cos\left(\frac{x}{2}\right)$$

Therefore,

$$f\left(\frac{\pi}{2}\right)=2\cos\left(\frac{19\pi}{4}\right)\cos\left(\frac{\pi}{4}\right)$$

$$f\left(\frac{\pi}{2}\right)=2\times\frac{-1}{\sqrt{2}}\times\frac{1}{\sqrt{2}}=-1$$

4. It is a fundamental concept. The function is a homogeneous function of degree –1.

5. Multiplication of rational number and irrational number is irrational number. For example, $x=2, y=\sqrt{3}$. Thus, $2\sqrt{3}$ is an irrational number.

6. We have

$$\left|\frac{3x^3+1}{2x^2+2}\right|_{x=-3}=\left|\frac{3(-27)+1}{2(9)+2}\right|=\left|\frac{-80}{20}\right|=4$$

7. We have

$$f(x)=(x-1)(x-2)(x-3)$$

and $\quad f(1)=f(2)=f(3)=0$

which implies that $f(x)$ is not one-to-one. For each $y\in R$, there exists $x\in R$ such that $f(x)=y$. Therefore, f is onto. Hence, $f:R\to R$ is onto but not one-to-one.

8. Here, $|x|$ is not one-one and

$$\lim_{x\to\infty}\frac{4x}{(\sqrt{x^2+8x+3}+\sqrt{x^2+4x+3}}$$

is not one-to-one. Also, x^2+1 is not one-to-one. However, $2x-5$ is one-to-one because

$$f(x)=f(y)\Rightarrow 2x-5=2y-5\Rightarrow x=y$$

Now, $f(x)=2x-5$ is onto and therefore, $f(x)=2x-5$ is bijective.

9. Let us consider

$$\mathrm{L}f'(2)\neq \mathrm{R}f'(2)$$

$$\Rightarrow \frac{x^2-4}{x^2+4}=\frac{y^2-4}{y^2+4}$$

$$\Rightarrow \frac{x^2-4}{x^2+4}-1=\frac{y^2-4}{y^2+4}-1 \Rightarrow x^2+4=y^2+4$$

$$\Rightarrow x=\pm y$$

Therefore, $f(x)$ is many-to-one. Now, for each $y\in(-1,1)$, there does not exist $x\in X$ such that $f(x)=y$. Hence, f is into.

10. We have $f'(x)=2+\cos x>0$. So, $f(x)$ is strictly monotonically increasing and so $f(x)$ is one-to-one and onto.

11. $f(x)=\dfrac{x}{1+x}$ is one-to-one and into function. $x_1,x_2\in[0,\infty)$, for one-to-one function.
Now,

$$f(x_1)=f(x_2)\Rightarrow\frac{x_1}{1+x_1}=\frac{x_2}{1+x_2}\Rightarrow x_1x_2+x_1=x_1x_2+x_2\Rightarrow x_1=x_2$$

Range of $f(x)\in[0,1)$ and range of $f(x)$ is not equal to co-domain. Hence, $f(x)$ is not onto function.

12. We have

$$-\sqrt{1+(-\sqrt{3})^2}\le(\sin x-\sqrt{3}\cos x)\le\sqrt{1+(-\sqrt{3})^2}$$

$$-2 \le (\sin x - \sqrt{3}\cos x) \le 2$$

$$-2+1 \le (\sin x - \sqrt{3}\cos x + 1) \le 2+1$$

$$-1 \le (\sin x - \sqrt{3}\cos x + 1) \le 3$$

That is, the range is [–1, 3]. Therefore, for f to be onto, $S = [-1, 3]$.

13. We have

$$f(x) = x + \sqrt{x^2} = x + |x|$$

Now, f is not one-to-one as

$$f(-1) = f(-2) = 0$$

But $-1 \ne 2$ and also f is not onto since $f(x) \ge 0, \forall x \in R$. Also, the range of f is $(0,\infty) \subset R$.

14. $f(x, y) \to (x/y)$ is a surjective function.

15. We have

$$f(x) = \frac{\sin^{-1}(3-x)}{\log[|x|-2]}$$

Let us consider

$$g(x) = \sin^{-1}(3-x) \Rightarrow -1 \le 3 - x \le 1$$

The domain of $g(x)$ is [2, 4] and let us consider

$$h(x) = \log[|x|-2] \Rightarrow |x| - 2 > 0$$

$$\Rightarrow |x| > 2 \Rightarrow x < -2 \text{ or } x > 2 \Rightarrow (-\infty, -2) \cup (2, \infty)$$

We know that

$$(f/g)(x) = \frac{f(x)}{g(x)} \forall x \in D_1 \cap D_2 - \{x \in R : g(x) = 0\}$$

Therefore, the domain of $f(x)$ is

$$(2, 4] - \{3\} = (2, 3) \cup (3, 4]$$

16. Here, $x + 3 > 0$ and $x^2 + 3x + 2 \ne 0$. Therefore,

$$x > -3 \text{ and } (x+1)(x+2) \ne 0$$

That is, $x \ne -1, -2$. Therefore, the domain is

$$(-3, \infty) - \{-1, -2\}$$

17. We have

$$f(x) = \begin{cases} (1/2)(-x-1), & x < -1 \\ \tan^{-1} x, & -1 \le x \le 1 \\ (1/2)(x-1), & x > 1 \end{cases}$$

and
$$f'(x) = \begin{cases} -1/2, & x < -1 \\ 1/1 + x^2, & -1 < x < 1 \\ 1/2, & x > 1 \end{cases}$$

Now, $\lim_{h\to 0} f'(-1-h) = -\frac{1}{2}$; $\lim_{h \to 0} f'(-1+h) = \frac{1}{1+(-1+0)^2} = \frac{1}{2}$

That is,

$$\lim_{h\to 0} f'(1-h) = \frac{1}{1+(1-0)^2} = \frac{1}{2}; \lim_{h\to 0} f'(1+h) = \frac{1}{2}$$

Therefore, $f'(-1)$ does not exist and therefore, the domain of $f'(x)$ is $R - \{-1\}$.

18. The function $f(x)$ is to be defined when $x^2 - 1 > 0$. So,

$$x^2 > 1, \Rightarrow x < -1 \text{ or } x > 1 \text{ and } 3 + x > 0$$

Therefore, $x > -3$ and $x \ne -2$ and hence,

$$D_f = (-3, -2) \cup (-2, -1) \cup (1, \infty)$$

19. The quantity which is under root is positive, when

$$-1 - \sqrt{3} \le x \le -1 + \sqrt{3}$$

20. Obviously, here $|x| > 2$ and $x \ne 1$, that is, $x \in (-\infty, -2) \cup (2, \infty)$.

21. The function $f(x) = \sqrt{\log(x^2 - 6x + 6)}$ is defined when $\log(x^2 - 6x + 6) \ge 0$.

$$x^2 - 6x + 6 \ge 1 \Rightarrow (x-5)(x-1) \ge 0$$

This inequality holds if $x \le 1$ or $x \ge 5$. Hence, the domain of the function is $(-\infty, 1] \cup [5, \infty)$.

22. We have

$$-1 \le 1 + 3x + 2x^2 \le 1$$

Case 1: Here, $2x^2 + 3x + 1 \ge -1$; $2x^2 + 3x + 2 \ge 0$.

Therefore,

$$x = \frac{-3 \pm \sqrt{9-16}}{6} = \frac{-3 \pm i\sqrt{7}}{6}$$

which is imaginary.

Case 2: Here, $2x^2 + 3x + 1 \le 1$

Therefore,

$$2x^2 + 3x \le 0 \Rightarrow 2x\left(x + \frac{3}{2}\right) \le 0$$

$$\Rightarrow \frac{-3}{2} \le x \le 0 \Rightarrow x \in \left[-\frac{3}{2}, 0\right]$$

In Case 1, we get imaginary value; hence it is rejected. Therefore, the domain of the function is

$$\left[\frac{-3}{2}, 0\right]$$

23. Here, $|x| > 1$; therefore, $x \in (-\infty, -1) \cup (1, \infty)$.

24. It should be $|x| - x > 0$. That is,

$$|x| > x$$

However, $|x| = x$ for x positive and $|x| > x$ for x negative. So, the domain is $(-\infty, 0)$.

25. Here,

$$f(x) = \sec\left(\frac{\pi}{4}\cos^2 x\right)$$

We know that $0 \le \cos^2 x \le 1$ at $\cos x = 0, f(x) = 1$ and at $\cos x = 1, = \alpha(1) - \beta(1) = \alpha - \beta$.

Therefore,

$$1 \le x \le \sqrt{2} \Rightarrow x \in [1, \sqrt{2}].$$

26. We have

$$f(x) = 1 + \frac{1}{[x + (1/2)]^2 + (3/4)}$$

Therefore, the range is $(1, 7/3]$.

27. We have

$$f(x) = \sqrt{2}\left[\sin\left(\frac{\pi}{4} - 2x\right)\right]$$

Therefore,

$$-\sqrt{2} \le f(x) \le \sqrt{2}$$

and
$$[-1, 1] \subset [-\sqrt{2}, \sqrt{2}]$$

28. Here, $\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}=\pi/2$ holds x which lies in [0, 1].

29. Here,

$$-1\le\cos\theta\le1\Rightarrow=-\frac{1}{a^2}$$

and $$\sec^2\theta\ge1 \text{ for } \theta>\frac{\pi}{3}$$

$$\sec\theta\ge2\Rightarrow\sec^2\theta\ge4$$

Therefore, the required interval is [2, ∞).

30. We have

$$f(x)=\sin[\log(x+\sqrt{1+x^2})]$$

$$\Rightarrow f(-x)=\sin[\log(-x+\sqrt{1+x^2})]$$

$$\Rightarrow f(-x)=\sin\log\left((\sqrt{1+x^2}-x)\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}+x}\right)$$

$$\Rightarrow f(-x)=\sin\log\left[\frac{1}{(x+\sqrt{1+x^2})}\right]$$

$$\Rightarrow f(-x)=\sin\left[\log(x+\sqrt{1+x^2})^{-1}\right]$$

$$\Rightarrow f(-x)=\sin\left[-\log(x+\sqrt{1+x^2})\right]$$

$$\Rightarrow f(-x)=-\sin\left[\log(x+\sqrt{1+x^2})\right]$$

$$\Rightarrow f(-x)=-f(x)$$

Therefore, $f(x)$ is an odd function.

31. We have

$$y=\frac{x+2}{x-1}\Rightarrow x=\frac{3}{y-1}+1=\frac{y+2}{y-1}=f(y)$$

32. It is given that

$$f(x)=2^{x(x-1)}\Rightarrow x(x-1)=\log_2 f(x)$$

$$\Rightarrow x^2-x-\log_2 f(x)=0\Rightarrow x=\frac{1\pm\sqrt{1+4\log_2 f(x)}}{2}$$

Only $x=\dfrac{1+\sqrt{1+4\log_2 f(x)}}{2}$ lies in the domain.

33. Let us consider

$$f(x)=y\Rightarrow x=f^{-1}(y)$$

Hence,

$$f(x)=y=3x-5\Rightarrow x=\frac{y+5}{3}\Rightarrow f^{-1}(y)=x=\frac{y+5}{3}$$

Therefore,

$$f^{-1}(x)=\frac{x+5}{3}$$

Also f is one-to-one and onto and hence f^{-1} exists and is given by

$$f^{-1}(x)=\frac{x+5}{3}$$

34. Here,

$$f(\theta)=\sin\theta(\sin\theta+\sin3\theta)$$
$$=\sin\theta(\sin\theta+3\sin\theta-4\sin^3\theta)=4\sin^2\theta(1-\sin^2\theta)$$
$$=4\sin^2\theta\cos^2\theta=(\sin2\theta)^2$$

Therefore, for all real θ, we have

$$f(\theta)\ge0$$

35. By the definition of composition of function, we have

$$g[f(x)]=(\sin x+\cos x)^2-1\Rightarrow g[f(x)]=\sin2x$$

We know that $\sin x$ is bijective only when

$$x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

Thus, $g(x)$ is bijective if we have

$$-\frac{\pi}{2}\le2x\le\frac{\pi}{2}\Rightarrow\frac{-\pi}{4}\le x\le\frac{\pi}{4}$$

36. Here,

$$g(x)=1+n-n=1, x=n\in Z$$

Now,

$$1+n+k-n=1+k,\ x=n+k$$

where $n\in Z$, $0<k<1$. Now,

$$f[g(x)]=\begin{cases}-1, & g(x)<0\\ 0, & g(x)=0\\ 1, & g(x)>0\end{cases}$$

Therefore, $g(x)>0\ \forall x$ and thus $f(g(x))=1\ \forall x$.

37. We have

$$f[f(x)]=\frac{\alpha f(x)}{f(x)+1}=\frac{\alpha[\alpha x/(x+1)]}{[\alpha x/(x+1)]+1}=\frac{\alpha^2x}{\alpha x+x+1}$$

Therefore,

$$x=\frac{\alpha^2\cdot x}{(\alpha+1)x+1}$$

or $$x[(\alpha+1)x+1-\alpha^2]=0$$

or $$(\alpha+1)x^2+(1-\alpha^2)x=0$$

which should hold $\forall x$. Therefore,

$$\alpha+1=0, 1-\alpha^2=0$$

and this is equal to

$$\lim_{h\to0}\frac{h[(e^{-1/h}-e^{1/h})/(e^{-1/h}+e^{1/h})]-0}{-h}=-1$$

38. We have

$$f[f(x)]=[a-\{f(x)\}^n]^{1/n}=[a-(a-x^n)]^{1/n}=x$$

39. The set B satisfied the given definition of function f and hence, option (C) is correct.

40. We have

$$(f-g)(x)=\begin{cases}x, & x\in Q\\ -x, & x\notin Q\end{cases}$$

41.

$$h(x)=[f(x)+f(-x)]\,[g(x)-g(-x)]$$
$$h(-x)=[f(-x)+f(x)]\,[g(-x)-g(x)]=-h(-x)$$

Therefore, $h(x)$ is an odd function.

42. We have

$$f(x)=\sqrt{4-x^2}+\frac{1}{\sqrt{|\sin x|-\sin x}}$$

(i) Here, $4-x^2\ge0\Rightarrow x^2\le4\Rightarrow-2\le x\le2$.

(ii) Here, $|\sin x|-\sin x>0$.

When $\sin x<0$, we have

$$|\sin x|=-\sin x\Rightarrow-2\sin x>0\Rightarrow\sin x<0 \text{ if } |\sin x|=\sin x$$

Then, we get $0>0$, which is not possible. Therefore, the domain is [–2, 0).

43. We have

$$f(x)=x^3+3x^2+12x-12x-2\sin x$$

Therefore,

$$f'(x)=3x^2+6x+12-2\cos x$$

Hence, $f'(x)>0\ \forall x$. Therefore, $f(x)$ is an increasing function and thus $f(x)$ is one-to-one and onto.

44. We have

$$f(x)=\ln\left(\frac{x^2+e}{x^2+1}\right)$$

$$\Rightarrow \ln\left(\frac{x^2+1-1+e}{x^2+1}\right)=\ln\left(1+\frac{e-1}{x^2+1}\right)$$

Therefore, the range is (0, 1].

45. $f(x+y)=f(xy)\ \forall\ x, y$

Put

$$x=y=1\Rightarrow f(2)=f(1)$$

Similarly,

$$f(2)=f(1)=f(3)=\cdots=f(2006)=-10$$

46. If function $f(x)$ is symmetrical about $x = 1$, then $f(1-\alpha) = f(1+\alpha)$ where $\alpha \in R$.

47. We have

$$f(x)=\sin^{-1}\left[x^2+\frac{1}{2}\right]+\cos^{-1}\left[x^2-\frac{1}{2}\right]$$

$$=\sin^{-1}\left[x^2+\frac{1}{2}\right]+\cos^{-1}\left[x^2+\frac{1}{2}-1\right]$$

$$=\sin^{-1}\left[x^2+\frac{1}{2}\right]+\cos^{-1}\left(\left[x^2+\frac{1}{2}\right]-1\right)$$

Since $x^2 + 1/2 \geq 1/2$, we get

$$\left[x^2+\frac{1}{2}\right]=0 \text{ or } 1$$

Since $\sin^{-1}[x^2+(1/2)]$ is defined only for these two values,

(i) when $[x^2+(1/2)]=0$, we get

$$f(x)=\sin^{-1}0+\cos^{-1}(-1)=\pi$$

(ii) when $[x^2+(1/2)]=1$, we get

$$f(x)=\sin^{-1}1+\cos^{-1}0=\pi$$

Therefore, the range of $f(x)=\{\pi\}$.

48. We have

$$fog=f[g(x)]=|\sin x|=\sqrt{\sin^2 x}$$

Also, we have

$$gof=g[f(x)]=\sin^2\sqrt{x}$$

Obviously,

$$\sqrt{\sin^2 x}=\sqrt{g(x)}$$

and

$$\sin^2\sqrt{x}=\sin^2[f(x)]$$

That is,

$$g(x)=\sin^2 x \text{ and } f(x)=\sqrt{x}$$

49. We have

$$f(x)=0\ \forall x\in R\Rightarrow f(3)-f(2)=0$$

50. Let us consider

$$f_1=\sqrt{\log_{1/4}\left(\frac{5x-x^2}{4}\right)}$$

and $f_2={}^{10}C_x$. Therefore, f_1 is defined for

$$\log_{1/4}\left(\frac{5x-x^2}{4}\right)\geq 0$$

$$\Rightarrow 0<\frac{5x-x^2}{4}\leq 1$$

$$\Rightarrow \frac{5x-x^2}{4}>0 \text{ and } \frac{5x-x^2}{4}\leq 1$$

$$\Rightarrow x(x-5)<0 \text{ and } x^2-5x+4\geq 0$$

$$\Rightarrow x\in(0,5) \text{ and } x\in(-\infty,1]\cup[4,\infty)$$

Therefore, f_1 is defined for

$$x\in(0,1]\cup[4,5)$$

and f_2 is defined for

$$x\in\{0,1,2,3,4,5,6,7,8,9,10\}$$

Thus, $f(x)$ is defined for

$$x\in D_{f_1}\cap D_{f_2}=\{1,4\}$$

51.

$$f'(x)=a-\sin x$$

Now,

$$f'(x)\geq 0\Rightarrow a\geq\sin x\Rightarrow a\geq 1$$

$$f'(x)\leq 0\Rightarrow a\leq\sin x\Rightarrow a\leq -1$$

Therefore, $f(x)$ is bijective if $a\in R-(-1,1)$.

52. We know that $f(x+2)=2f(x)-f(x+1)$. Substituting $x = 0$, we get

$$f(2)=4-3\Rightarrow f(2)=1$$

Substituting $x = 1$, we get

$$f(3)=6-1\Rightarrow f(3)=5$$

Substituting $x = 2$, we get

$$f(4)=2-5\Rightarrow f(4)=-3$$

Substituting $x = 3$, we get

$$f(5)=10+3\Rightarrow f(5)=13$$

53.

$$3^x+4^x+5^x-6^x=0\Rightarrow\left(\frac{3}{6}\right)^x+\left(\frac{4}{6}\right)^x+\left(\frac{5}{6}\right)^x=1$$

Now,

$$f(x)=\left(\frac{3}{6}\right)^x+\left(\frac{4}{6}\right)^x+\left(\frac{5}{6}\right)^x$$

$$f(x)\to 0,\ x\to\infty$$

$$f(0)=3,\ x=0$$

$$f(x)\to\infty,\ x\to-\infty$$

Therefore, $f(x)=1$ will have one real root.

54. We have $0\leq\{x\}<1$. That is,

$$0\leq\sin^{-1}\{x\}<\frac{\pi}{2}$$

$$0\leq\sin[\sin^{-1}\{x\}]<1$$

55.

$$f(x)=\frac{3}{x-2}+\frac{4}{x-3}+\frac{5}{x-4}$$

$$\Rightarrow f(x)<0\ \forall\ x\in(-\infty,2),\ f(x)>0\ \forall\ x\in(4,\infty)$$

$$\Rightarrow f(2^+)>0 \text{ and } f(3^-)<0$$

So $f(x)$ has exactly one real root $\in(2,3)$.

56. We have

$$\frac{2\pi}{\sqrt{[a]}} = \pi \Rightarrow 2 = \sqrt{[a]} \Rightarrow 4 = [a]$$

Therefore, $a \in [4, 5)$.

57. We have

$$fof(x) = \begin{cases} f(x), & \text{when } f(x) \text{ is rational} \\ 1-f(x), & \text{when } f(x) \text{ is irrational} \end{cases}$$

$$= \begin{cases} x, & \text{when } x \text{ is rational} \\ 1-(1-x), & \text{when } x \text{ is irrational} \end{cases}$$

That is, it $fof(x) = x$.

58. The function $f(x)$ is defined on $[0, 1] \Rightarrow 0 \le x \le 1$. Now, $f(2\sin x)$ shall be defined if $0 \le 2\sin x \le 1$. Therefore,

$$0 \le \sin x \le \frac{1}{2} \Rightarrow x \in \bigcup_{n \in I} \left\{\left[2n\pi, 2n\pi + \frac{\pi}{6}\right] \cup \left[2n\pi + \frac{5\pi}{6}, (2n+1)\pi\right]\right\}$$

59. As f and g are mirror images of each other above the line $y = a$. So,

$$g(x) = 2a - f(x)$$

Now

$$-h(x) = f(x) + g(x) = 2a$$

60. We have

$$f(x) = \cos|x| + \left[\left|\frac{\sin x}{2}\right|\right]$$

$$0 \le \left|\frac{\sin x}{2}\right| \le 1/2 \Rightarrow \left[\left|\frac{\sin x}{2}\right|\right] = 0$$

Therefore,

$$f(x) = \cos|x| = \cos x \ \forall x \in R$$

61. For $f(x)$ to be defined, we need to have

$$x^2 + 4x \ge 0 \Rightarrow x \in (-\infty, -4] \cup [0, \infty)$$

Also,

$$(x^2 + 4x) - (2x^2 + 3) \ge 0 \Rightarrow x \in [1, 3]$$

However, ${}^nC_r$ is defined only for the non-negative integral values of n and r and thus the domain is {1, 2, 3} and the range is $\{f(1), f(2), f(3)\} = \{1, 2\sqrt{3}\}$.

62. Let

$$f(x) = \log_{\sin x} 2^{\tan x}$$

$$\Rightarrow f(x) = \ln 2\left(\frac{\tan x}{\ln \sin x}\right)$$

For $f(x) > 0$, we must have

$$\tan x > 0$$

and

$$\ln \sin x > 0$$

Now,

$$x \in \left(0, \frac{\pi}{2}\right)$$

$$\tan x > 0$$

and

$$0 < \sin x < 1$$

$$\Rightarrow \ln \sin x < 0 \ \text{(As } \ln x < 0, \text{ for } 0 < x < 1)$$

Therefore, there is no solution for $f(x) > 0$.

63. We have

$$\left(\frac{5}{13}\right)^x + \left(\frac{12}{13}\right)^x \ge 1$$

Therefore,

$$(\cos\alpha)^x + (\sin\alpha)^x \ge 1$$

where $\cos\alpha = 5/13$. Equality holds for $x = 2$. Now,

$$\left(\frac{5}{13}\right)^x + \left(\frac{12}{13}\right)^x$$

decreases as x increases. Therefore, $x \in (-\infty, 2]$.

64. We should have

$$-1 \le [x] - x^2 + 4 \le 2 = x^2 - 5 \le [x] \le x^2 - 2$$

On solving it by graph, we get

$$x \in [-\sqrt{3}, -1] \cup (\sqrt{3}, \sqrt{7})$$

65. We have

$$f(x) = -\{(5x+7) - [5x+7]\} + \cos\pi x + 7 = -\{5x+7\} + \cos\pi x + 7$$

The period of $(5x + 7)$ is 1/5 and the period of $\cos\pi x$ is 2. Therefore, the period of $f(x)$ is 2.

66. The function $f(x)$ can be written as

$$[x] + [2x] + \cdots + [nx] - (x + 2x + 3x + \cdots + nx)$$

$$= -(\{x\} + \{2x\} + \cdots + \{nx\})$$

The period of $f(x)$ is

$$\text{LCM}\left(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}\right) = 1$$

67. All numbers $r!$ ($r \ge 7$) are the multiples of 7 and hence for remainder. Let us consider

$$S_6 = 1 + 2 + 6 + 24 + 120 + 720$$

which gives remainder 5 when divided by 7.

68. We have

$$f(x) = \frac{1}{3}[\sin 3x + |\sin 3x| + (\sin 3x)]$$

The period of $\sin 3x$ is $2\pi/3$; the period of $|\sin 3x|$ is $\pi/3$; the period of $[\sin 3x]$ is $2\pi/3$. Hence, the period of $f(x)$ is $2\pi/3$.

69.

$$f(x) = f(-x)$$

Hence, $f(x)$ is an even function.

$$1 \le |\sin x| + |\cos x| \le \sqrt{2} \Rightarrow [|\sin x| + |\cos x|] = 1 \Rightarrow f(x) = 1$$

70. We have

$$f(x) = \sqrt{\sec^{-1}\left(\frac{2-|x|}{4}\right)}$$

Now,

$$\sec^{-1}\left(\frac{2-|x|}{4}\right) \ge 0$$

Therefore,

$$\frac{2-|x|}{4} \ge 1 \Rightarrow 2 - |x| \ge 4 \Rightarrow |x| \le -2$$

is not possible. Also,

$$\frac{2-|x|}{4} \le -1 \Rightarrow 2 + 4 \le |x| \Rightarrow |x| \ge 6$$

Therefore, $x \ge 6$ or $x \le -6$ and thus the domain is $(-\infty, -6] \cup [6, \infty)$.

71. See Fig. 1.75. The following cases arise:

(i) If $[x] = -1$, then $x^2 - 2 = -2 \Rightarrow x = 0$ which is not possible.

(ii) If $[x] = 0$, then $x^2 - 2 = 0 \Rightarrow x = \pm\sqrt{2}$ which is not possible.

(iii) If $[x] = 1$, then $x^2 - 2 = 2 \Rightarrow x = \pm 2$ which is not possible.

(iv) If $[x] = 2$. Then $x^2 - 2 = 4 \Rightarrow x = \pm\sqrt{6}$ and the only possible solution is $\sqrt{6}$.

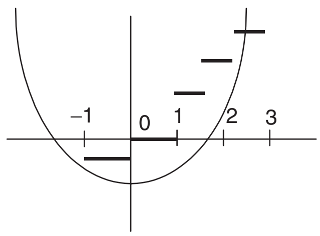


Figure 1.75

72. Here, $\sqrt{16-x^2}$ is defined in [–4, 4]. Now, $\sin^{-1}[1 + \cos x]$ is defined if $-1 \le 1+\cos x < 2$. That is,

$$\cos x \ne 1 \Rightarrow x \in [-4, 0) \cup (0, 4]$$

73. Let $g(x)$ be the inverse of f, then $f[g(x)] = x$. This implies that

$$3^{g(x)[g(x)-2]} = x \Rightarrow [g(x)]^2 - 2g(x) - \log_3 x = 0$$

$$\Rightarrow g(x) = \frac{2 \pm \sqrt{4+4\log_3 x}}{2} = 1 \pm \sqrt{1+\log_3 x}$$

Since g: [1, ∞] → [2, ∞], we get

$$g(x) = 1 + \sqrt{1+\log_3 x}$$

74. For $f(x)$ to be odd, then $[x^2/|a|]$ should depend upon the value of x.

$$x \in [-4, 4] \Rightarrow 0 \le x^2 \le 16$$

Now, $[x^2/|a|] = 0$ if

$$|a| > 16 \Rightarrow a \in (-\infty, -16) \cup (16, \infty)$$

75. The number of onto function from A → B is the coefficient of

x^4 in $4!\left[x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right]^3$, that is,

$$\text{Coefficient of } x^4 \text{ in } 4![e^{3x} - 3e^{2x} + 3e^x - 1] = 4!\left[\frac{3^4}{4!} - \frac{3.2^4}{4!} + \frac{3}{4!}\right]$$

$$= 3^4 - 3.2^4 + 3 = 81 - 48 + 3 = 36$$

The total number of functions is 81. Hence, the required number of functions is

$$81 - 36 = 45$$

76. See Fig. 1.76. Obviously, there are two solutions.

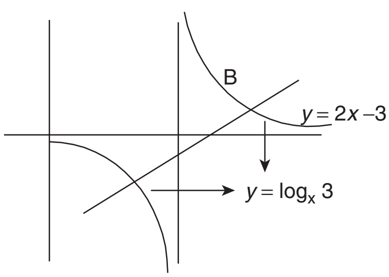


Figure 1.76

77. For $\frac{x-1}{x-2\{x\}} \ge 0$, we have the following two cases:

(i) $x \ge 1 \Rightarrow x > 2\{x\} \Rightarrow x \ge 2 \Rightarrow x \in [2, \infty)$

(ii) $x \le 1 \Rightarrow x < 2\{x\} \Rightarrow x < 1, x \ne 0$

The common part is $x \in (-\infty, 0) \cup (0, 1)$. Finally, $x = 1$ is also a part of the domain.

78. We should have

$$[\sin 2x] \ge [\cos 2x]$$

which implies that we can have

$$[\sin 2x] = 1, [\cos 2x] = 1, 0, -1$$

Now,

$$[\sin 2x] = 0, [\cos 2x] = 0, -1$$

That is,

$$[\sin 2x] = -1, [\cos 2x] = -1$$

However, $[\sin 2x] = 1$, $[\cos 2x] = 1$ and $[\sin 2x] = 1$, $[\cos 2x] = -1$ are not possible. Hence, the range is {0, 1}.

79. See Fig. 1.77. The number of points of intersection of the curves $|y| = \ln|x|$ and $(x-1)^2 + y^2 = 4$ is 3.

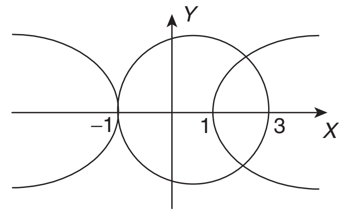


Figure 1.77

80. $$f(x+y, x-y) = xy$$

Let $X = x+y, Y = x-y$. Then

$$f(X,Y) = \frac{X^2 - Y^2}{4} \text{ and } Y = x+y, X = x-y$$

$$\Rightarrow f(Y,X) = \frac{Y^2 - X^2}{4}$$

Therefore, A.M. of $f(X,Y)$ and $f(Y,X)$ is 0.

81. $f(x) = x + a - [x + a] - b - a \Rightarrow f(x) = \{x + a\} - b - a$
Hence, $f(x)$ is periodic with 1.

82. Domain of $f(x) = \sqrt{x^{12} - x^3 + x^4 - x + 1}$

$$x^{12} - x^3 + x^4 - x + 1 \ge 0$$

$$\Rightarrow x^{12} + x^4 + 1 \ge x^3 + x \quad \forall x \in R$$

83. Domain of $f(x) = \sqrt{4^x + (64)^{(x-2)/3} - [(1/2)(72 + 2^{2x})]}$

$$\Rightarrow 4^x + (64)^{(x-2)/3} - [(1/2)(72 + 2^{2x})] \ge 0$$

$$\Rightarrow \frac{17.4^x}{16} - \frac{(72+4^x)}{2} \ge 0 \Rightarrow 17.4^x - 8.72 - 8.4^x \ge 0$$

$$\Rightarrow 9.4^x - 8.72 \ge 0 \Rightarrow 4^x - 64 \ge 0 \Rightarrow x \ge 3$$

84. $$f(x) = \int_0^x \log\frac{1-\tan t}{1+\tan t}\,dt$$

Replacing x by $-x$, we get

$$f(-x) = \int_0^{-x} \log\frac{1-\tan t}{1+\tan t}\,dt$$

Let

$$t = -z \Rightarrow dt = -dz$$

$$\Rightarrow f(-x) = -\int_0^z \log\frac{1+\tan z}{1-\tan z}\,dz = \int_0^x \log\frac{1-\tan t}{1+\tan t}\,dt = f(x)$$

Therefore, $f(x)$ is an even function.

85. $f(x+y) = f(x) + f(y) - xy - 1 \quad \forall x, y \in R$

Put $x = y = 1$, $f(2) = 2f(1) - 2 = 0$

Put $x = 1, y = 2$, $f(3) = f(1) + f(2) - 2 - 1 = 3 - 3 = 0$

Put $x = 1, y = 3$, $f(4) = f(1) + f(3) - 3 - 1 = -3$

Hence, only one solution for $f(n) = n$.

86.
$$x^2f(x) - 2f(1/x) = g(x)$$
where $f(x)$ is an even function and $g(x)$ is an odd function. Replacing x by $-x$, we get
$$x^2f(-x) - 2f\left(-\frac{1}{x}\right) = g(-x)$$
and
$$g(x) + g(-x) = 0$$
$$\Rightarrow x^2f(x) = 2f\left(\frac{1}{x}\right) \Rightarrow f(1) = 0$$
Replacing x by $1/x$, we get
$$\left(\frac{1}{x}\right)^2 f\left(\frac{1}{x}\right) = 2f(x)$$
$$\Rightarrow \left(\frac{1}{x}\right)^2 \frac{x^2}{2} f(x) = 2f(x) \Rightarrow f(x) = 0$$

87. The given data is shown in Fig. 1.78. Since $f^{-1}(d) = x$, we get
$$f(x) = d$$
Now, if $a \subset x$, $f(a) \subset d$, we get $f^{-1}[f(a)] = a$.

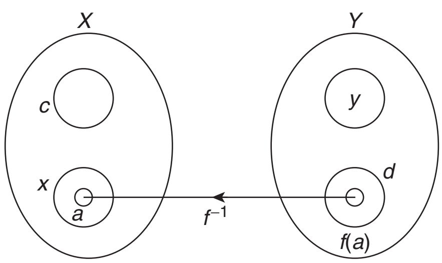


Figure 1.78

88.
$$f(x) = x/\sqrt{1+x^2}$$
$$\Rightarrow f(f(x)) = f(x)/\sqrt{1+f^2(x)} = x/\sqrt{1+2x^2}$$
$$\Rightarrow f(f(f(x))) = x/\sqrt{1+3x^2}$$

89. $y = (x^2+2x+c)/(x^2+4x+3c), \quad y \in R\ \forall x \in R$

$\Rightarrow y(x^2+4x+3c) = (x^2+2x+c)$

$\Rightarrow x^2(y-1) + 2x(2y-1) + c(3y-1) = 0$ has real solution

Now,
$$D \geq 0 \Rightarrow 4(2y-1)^2 - 4(y-1)(3y-1) \geq 0$$
$$\Rightarrow 4y^2 + 1 - 4y - c(3y^2 + 1 - 4y) \geq 0$$
$$\Rightarrow (4-3c)y^2 - 4y(1-c) + 1 - c \geq 0 \quad \forall y \in R$$
$$\Rightarrow 4 - 3c > 0, \quad 16(1-c)^2 - 4(4-3c)(1-c) \leq 0$$
$$\Rightarrow \frac{4}{3} > c, \quad (4c^2 + 4 - 8c) - (3c^2 - 7c + 4) \leq 0$$
$$\Rightarrow \frac{4}{3} > c, (c^2 - c) \leq 0$$
$$\Rightarrow c < \frac{4}{3}, \quad c \in [0,1]$$
But for $c = 0$, range of $f(x)$ is not R. Hence, $c \in (0,1]$.

90. $n(A \times A) = n(A) \cdot n(A) = 3^2 = 9$
So, the total number of subsets of $A \times A$ is 2^9 and a subset of $A \times A$ is a relation over the set A.

91. Here, $n(A \times B) = 2 \times 3 = 6$.
Since every subset of $A \times B$ defines a relation from A to B, number of relation from A to B is equal to number of subsets of $A \times B = 2^6 = 64$.

92. $R = \{(a,b) : a,b \in N, a-b = 3\} = \{((n+3),n) : n \in N\}$
$= \{(4,1),(5,2),(6,3),...\}$.

93. $R = \{(2, 1), (4, 2), (6, 3), ...\}$.
So, $R^{-1} = \{(1, 2), (2, 4), (3, 6), ...\}$.

94. Since (1, 1); (2, 2); (3, 3) $\in R$, R is reflexive. (1, 2) $\in R$ but (2, 1) $\notin R$, therefore R is not symmetric. It can be easily seen that R is transitive.

95. Since $x < y, y < z \Rightarrow x < z \ \forall\ x, y, z \in N$
Therefore,
$$xRy, yRz \Rightarrow xRz$$
$\Rightarrow$ Relation is transitive,
$\Rightarrow$ $x < y$ does not give $y < x$
$\Rightarrow$ Relation is not symmetric.
Since $x < x$ does not hold, relation is not reflexive.

96. Obviously, the relation is not reflexive and transitive, but it is symmetric because $x^2 + y^2 = 1 \Rightarrow y^2 + x^2 = 1$.

97. Since R is an equivalence relation on set A, $(a, a) \in R$ for all $a \in A$. Hence, R has at least n ordered pairs.

98. For any $x \in R$, we have $x - x + \sqrt{2} = \sqrt{2}$ an irrational number.
Therefore, xRx for all x. So, R is reflexive.
R is not symmetric because $\sqrt{2}R1$ but $1 \not R \sqrt{2}$, R is not transitive also because $\sqrt{2}R1$ and $1R2\sqrt{2}$ but $\sqrt{2} \not R 2\sqrt{2}$.

99. Clearly, the relation is symmetric but it is neither reflexive nor transitive.

100. We have
$$R = \{(1, 3); (1, 5); (2, 3); (2, 5); (3, 5); (4, 5)\}$$
$$R^{-1} = \{(3, 1), (5, 1), (3, 2), (5, 2); (5, 3); (5, 4)\}$$
Hence, $RoR^{-1} = \{(3, 3); (3, 5); (5, 3); (5, 5)\}$

101. A relation from P to Q is a subset of $P \times Q$.

102. $R = A \times B$.

103. Number of relations on the set A = Number of subsets of $A \times A = 2^{n^2}$ [Because $n(A \times A) = n^2$].

104. Since R is reflexive relation on A, therefore $(a,a) \in R$ for all $a \in A$. The minimum number of ordered pairs in R is n. Hence, $m \geq n$.

105. Here $R = \{(x, y) : |x^2 - y^2| < 16\}$
and given $A = \{1,2,3,4,5\}$
Therefore, $R = \{(1,2)(1,3)(1,4);(2,1)(2,2)(2,3)(2,4);(3,1)(3,2)$
$(3,3)(3,4);(4,1)(4,2)(4,3);(4,4)(4,5),(5,4)(5,5)\}$.

106. Given, $xRy \Rightarrow x$ is relatively prime to y.
Therefore, domain of $R = \{2,3,4,5\}$.

107. R be a relation on N defined by $x+2y=8$.

Therefore, $R\{(2,3);(4,2);(6,1)\}$

Hence, domain of $R=\{2,4,6\}$

108. As $R=\{(x,y)\,|\,x,y\in Z, x^2+y^2\le 4\}$

Therefore, $R=\{(-2,0),(-1,0),(-1,1),(0,-1)(0,1),(0,2),(0,-2)$
$(1,0),(1,1),(2,0)\}$

Hence, domain of $R=\{-2,-1,0,1,2\}$.

109. R is a relation from {11, 12, 13} to {8, 10, 12} defined by
$$y=x-3\Rightarrow x-y=3$$

Therefore, $R=\{11,8\},\{13,10\}$.

Hence, $R^{-1}=\{8,11\};\{10,13\}$.

Practice Exercise 2

1. As
$$f'(x)\cdot g(x)-g'(x)\cdot f(x)=(x^2+1)^2+9 \qquad (1)$$
$$\Rightarrow f'(x)g(x)-g'(x)f(x)>0$$

If α,β are consecutive roots of $f(x)=0$, then
$$f'(\alpha)\cdot g(\alpha)>0,\ f'(\beta)\cdot g(\beta)>0$$
$$\Rightarrow g(\alpha)\cdot g(\beta)<0 \qquad (2)$$

Hence, there exists at least one root in (α,β) also from Eq. (1).
$$f''(x)g(x)-g''(x)\cdot f(x)=4x(x^2+1)$$
$$\Rightarrow f''(\alpha)g(\alpha)=4\alpha(\alpha^2+1),\ f''(\beta)g(\beta)=4\beta(\beta^2+1)$$
$$\Rightarrow 16\alpha\beta(\alpha^2+1)(\beta^2+1)=f''(\alpha)\cdot f''(\beta)\cdot g(\alpha)\cdot g(\beta)$$

As $f''(x)>0$ and $g(\alpha)\cdot g(\beta)<0$, we can conclude that $\alpha\beta<0$.

2. By solving, we get $x=2^{1/2}$, $y=3^{1/3}$, $z=5^{1/5}$

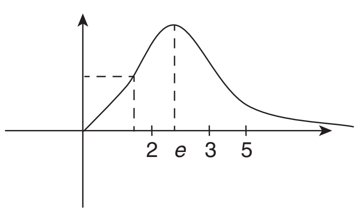


Figure 1.79

See Fig. 1.79. Using graph of $x^{1/x}$
$$3^{1/3}>5^{1/5}$$

Also $\quad 2^{1/2}<3^{1/3}$ as $2^3<3^2$ | $2^{1/2}>5^{1/5}$, $2^5>5^2$
$$\Rightarrow y>x>z$$

3.
$$(x-y)f(x+y)-(x+y)f(x-y)=2y(x-y(x+y))$$

Let $x-y=u;\ x+y=v$. Then
$$uf(v)-vf(u)=2uv(v-u)$$
$$\Rightarrow \frac{f(v)}{v}-\frac{f(u)}{u}=v-u$$
$$\Rightarrow\left(\frac{f(v)}{v}-v\right)=\left(\frac{f(u)}{u}-u\right)=\text{constant}$$

Let $\frac{f(x)}{x}-x=\lambda$. Then
$$f(x)=(\lambda x+x^2)$$

Now,
$$f(1)=2$$

Therefore,
$$\lambda+1=2\Rightarrow\lambda=1$$

Hence, $f(x)=x^2+x$

Now,
$$f(3)=3^2+3=12$$
and
$$f(0)=0$$

4.
$$f(x)\cdot g(y)+f'(x)\cdot g(y)=g'(y)$$
$$\Rightarrow [f(x)+f'(x)]g(y)=g'(y)$$
$$\Rightarrow f(x)+f'(x)=\frac{g'(y)}{g(y)}=\text{ constant}$$

Put $y=0$, then
$$\frac{g'(0)}{g(0)}=\frac{1}{1}=1$$

So, $f(x)+f'(x)=1$ and $\frac{g'(y)}{g(y)}=1$

$$\Rightarrow e^xf(x)=e^x+\lambda$$
$$\Rightarrow f(x)=1+\lambda e^{-x}$$
$$f'(x)=-\lambda e^{-x}$$
$$f'(0)=-\lambda=-5\Rightarrow\lambda=5$$
$$f(x)=1+5e^{-x}$$
$$f(0)=6$$
$$f(1)=1+\frac{5}{e}$$

$$\ln g(y)=y+\ln b$$
$$g(y)=be^y$$
$$g(0)=b=1\Rightarrow b=1$$
$$g(y)=e^y$$
$$g(1)=e,\ g(-1)=\frac{1}{e}$$

5. Let $[x]=I$ and $\{x\}=f$. Then
$$13I+25f=271\Rightarrow f=\frac{271-13I}{25}$$

Now,
$$0\le\frac{271-13I}{25}<1$$
$$\Rightarrow 246<13I\le 271$$
$$\Rightarrow\frac{246}{13}<I\le\frac{271}{13}\Rightarrow I=19,20$$

6.
$$\sin x=1+[\sin x]+1+[-\cos x]$$
$$\sin x=2+[\sin x]+[-\cos x]$$

At $x=\frac{-\pi}{2}$, $\sin x=1$,

Solution is not possible at $x = -\dfrac{\pi}{2}$.

If $x \in \left(\dfrac{-\pi}{2}, 0\right)$, then $\sin x = 0$, not possible.

If $x \in \left(0, \dfrac{\pi}{2}\right)$, then $\sin x = 1$ impossible

Similarly for other cases. Hence, (A), (B), (C) and (D) are the correct answers.

7. See Fig. 1.80.

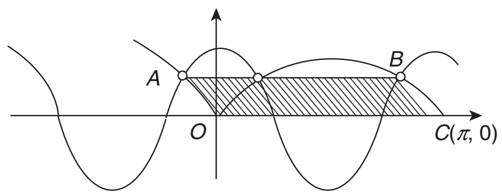


Figure 1.80

$$\cos 2x = |\sin x| \Rightarrow x = \frac{\pi}{6}, -\frac{\pi}{6}, \frac{5\pi}{6}$$

That is,

$$A \equiv \left(-\frac{\pi}{6}, \frac{1}{2}\right)$$

and $$B \equiv \left(\frac{5\pi}{6}, \frac{1}{2}\right)$$

Clearly, $AB = OC = \pi$ and $AB \parallel OC$

Also, $OA = BC$ and $OA \parallel BC$

That is, $OABC$ is a parallelogram.

$$\text{Area } OABC = \pi \cdot \frac{1}{2} = \frac{\pi}{2} \text{ sq. units}$$

8. See Fig. 1.81. $f(x)$ is a concave downward function for $0 < x < 6$, while $g(x)$ is a concave upward function for $x > 0$.

Moreover, $g(3) > f(3)$

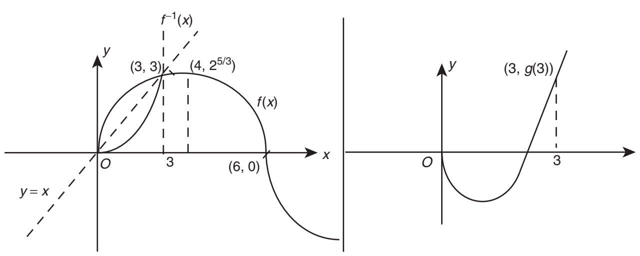


Figure 1.81

Hence, (A), (C) and (D) are the correct answers.

9. $$f(x+13) + f(x+630) = 0 \quad (1)$$

Putting $x = x + 617$

$$f(x+630) + f(x+1247) = 0$$

Subtract Eq. (1) from Eq. (2), we get

$$f(x+1247) - f(x+13) = 0$$

$$\Rightarrow f(x+1247) = f(x+13)$$

Putting $x = x - 13$, $f(x+1234) = f(x)$

Therefore, $f(x)$ is periodic with period 1234. So, $T = 1234$.

Since, $\int_p^{p+T} f(x)dx = \int_0^T f(x)dx$, if $f(x)$ is periodic with period T

$$\sum_{p=1}^{777} \int_p^{p+T} f(x)dx = \sum_{p=1}^{777} \int_0^T f(x)dx = 777\int_0^T f(x)dx$$

$$\Rightarrow \lambda = 777$$

Thus,

$$\lambda + T = 777 + 1234 = 2011$$

Therefore, $[\lambda + 1] + [T + 1] = 2013$

10. Since fundamental period of $f(x)$ is π, therefore, $[a] = 1$.

Therefore, $1 \le a < 2$. Hence, (A) and (B) are the correct answers.

11. $f(x) = [x]^2 + [x+1] - 3 = \{[x] + 2\}\{[x] - 1\}$

So, $x = 1, 1\cdot1, 1\cdot2, \ldots \Rightarrow f(x) = 0$

Therefore, $f(x)$ is many one.

Only integral values will be attained.

Therefore, $f(x)$ is into. Hence, (A) and (B) are the correct answers.

12. $f(x) = \begin{cases} 0 & x < 0 \\ e^{-x} - e^{x} & x \ge 0 \end{cases}$

Range $= (-\infty, 0]$

Therefore, many one into. Hence, (A) and (D) are the correct answers.

13. **(A)** Domain of f and g both are 'R'.

$$f(x) = \cos(2\tan^{-1}x) = \frac{1 - \tan^2(\tan^{-1}x)}{1 + \tan^2(\tan^{-1}x)} = \frac{1 - x^2}{1 + x^2} = g(x)$$

(B) Domain of f and g both are 'R'.

$$g(x) = \sin(2\cot^{-1}x) = \frac{2\tan(\cot^{-1}x)}{1 + \tan^2(\cot^{-1}x)} = \frac{2 \times \frac{1}{x}}{1 + \frac{1}{x^2}} = \frac{2x}{1 + x^2} = f(x)$$

(C) Domain of f and g are $R - \left\{n\pi, (2n+1)\dfrac{\pi}{2}\right\}e, n \in \mathrm{I}$

$$f(x) = \tan x + \cot x = \tan x + \frac{1}{\tan x} = \frac{\sec^2 x}{\tan x} = \frac{\cos x}{\sin x \cdot \cos^2 x}$$

$$= \frac{1}{\cos x \sin x} = 2\,\text{cosec}\,2x = g(x)$$

(D) Domain of f:

$$\text{sgn}(\cot^{-1}x) > 0 \Rightarrow \text{sgn}(\cot^{-1}x) = 1$$
$$\Rightarrow \cot^{-1}x > 0 \Rightarrow x \in R$$

Domain of g:

$$[1 + \{x\}] > 0 \Rightarrow [\{x\}] > 0 \Rightarrow 0 \le \{x\} < 1 \Rightarrow x \in R$$

Now,

$f(x) = e^{\ln(\text{sgn}\cot^{-1}x)} = \text{sgn}(\cot^{-1}x) \ldots (\text{Since}, 0 < \cot^{-1}x < \pi) = 1$

$g(x) = e^{\ln[1+\{x\}]} = [1 + \{x\}] = 1 + [\{x\}]$

Since, $0 \le \{x\} < 1 = 1$

Therefore, $f(x)$ and $g(x)$ are identical functions.

14. $y = f(x) = \sin\{[x+5] + \{x - \{x - \{x\}\}\}\} = \sin\{x - \{x - \{x\}\}\}$

$= \sin\{x - \{[x]\}\} = \sin\{x - 0\} = \sin\{x\} = \sin x$

Since,

$$0 < x < \frac{\pi}{4}$$

Therefore,

$$x = \sin^{-1}y$$

or, $f^{-1}(x) = \sin^{-1} x$

Hence, (A), (B) and (C) are the correct answers.

15. We know that

$$-\sqrt{5} \le 2\sin x + \cos x \le \sqrt{5},\ \forall x \in R$$

$$\Rightarrow -5 \le \sqrt{5}\,(2\sin x + \cos x) \le 5$$

$$\Rightarrow 0 \le \sqrt{5}\,(2\sin x + \cos x) + 5 \le 10$$

$$\Rightarrow -\infty < \log_{\sqrt[3]{10}} e\,(\sqrt{5}(2\sin x + \cos x) + 5) \le 3$$

Hence, range is $(-\infty, 3]$.

16. The normals to the planes and the given line are coplanar. Hence, applying the condition, the functional equation obtained is

$$2tf\left(\frac{1}{t^2}\right) + t^2 f(-t) + f(t) = 0 \qquad (1)$$

Also,

$$-2tf\left(\frac{1}{t^2}\right) + t^2 f(t) + f(-t) = 0 \qquad (2)$$

Adding Eqs. (1) and (2) $\Rightarrow f(t)$ is an odd function.

17. From Eqs. (1) and (2) of Solution 16, we have

$$2tf\left(\frac{1}{t^2}\right) + t^2 f(-t) + f(t) = 0$$

$$\Rightarrow f(t) = \frac{-2t}{1-t^2} f\left(\frac{1}{t^2}\right),\ (f(-t) = f(t))$$

Putting $t = \tan\theta$, we get

$$f(\tan\theta) = \frac{-2\tan\theta}{1-(\tan\theta)^2} f\left(\frac{1}{(\tan\theta)^2}\right)$$

$$\Rightarrow f(\tan\theta) = -\tan(2\theta)\cdot f(\cot^2\theta)$$

18. By the given data, we can trace the graph (see Fig. 1.82) of $y = f(x)$.

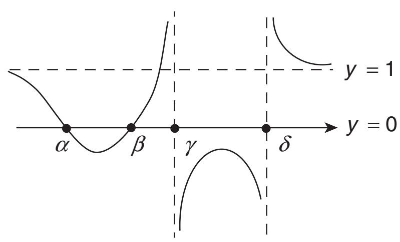


Figure 1.82

Hence, (C) is the correct answer.

19. When δ is replace by $-\delta$, then $-\delta < \alpha < \beta < \gamma$. Therefore, the graph (see Fig. 1.83) will be

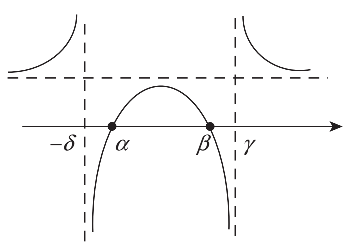


Figure 1.83

Hence, (B) is the correct answer.

20. See Fig. 1.84. $f^{-1}(x)$ remains in the 2nd and 4th quadrants. So, $f(x)$ is defined as

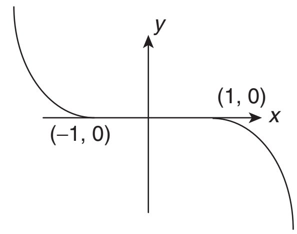


Figure 1.84

Hence, (C) is the correct answer.

21. See Fig. 1.85. $f^{-1}(x)$ is defined as

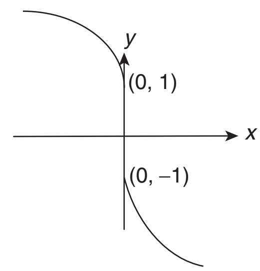


Figure 1.85

Hence, (B) is the correct answer.

22. **(A)** Fundamental period of $\sin^{-1}(\sin kx)$ is $\frac{2\pi}{|k|} = \frac{\pi}{2}$.

That is,

$$|k| = 4$$
$$\Rightarrow k = \pm 4$$

(B) Domain of $\tan^{-1} x$ is R, domain of $\sin^{-1}x$ is $[-1, 1]$, domain of $\sec^{-1}x$ is $(-\infty, -1] \cup [1, \infty)$. Therefore, domain of $f(x)$ is $\{-1, 1\}$

(C) π is a period of $\sin x \cdot \operatorname{cosec} x$

Therefore, $\pi \times \frac{2}{\pi}$ is a period of $\sin\frac{\pi x}{2}\cdot\operatorname{cosec}\frac{\pi x}{2}$

That is, 2 is a period of $\sin\frac{\pi x}{2}\ \operatorname{cosec}\frac{\pi x}{2}$

$$f(x+1) = \sin\frac{\pi}{2}(x+1)\operatorname{cosec}\frac{\pi}{2}(x+1) = \cos\frac{\pi}{2}x\cdot\sec\frac{\pi}{2}x \ne f(x)$$

(D) $f(x) = \cos^{-1}[5x]$

$[5x]$ can take the values $-1, 0, 1$

Therefore,

$$\text{range} = \left\{\pi, \frac{\pi}{2}, 0\right\}$$

Therefore,

$$a + b + c = \pi + \frac{\pi}{2} + 0 = \frac{3\pi}{2}$$

23. **(A)** $$f'(x) = \frac{1}{2\sqrt{\sin x}}\cos x$$

$f'(x)$ is positive if $x \in \left[0, \frac{\pi}{3}\right]$

f is one-to-one function.

Since

$$0 \le x \le \frac{\pi}{3}$$

$$0 \le \sin x \le \frac{\sqrt{3}}{2}$$

$$0 \le \sqrt{\sin x} \le \sqrt{\frac{\sqrt{3}}{2}} < 1$$

Hence, f is into function.

(B) $$f(x) = \frac{x+3}{x-1}$$

$$f'(x) = \frac{(x-1)\cdot 1-(x+3)\cdot 1}{(x-1)^2}$$

$$f'(x) = \frac{-4}{(x-1)^2}$$

$$f'(x) < 0$$

Hence, $f(x)$ is one to one.
Since, $x > 1$.
Therefore, range of $y = \frac{x+3}{x-1}$ is $(1, \infty)$.
Hence, f is onto function.

(C) See Fig. 1.86.

$$-\frac{\pi}{2} \le x \le \frac{4\pi}{3}$$

$$f(x) = \sin x$$

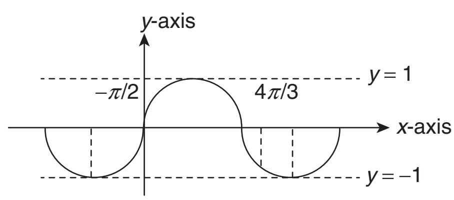


Figure 1.86

From graph, $f(x)$ is many-one and onto.

(D) $$f(x) = \frac{x^2}{x-2}$$

$$f'(x) = \frac{(x-2)\cdot 2x - x^2}{(x-2)^2}$$

$$f'(x) = \frac{x^2-4x}{(x-2)^2}$$

Therefore, $f'(x) < 0$ if $2 < x < 4$ and $f'(x) > 0$ if $x > 4$.
$f(x)$ is many-one.
$f(4) = 8$ (is the least value of $f(x)$)
Therefore, range $= [8, \infty)$
Therefore, $f(x)$ is onto.

24. (A) $$x^2 - x - \pi + 2 < 0$$

$$x = \frac{1 \pm \sqrt{4\pi - 7}}{2}$$

$$\frac{1-\sqrt{4\pi-7}}{2} < x < \frac{1+\sqrt{4\pi-7}}{2}$$

$$\lambda = 1$$

(B) $2[x] = x + 2\{x\}$

(i) If x is an integer, then the equation becomes $2x = x + 0$
That is, $x = 0$ is a solution

(ii) If $x\ I$, the equation becomes

$$2[x] = [x] + \{x\} + 2\{x\}$$

$$\Rightarrow \{x\} = \frac{1}{3}[x]$$

Therefore,

$$0 < \frac{[x]}{3} < 1 \Rightarrow 0 < [x] < 3$$

Therefore, possible values of $[x]$ are 1, 2.

If $[x] = 1$, then $\{x\} = \frac{1}{3}$.

Therefore,

$$x = 1 + \frac{1}{3} = \frac{4}{3}$$

If $[x] = 2$, then $\{x\} = \frac{2}{3}$.

Therefore, $x = \frac{8}{3}$.

Therefore, there are 3 solutions

(C) Let $x = \cos\theta, y = \sin\theta$.
Therefore, $x + y = \cos\theta + \sin\theta$.
Therefore, maximum value of $x + y$ is $\sqrt{2}$.

(D) $$f\left(x+\frac{1}{2}\right) + f\left(x-\frac{1}{2}\right) = f(x)$$

$$\Rightarrow f(x+1) + f(x) = f\left(x+\frac{1}{2}\right) \Rightarrow f(x+1) + f\left(x-\frac{1}{2}\right) = 0$$

$$\Rightarrow f\left(x+\frac{3}{2}\right) = -f(x) \Rightarrow f(x+3) = -f\left(x+\frac{3}{2}\right) = f(x)$$

Therefore, $f(x)$ is periodic with period 3.

25. $$\sum_{r=0}^{200} \alpha_r x^r = \sum_{r=0}^{200} \beta_r (1+x)^r$$

$$\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_{200} x^{200}$$

$$= \beta_0 + \beta_1(1+x) + \cdots + \beta_{200}(1+x)^{200}$$

Equating coefficient of x^{100}, we get

$$\alpha_{100} = {}^{100}C_{100} + {}^{101}C_{100} + \cdots {}^{200}C_{100} = {}^{201}C_{101}$$

Similarly, we can find $\alpha_{101}, \ldots \alpha_{200}$.

$$\sum_{r=100}^{200} \alpha_r = {}^{201}C_{101} + {}^{201}C_{102} + \cdots {}^{201}C_{201}$$

$$A = 2^{200}$$

When A is divided by 15 remainder is 1.

26. Given

$$g(x_1)g(x_2)g(x_3)g(x_4)g(x_5) = A$$

$$= (x_1^2-2)(x_2^2-2)(x_3^2-2)(x_4^2-2)(x_5^2-2)$$

$$= -(2-x_1^2)(2-x_2^2)(2-x_3^2)(2-x_4^2)(2-x_5^2) \quad (1)$$

$$=-[2^5-\left(\sum x_1^2\right)2^4+\sum x_1^2\cdot x_2^2\cdot 2^3-\sum x_1^2\cdot x_2^2\cdot x_3^2\cdot 2^2$$

$$+\sum x_1^2\cdot x_2^2\cdot x_3^2\cdot x_4^2\cdot 2-x_1^2\cdot x_2^2\cdot x_3^2\cdot x_4^2\cdot x_5^2]$$

$p(x)=x^5+x^2+1=0$ has roots $x_1, x_2, ..., x_5$, then that equation $q(x)$ whose roots are square of the roots of $p(x)$ is

$$q(x)=(\sqrt{y})^5+(\sqrt{y})^2+1=0;\ \alpha=x \text{ and } y=\alpha^2$$

$$\Rightarrow (y+1)^2=(-\sqrt{y})^{5\times 2}$$

$$\Rightarrow y^2+2y+1=y^5 \Rightarrow q(x)=y^5-y^2-2y-1=0$$

Then,

$$\sum x_1^2=\sum y_1=0$$

$$\sum x_1^2.x_2^2=\sum y_1.y_2=0$$

$$\sum x_1^2\cdot x_2^2\cdot x_3^2=\sum y_1\cdot y_2\cdot y_3=1$$

$$\sum x_1^2\cdot x_2^2\cdot x_3^2\cdot x_4^2=\sum y_1\cdot y_2\cdot y_3\cdot y_4=-2$$

$$\sum x_1^2\cdot x_2^2\cdot x_3^2\cdot x_4^2\cdot x_5^2=\sum y_1\cdot y_2\cdot y_3\cdot y_4\cdot y_5=1, \text{ then}$$

$$A=-[2^5-0+0-2^2-2\cdot 2-1]=-[32-4-4-1]=-[32-9]$$

$$=-23$$

$$x_1x_2x_3x_4x_5=-1 \Rightarrow g(x_1x_2...x_5)=-1$$

$$\Rightarrow g(x_1)g(x_2)...g(x_5)-30g(x_1x_2...x_5)=7$$

Alternative solution:

Let us form that equation having roots $y=g(x_i)$. Then

$$y=x^2-2$$

$$x=\sqrt{y+2}$$

$$\Rightarrow (\sqrt{y+2})^5+(\sqrt{y+2})^2+1=0$$

$$\Rightarrow y^5+20y^4+40y^3+79y^2+74y+23=0$$

Therefore,

$$g(x_1)...g(x_5)=\text{Product of roots}$$

$$=-23$$

$$x_1x_2x_3x_4x_5=-1 \Rightarrow g(x_1x_2...x_5)=-1$$

$$\Rightarrow g(x_1)g(x_2)...g(x_5)-30g(x_1x_2...x_5)=7$$

27. See Fig. 1.87.

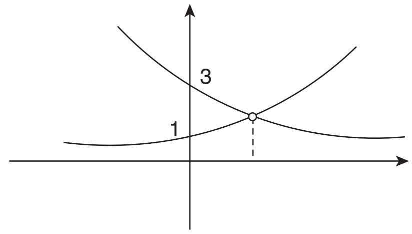


Figure 1.87

Let

$$f^{-1}(x) \text{ beg } \Rightarrow \ln(g+\sqrt{g^2+1})=x$$

$$\Rightarrow g+\sqrt{g^2+1}=e^x \quad (1)$$

and $$-g+\sqrt{g^2+1}=e^{-x} \quad (2)$$

Now, $|e^x-e^{-x}|=2e^{-|x|}$.

Case I: $x>0$; $e^{-|x|}=e^{-x}$ and $e^x>e^{-x}$

$$e^x-e^{-x}=2e^{-x} \Rightarrow e^x=3e^{-x}$$

Case II: $x<0$; $e^{-|x|}=e^x$ and $e^x<e^{-x}$

$$e^{-x}-e^x=2e^x \Rightarrow e^{-x}=3e^x$$

Therefore, two solutions.

28. $$f(x)=30-2x-x^3$$

$f'(x)=-2-3x^2<0 \Rightarrow f(x)$ is decreasing function.

Thus,

$$f(f(f(x)))>f(f(-x)) \Rightarrow f(f(x))<f(-x)$$

$$\Rightarrow f(x)>-x$$

$$\Rightarrow 30-2x-x^3>-x \Rightarrow x^3+x-30<0$$

$$\Rightarrow (x-3)(x^2+3x+10)<0$$

$$\Rightarrow x<3$$

29. $$f(x)=(x^2+5x+4)(x^2+5x+6)+5$$

$$=[(x^2+5x+5)-1][(x^2+5x+5)+1]+5$$

$$=(x^2+5x+5)^2-1+5=(x^2+5x+5)^2+4$$

Therefore, minimum value of $f(x)=4$ and maximum value occurs at $x=6$.

$$f(x)_{\max}=(36+30+5)^2+4=5045$$

Now, $a=4$, $b=5045$. Hence, $\dfrac{a+b}{1683}=3$.

Solved JEE 2017 Questions

JEE Main 2017

1. The function $f:\mathbb{R}\to\left[-\frac{1}{2},\frac{1}{2}\right]$ defined as $f(x)=\frac{x}{1+x^2}$, is:

(A) injective but not surjective.
(B) surjective but not injective.
(C) neither injective nor surjective.
(D) invertible.

(OFFLINE)

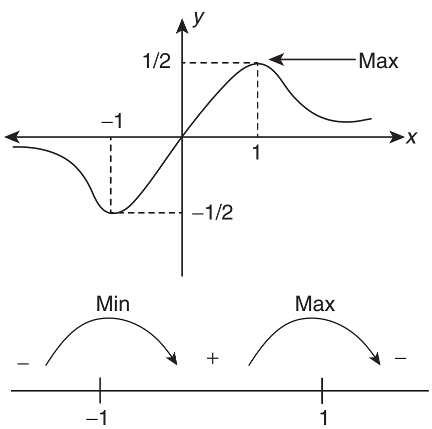
Solution: The given function is defined as

$$f(x)=\frac{x}{1+x^2}$$

Now,

$$\frac{dy}{dx}=\frac{(1+x^2)\times 1-x(0+2x)}{(1+x^2)^2}=\frac{1-x^2}{(1+x^2)^2}=0$$

Therefore, x = 1, –1, which is odd function and there is symmetry about the origin – that is, the function is non-monotonic and non-injective – in the resultant curve as shown in the following figures:



Any line parallel to x-axis cuts the graph more than one point; hence, the function is many-to-one. Now,

$$y=\frac{x}{1+x^2}$$

$$\Rightarrow x^2(y)-x+y=0$$

Now, $D > 0; 1-4y^2\geq 0$. That is, the range is

$$y\in\left[-\frac{1}{2},\frac{1}{2}\right]=\text{codomain}$$

Hence, the function is onto. Therefore, the function is surjective but not injective.
Hence, the correct answer is option (B).

2. The following statement $(p\to q)\to[\sim p\to q)\to q]$ is:

(A) equivalent to $\sim p\to q$.
(B) equivalent to $p\to \sim q$.
(C) a fallacy.
(D) a tautology.

(OFFLINE)

Solution: See the following table for the given statement:

q	p	$\sim p$	$(p\to q)$	$(\sim p\to q)$	$(\sim p\to q)\to q$	$(p\to q)\to[(\sim p\to q)\to q]$
T	T	F	T	T	T	T
F	T	F	F	T	F	T
T	F	T	T	T	T	T
F	F	T	T	F	T	T

From this table, we can confirm that the given statement $(p\to q)\to[\sim p\to q)\to q]$ is a tautology.
Hence, the correct answer is option (D).

3. Let $f(x)=2^{10}\times x+1$ and $g(x)=3^{10}\times x-1$. If $(fog)(x)=x$, then x is equal to

(A) $\frac{3^{10}-1}{3^{10}-2^{-10}}$ **(B)** $\frac{2^{10}-1}{2^{10}-3^{-10}}$
(C) $\frac{1-3^{-10}}{2^{10}-3^{-10}}$ **(D)** $\frac{1-2^{-10}}{3^{10}-2^{-10}}$

(ONLINE)

Solution: It is given that

$$(fog)(x)=x$$

That is,

$$f(g(x))=x$$

$$\Rightarrow f(3^{10}\times x-1)=x\Rightarrow 2^{10}\times(3^{10}\times x-1)+1=x$$

$$\Rightarrow 2^{10}3^{10}x-2^{10}+1=x\Rightarrow 6^{10}x-x=2^{10}-1\Rightarrow x(6^{10}-1)=2^{10}-1$$

$$\Rightarrow x=\frac{2^{10}-1}{6^{10}-1}=\frac{2^{10}-1}{2^{10}\cdot 3^{10}-1}\Rightarrow x=\frac{2^{10}(1-2^{-10})}{2^{10}(3^{10}-2^{-10})}\Rightarrow x=\frac{1-2^{-10}}{2^{10}-2^{-10}}$$

Hence, the correct answer is option (D).

4. The proposition $(\sim p)\vee(p\wedge\sim q)$ is equivalent to

(A) $p\wedge\sim q$ **(B)** $p\vee\sim q$
(C) $p\to\sim q$ **(D)** $q\to p$

(ONLINE)

Solution: This can be explained with the help of the following truth tables ('∧' symbol stands for AND and '∨' symbol stands for OR):

p	q	$\sim p$	$\sim q$	$p\wedge\sim q$	$(\sim p)\vee(p\wedge\sim q)$
F	F	T	T	F	T
T	F	F	T	T	T
F	T	T	F	F	T
T	T	F	F	F	F

p	q	$\sim q$	$p\wedge\sim q$	$p\vee\sim q$	$p\to\sim q$
F	F	T	F	T	T
T	F	T	T	T	T
F	T	F	F	F	T
T	T	F	F	T	F

p	q	$q \to p$
F	F	T
T	F	T
F	T	F
T	T	T

Thus from the truth table, we conclude that $(\sim p) \vee (p \wedge \sim q)$ is equivalent to $p \to (\sim q)$.

Hence, the correct answer is option (C).

5. The function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = x - 5\left[\frac{x}{5}\right]$, where $\mathbb{N}$ is the set of natural numbers and $[x]$ denotes the greatest integer less than or equal to x, is

(A) one-one and onto.
(B) onto but not one-one.
(C) neither one-one nor onto.
(D) one-one but not onto.

(ONLINE)

Solution: The given function is

$$f(x) = x - 5\left[\frac{x}{5}\right]$$

Taking x in interval of five natural numbers, we have the following:

$$f(x) = \begin{cases} x - 5(0), & 0 \le x \le 5 \\ x - 5(1), & 5 \le x \le x10 \\ x - 5(2), & 10 \le x \le 15 \\ x - 5(3), & 15 \le x \le 20 \end{cases}$$

Therefore, here, $x \in \mathbb{N}$. hence, $f(x)$ is neither a one-one function nor onto function.

Hence, the correct answer is option (C).

JEE Advanced 2017

1. Let $S = \{1, 2, 3, \ldots, 9\}$. For $k = 1, 2, \ldots, 5$, let N_K be the number of subsets of S, each containing five elements out of which exactly k are odd. Then $N_1 + N_2 + N_3 + N_4 + N_5 =$ ______.

(A) 210 **(B)** 252
(C) 125 **(D)** 126

Solution: If N_k be number of subjects of S containing 5 elements each out of which exactly k are odd elements. Therefore,

$$N_1 = {}^5C_1 \times {}^4C_4 = \frac{5!}{4! \times 1!} \times \frac{4!}{4! \times 0!} = 5$$

$$N_2 = {}^5C_2 \times {}^4C_3 = \frac{5!}{3! \times 2!} \times \frac{4!}{3! \times 1!} = \frac{5 \times 4 \times 4}{2} = 40$$

$$N_3 = {}^5C_3 \times {}^4C_2 = \frac{5!}{3! \times 2!} \times \frac{4!}{2! \times 2!} = \frac{5 \times 4}{2} \times \frac{4 \times 3}{2} = 60$$

$$N_4 = {}^5C_4 \times {}^4C_1 = \frac{5!}{4! \times 1!} \times \frac{4!}{3! \times 1!} = 5 \times 4 = 20$$

$$N_5 = {}^5C_5 \times {}^4C_0 = \frac{5!}{5! \times 0!} \times \frac{4!}{4! \times 0!} = 1$$

Therefore, $N_1 + N_2 + N_3 + N_4 + N_5 = 5 + 40 + 60 + 20 + 1 = 126$.

Hence, the correct answer is option (D).

2 Trigonometric Ratios and Identities

2.1 Introduction

Trigonometry is a branch of Mathematics that relates to the study of angles, measurement of angles and units of measurement. It also concerns itself with the six ratios for a given angle and the relations satisfied by these ratios.

In an extended way, it is also a study of the angles forming the elements of a triangle. Logically, a discussion of the properties of a triangle, solving problems related to triangles, physical problems in the area of heights and distances using the properties of a triangle – all constitute a part of the study. It also provides a method of solution of trigonometric equations.

2.2 Definitions

1. **Angle:** The motion of any revolving line in a plane from its initial position (initial side) to the final position (terminal side) is called angle (Fig. 2.1). The end point O about which the line rotates is called the **vertex of the angle**.

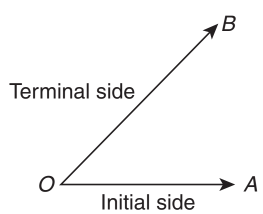


Figure 2.1

2. **Measure of an angle:** The measure of an angle is the amount of rotation from the initial side to the terminal side.
3. **Sense of an angle:** The sense of an angle is determined by the direction of rotation of the initial side into the terminal side. The sense of an angle is said to be positive or negative according to the rotation of the initial side in anticlockwise or clockwise direction to get to the terminal side (Fig. 2.2).

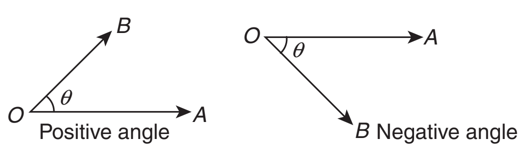


Figure 2.2

4. **Right angle:** If the revolving ray, starting from its initial position to final position, describes one quarter of a circle, then we say that the measure of the angle formed is a right angle.
5. **Quadrants:** Let *X′OX* and *YOY′* be two lines at right angles in the plane of the paper (Fig. 2.3). These lines divide the plane of paper into four equal parts known as quadrants. The lines *X′OX* and *YOY′* are known as *x*-axis and *y*-axis, respectively. These two lines taken together are known as **coordinate axes**.

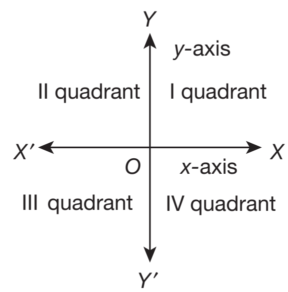


Figure 2.3

6. **Angle in standard position:** An angle is said to be in standard position if its vertex coincides with the origin *O* and the initial side coincides with *OX*, that is, the positive direction of *x*-axis.
7. **Angle in a quadrant:** An angle is said to be in a particular quadrant if the terminal side of the angle in standard position lies in that quadrant.
8. **Quadrant angle:** An angle in standard position is said to be a quadrant angle if the terminal side coincides with one of the axes.

2.3 Measurement of Angles

There are three systems for measuring angles.

1. **Sexagesimal or English system:** Here a right angle is divided into 90 equal parts known as **degrees**. Each degree is divided into 60 equal parts called **minutes** and each minute is further divided into 60 equal parts called **seconds**. Therefore,

$$1 \text{ right angle} = 90 \text{ degree } (= 90^\circ)$$
$$1^\circ = 60 \text{ min } (= 60')$$
$$1' = 60 \text{ s } (= 60'')$$

2. **Centesimal or French system:** It is also known as French system. Here a right angle is divided into 100 equal parts called **grades** and each grade is divided into 100 equal parts called **minutes** and each minute is further divided into 100 equal parts called **seconds**. Therefore,

$$1 \text{ right angle} = 100 \text{ grades} = (100^g)$$

1 grade = 100 min (= 100′)
1 min = 100 s (= 100″)

3. **Circular system:** In this system, the unit of measurement is **radian**. One radian, written as 1^c, is the measure of an angle subtended at the centre of a circle by an arc of length equal to the radius of the circle.

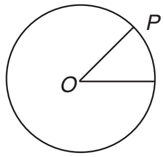


Figure 2.4

Consider a circle of radius r having centre at O (Fig. 2.4). Let A be a point on the circle. Now cut off an arc AP whose length is equal to the radius r of the circle. Then by definition the measure of $\angle AOP$ is 1 radian $(= 1^c)$.

2.4 Relation Between Three Systems of Measurement and Angle

Let D be the number of degrees, R be the number of radians and G be the number grades in an angle θ. Now

$$90° = 1 \text{ right angle} \Rightarrow 1° = \frac{1}{90} \text{ right angle}$$

$$\Rightarrow D° = \frac{D}{90} \text{ right angles} \Rightarrow \theta = \frac{D}{90} \text{ right angles} \quad (2.1)$$

Again, π radians = 2 right angles $\Rightarrow$ 1 radian $= \frac{2}{\pi}$ right angles

$$\Rightarrow R \text{ radians} = \frac{2R}{\pi} \text{ right angles} \Rightarrow \theta = \frac{2R}{\pi} \text{ right angles} \quad (2.2)$$

And 100 grades = 1 right angle $\Rightarrow$ 1 grade $= \frac{1}{100}$ right angle

$$\Rightarrow G \text{ grades} = \frac{G}{100} \text{ right angles} \Rightarrow \theta = \frac{G}{100} \text{ right angles} \quad (2.3)$$

From Eqs. (2.1)–(2.3), we get

$$\frac{D}{90} = \frac{G}{100} = \frac{2R}{\pi}$$

This is the required relation between the three systems of measurement of an angle.

2.5 Relation Between Arc and Angle

If s is the length of an arc of a circle of radius r, then the angle θ (in radians) subtended by this arc at the centre of the circle (Fig. 2.5) is given by

$$\theta = \frac{s}{r} \text{ or } s = r\theta$$

Arc = Radius × Angle in radians

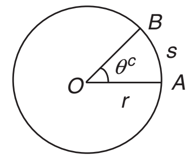


Figure 2.5

Sectorial area: Let OAB be a sector having central angle θ^c and radius r. Then area of the sector OAB is given by

$$\frac{1}{2}r^2\theta^c$$

Note: π is a real number whereas π^c stands for 180°.

Remember the relation,

$$\pi \text{ radians} = 180° = 200^g$$

$$1 \text{ radian} = \frac{2}{\pi} \times \text{right angle} = \frac{180°}{\pi}$$

$$= 180° \times 0.3183098862\ldots = 57.2957795°$$

$$= 57°17'44.8'' \text{ (nearly)}$$

Illustration 2.1 Find the radian measure corresponding to –37° 30′.

Solution:

We know that 60′ = 1°. Therefore

$$30' = \left(\frac{1}{2}\right)^\circ; \ -37°\ 30' = -\left(37\frac{1}{2}\right)^\circ = -\left(\frac{75}{2}\right)^\circ$$

As 360° = 2π radians, we have

$$-\left(\frac{75}{2}\right)\frac{\pi}{180} \text{ radians} = -\frac{5\pi}{24} \text{ radians}$$

Illustration 2.2 The minute hand of a clock is 10-cm long. How far does the tip of the hand move in 20 min?

Solution:

The minute hand moves through 120° in 20 min or moves through $2\pi/3$ radians. Since the length of the minute hand is 10 cm, the distance moved by the tip of the hand is given by the formula

$$l = r\,\theta = 10 \cdot \frac{2\pi}{3} = \frac{20\pi}{3} \text{ cm}$$

Illustration 2.3 A rail road curve is to be laid out on a circle. What radius should be used if the track is to change direction by 25° in a distance of 40 meters?

Solution:

The angle in radian measure $= \frac{25\pi}{180} = \frac{5\pi}{36}$

If r is the radius of the circle, using $l = r\,\theta$, we have

$$r = \frac{l}{\theta} = \frac{40}{\frac{5\pi}{36}} = \frac{288}{\pi} = 91.636 \text{ m}$$

Illustration 2.4 The circular wire of radius 7 cm is cut and bend again into an arc of a circle of radius 12 cm. The angle subtended by an arc at the centre of the circle is _____.

Solution:

Given the diameter of circular wire = 14 cm. Therefore, length of wire = 14π cm. Hence,

$$\text{Required angle} = \frac{\text{Arc}}{\text{Radius}} = \frac{14\pi}{12} = \frac{7\pi}{6} \text{ radian}$$

Illustration 2.5 The angles of a quadrilateral are in AP and the greatest angle is 120°. The angles in radians are _____.

Solution:

Let the angles in degrees be $\alpha - 3\delta,\ \alpha - \delta,\ \alpha + \delta,\ \alpha + 3\delta$.

Sum of the angles $= 4\alpha = 360° \Rightarrow \alpha = 90°$

Greatest angle $= \alpha + 3\delta = 120°$

Hence,

$$3\delta = 120° - \alpha = 120° - 90° = 30°$$
$$\Rightarrow \delta = 10°$$

Hence, the angles in degrees are

$$90° - 30° = 60°;\ 90° - 10° = 80°$$
$$90° + 10° = 100°;\ 90° + 30° = 120°$$

In terms of radians, the angles are $\frac{\pi}{3}, \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{2\pi}{3}$.

2.6 Trigonometric Ratio or Function

The six trigonometric ratios sine, cosine, tangent, cotangent, secant and cosecant of an angle θ, $0° < \theta < 90°$, are defined as the ratios of two sides of a right-angled triangle with θ as the angle between base and hypotenuse. However, these can be defined through a unit circle more elegantly.

Draw a unit circle and take any two diameters at right angle as X and Y (Fig. 2.6). Taking OX as the initial line, let $\overline{OP}$ be the radius vector corresponding to an angle θ, where P lies on the unit circle. Let (x, y) be the coordinates of P.

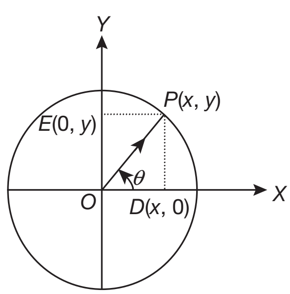


Figure 2.6

Then by definition

$$\cos\theta = x, \text{ the } x\text{-coordinate of } P$$
$$\sin\theta = y, \text{ the } y\text{-coordinate of } P$$
$$\tan\theta = \frac{y}{x}, x \neq 0$$
$$\cot\theta = \frac{x}{y}, y \neq 0$$
$$\sec\theta = \frac{1}{x}, x \neq 0$$
$$\operatorname{cosec}\theta = \frac{1}{y}, y \neq 0$$

Angles measured anticlockwise from the initial line OX are deemed to be positive and angles measured clockwise are considered to be negative.

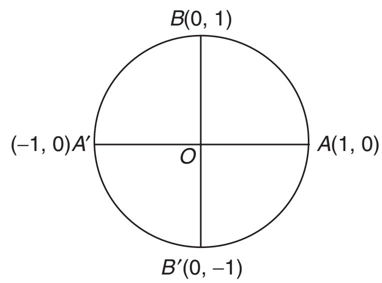


Figure 2.7

Since we can associate a unique radius vector $\overline{OP}$ and a unique point P with each angle θ, we say x and y and their ratios are functions of θ. This justifies the term 'trigonometric function'. This definition holds good for all angles positive, negative, acute or not acute (irrespective of the magnitude of the angle).

This definition also helps us to write the sine and cosine of four important angles 0°, 90°, 180° and 270° easily (see Fig. 2.7).

$$\theta = 0° \Rightarrow A(1, 0)$$
$$\theta = 90° \Rightarrow B(0, 1)$$
$$\theta = 180° \Rightarrow A'(-1, 0)$$
$$\theta = 270° \Rightarrow B'(0, -1)$$

$$\left.\begin{matrix}\cos 0° = 1\\ \sin 0° = 0\end{matrix}\right\} \quad \left.\begin{matrix}\cos 90° = 0\\ \sin 90° = 1\end{matrix}\right\} \quad \left.\begin{matrix}\cos 180° = -1\\ \sin 180° = 0\end{matrix}\right\} \quad \left.\begin{matrix}\cos 270° = 0\\ \sin 270° = -1\end{matrix}\right\}$$

We can also infer the quadrant rule for sine, cosine and tangent easily.

I quadrant	**II quadrant**	**III quadrant**	**IV quadrant**
sin, cosine and tangent **is positive**	sin alone **is positive**	tangent alone **is positive**	cosine alone **is positive**

$$90° \to \text{Point } B(0, 1)$$

Since, $\tan\theta = y/x$, $x \neq 0$, $\tan 90° = 1/0$ and hence undefined. However, as θ increases from 0 to 90°, $\tan\theta$ increases from 0 to $+\infty$.

Similarly, sec 90°, cot 0°, cosec 0° are also undefined. 360° and 0° correspond to one and the same point $A(1, 0)$. Therefore, the trigonometric functions of 360° are the same as trigonometric functions of 0°.

$$\sin 360° = 0, \cos 360° = 1 \text{ and } \tan 360° = 0$$

Since $\theta, 2\pi + \theta, 4\pi + \theta, 6\pi + \theta, \ldots, 2n\pi + \theta$ and $\theta - 2\pi, \theta - 4\pi, \theta - 6\pi, \ldots, \theta - 2n\pi$, all correspond to the same radius vector, the trigonometric functions of all these angles are the same as those of θ. Therefore,

$$\sin(2n\pi + \theta) = \sin\theta \text{ and } \sin(\theta - 2n\pi) = \sin\theta$$
$$\cos(2n\pi + \theta) = \cos\theta \text{ and } \cos(\theta - 2n\pi) = \cos\theta$$
$$\tan(2n\pi + \theta) = \tan\theta \text{ and } \tan(\theta - 2n\pi) = \tan\theta$$

The range of the trigonometric ratios in the four quadrants is depicted in the following table.

In the second quadrant (Y)

sine	decreases from	1 to 0
cosine	decreases from	0 to –1
tangent	increases from	–∞ to 0
cotangent	decreases from	0 to – ∞
secant	increases from	–∞ to – 1
cosecant	increases from	1 to ∞

In the first quadrant

sine	increases from	0 to 1
cosine	decreases from	1 to 0
tangent	increases from	0 to ∞
cotangent	decreases from	∞ to 0
secant	increases from	1 to ∞
cosecant	decreases from	∞ to 1

X′ *O* *X*

In the third quadrant

sine	decreases from	0 to –1
cosine	increases from	–1 to 0
tangent	increases from	0 to ∞
cotangent	decreases from	∞ to 0
secant	decreases from	–1 to –∞
cosecant	increases from	–∞ to –1

In the fourth quadrant

sine	increases from	–1 to 0
cosine	increases from	0 to 1
tangent	increases from	–∞ to 0
cotangent	decreases from	0 to –∞
secant	decreases from	∞ to 1
cosecant	decreases from	–1 to –∞

Y′

2.6.1 Trigonometric Functions of $-\theta$

Let *OP* and *OP′* be the radii vectors on the unit circle corresponding to θ and $-\theta$. If (x, y) are the coordinates of *P*, then $(x, -y)$ would be the coordinates of *P′*. Now, $\sin\theta = y$ and $\sin(-\theta) = -y$. Hence,

$$\sin(-\theta) = -\sin\theta$$

Similarly,

$$\cos(-\theta) = \cos\theta \text{ and } \tan(-\theta) = -\tan\theta$$

2.6.2 Circular Functions of Allied Angles

When θ is an acute angle, $90° - \theta$ is called the **angle complementary** to θ. Trigonometric functions of $90° - \theta$ are related to trigonometric functions of θ as follows:

$\sin(90° - \theta) = \cos\theta$ $\quad$ $\text{cosec}(90° - \theta) = \sec\theta$

$\cos(90° - \theta) = \sin\theta$ $\quad$ $\sec(90° - \theta) = \text{cosec}\,\theta$

$\tan(90° - \theta) = \cot\theta$ $\quad$ $\cot(90° - \theta) = \tan\theta$

When θ is acute, θ and $180° - \theta$ are called **supplementary angles**.

$\sin(180° - \theta) = \sin\theta$ $\quad$ $\text{cosec}(180° - \theta) = \text{cosec}\,\theta$

$\cos(180° - \theta) = -\cos\theta$ $\quad$ $\sec(180° - \theta) = -\sec\theta$

$\tan(180° - \theta) = -\tan\theta$ $\quad$ $\cot(180° - \theta) = -\cot\theta$

Formulae for the functions of $180° + \theta$, $270° - \theta$, $270° + \theta$, $360° - \theta$ can all be derived with the help of unit circle definition.

There is an easy way to remember these formulae. First of all think of θ as an acute angle. Angles like $180° \pm \theta$, $360° \pm \theta$, $-\theta$ can be considered as angles associated with the horizontal line, angles like $90° - \theta$, $90° + \theta$, $270° \mp \theta$ can be considered as angles associated with vertical line. When associated with the horizontal line, the magnitude of the function does not change, whereas with the vertical line the function changes to the corresponding complementary value. For example, $\sin(180° + \theta)$ will be only $\sin\theta$ (in magnitude) plus or minus and $\cos(180° - \theta)$ will be cosine θ only in magnitude.

To decide upon the sign, consider the quadrant in which the angle falls and decide the sign by the quadrant rule.

For example, $\sin(180° + \theta)$ is $\sin\theta$ (in magnitude), $(180° + \theta)$ lies in third quadrant and hence $\sin(180° + \theta)$ is negative. Therefore

$$\sin(180° + \theta) = -\sin\theta$$

Now consider $\cos(360° - \theta)$: first of all, it should be $\cos\theta$ (in magnitude); since $(360° - \theta)$ lies in IV quadrant, its cosine is positive. Hence,

$$\cos(360° - \theta) = \cos\theta$$

Again consider $\tan(90° + \theta)$: This should be $\cot\theta$ and must have a negative sign since $(90° + \theta)$ is in II quadrant and hence $\tan(90° + \theta)$ is negative. Hence,

$$\tan(90° + \theta) = -\cot\theta$$

Following is the table of formulae for allied angles.

	$180° - \theta$	$180° + \theta$	$360° - \theta$	$-\theta$	$90° - \theta$	$90° + \theta$	$270° - \theta$	$270° + \theta$
sin	$\sin\theta$	$-\sin\theta$	$-\sin\theta$	$-\sin\theta$	$\cos\theta$	$\cos\theta$	$-\cos\theta$	$-\cos\theta$
cos	$-\cos\theta$	$-\cos\theta$	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\sin\theta$	$\sin\theta$
tan	$-\tan\theta$	$\tan\theta$	$-\tan\theta$	$-\tan\theta$	$\cot\theta$	$-\cot\theta$	$\cot\theta$	$-\cot\theta$

These formulae are not memorized but derived as and when the occasion demands according to the rule explained above.

Trigonometric ratios of 30°, 45° and 60° are of great importance in solving problems on heights and distances. These along with 0° and 90° are written in tabular form and remembered.

ANGLE / RATIO	0°	30°	45°	60°	90°
sine	0	1/2	$1/\sqrt{2}$	$\sqrt{3}/2$	1
cosine	1	$\sqrt{3}/2$	$1/\sqrt{2}$	1/2	0
tangent	0	$1/\sqrt{3}$	1	$\sqrt{3}$	undefined
cotangent	undefined	$\sqrt{3}$	1	$1/\sqrt{3}$	0
secant	1	$2/\sqrt{3}$	$\sqrt{2}$	2	undefined
cosecant	undefined	2	$\sqrt{2}$	$2/\sqrt{3}$	1

2.6.3 Important Facts of Trigonometric Functions

The following points may be noted:

1. For any power n, $(\sin A)^n$ is written as $\sin^n A$. Similarly, for all other trigonometric ratios.
2. cosec A, sec A and cot A are, respectively, the reciprocals of sin A, cos A and tan A.
3. (a) $\sin^2 A + \cos^2 A = 1$
   (b) $1 + \tan^2 A = \sec^2 A$
   (c) $1 + \cot^2 A = \text{cosec}^2 A$.
4. sec A − tan A and sec A + tan A are reciprocals. So also are cosec A − cot A and cosec A + cot A.
   Whenever sec A or tan A is thought of for an angle A, it is necessary to stress that, $A \neq \pi/2$ particularly, and generally $A \neq n\pi + \pi/2$, $n \in \boldsymbol{N}$, where $\boldsymbol{N}$ is the set of natural numbers.
5. sin A and cos A are bounded functions which can be seen from the following inequalities:
   (a) $|\sin A| \leq 1 \Rightarrow -1 \leq \sin A \leq 1$
   (b) $|\cos A| \leq 1 \Rightarrow -1 \leq \cos A \leq 1$
   (c) $|\text{cosec}\, A| \geq 1 \Rightarrow \text{cosec}\, A \geq 1$ or $\text{cosec}\, A \leq -1$
   (d) $|\sec A| \geq 1 \Rightarrow \sec A \geq 1$ or $\sec A \leq -1$

   (a) $\sin\left(\dfrac{\pi}{2} - A\right) = \sin\left(\dfrac{\pi}{2} + A\right) = \cos A$

   (b) $\cos\left(\dfrac{\pi}{2} - A\right) = -\cos\left(\dfrac{\pi}{2} + A\right) = \sin A$

   (c) $\sin(\pi - A) = -\sin(\pi + A) = \sin A$
   (d) $\cos(\pi - A) = \cos(\pi + A) = -\cos A$
   (e) $\tan(\pi - A) = -\tan(\pi + A) = -\tan A$
6. The trigonometric ratios are also called trigonometric functions. They are also sometimes called **circular functions**.

   The trigonometric functions, apart from possessing many other properties, exhibit a property of the values being repeated when the angle is changed (increased or decreased) by a constant value. Such a property is referred to as **periodicity**. Thus,

$$\sin x = \sin(x + 2\pi) = \sin(x + 4\pi)$$
$$= \sin(x - 2\pi) = \sin(x + 2k\pi),\ k \text{ an integer}$$
$$\cos x = \cos(x + 2\pi) = \cos(x + 4\pi)$$
$$= \cos(x - 2\pi) = \cos(x + 2k\pi),\ k \text{ an integer}$$

Hence, both $\sin x$ and $\cos x$ are periodic functions of period 2π radians. From point 5, it is clear that they are also bounded functions.
Note that:
1. cosec x and sec x, whenever they exist, are also periodic of period 2π radians.
2. tan x and cot x, when they exist, are periodic of period π radians.
3. tan x, sec x, cosec x and cot x are unbounded functions.

2.6.4 Graphs of Trigonometric Functions

1. $y = \sin x$ (Fig. 2.8)

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
sin x	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	−1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0

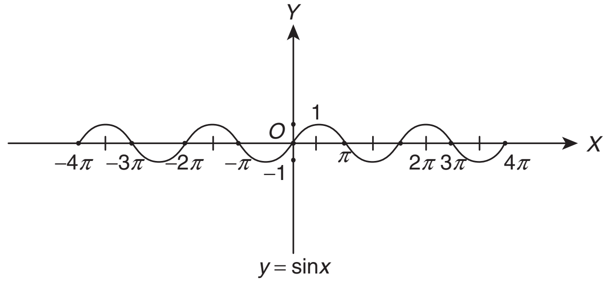


Figure 2.8

2. $y = \cos x$ (Fig. 2.9)

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
cos x	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{3}}{2}$	−1	$\frac{-\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1

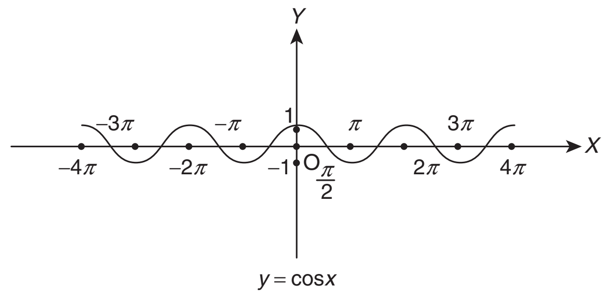


Figure 2.9

3. $y = \tan x$ (Fig. 2.10)

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
tan x	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$-\sqrt{3}$	−1	$-\frac{1}{\sqrt{3}}$	0

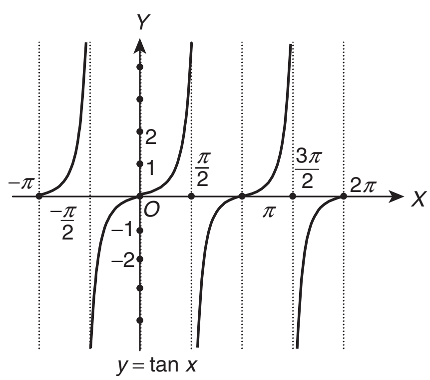


Figure 2.10

4. $y = \cot x$ (Fig. 2.11)

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
cot x	undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	−1	$-\sqrt{3}$	undefined

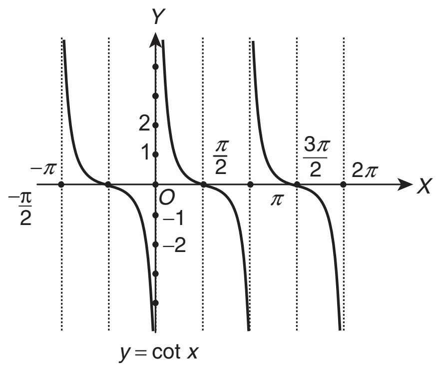


Figure 2.11

5. $y = \sec x$ (Fig. 2.12)

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
sec x	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	–2	$-\sqrt{2}$	$\frac{-2}{\sqrt{3}}$	–1

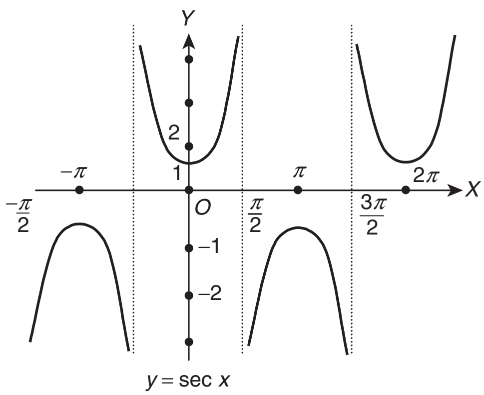


Figure 2.12

6. $y = \text{cosec}\, x$ (Fig. 2.13)

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
cosec x	undefined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	undefined

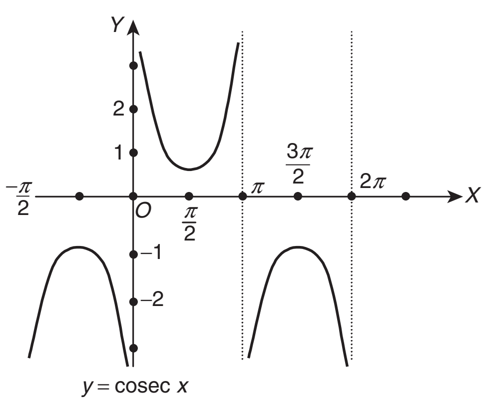


Figure 2.13

Illustration 2.6 Evaluate:

1. sin (1560°)
2. cos (–3030°)

Solution:

1. $\sin(1560°) = \sin(4 \times 360° + 120°) = \sin 120° = \sin(180° - 60°)$
   $= \sin 60° = \frac{\sqrt{3}}{2}$
2. $\cos(-3030°) = \cos(3030°)$ [using $\cos(-\theta) = \cos\theta$]
   $= \cos(8 \times 360° + 150°) = \cos 150° = \cos(180° - 30°)$
   $= -\cos 30° = -\frac{\sqrt{3}}{2}$

Illustration 2.7 Prove that $(1 - \sin\theta + \cos\theta)^2 = 2(1 - \sin\theta)(1 + \cos\theta)$.

Solution:

$$\begin{aligned}\text{L.H.S.} &= [(1 - \sin\theta) + \cos\theta]^2 = (1 - \sin\theta)^2 + \cos^2\theta + 2\cos\theta(1 - \sin\theta)\\ &= (1 - \sin\theta)^2 + (1 - \sin^2\theta) + 2\cos\theta(1 - \sin\theta)\\ &= (1 - \sin\theta)\cdot[(1 - \sin\theta) + (1 + \sin\theta) + 2\cos\theta]\\ &= (1 - \sin\theta)\cdot(2 + 2\cos\theta) = 2(1 - \sin\theta)(1 + \cos\theta)\end{aligned}$$

Illustration 2.8 Prove that $\text{cosec}^4\theta(1 - \cos^4\theta) = 1 + 2\cot^2\theta$.

Solution:

$$\text{cosec}^4\theta\,(1 - \cos^4\theta) - 2\cot^2\theta = \frac{\text{cosec}^2\theta(1-\cos^2\theta)(1+\cos^2\theta)}{\sin^2\theta} - 2\cot^2\theta$$

$$= \text{cosec}^2\theta(1 + \cos^2\theta) - 2\cot^2\theta = \text{cosec}^2\theta + \cot^2\theta - 2\cot^2\theta$$
$$= 1 + 2\cot^2\theta - 2\cot^2\theta = 1$$

Illustration 2.9 Find the minimum and maximum values of $\sin^2\theta + \cos^4\theta$.

Solution: The given expression can be written as

$$\sin^2\theta + \cos^4\theta = 1 - \cos^2\theta + \cos^4\theta$$

It can be considered as a quadratic in $\cos^2\theta$. So we have

$$\sin^2\theta + \cos^4\theta = 1 - \cos^2\theta + \cos^4\theta.$$
$$= 1 + \left(\cos^2\theta - \frac{1}{2}\right)^2 - \frac{1}{4}$$
$$= \frac{3}{4} + \left(\cos^2\theta - \frac{1}{2}\right)^2 \ge \frac{3}{4}$$

Hence, the expression has a minimum value 3/4.
Also

$$\sin^2\theta + \cos^4\theta = \sin^2\theta + \cos^2\theta\cos^2\theta \le \sin^2\theta + \cos^2\theta = 1$$

Therefore, maximum value = 1.

Illustration 2.10 For any real θ, find the maximum value of $\cos^2(\cos\theta) + \sin^2(\sin\theta)$.

Solution:

$$-1 \le \cos\theta \le 1 \Rightarrow \cos 1 \le \cos(\cos\theta) \le 1$$
$$\Rightarrow \cos^2 1 \le \cos^2(\cos\theta) \le 1 \quad (1)$$
$$-1 \le \sin\theta \le 1 \Rightarrow -\sin 1 \le \sin(\sin\theta) \le \sin 1$$
$$\Rightarrow 0 \le \sin^2(\sin\theta) \le \sin^2 1 \quad (2)$$

From Eqs. (1) and (2) we can see that maximum value of $\cos^2(\cos\theta) + \sin^2(\sin\theta)$ exists at $\theta = \pi/2$ which is $1 + \sin^2 1$.
Hence, maximum value is $1 + \sin^2 1$.

Illustration 2.11 If $2\tan^2\alpha\tan^2\beta\tan^2\gamma + \tan^2\alpha\tan^2\beta + \tan^2\beta\tan^2\gamma + \tan^2\gamma\tan^2\alpha = 1$, prove that $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 1$.

Solution: We have

$2\tan^2\alpha\tan^2\beta\tan^2\gamma + \tan^2\alpha\tan^2\beta + \tan^2\beta\tan^2\gamma + \tan^2\gamma\tan^2\alpha = 1.$

So dividing both sides by $\tan^2\alpha\tan^2\beta\tan^2\gamma$, we get

$$2 + \cot^2\gamma + \cot^2\alpha + \cot^2\beta = \cot^2\alpha\cot^2\beta\cot^2\gamma$$

$$\Rightarrow \text{cosec}^2\alpha + \text{cosec}^2\beta + \text{cosec}^2\gamma - 1 = (\text{cosec}^2\alpha - 1)(\text{cosec}^2\beta - 1)(\text{cosec}^2\gamma - 1)$$

$$\Rightarrow \text{cosec}^2\alpha + \text{cosec}^2\beta + \text{cosec}^2\gamma - 1 = -1 + \text{cosec}^2\alpha + \text{cosec}^2\beta + \text{cosec}^2\gamma - (\text{cosec}^2\alpha\,\text{cosec}^2\beta + \text{cosec}^2\beta\,\text{cosec}^2\gamma + \text{cosec}^2\gamma\,\text{cosec}^2\alpha + \text{cosec}^2\alpha\cdot\text{cosec}^2\beta\cdot\text{cosec}^2\gamma\,\text{cosec}^2\alpha\,\text{cosec}^2\beta + \text{cosec}^2\beta\cdot\text{cosec}^2\gamma + \text{cosec}^2\gamma\,\text{cosec}^2\alpha$$

$$= \text{cosec}^2\alpha\,\text{cosec}^2\beta\cdot\text{cosec}^2\gamma$$

$$\Rightarrow \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 1$$

Illustration 2.12 Simplify $\dfrac{\sin\left(\dfrac{3\pi}{2}-\theta\right)\cos\left(\dfrac{\pi}{2}+\theta\right)}{\tan\left(\dfrac{\pi}{2}+\theta\right)} - \dfrac{\sin\left(\dfrac{3\pi}{2}-\theta\right)}{\sec(\pi+\theta)}$.

Solution:

The expression can be rewritten as

$$\frac{(-\cos\theta)(-\sin\theta)}{(-\cot\theta)} - \frac{(-\cos\theta)}{(-\sec\theta)} = -\sin^2\theta - \cos^2\theta = -1$$

2.6.5 Circular Function of Compound Angle

An equation involving trigonometric functions, which is true for all those values of θ for which the functions are defined, is called a trigonometric identity; otherwise it is a trigonometric equation.

We shall now derive some results which are useful in simplifying trigonometric equations.

To prove:

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi \quad (2.4)$$

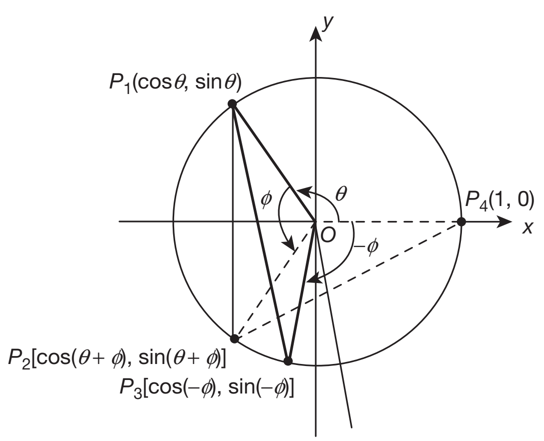


Figure 2.14

Consider a unit circle with origin as its centre (Fig. 2.14). Let

$$\angle P_4OP_1 = \theta \text{ and } \angle P_1OP_2 = \phi$$

Hence,

$$\angle P_4OP_2 = \theta + \phi \text{ and } \angle P_4OP_3 = -\phi$$

Coordinates of P_1, P_2, P_3, P_4 are

$$P_1(\cos\theta, \sin\theta)$$
$$P_2[\cos(\theta+\phi), \sin(\theta+\phi)]$$
$$P_3[\cos(-\phi), \sin(-\phi)]$$
$$P_4(1, 0)$$

ΔP_1OP_3 is congruent to ΔP_2OP_4.

Since $OP_1 = OP_4 = OP_3 = OP_2$ = Radius of the circle

$$\angle P_1OP_3 = \angle P_2OP_4 = 360^\circ - (\theta + \phi)$$

Therefore, by side angle, the triangles are congruent. Hence,

$$P_1P_3 = P_2P_4$$

Applying the distance formula,

$$P_1P_3{}^2 = [\cos\theta - \cos(-\phi)]^2 + [\sin\theta - \sin(-\phi)]^2$$
$$= (\cos\theta - \cos\phi)^2 + (\sin\theta + \sin\phi)^2$$
$$[\text{using } \cos(-\phi) = \cos\phi \text{ and } \sin(-\phi) = -\sin\phi]$$
$$= \cos^2\theta + \cos^2\phi - 2\cos\theta\cos\phi + \sin^2\theta + \sin^2\phi + 2\sin\theta\sin\phi$$
$$= 2 - 2(\cos\theta\cos\phi - \sin\theta\sin\phi)$$
$$P_2P_4^2 = [1 - \cos(\theta+\phi)]^2 + [0 - \sin(\theta+\phi)]^2$$
$$= 1 - 2\cos(\theta+\phi) + \cos^2(\theta+\phi) + \sin^2(\theta+\phi)$$
$$= 2 - 2\cos(\theta+\phi)$$

Since $P_1P_3 = P_2P_4$, we have $P_1P_3^2 = P_2P_4^2$. Therefore

$$2 - 2(\cos\theta\cos\phi - \sin\theta\sin\phi) = 2 - 2\cos(\theta+\phi)$$

Hence,

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

Replacing ϕ by $-\phi$ in Eq. (2.4), we get

$$\cos(\theta-\phi) = \cos\theta\cos(-\phi) - \sin\theta\sin(-\phi)$$

or

$$\cos(\theta-\phi) = \cos\theta\cos\phi + \sin\theta\sin\phi \quad (2.5)$$

2.7 Formulae for Trigonometric Ratios of Sum and Differences of Two or More Angles

1. $\sin(A+B) = \sin A\cos B + \cos A\sin B$
2. $\sin(A-B) = \sin A\cos B - \cos A\sin B$
3. $\cos(A+B) = \cos A\cos B - \sin A\sin B$
4. $\cos(A-B) = \cos A\cos B + \sin A\sin B$
5. $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$
6. $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$
7. $\cot(A+B) = \dfrac{\cot A\cot B - 1}{\cot A + \cot B}$
8. $\cot(A-B) = \dfrac{\cot A\cot B + 1}{\cot B - \cot A}$
9. $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

10. $\cos(A+B)\cos(A-B)=\cos^2 A-\sin^2 B=\cos^2 B-\sin^2 A$

11. $\sin(A+B+C)=\sin A\cos B\cos C+\sin B\cos A\cos C$
$+\sin C\cos A\cos C-\sin A\sin B\sin C$
$=\cos A\cos B\cos C(\tan A+\tan B+\tan C$
$-\tan A\tan B\tan C)$

12. $\cos(A+B+C)=\cos A\cos B\cos C-\sin A\sin B\cos C$
$-\sin A\cos B\sin C-\cos A\sin B\sin C$
$=\cos A\cos B\cos C(1-\tan A\tan B-\tan B\tan C$
$-\tan C\tan A)$

13. $\tan(A+B+C)=\dfrac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan C\tan B-\tan A\tan C}$

14. $\cot(A+B+C)=\dfrac{\cot A\cot B\cot C-\cot A-\cot B-\cot C}{\cot A\cot B+\cot B\cot C+\cot C\cot A-1}$

Illustration 2.13 $\dfrac{\cos 12°-\sin 12°}{\cos 12°+\sin 12°}+\dfrac{\sin 147°}{\cos 147°}=$ ____.

Solution:

$$\frac{\cos 12°-\sin 12°}{\cos 12°+\sin 12°}+\frac{\sin 147°}{\cos 147°}=\frac{1-\tan 12°}{1+\tan 12°}+\tan 147°$$
$$=\tan(45°-12°)+\tan(180°-33°)$$
$$=\tan 33°+(-\tan 33°)=0$$

Illustration 2.14 Solve $\dfrac{\sin^2 A-\sin^2 B}{\sin A\cos A-\sin B\cos B}$.

Solution:

$$\frac{2(\sin^2 A-\sin^2 B)}{2\sin A\cos A-2\sin B\cos B}=\frac{2\sin(A+B)\cdot\sin(A-B)}{\sin 2A-\sin 2B}$$
$$=\frac{2\sin(A+B)\sin(A-B)}{2\sin(A-B)\cos(A+B)}=\tan(A+B)$$

Illustration 2.15 If $\tan\theta-\cot\theta=a$ and $\sin\theta+\cos\theta=b$, then solve $(b^2-1)^2(a^2+4)$.

Solution:

Given that

$$\tan\theta-\cot\theta=a \quad (1)$$

and

$$\sin\theta+\cos\theta=b \quad (2)$$

Now,

$$(b^2-1)^2(a^2+4)=[(\sin\theta+\cos\theta)^2-1]^2[(\tan\theta-\cot\theta)^2+4]$$
$$=[1+\sin 2\theta-1]^2[\tan^2\theta+\cot^2\theta-2+4]=\sin^2 2\theta(\operatorname{cosec}^2\theta+\sec^2\theta)$$
$$=4\sin^2\theta\cos^2\theta\left[\frac{1}{\sin^2\theta}+\frac{1}{\cos^2\theta}\right]=4$$

Trick: Obviously the value of expression $(b^2-1)^2(a^2+4)$ is independent of θ, therefore put any suitable value of θ. Let $\theta=45°$. We get $a=0$, $b=\sqrt{2}$ so that $[(\sqrt{2})^2-1]^2(0^2+4)=4$.

Illustration 2.16 If $\sin\theta=\dfrac{8}{17}$ and $\cos\beta=\dfrac{9}{41}$, find $\sin(\theta+\beta)$, $\cos(\theta+\beta)$, $\sin(\theta-\beta)$ and $\cos(\theta-\beta)$, where θ is an obtuse angle and β is an acute angle.

Solution:

Since $\sin\theta=\dfrac{8}{17}$, we have

$$\cos^2\theta=1-\sin^2\theta=1-\left(\frac{8}{17}\right)^2=1-\frac{64}{289}=\frac{225}{289}$$

Therefore,

$$\cos\theta=\pm\frac{15}{17}$$

As θ is obtuse, $\cos\theta$ is negative. Therefore, $\cos\theta=-\dfrac{15}{17}$.

Now $\cos\beta=9/41$ and $\sin^2\beta=\cos^2\beta-1$. So

$$\sin^2\beta=1-\frac{81}{1681}=\frac{1600}{1681}$$
$$\Rightarrow \sin\beta=\pm\frac{40}{41}$$

As β is acute, $\sin\beta$ is positive. Hence

$$\sin\beta=+\frac{40}{41}$$

Now

$$\sin(\theta+\beta)=\sin\theta\cos\beta+\cos\theta\sin\beta$$
$$=\frac{8}{17}\cdot\frac{9}{41}+\left(-\frac{15}{17}\right)\cdot\frac{40}{41}=-\frac{528}{697}$$
$$\cos(\theta+\beta)=\cos\theta\cos\beta-\sin\theta\sin\beta$$
$$=\left(-\frac{15}{17}\right)\cdot\frac{9}{41}-\left(\frac{8}{17}\right)\cdot\frac{40}{41}=-\frac{455}{697}$$
$$\sin(\theta-\beta)=\sin\theta\cos\beta-\cos\theta\sin\beta$$
$$=\left(\frac{8}{17}\right)\cdot\frac{9}{41}-\left(-\frac{15}{17}\right)\cdot\frac{40}{41}=\frac{672}{697}$$
$$\cos(\theta-\beta)=\cos\theta\cos\beta+\sin\theta\sin\beta$$
$$=\left(-\frac{15}{17}\right)\cdot\frac{9}{41}+\left(\frac{8}{17}\right)\cdot\frac{40}{41}=\frac{185}{697}$$

Illustration 2.17 Consider triangle ABC in which $A+B+C=\pi$. Prove that

1. $\tan A+\tan B+\tan C=\tan A\tan B\tan C$
2. $\tan(B/2)\tan(C/2)+\tan(C/2)\tan(A/2)+\tan(A/2)\tan(B/2)=1$

Solution:

1. We have $A+B=\pi-C=180°-C$
$\Rightarrow \tan(A+B)=\tan(180°-C)=-\tan C$
$\Rightarrow \dfrac{\tan A+\tan B}{1-\tan A\tan B}=-\tan C$
$\Rightarrow \tan A+\tan B=-\tan C(1-\tan A\tan B)$
$\Rightarrow \tan A+\tan B+\tan C=\tan A\tan B\tan C$

2. We have $(A/2+B/2)=\pi/2-C/2=90°-C/2$
$\Rightarrow \tan(A/2+B/2)=\tan(\pi/2-C/2)=\cot(C/2)$

$$\Rightarrow \frac{\tan A/2+\tan B/2}{1-\tan A/2\tan B/2}=\cot(C/2)$$

$$\Rightarrow \tan\frac{C}{2}(\tan A/2+\tan B/2)=1-\tan\frac{A}{2}\tan\frac{B}{2}$$

Therefore, we get

$$\tan(C/2)\tan(A/2)+\tan(B/2)\tan(C/2)+\tan(A/2)\tan(B/2)=1$$

Your Turn 1

1. If $\dfrac{\sin^3\theta}{\sin(2\theta+\alpha)}=\dfrac{\cos^3\theta}{\cos(2\theta+\alpha)}$, prove that $\tan 2\theta = 2\tan(3\theta+\alpha)$.
2. If $\tan\alpha=\dfrac{m}{m+1}$ and $\tan\beta=\dfrac{1}{2m+1}$ then find $\alpha+\beta$.

   Ans. $\alpha+\beta=\dfrac{\pi}{4}$
3. Prove that tan(112A) tan(99A) tan(13A) = tan(112A) – tan(99A) – tan(13A).
4. If $\dfrac{x}{\cos\theta}=\dfrac{y}{\cos\left(\theta+\frac{2\pi}{3}\right)}=\dfrac{z}{\cos\left(\theta+\frac{4\pi}{3}\right)}$, then $x+y+z$ is equal to ____.

   Ans. 0
5. If $\sin\theta_1+\sin\theta_2+\sin\theta_3=3$, then $\cos\theta_1+\cos\theta_2+\cos\theta_3=$ ____.

   Ans. 0

2.8 Formulae to Transform Product into Sum or Difference

1. $2\sin A\cos B=\sin(A+B)+\sin(A-B)$
2. $2\cos A\sin B=\sin(A+B)-\sin(A-B)$
3. $2\cos A\cos B=\cos(A+B)+\cos(A-B)$
4. $2\sin A\sin B=\cos(A-B)-\cos(A+B)$

Illustration 2.18 Show that 8 sin 10° sin 50° sin 70° = 1.

Solution:

$$\begin{aligned}\text{L.H.S.} &= 4\,(2\sin 50^\circ \sin 10^\circ)\sin 70^\circ\\ &= 4[\cos(50^\circ-10^\circ)-\cos(50^\circ+10^\circ)]\sin 70^\circ\\ &= 2\cdot(2\sin 70^\circ\cdot\cos 40^\circ)-4\cos 60^\circ\sin 70^\circ\\ &= 2\sin 70^\circ+2\sin 30^\circ-2\sin 70^\circ\\ &= 2\sin 30^\circ = 1\end{aligned}$$

Illustration 2.19 Show that $\sin(45^\circ+A)\sin(45^\circ-A)=\frac{1}{2}\cos 2A$

Solution:

$$\sin(45^\circ+A)\sin(45^\circ-A)=\frac{1}{2}\{[\cos(2A)-\cos(90^\circ)]\}=\frac{1}{2}\cos(2A)$$

2.9 Formulae to Transform Sum or Difference into Product

Let $A+B=C$ and $A-B=D$. Then

$$A=\frac{C+D}{2}\text{ and }B=\frac{C-D}{2}$$

Therefore,

1. $\sin C+\sin D=2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$
2. $\sin C-\sin D=2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$
3. $\cos C+\cos D=2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$
4. $\cos C-\cos D=-2\sin\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

Illustration 2.20 If $\sin B=\frac{1}{5}\sin(2A+B)$, then $\dfrac{\tan(A+B)}{\tan A}=$ ____.

Solution:

$$\frac{\sin(2A+B)}{\sin B}=\frac{5}{1}$$

By Componendo and Dividendo, we have

$$\frac{\sin(2A+B)+\sin B}{\sin(2A+B)-\sin B}=\frac{5+1}{5-1}$$

$$\frac{2\sin(A+B)\cdot\cos A}{2\cos(A+B)\cdot\sin A}=\frac{6}{4}\Rightarrow\frac{\tan(A+B)}{\tan A}=\frac{3}{2}$$

Illustration 2.21 Solve $\dfrac{\sin 70^\circ+\cos 40^\circ}{\cos 70^\circ+\sin 40^\circ}$.

Solution:

$$\frac{\sin 70^\circ+\cos 40^\circ}{\cos 70^\circ+\sin 40^\circ}=\frac{\sin 70^\circ+\sin 50^\circ}{\sin 20^\circ+\sin 40^\circ}=\frac{2\sin 60^\circ\cos 10^\circ}{2\sin 30^\circ\cos(-10^\circ)}$$

$$=\frac{\sin 60^\circ}{\sin 30^\circ}=\frac{\sqrt{3}}{2}\cdot\frac{2}{1}=\sqrt{3}$$

Illustration 2.22 Show that

$$\frac{\sin 7x-\sin 3x-\sin 5x+\sin x}{\cos 7x+\cos 3x-\cos 5x-\cos x}=\tan 2x$$

Solution:

Numerator = (sin7x + sin$x$) – (sin5x + sin3x)
= 2sin4x·cos3x – 2sin4x·cos$x$ (using C.D. formula)
= 2sin4x (cos3x – cos$x$)

Denominator = (cos 3x – cos5x) – (cos$x$ – cos7x)
= 2sin4x sin$x$ – 2sin4x sin3x
= 2sin4x (sin$x$ – sin3x)

Therefore, the given expression is

$$\frac{\cos 3x-\cos x}{\sin x-\sin 3x}=\frac{2\sin 2x\sin x}{2\cos 2x\sin x}=\tan 2x$$

Illustration 2.23 Solve $\sin 47^\circ+\sin 61^\circ-\sin 11^\circ-\sin 25^\circ$.

Solution:

$$\sin 47^\circ+\sin 61^\circ-(\sin 11^\circ+\sin 25^\circ)=2\sin 54^\circ\cdot\cos 7^\circ-2\sin 18^\circ\cos 7^\circ$$

$$=2\cos 7^\circ(\sin 54^\circ-\sin 18^\circ)=2\cos 7^\circ\cdot 2\cos 36^\circ\cdot\sin 18^\circ$$

$$=4\cdot\cos 7^\circ\cdot\frac{\sqrt{5}+1}{4}\cdot\frac{\sqrt{5}-1}{4}=\cos 7^\circ$$

Illustration 2.24 If $\cos(A+B)\sin(C+D) = \cos(A-B)\sin(C-D)$, prove that $\cot A \cot B \cot C = \cot D$.

Solution:

We have

$$\cos(A+B)\sin(C+D) = \cos(A-B)\sin(C-D)$$
$$\Rightarrow \frac{\cos(A-B)}{\cos(A+B)} = \frac{\sin(C+D)}{\sin(C-D)}$$
$$\Rightarrow \frac{\cos(A-B)+\cos(A+B)}{\cos(A-B)-\cos(A+B)} = \frac{\sin(C+D)+\sin(C-D)}{\sin(C+D)-\sin(C-D)}$$
$$\Rightarrow \frac{2\cos A\cos B}{2\sin A\sin B} = \frac{2\sin C\cos D}{2\cos C\sin D}$$
$$\Rightarrow \cot A\cot B = \tan C\cot D$$
$$\Rightarrow \cot A\cot B\cot C = \cot D$$

Illustration 2.25 If A, B, C and D are angles of a quadrilateral and $\sin\frac{A}{2}\sin\frac{B}{2}\cdot\sin\frac{C}{2}\cdot\sin\frac{D}{2} = \frac{1}{4}$, prove that $A = B = C = D = \pi/2$.

Solution:

Given that

$$4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{D}{2} = 1$$
$$\Rightarrow \left[\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right]\left[\cos\left(\frac{C-D}{2}\right) - \cos\left(\frac{C+D}{2}\right)\right] = 1$$

Since, $A + B = 2\pi - (C+D)$, the above equation becomes

$$\left[\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right]\left[\cos\left(\frac{C-D}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right] = 1$$
$$\Rightarrow \cos^2\left(\frac{A+B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\left[\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{C-D}{2}\right)\right]$$
$$+1 - \cos\left(\frac{A-B}{2}\right)\cos\left(\frac{C-D}{2}\right) = 0$$

This is quadratic equation in cosine which has real roots. So

$$\left[\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{C-D}{2}\right)\right]^2 - 4\left[1 - \cos\left(\frac{A-B}{2}\right)\cos\left(\frac{C-D}{2}\right)\right] \ge 0$$
$$\Rightarrow \left[\cos\left(\frac{A-B}{2}\right) + \cos\left(\frac{C-D}{2}\right)\right]^2 \ge 4$$
$$\Rightarrow \left[\cos\left(\frac{A-B}{2}\right) + \cos\left(\frac{C-D}{2}\right)\right] \ge 2$$

Now both $\cos\left(\frac{A-B}{2}\right)$ and $\cos\left(\frac{C-D}{2}\right) \le 1$. So

$$\cos\left(\frac{A-B}{2}\right) = 1 = \cos\left(\frac{C-D}{2}\right) \Rightarrow \frac{A-B}{2} = 0 = \frac{C-D}{2}$$
$$\Rightarrow A = B, C = D$$

Similarly, $A = C, B = D \Rightarrow A = B = C = D = \pi/2$.

Your Turn 2

1. Solve $\dfrac{\cos 10° - \sin 10°}{\cos 10° + \sin 10°}$ **Ans.** cot 55°
2. If $\tan(A+B) = p$ and $\tan(A-B) = q$ then the value of $\tan 2A =$ _____.
   Ans. $\tan 2A = \dfrac{p+q}{1-pq}$
3. Solve $\sin 163°\cos 347° + \sin 73°\sin 167°$. **Ans.** 1/2
4. The value of $\cot 70° + 4\cos 70°$ is _____. **Ans.** $\sqrt{3}$
5. If $\tan\alpha = (1+2^{-x})^{-1}$, $\tan\beta = (1+2^{x+1})^{-1}$, then $\alpha + \beta$ equals _____.
   Ans. $\alpha + \beta = \dfrac{\pi}{4}$
6. The value of $\dfrac{\tan 70° - \tan 20°}{\tan 50°}$. **Ans.** 2

2.10 Trigonometric Ratio of Multiple of Angles

1. $\sin 2A = 2\sin A\cos A = \dfrac{2\tan A}{1+\tan^2 A}$
2. $\cos 2A = 1 - 2\sin^2 A = 2\cos^2 A - 1 = \cos^2 A - \sin^2 A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$
3. $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$
4. $\sin 3A = 3\sin A - 4\sin^3 A = 4\sin(60° - A)\sin A\sin(60° + A)$
5. $\cos 3A = 4\cos^3 A - 3\cos A = 4\cos(60° - A)\cos A\cos(60° + A)$
6. $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A} = \tan(60° - A)\tan A\tan(60° + A)$
7. $\sin 4A = 4\sin A\cos^3 A - 4\cos A\sin^3 A$
8. $\cos 4A = 8\cos^4 A - 8\cos^2 A + 1$
9. $\tan 4A = \dfrac{4\tan A - 4\tan^3 A}{1 - 6\tan^2 A + \tan^4 A}$
10. $\sin(A_1 + A_2 + A_3 + \cdots + A_n) = \cos A_1\cos A_2\cdots\cos A_n(S_1 - S_3 + S_5 - S_7\cdots)$
11. $\cos(A_1 + A_2 + A_3 + \cdots + A_n) = \cos A_1\cos A_2\cdots\cos A_n(1 - S_2 + S_4 - S_6\cdots)$
12. $\tan(A_1 + A_2 + A_3 + \cdots + A_n) = \dfrac{S_1 - S_3 + S_5 - S_7 + \cdots}{1 - S_2 + S_4 - S_6 + \cdots}$

    where

    $S_1 = \tan A_1 + \tan A_2 + \cdots + \tan A_n$
    = The sum of the tangents of the separate angles

    $S_2 = \tan A_1\cdot\tan A_2 + \tan A_2\cdot\tan A_3\cdots$
    = The sum of the tangents taken two at a time

    $S_3 = \tan A_1\cdot\tan A_2\cdot\tan A_3 + \tan A_2\cdot\tan A_3\tan A_4 + \cdots$
    = Sum of tangents three at a time, and so on

If $A_1 = A_2 = A_3 = \cdots = A_n$ then $S_1 = n\tan A, S_2 = {}^nC_2\tan^2 A$, $S_3 = {}^nC_3\tan^3 A\ldots$

Illustration 2.26 If $\dfrac{\tan 3\theta}{\tan\theta} = 4$, then find the value of $\dfrac{\sin 3\theta}{\sin\theta}$.

Solution:

$$\frac{\tan 3\theta}{\tan\theta} = \frac{(3\tan\theta - \tan^3\theta)}{(1-3\tan^2\theta)\tan\theta} = 4$$

$$\Rightarrow \frac{3-\tan^2\theta}{1-3\tan^2\theta} = 4 \Rightarrow \tan^2\theta = \frac{1}{11}$$

Now,

$$\frac{\sin 3\theta}{\sin\theta} = 3 - 4\sin^2\theta = 3 - 4\left(\frac{1}{1+\cot^2\theta}\right) = \frac{8}{3}$$

Illustration 2.27 Prove that $\dfrac{\tan\left(\dfrac{\pi}{4}+A\right)}{\tan\left(\dfrac{\pi}{4}-A\right)} = \dfrac{2\cos A + \sin A + \sin 3A}{2\cos A - \sin A - \sin 3A}$.

Solution:

$$\text{L.H.S.} = \frac{\tan\left(\frac{\pi}{4}+A\right)}{\tan\left(\frac{\pi}{4}-A\right)} = \frac{\frac{1+\tan A}{1-\tan A}}{\frac{1-\tan A}{1+\tan A}} = \left(\frac{1+\tan A}{1-\tan A}\right)^2 = \left(\frac{\cos A + \sin A}{\cos A - \sin A}\right)^2$$

$$= \left(\frac{\cos^2 A + \sin^2 A + 2\sin A\cos A}{\cos^2 A + \sin^2 A - 2\sin A\cos A}\right) = \left(\frac{1+\sin 2A}{1-\sin 2A}\right)$$

$$\text{R.H.S.} = \frac{2\cos A + 2\sin 2A\cos A}{2\cos A - 2\sin 2A\cos A} = \left(\frac{1+\sin 2A}{1-\sin 2A}\right)$$

Hence, both sides reduce to the same result.

Illustration 2.28 $\dfrac{\sec 8A - 1}{\sec 4A - 1}$ equals to ______.

Solution:

$$\frac{1-\cos 8A}{\cos 8A}\cdot\frac{\cos 4A}{1-\cos 4A} = \frac{2\sin^2 4A}{\cos 8A}\cdot\frac{\cos 4A}{2\sin^2 2A}$$

$$= \frac{2\sin 4A\cdot\cos 4A\cdot\sin 4A}{\cos 8A\cdot 2\sin^2 2A} = \frac{\sin 8A\cdot 2\sin 2A\cdot\cos 2A}{\cos 8A\cdot 2\sin^2 2A} = \frac{\tan 8A}{\tan 2A}$$

2.11 Trigonometric Ratio of Sub-Multiple of Angles

1. $\left|\cos\dfrac{A}{2} + \sin\dfrac{A}{2}\right| = \sqrt{1+\sin A}$ or $\cos\dfrac{A}{2} + \sin\dfrac{A}{2} = \pm\sqrt{1+\sin A}$

   That is, $\begin{cases} +, \text{ if } 2n\pi - \dfrac{\pi}{4} \le \dfrac{A}{2} \le 2n\pi + \dfrac{3\pi}{4} \\ -, \text{ otherwise} \end{cases}$

2. $\left|\sin\dfrac{A}{2} - \cos\dfrac{A}{2}\right| = \sqrt{1-\sin A}$ or $\sin\dfrac{A}{2} - \cos\dfrac{A}{2} = \pm\sqrt{1-\sin A}$

   That is, $\begin{cases} +, \text{ if } 2n\pi + \dfrac{\pi}{4} \le \dfrac{A}{2} \le 2n\pi + \dfrac{5\pi}{4} \\ -, \text{ otherwise} \end{cases}$

3. **(i)** $\tan\dfrac{A}{2} = \pm\sqrt{\dfrac{1-\cos A}{1+\cos A}} = \dfrac{1-\cos A}{\sin A}$, where $A \ne (2n+1)\pi$

   (ii) $\cot\dfrac{A}{2} = \pm\sqrt{\dfrac{1+\cos A}{1-\cos A}} = \dfrac{1+\cos A}{\sin A}$, where $A \ne 2n\pi$

See Fig. 2.15.

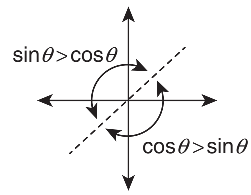


Figure 2.15

The ambiguities of signs are removed by locating the quadrants in which $\dfrac{A}{2}$ lies or you can follow Fig. 2.16.

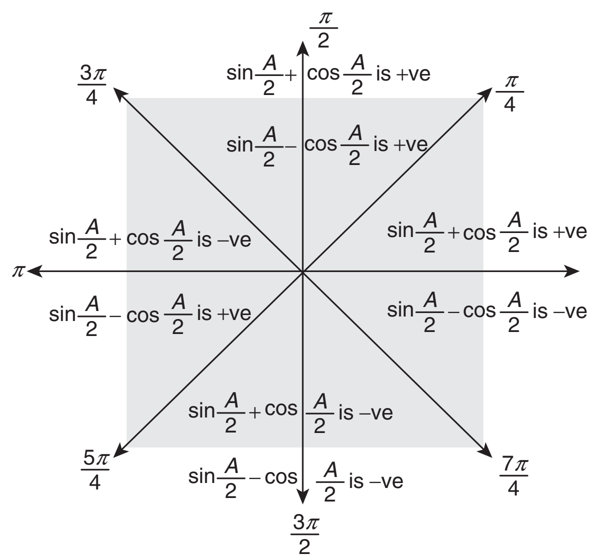


Figure 2.16

4. $\tan^2\dfrac{A}{2} = \dfrac{1-\cos A}{1+\cos A}$, where $A \ne (2n+1)\pi$

5. $\cot^2\dfrac{A}{2} = \dfrac{1+\cos A}{1-\cos A}$, where $A \ne 2n\pi$

Your Turn 3

1. Show that $\dfrac{\sin 2A}{1+\cos 2A} = \tan A$.

2. Show that $\dfrac{\sin A + \sin 2A}{1+\cos A+\cos 2A} = \tan A$.

3. Show that $\sin 3A + \sin 2A - \sin A = 4\sin A\cos\dfrac{A}{2}\cos\dfrac{3A}{2}$.

4. Show that $\tan\left(45° + \dfrac{A}{2}\right) = \sqrt{\dfrac{1+\sin A}{1-\sin A}}$.

5. If $\tan\theta = \dfrac{a}{b}$, then show that

$$\frac{\sin\theta}{\cos^8\theta} + \frac{\cos\theta}{\sin^8\theta} = \pm\frac{(a^2+b^2)^4}{\sqrt{a^2+b^2}}\left[\frac{a}{b^8} + \frac{b}{a^8}\right]$$

2.12 Maximum and Minimum Values of $a\cos\theta + b\sin\theta$

Let

$$a = r\cos\alpha \quad (2.6)$$

$$b = r\sin\alpha \quad (2.7)$$

Squaring and adding Eqs. (2.6) and (2.7) we get

$$a^2 + b^2 = r^2 \text{ or } r = \sqrt{a^2+b^2}$$

Therefore,

$$a\sin\theta + b\cos\theta = r(\sin\theta\cos\alpha + \cos\theta\sin\alpha) = r\sin(\theta+\alpha)$$

But $-1 \le \sin\theta \le 1 \Rightarrow -r \le r\sin(\theta+\alpha) \le r$. Hence

$$-\sqrt{a^2+b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2+b^2}$$

So the greatest and least values of $a\sin\theta + b\cos\theta$, respectively, are

$$\sqrt{a^2+b^2} \text{ and } -\sqrt{a^2+b^2}$$

Illustration 2.29

1. Prove that $5\cos x + 3\cos(x + \pi/3) + 3$ lies between -4 and 10.
2. Show that, whatever be the value of θ, the expression $a\sin^2\theta + b\sin\theta\cos\theta + c\cos^2\theta$ lies between

$$\frac{(a+c)}{2} - \frac{\sqrt{b^2+(a-c)^2}}{2} \text{ and } \frac{(a+c)}{2} + \frac{\sqrt{b^2+(a-c)^2}}{2}$$

Solution:

1. $5\cos x + 3\left(\cos x\cos\frac{\pi}{3} - \sin x\sin\frac{\pi}{3}\right) + 3$

$$= \left(5+\frac{3}{2}\right)\cos x - \frac{3\sqrt{3}}{2}\sin x + 3$$

$$= \frac{13}{2}\cos x - \frac{3\sqrt{3}}{2}\sin x + 3$$

$$= \sqrt{\frac{169}{4}+\frac{27}{4}}\left\{\frac{13}{2\sqrt{\frac{169}{4}+\frac{27}{4}}}\cos x - \frac{3\sqrt{3}}{2\sqrt{\frac{169}{4}+\frac{27}{4}}}\sin x\right\} + 3$$

$$= 7(\cos\alpha\cos x - \sin\alpha\sin x) + 3 \quad \left[\text{where } \tan\alpha = \frac{3\sqrt{3}}{13}\right]$$

$= 7\cos(\alpha + x) + 3$
$= -1 \le \cos(\alpha + x) \le 1$
$= -7 + 3 \le 7\cos(\alpha + x) + 3 \le 7 + 3$
$= -4 \le 7\cos(\alpha + x) + 3 \le 10$

2. Let $f(\theta) = a\sin^2\theta + b\sin\theta\cos\theta + c\cos^2\theta$. Then

$$f(\theta) = a\frac{(1-\cos 2\theta)}{2} + b\frac{\sin 2\theta}{2} + c\frac{(1+\cos 2\theta)}{2}$$

$$= \frac{1}{2}\left\{(a+c) + \frac{\sqrt{b^2+(a-c)^2}}{2}(\sin 2\theta\cos\alpha - \cos 2\theta\sin\alpha)\right\}$$

$$= \frac{1}{2}\left\{(a+c) + \frac{\sqrt{b^2+(a-c)^2}}{2}\sin(2\theta-\alpha)\right\},\ -1 \le \sin(2\theta-\alpha) \le 1$$

Therefore, $\frac{(a+c)}{2} - \frac{\sqrt{b^2+(a-c)^2}}{2} \le f(\theta) \le \frac{(a+c)}{2} + \frac{\sqrt{b^2+(a-c)^2}}{2}$

Illustration 2.30 The greatest and least values of $\sin x\cos x$ are ____.

Solution:

$$\sin x\cos x = \frac{1}{2}(2\sin x\cos x)$$
$$= \frac{\sin 2x}{2}$$
$$\Rightarrow -1 \le \sin 2x \le 1$$
$$\Rightarrow -\frac{1}{2} \le \frac{\sin 2x}{2} \le \frac{1}{2}$$

Maximum value is 1/2 and minimum value is –1/2

Illustration 2.31 The maximum value of $4\sin^2 x + 3\cos^2 x$ is ____.

Solution:

$$f(x) = 4\sin^2 x + 3\cos^2 x = \sin^2 x + 3 \text{ and } 0 \le |\sin x| \le 1$$

Therefore, maximum value of $4\sin^2 x + 3\cos^2 x$ is 4.

Illustration 2.32 If $A = \cos^2\theta + \sin^4\theta$, then for all values of θ find the range of A.

Solution:

$$A = \cos^2\theta + \sin^4\theta \Rightarrow A = \cos^2\theta + \sin^2\theta\sin^2\theta$$

$$\Rightarrow A \le \cos^2\theta + \sin^2\theta \quad [\because \sin^2\theta \le 1]$$

$$\Rightarrow A \le 1$$

Again

$$A = \cos^2\theta + \sin^4\theta = (1-\sin^2\theta) + \sin^4\theta$$

$$\Rightarrow A = \left(\sin^2\theta - \frac{1}{2}\right)^2 + \frac{3}{4} \ge \frac{3}{4}$$

Hence,

$$\frac{3}{4} \le A \le 1$$

2.13 Trigonometric Series

If we have a cosine series in its product form where the angles are in G.P. with common ratio 2, then multiply both numerator and denominator by 2 sin (least angle).

Illustration 2.33 Simplify the product $\cos A\cos 2A\cos 2^2A \cdots \cos 2^{n-1}A$.

Solution:

$$\cos A\cdot\cos 2A \cdots \cos 2^{n-1}A = \frac{1}{2\sin A}\cdot(\sin A\cdot\cos A)\cdot\cos 2A \cdots \cos 2^{n-1}A$$

$$= \frac{1}{2\sin A}\cdot(\sin 2A\cdot\cos 2A) \cdots \cos 2^{n-1}A$$

$$= \frac{1}{4\sin A} \cdot (\sin 2A \cdot \cos 2A) \cdots \cos 2^{n-1}A$$

Continuing like this we have

$$\cos A \cdot \cos 2A \cdots \cos 2^{n-1}A = \frac{\sin 2^n A}{2^n \sin A}$$

Note:

- $\prod_{r=0}^{n-1} \cos 2^r A = \frac{\sin 2^n A}{2^n \sin A}$ where $\prod$ denotes products.
- If we have a cosine series or a sine series in its sum form where the angles are in AP, then multiply both numerator and denominator with $2\sin\left(\frac{\text{common difference}}{2}\right)$.
- $\sum_{r=1}^{n} \sin[A+(r-1)B] = \frac{\sin\left[A+\frac{(n-1)}{2}B\right]\sin\frac{nB}{2}}{\sin\frac{B}{2}}$.
- $\sum_{r=1}^{n} \cos[A+(r-1)B] = \frac{\cos\left[A+\frac{(n-1)}{2}B\right]\sin\frac{nB}{2}}{\sin\frac{B}{2}}$, where $\sum$ denotes summation.

Illustration 2.34 Prove that $\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7} = -\frac{1}{2}$.

Solution:

$$\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7} = \frac{2\sin\frac{\pi}{7}}{2\sin\frac{\pi}{7}}\left(\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}\right)$$

$$= \frac{\left(\sin\frac{3\pi}{7}-\sin\frac{\pi}{7}+\sin\frac{5\pi}{7}-\sin\frac{3\pi}{7}+\sin\pi-\sin\frac{5\pi}{7}\right)}{2\sin\frac{\pi}{7}} = -\frac{1}{2}$$

Illustration 2.35 Sum to n-terms of the series

$$\sin\alpha - \sin(\alpha+\beta) + \sin(\alpha+2\beta) - \sin(\alpha+3\beta) + \cdots$$

Solution:

Since,

$$\sin(\pi+\alpha) = -\sin\alpha \text{ and } \sin(2\pi+\alpha) = \sin\alpha$$

Therefore,

$$-\sin(\alpha+\beta) = \sin(\pi+\alpha+\beta)$$
$$\sin(\alpha+2\beta) = \sin(2\pi+\alpha+2\beta)$$
$$-\sin(\alpha+3\beta) = \sin(3\pi+\alpha+3\beta) \text{ and so on.}$$

Using these results, the required sum is

$$S = \sin\alpha + \sin(\pi+\alpha+\beta) + \sin(2\pi+\alpha+2\beta) + \sin(3\pi+\alpha+3\beta) + \cdots$$

upto n terms

$$S = \frac{\sin n\frac{\pi+\beta}{2}}{\sin\frac{\pi+\beta}{2}} \cdot \sin\left[\alpha+(n-1)\frac{\pi+\beta}{2}\right]$$

Illustration 2.36 Show that $\cos\frac{\pi}{11}\cos\frac{2\pi}{11}\cos\frac{3\pi}{11}\cos\frac{4\pi}{11}\cos\frac{5\pi}{11} = \frac{1}{32}$.

Solution:

$$\cos\frac{\pi}{11}\cos\frac{2\pi}{11}\cos\frac{3\pi}{11}\cos\frac{4\pi}{11}\cos\frac{5\pi}{11} = \frac{1}{32}$$

$$\text{LHS} = \cos\frac{\pi}{11}\cos\frac{2\pi}{11}\cos\frac{3\pi}{11}\cos\frac{4\pi}{11}\cos\frac{5\pi}{11}$$

Let $\frac{\pi}{11} = \alpha$. Then the above equation can be written as

$$\cos\alpha\cos 2\alpha\cos 3\alpha\cos 4\alpha\cos 5\alpha$$
$$= -\cos\alpha\cos 2\alpha\cos 4\alpha\cos 8\alpha\cos 5\alpha \quad (11\alpha=\pi \Rightarrow 3\alpha=\pi-8\alpha)$$
$$= -\cos 2^0\alpha\cos 2^1\alpha\cos 2^2\alpha\cos 2^3\alpha\cos 5\alpha$$

Using formula $\cos\alpha\cos 2\alpha\ \cos 4\alpha - \cos 2^{n-1}\alpha = \frac{\sin 2^n\alpha}{2^n\sin\alpha}$

$$= -\frac{\sin 2^4}{2^4\sin\alpha}\cos 5\alpha = -\frac{\sin\frac{16\pi}{11}\cos\frac{5\pi}{11}}{16\sin\frac{\pi}{11}}$$

$$= \frac{2\sin\frac{5\pi}{11}\cos\frac{5\pi}{11}}{32\sin\frac{\pi}{11}} = \frac{1}{32}$$

2.14 Conditional Trigonometrical Identities

1. **Identities:** A trigonometric equation is an identity if it is true for all values of the angle or angles involved.
2. **Conditional identities:** When the angles involved satisfy a given relation, the identity is called conditional identity. In proving these identities we require properties of complementary and supplementary angles.

2.14.1 Important Conditional Identities

(A) If $A+B+C=\pi$, then

1. $\sin 2A+\sin 2B+\sin 2C = 4\sin A\sin B\sin C$
2. $\sin 2A+\sin 2B-\sin 2C = 4\cos A\cos B\sin C$
3. $\sin(B+C-A)+\sin(A+C-B)+\sin(B+A-C)$
   $= 4\sin A\sin B\sin C$
4. $\cos 2A+\cos 2B+\cos 2C = -1-4\cos A\cos B\cos C$
5. $\cos 2A+\cos 2B-\cos 2C = 1-4\sin A\sin B\cos C$
6. $\sin A+\sin B+\sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$
7. $\sin A+\sin B-\sin C = 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$
8. $\cos A+\cos B+\cos C = 1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
9. $\cos A+\cos B-\cos C = -1+4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$
10. $\frac{\cos A}{\sin B\sin C}+\frac{\cos B}{\sin A\sin C}+\frac{\cos C}{\sin B\sin A} = 2$

11. $\sin^2 A+\sin^2 B-\sin^2 C=2\sin A\sin B\cos C$

12. $\cos^2 A+\cos^2 B+\cos^2 C=1-2\cos A\cos B\cos C$

13. $\sin^2 A+\sin^2 B+\sin^2 C=2+2\cos A\cos B\cos C$

14. $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

15. $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}=2+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

16. $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin^2\frac{C}{2}=1-2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

17. $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}-\cos^2\frac{C}{2}=2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

18. $\tan A+\tan B+\tan C=\tan A\tan B\tan C$

19. $\cot A\cot B+\cot B\cot C+\cot C\cot A=1$

20. $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$

21. $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cdot\cot\frac{B}{2}\cdot\cot\frac{C}{2}$

(B) If $x+y+z=\frac{\pi}{2}$, then

22. $\sin^2 x+\sin^2 y+\sin^2 z=1-2\sin x\cdot\sin y\cdot\sin z$

23. $\cos^2 x+\cos^2 y+\cos^2 z=2+2\sin x\cdot\sin y\cdot\sin z$

24. $\sin 2x+\sin 2y+\sin 2z=4\cos x\cdot\cos y\cdot\cos z$

Illustration 2.37 If $A+B+C=\pi$, then $\cos^2 A+\cos^2 B-\cos^2 C$ is equal to ______.

Solution:

$$\cos^2 A+\cos^2 B-\cos^2 C=\cos^2 A+1-\sin^2 B-\cos^2 C$$
$$=1+\cos^2 A-\sin^2 B-\cos^2 C=1+\cos(A+B)\cos(A-B)-\cos^2 C$$
$$=1+\cos(\pi-C)\cos(A-B)-\cos^2 C=1-\cos C\cos(A-B)-\cos^2 C$$
$$=1-\cos C[\cos(A-B)+\cos C]=1-\cos C[\cos(A-B)-\cos(A+B)]$$
$$=1-2\sin A\sin B\cos C$$

Illustration 2.38 If $A+B+C=\pi$, then the value of $(\cot A+\cot B)$ $(\cot C+\cot B)$ $(\cot A+\cot C)$ will be _____.

Solution:

$$\cot A+\cot B=\frac{\sin A\cos B+\sin B\cos A}{\sin A\sin B}=\frac{\sin(A+B)}{\sin A\sin B}=\frac{\sin C}{\sin A\sin B}$$

Similarly,

$$\cot C+\cot B=\frac{\sin A}{\sin C\sin B}$$

and

$$\cot C+\cot A=\frac{\sin B}{\sin C\sin A}$$

Therefore,

$$(\cot A+\cot B)(\cot C+\cot B)(\cot A+\cot C)$$
$$=\frac{\sin C}{\sin A\sin B}\cdot\frac{\sin A}{\sin C\sin B}\cdot\frac{\sin B}{\sin C\sin A}=\operatorname{cosec} A\cdot\operatorname{cosec} B\cdot\operatorname{cosec} C$$

Illustration 2.39 If A,B and C are angles of a triangle, then $\sin 2A+\sin 2B-\sin 2C$ is equal to ______.

Solution:

$$\sin 2A+\sin 2B-\sin 2C=2\sin A\cos A+2\cos(B+C)\sin(B-C) \quad (1)$$

Since, $A+B+C=\pi$. We have $B+C=\pi-A$. Hence

$$\cos(B+C)=-\cos A \text{ and } \sin(B+C)=\sin A$$

Taking the RHS of Eq. (1) and substituting $\cos(B+C)=-\cos A$, $\sin(B+C)=\sin A$ we get

$$2\cos A[\sin A-\sin(B-C)]=2\cos A[\sin(B+C)-\sin(B-C)]$$
$$=4\cos A\cos B\sin C$$

Illustration 2.40 If $x+y+z=xyz$, prove that

$$\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}=\frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}$$

Solution:

Let $x=\tan A$, $y=\tan B$, $z=\tan C$. Therefore

$$\tan A+\tan B+\tan C=\tan A\cdot\tan B\cdot\tan C$$
$$\Rightarrow A+B+C=\pi$$

Hence,

$$\tan(2A+2B)=\tan(2\pi-2C)$$
$$\Rightarrow \tan(2A+2B)=-\tan 2C$$
$$\Rightarrow \tan 2A+\tan 2B+\tan 2C=\tan 2A\cdot\tan 2B\cdot\tan 2C$$
$$\Rightarrow \frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}=\frac{8xyz}{(1-x^2)(1-y^2)(1-z^2)}$$
$$\left(\tan 2A=\frac{2\tan A}{1-\tan^2 A}\right)$$
$$\Rightarrow \frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}=\frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}$$

Illustration 2.41 If $A+B+C=180°$, prove that

$$\sin(B+C-A)+\sin(C+A-B)+\sin(A+B-C)=4\sin A\sin B\sin C$$

Solution:

$$\text{LHS}=\sin(B+C-A)+\sin(C+A-B)+\sin(A+B-C)$$
$$=\sin(\pi-A-A)+\sin(\pi-B-B)+\sin(\pi-C-C)$$
$$(\because A+B+C=\pi)$$
$$=\sin 2A+\sin 2B+\sin 2C$$
$$=4\sin A\sin B\sin C$$

Illustration 2.42 If in ΔABC, $\cos^3 A+\cos^3 B+\cos^3 C=3\cos A\cos B\cos C$, then prove that the triangle is equilateral.

Solution:

Given that $\cos^3 A+\cos^3 B+\cos^3 C-3\cos A\cos B\cos C=0$. So

$$(\cos A+\cos B+\cos C)(\cos^2 A+\cos^2 B+\cos^2 C-\cos A\cos B-\cos B\cos C-\cos C\cos A)=0$$
$$\Rightarrow \cos^2 A+\cos^2 B+\cos^2 C-\cos A\cos B-\cos B\cos C-\cos C\cos A=0$$

(as $\cos A + \cos B + \cos C = 1 + 4 \sin A/2 \sin B/2 \sin C/2 \neq 0$)

$\Rightarrow (\cos A - \cos B)^2 + (\cos B - \cos C)^2 + (\cos C - \cos A)^2 = 0$

$\Rightarrow \cos A = \cos B = \cos C$

$\Rightarrow A = B = C$, ($\because\ 0 < A, B, C < \pi$)

$\Rightarrow \Delta ABC$ is equilateral.

Illustration 2.43 If A, B and C are angles of a triangle, prove that

$$E = \frac{\cos\left(\frac{B-C}{2}\right)}{\cos\left(\frac{B+C}{2}\right)} + \frac{\cos\left(\frac{C-A}{2}\right)}{\cos\left(\frac{C+A}{2}\right)} + \frac{\cos\left(\frac{A-B}{2}\right)}{\cos\left(\frac{A+B}{2}\right)}$$

Solution:

Since $A + B + C = \pi$

$$E = \frac{\cos\left(\frac{B-C}{2}\right)}{\cos\left(\frac{B+C}{2}\right)} + \frac{\cos\left(\frac{C-A}{2}\right)}{\cos\left(\frac{C+A}{2}\right)} + \frac{\cos\left(\frac{A-B}{2}\right)}{\cos\left(\frac{A+B}{2}\right)}$$

$$= \frac{\cos\left(\frac{B-C}{2}\right)}{\sin\left(\frac{A}{2}\right)} + \frac{\cos\left(\frac{C-A}{2}\right)}{\sin\left(\frac{B}{2}\right)} + \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\left(\frac{C}{2}\right)}$$

$$= \frac{2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)}{\sin A} + \frac{2\cos\left(\frac{B}{2}\right)\cos\left(\frac{C-A}{2}\right)}{\sin B} + \frac{2\cos\left(\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right)}{\sin C}$$

$$= \frac{\sin B + \sin C}{\sin A} + \frac{\sin A + \sin C}{\sin B} + \frac{\sin B + \sin A}{\sin C}$$

$$= \left(\frac{\sin B}{\sin A} + \frac{\sin A}{\sin B}\right) + \left(\frac{\sin C}{\sin B} + \frac{\sin B}{\sin C}\right) + \left(\frac{\sin C}{\sin A} + \frac{\sin A}{\sin C}\right)$$

As A, B, C are angles of a triangle, hence $0 < A, B, C < \pi$. So $\sin A, \sin B, \sin C > 0$

$$\Rightarrow E \geq 2 + 2 + 2 \left(\text{as } x + \frac{1}{x} \geq 2 \text{ if } x > 0\right)$$

2.15 Height and Distance

Angle of elevation: Let 'O' be the observer's eye and OA be the horizontal line through O. If the object B is at a higher level than eye, then angle $AOB = \theta$ is called the angle of elevation (Fig. 2.17).

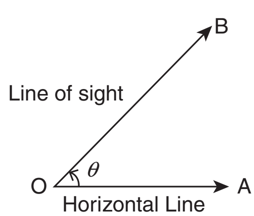


Figure 2.17

Angle of depression: If the object B is at a lower level than O, then angle AOB is called the angle of depression (Fig. 2.18).

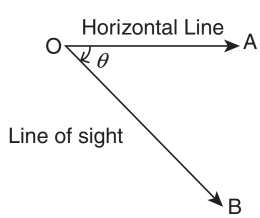


Figure 2.18

Illustration 2.44 A man is standing away from a tower of 150 meter height. At the top of the tower, the angle of depression of the man changes from 60° to 45° when man moves towards the tower. Find the distance travelled by the man.

Solution:

Let PQ be the tower, where $PQ = 150$ meters (Fig. 2.19). Let the two positions of the man be at A and B so that

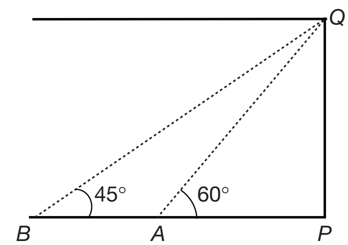


Figure 2.19

$$AP = h \cot 60° = 150\sqrt{3}$$
$$BP = 150 \cot 45° = 150$$

Now

$$AB = 150 - \frac{150}{\sqrt{3}} = \frac{150}{\sqrt{3}}(\sqrt{3} - 1)$$

Hence, the distance traveled by the man is $\frac{150}{\sqrt{3}}(\sqrt{3} - 1)$ m.

Additional Solved Examples

1. $\sqrt{3}$ cosec 20° – sec 20° = _____.

(A) 1 **(B)** 2

(C) 3 **(D)** 4

Solution:

$$\frac{4 \cdot \left(\frac{\sqrt{3}}{2}\cos 20° - \frac{1}{2}\sin 20°\right)}{2\sin 20° \cos 20°} = 4 \cdot \frac{\sin 60° \cos 20° - \cos 60° \sin 20°}{\sin 40°}$$
$$= 4$$

Hence, the correct answer is option (D).

2. $\tan A + 2 \tan 2A + 4 \tan 4A + 8 \cot 8A =$ _____.

(A) $\cot A$ **(B)** $\tan 6A$

(C) $\cot 4A$ **(D)** None of these

Solution:

$$\tan A + 2\tan 2A + 4\tan 4A + 8\cot 8A$$
$$= \tan A + 2\tan 2A + 4\tan 4A + 8\,\frac{1-\tan^2 4A}{2\tan 4A}$$
$$= \tan A + 2\tan 2A + 4\cot 4A = \tan A + 2\tan 2A + 4\,\frac{1-\tan^2 2A}{2\tan 2A}$$
$$= \tan A + 2\cot 2A = \tan A + \frac{1-\tan^2 A}{\tan A} = \cot A$$

Hence, the correct answer is option (A).

3. The value of sin 12°·sin48°·sin54° = _____.

(A) 1/8 **(B)** 1/6

(C) 1/4 **(D)** 1/2

Solution:

$$\sin 12^\circ \cdot \sin 48^\circ \cdot \sin 54^\circ$$

$$= \frac{1}{2}[\cos 36^\circ - \cos 60^\circ]\sin 54^\circ = \frac{1}{2}\left[\cos 36^\circ \sin 54^\circ - \frac{1}{2}\sin 54^\circ\right]$$

$$= \frac{1}{4}[2\cos 36^\circ \sin 54^\circ - \sin 54^\circ] = \frac{1}{4}[\sin 90^\circ + \sin 18^\circ - \sin 54^\circ]$$

$$= \frac{1}{4}[1 - (\sin 54^\circ - \sin 18^\circ)] = \frac{1}{4}[1 - 2\sin 18^\circ \cos 36^\circ]$$

$$= \frac{1}{4}\left[1 - 2\frac{\sin 18^\circ \cos 18^\circ \cos 36^\circ}{\cos 18^\circ}\right] = \frac{1}{4}\left[1 - \frac{\sin 36^\circ \cos 36^\circ}{\cos 18^\circ}\right]$$

$$= \frac{1}{4}\left[1 - \frac{2\sin 36^\circ \cos 36^\circ}{2\cos 18^\circ}\right] = \frac{1}{4}\left[1 - \frac{\sin 72^\circ}{2\sin 72^\circ}\right] = \frac{1}{4}\left[1 - \frac{1}{2}\right] = \frac{1}{8}$$

Alternative Method

Let $\theta = 12^\circ$

$$\sin 12^\circ \cdot \sin 48^\circ \cdot \sin 54^\circ = \frac{\sin 12^\circ \sin 48^\circ \sin 72^\circ \sin 54^\circ}{\sin 72^\circ}$$

$$= \frac{\sin(3\cdot 12)^\circ \sin 54^\circ}{4\sin 72^\circ} = \frac{\sin(36)^\circ \sin 54^\circ}{8\sin(36)^\circ \cos(36)^\circ} = \frac{\cos 36^\circ}{8\cos(36)^\circ} = \frac{1}{8} = \frac{1}{8}$$

Hence, the correct answer is option (A).

4. If $\sin\theta = 3\sin(\theta + 2\alpha)$, then the value of $\tan(\theta + \alpha) + 2\tan\alpha$ is

(A) 0 **(B)** 2
(C) 4 **(D)** 1

Solution: Given $\sin\theta = 3\sin(\theta + 2\alpha)$. Now

$$\sin(\theta + \alpha - \alpha) = 3\sin(\theta + \alpha + \alpha)$$
$$\Rightarrow \sin(\theta + \alpha)\cos\alpha - \cos(\theta + \alpha)\sin\alpha$$
$$= 3\sin(\theta + \alpha)\cos\alpha + 3\cos(\theta + \alpha)\sin\alpha$$
$$\Rightarrow -2\sin(\theta + \alpha)\cos\alpha = 4\cos(\theta + \alpha)\sin\alpha$$
$$\Rightarrow \frac{-\sin(\theta+\alpha)}{\cos(\theta+\alpha)} = \frac{2\sin\alpha}{\cos\alpha}$$
$$\Rightarrow \tan(\theta + \alpha) + 2\tan\alpha = 0$$

Hence, the correct answer is option (A).

5. The minimum value of $3\tan^2\theta + 12\cot^2\theta$ is

(A) 6 **(B)** 8
(C) 10 **(D)** None of these

Solution:

$$\text{A.M.} \geq \text{G.M} \Rightarrow \frac{1}{2}(3\tan^2\theta + 12\cot^2\theta) \geq 6$$

$\Rightarrow 3\tan^2\theta + 12\cot^2\theta$ has minimum value 12.

Hence, the correct answer is option (D).

6. Prove that $3(\sin x - \cos x)^4 + 4(\sin^6 x - \cos^6 x) + 6(\sin x + \cos x)^2 = 13$.

Solution: Let t_1, t_2, t_3 denote the three expressions on the left.

$t_1 = 3[(\sin x - \cos x)^2]^2 = 3(\sin^2 x + \cos^2 x - 2\sin x\cos x)^2$
$= 3(1 - 2\sin x\cos x)^2 = 3(1 + 4\sin^2 x\cos^2 x - 4\sin x\cos x)$

$t_2 = 4(\sin^6 x - \cos^6 x) = 4(\sin^2 x + \cos^2 x)(\sin^4 x + \cos^4 x - \sin^2 x\cos^2 x)$
$= 4[(\sin^2 x + \cos^2 x)^2 - 2\cos^2 x\sin^2 x - \sin^2 x\cos^2 x]$
$= 4(1 - 3\sin^2 x\cos^2 x)$

$t_3 = 6(\sin^2 x + \cos^2 x + 2\sin x\cos x) = 6(1 + 2\sin x\cos x)$

Therefore, $t_1 + t_2 + t_3 = 3 + 4 + 6 = 13$

7. Prove that $\sin^2\frac{\pi}{8} + \sin^2\frac{3\pi}{8} + \sin^2\frac{5\pi}{8} + \sin^2\frac{7\pi}{8} = 2$.

Solution:

$$\sin\frac{7\pi}{8} = \sin\left(\pi - \frac{\pi}{8}\right) = \sin\frac{\pi}{8}$$
$$\sin\frac{5\pi}{8} = \sin\left(\pi - \frac{3\pi}{8}\right) = \sin\frac{3\pi}{8}$$
$$\sin\frac{3\pi}{8} = \sin\left(\frac{\pi}{2} - \frac{\pi}{8}\right) = \cos\frac{\pi}{8}$$

So we have

$$\sin^2\frac{\pi}{8} + \sin^2\frac{3\pi}{8} + \sin^2\frac{5\pi}{8} + \sin^2\frac{7\pi}{8} = 2\left(\sin^2\frac{\pi}{8} + \sin^2\frac{3\pi}{8}\right)$$
$$= 2\left(\sin^2\frac{\pi}{8} + \cos^2\frac{\pi}{8}\right) = 2$$

8. Prove the identity:

$(\cos A + \cos B)(\cos 2A + \cos 2B)(\cos 2^2A + \cos 2^2B) \cdots (\cos 2^{n-1}A + \cos 2^{n-1}B) = \dfrac{(\cos 2^n A - \cos 2^n B)}{2^n(\cos A - \cos B)}$.

Solution:

$$(\cos A - \cos B)(\cos A + \cos B) = \cos^2 A - \cos^2 B = \frac{1}{2}[(1 + \cos 2A) - (1 + \cos 2B)]$$
$$= \frac{1}{2}(\cos 2A - \cos 2B) \qquad (1)$$

Therefore,

$$(\cos A - \cos B)(\cos A + \cos B)(\cos 2A + \cos 2B)$$
$$= \frac{1}{2}[(\cos 2A - \cos 2B)(\cos 2A + \cos 2B)] = \frac{1}{2^2}(\cos 2^2A - \cos 2^2B)$$

Therefore,

$$(\cos A - \cos B)(\cos A + \cos B)(\cos 2A + \cos 2B)(\cos 2^2A + \cos 2^2B)$$

Proceeding in this manner, we get

$$(\cos A - \cos B)(\cos A + \cos B)(\cos 2A + \cos 2B)(\cos 2^2A + \cos 2^2B) \cdots (\cos 2^{n-1}A + \cos 2^{n-1}B)$$
$$= \frac{1}{2^n}(\cos 2^n A - \cos 2^n B)$$

Hence, the given identity follows.

9. Show that $\dfrac{\sin 8A}{2\sin A} = \cos A + \cos 3A + \cos 5A + \cos 7A$

Solution:

R.H.S. $= (\cos A + \cos 3A) + (\cos 5A + \cos 7A)$
$= [\cos(2A - A) + \cos(2A + A)] + [\cos(6A - A) + \cos(6A + A)]$
$= 2\cos A \cdot \cos 2A + 2\cos A \cdot \cos 6A$
$= 2\cos A[\cos(4A - 2A) + \cos(4A + 2A)] = 2\cos A \cdot 2\cos 2A \cdot \cos 4A$

$$= \frac{(2\sin A\cos A)}{\sin A} \cdot 2\cos 2A \cdot \cos 4A$$
$$= \frac{(2\sin 2A \cdot \cos 2A) \cdot \cos 4A}{\sin A} = \frac{\sin 4A \cdot \cos 4A}{\sin A} = \frac{\sin 8A}{2\sin A}$$

10. Prove that $\cos^2 x + \cos^2\left(\frac{\pi}{3} + x\right) - \cos x\cos\left(\frac{\pi}{3} + x\right)$ is independent of x.

Solution:

$$\cos^2 x+\cos^2\left(\frac{\pi}{3}+x\right)-\cos\left(\frac{\pi}{3}+x\right)\cos x$$

$$=\frac{1}{2}\left(2\cos^2 x+2\cos^2\left(\frac{\pi}{3}+x\right)-2\cos\left(\frac{\pi}{3}+x\right)\cos x\right)$$

$$=\frac{1}{2}\left(1+\cos\left(\frac{2\pi}{3}+2x\right)+1+\cos 2x-2\cos\left(\frac{\pi}{3}+x\right)\cos x\right)$$

$$=\frac{1}{2}\left(2+\cos\left(\frac{\pi}{3}+2x\right)+\cos 2x+\cos\frac{\pi}{3}-\cos\left(\frac{\pi}{3}+2x\right)\right)$$

$$=\frac{1}{2}\left(2-\frac{1}{2}+2\cos\left(\frac{\pi}{3}+2x\right)\cos\frac{\pi}{3}-\cos\left(\frac{\pi}{3}+2x\right)\right)\text{ since }\cos\frac{\pi}{3}=\frac{1}{2}$$

$$=\frac{3}{4}+\frac{1}{2}\cos\left(\frac{\pi}{3}+2x\right)-\frac{1}{2}\cos\left(\frac{\pi}{3}+2x\right)=\frac{3}{4}$$

and this does not contain x. Hence proved.

11. If $\frac{\cos^4 x}{\cos^2 y}+\frac{\sin^4 x}{\sin^2 y}=1$, then prove that $\frac{\cos^4 y}{\cos^2 x}+\frac{\sin^4 y}{\sin^2 x}=1$.

Solution: The given condition is

$$\cos^4 x\sin^2 y+\sin^4 x\cos^2 y=\sin^2 y\cos^2 y$$
$$=\sin^2 y(1-\sin^2 y)=\sin^2 y-\sin^4 y \quad (1)$$

Therefore,

$$\sin^4 y=\sin^2 y(1-\cos^4 x)-\sin^4 x\cos^2 y$$
$$=\sin^2 y(1-\cos^2 x)(1+\cos^2 x)-\sin^4 x\cos^2 y$$
$$=\sin^2 y\sin^2 x(1+\cos^2 x)-\sin^4 x\cos^2 y$$

Hence,

$$\frac{\sin^4 y}{\sin^2 x}=\sin^2 y+\sin^2 y\cos^2 x-\cos^2 y\sin^2 x \quad (2)$$

Similarly, on the R.H.S. of Eq. (1), replacing $\sin^2 y$ by $1-\cos^2 y$ and simplifying as shown above, we get

$$\frac{\cos^4 y}{\cos^2 x}=\cos^2 y+\cos^2 y\sin^2 x-\cos^2 x\sin^2 y \quad (3)$$

By adding Eqs. (2) and (3), we get the desired result.

12. Prove that:

1. $\tan A+\cot A=2\operatorname{cosec}2A$
2. $\cot A-\tan A=2\cot 2A$

Deduce that $\tan A+2\tan 2A+4\tan 4A+8\cot 8A=\cot A$ and more generally
$\tan A+2\tan 2A+2^2\tan 2^2A+\cdots+2^{n-1}\tan^{2n-1}A+2^n\cot 2^n A=\cot A$

Solution:

1. $\tan A+\cot A=\tan A+\frac{1}{\tan A}=\frac{1+\tan^2 A}{\tan A}$

$$=\frac{\sec^2 A}{\tan A}=\frac{2}{2\tan A\cos^2 A}=\frac{2}{\sin 2A}=2\operatorname{cosec}2A$$

2. $\cot A-\tan A=\frac{\cos A}{\sin A}-\frac{\sin A}{\cos A}=\frac{\cos^2 A-\sin^2 A}{\sin A\cos A}=\frac{2\cos 2A}{\sin 2A}=2\cot 2A$

Therefore,

$$\tan A=\cot A-2\cot 2A \quad (1)$$
$$\tan 2A=\cot 2A-2\cot 4A \text{ [changing } A \text{ to } 2A \text{ in Eq. (1)]} \quad (2)$$
$$\tan 4A=\cot 4A-2\cot 8A \text{ [similar change]} \quad (3)$$

Multiplying Eqs. (1) – (3) by 1, 2, 2^2 and adding, we get

$$\tan A+2\tan 2A+2^2\tan 4A=\cot A-8\cot 8A$$

Hence,

$$\tan A+2\tan 2A+2^2\tan 2^2A+2^3\cot 2^3A=\cot A$$

The general result can be obtained by repeating the above sequence of steps n times.

13. If $A+B+C=\pi$, and

$$\tan\left(\frac{A+B-C}{4}\right)\tan\left(\frac{B+C-A}{4}\right)\tan\left(\frac{A+C-B}{4}\right)=1$$

prove that $\sin A+\sin B+\sin C+\sin A\sin B\sin C=0$.

Solution:

$$\tan\left(\frac{A+B-C}{4}\right)=\tan\left(\frac{\pi-2C}{4}\right)=\tan\left(\frac{\pi}{4}-\frac{C}{2}\right)=\frac{1-\tan\frac{C}{2}}{1+\tan\frac{C}{2}}$$

$$=\frac{\left(\cos\frac{C}{2}-\sin\frac{C}{2}\right)^2}{\cos^2\frac{C}{2}-\sin^2\frac{C}{2}}=\frac{1-\sin C}{\cos C}=\frac{\cos C}{1+\sin C}$$

Similarly,

$$\tan\left(\frac{B+C-A}{4}\right)=\frac{1-\sin A}{\cos A}=\frac{\cos A}{1+\sin A}$$

and

$$\tan\left(\frac{C+A-B}{4}\right)=\frac{1-\sin B}{\cos B}=\frac{\cos B}{1+\sin B}$$

The given condition implies

$$\left(\frac{1-\sin A}{\cos A}\right)\left(\frac{1-\sin B}{\cos B}\right)\left(\frac{1-\sin C}{\cos C}\right)=1 \quad (1)$$

as well as

$$\left(\frac{\cos A}{1+\sin A}\right)\left(\frac{\cos B}{1+\sin B}\right)\left(\frac{\cos C}{1+\sin C}\right)=1 \quad (2)$$

From Eqs. (1) and (2), we get

$$\cos A\cos B\cos C=(1-\sin A)(1-\sin B)(1-\sin C)$$
$$=(1+\sin A)(1+\sin B)(1+\sin C)$$

Hence,

$$1-\Sigma\sin A+\Sigma\sin A\sin B-\sin A\sin B\sin C$$
$$=1+\Sigma\sin A+\Sigma\sin A\sin B+\sin A\sin B\sin C$$

Therefore, $\Sigma\sin A+\sin A\sin B\sin C=0$.

14. If $0 \le \theta \le \frac{\pi}{2}$, prove the inequality $\cos(\sin\theta) > \sin(\cos\theta)$.

Solution:

We have $\sin\theta + \cos\theta = \sqrt{2}\,\sin\left(\theta + \frac{\pi}{4}\right) \le \sqrt{2}$ since the maximum value of $\sin\left(\theta+\frac{\pi}{4}\right)=1$

But $\sqrt{2} < \pi/2$; ($\sqrt{2}$ is approximately 1.414 and $\pi/2$ is approximately 1.59). Therefore,

$$\sin\theta + \cos\theta < \frac{\pi}{2}$$

$$\Rightarrow \sin\theta < \frac{\pi}{2} - \cos\theta$$

$$\cos(\sin\theta) > \cos\left(\frac{\pi}{2} - \cos\theta\right) \text{ since } \alpha < \beta \Rightarrow \cos\alpha > \cos\beta$$

cosine being a decreasing function in first quadrant. That is

$$\cos(\sin\theta) > \sin(\cos\theta)$$

15. If $\tan(\pi/4 + y/2) = \tan^3(\pi/4 + x/2)$, prove that

$$\sin y = \sin x\left(\frac{3+\sin^2 x}{1+3\sin^2 x}\right)$$

Solution:

$$\tan\left(\frac{\pi}{4}+\frac{y}{2}\right) = \left(\frac{1+\tan\frac{y}{2}}{1-\tan\frac{y}{2}}\right)$$

Therefore, from the given condition,

$$\left(\frac{1+\tan\frac{y}{2}}{1-\tan\frac{y}{2}}\right) = \left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right)^3$$

Hence,

$$\left(\frac{\left(1+\tan\frac{y}{2}\right)-\left(1-\tan\frac{y}{2}\right)}{\left(1+\tan\frac{y}{2}\right)+\left(1-\tan\frac{y}{2}\right)}\right) = \left(\frac{\left(1+\tan\frac{x}{2}\right)^3-\left(1-\tan\frac{x}{2}\right)^3}{\left(1+\tan\frac{x}{2}\right)^3+\left(1-\tan\frac{x}{2}\right)^3}\right)$$

$$\left(\because \frac{a}{b}=\frac{c}{d} \Rightarrow \frac{a-b}{a+b}=\frac{c-d}{c+d}\right)$$

$$\Rightarrow \tan\frac{y}{2} = \left(\frac{3\tan\frac{x}{2}+\tan^3\frac{x}{2}}{1+3\tan^2\frac{x}{2}}\right)$$

$$\Rightarrow \tan\frac{y}{2} = \left(\frac{3t+t^3}{1+3t^2}\right) \qquad \left(\text{where } \tan\frac{x}{2}=t\right)$$

$$\text{LHS} = \sin y = \frac{2\tan\frac{y}{2}}{1+\tan^2\frac{y}{2}} = \frac{2\left(\frac{3t+t^3}{1+3t^2}\right)}{1+\left(\frac{3t+t^3}{1+3t^2}\right)^2} = \frac{2(3+t^2)(1+3t^2)}{(1+t^2)(1+14t^2+t^4)}$$

$$\left(\cos x = \frac{1-\tan^2 t}{1+\tan^2 t} \Rightarrow t^2 = \frac{1-\cos x}{1+\cos x}\right)$$

$$= \sin x \frac{2\left(3+\left(\frac{1-\cos x}{1+\cos x}\right)\right)\left(1+3\left(\frac{1-\cos x}{1+\cos x}\right)\right)}{\left(1+\left(\frac{1-\cos x}{1+\cos x}\right)\right)\left(1+14\left(\frac{1-\cos x}{1+\cos x}\right)+\left(\frac{1-\cos x}{1+\cos x}\right)^2\right)}$$

$$= \sin x \frac{2(4+2\cos x)(4-2\cos x)}{((1+\cos x)^2+14(1+\cos x)(1-\cos x)+(1-\cos x)^2)}$$

$$= \sin x \frac{3+\sin^2 x}{1+3\sin^2 x} = \text{RHS}$$

Previous Years' Solved JEE Main/AIEEE Questions

1. A body weighing 13 kg is suspended by two strings 5-m and 12-m long, their other ends being fastened to the extremities of a rod 13-m long. If the rod be so held that the body hangs immediately below the middle point. The tensions in the strings are:

(A) 12 kg and 13 kg **(B)** 5 kg and 5 kg
(C) 5 kg and 12 kg **(D)** 5 kg and 13 kg

[AIEEE 2007]

Solution: See Fig. 2.20. Since, $13^2 = 5^2 + 12^2$, therefore,

$$\angle AOB = \frac{\pi}{2}$$

$\angle AOB$ is the angle in a semicircle with diameter AB and centre C.

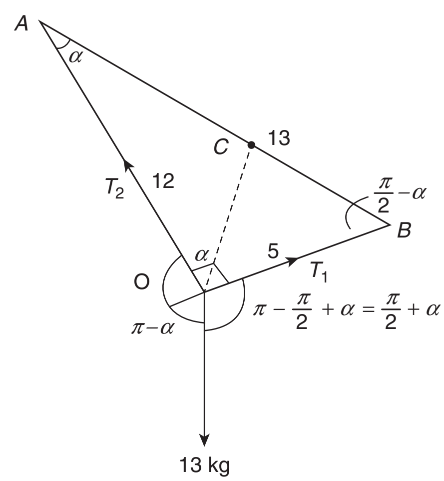


Figure 2.20

Therefore,

$$\sin\alpha = \frac{5}{13} \text{ and } \cos\alpha = \frac{12}{13}$$

Also due to equilibrium,

$$\frac{T_1}{\sin(\pi-\alpha)} = \frac{T_2}{\sin(\pi/2)+\alpha} = \frac{13}{\sin(\pi/2)}$$

$$\Rightarrow \sin\alpha = \frac{T_1}{13} \text{ and } \cos\alpha = \frac{T_2}{13}$$

Therefore, $T_1 = 5$ and $T_2 = 12$.

Hence, the correct answer is option (C).

2. A tower stands at the centre of a circular park. *A* and *B* are two points on the boundary of the park such that *AB* (= *a*) subtends an angle of 60° at the foot of the tower, and the angle of elevation of the top of the tower from *A* or *B* is 30°. The height of the tower is

(A) $\frac{2a}{\sqrt{3}}$ **(B)** $2a\sqrt{3}$

(C) $\frac{a}{\sqrt{3}}$ **(D)** $a\sqrt{3}$ **[AIEEE 2007]**

Solution: The situation is depicted in Fig. 2.21 in which it is clearly shown that ΔOAB is equilateral. Therefore,

$$OA = OB = AB = a$$

Now considering triangle *OBH*, $\tan 30° = \frac{h}{a} \Rightarrow h = \frac{a}{\sqrt{3}}$.

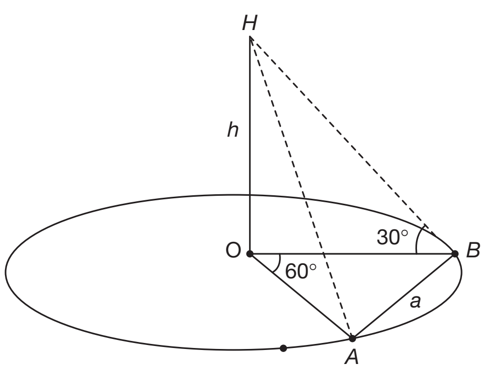


Figure 2.21

Hence, the correct answer is option (C).

3. Let $P = (-1, 0)$, $Q = (0, 0)$ and $R = (3, 3\sqrt{3})$ be three points. The equation of the bisector of the angle *PQR* is

(A) $\sqrt{3}x + y = 0$ **(B)** $x + \frac{\sqrt{3}}{2}y = 0$

(C) $\frac{\sqrt{3}}{2}x + y = 0$ **(D)** $x + \sqrt{3}y = 0$

[AIEEE 2007]

Solution: See Fig. 2.22.

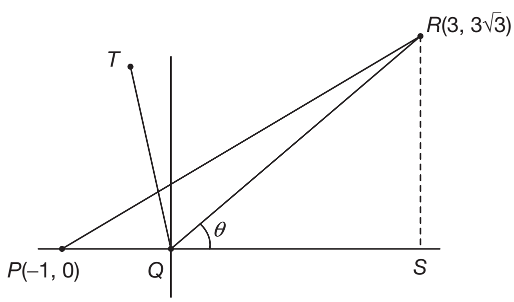


Figure 2.22

$$\tan\theta = \frac{3\sqrt{3}}{3} = \sqrt{3} \Rightarrow \theta = 60° \Rightarrow \angle RQS = 60°$$

$$\angle PQR = 180° - \angle RQS \Rightarrow \angle SQT = 60 + 60 = 120°$$

Therefore, equation of bisector is

$$y - 0 = [\tan 120°](x - 0) \Rightarrow y = -\sqrt{3}x$$

From Fig. 2.22, slope of the line *QM* is $\tan\frac{2\pi}{3} = -\sqrt{3}$.

Hence equation is line *QM* is $y = -\sqrt{3}x \Rightarrow y + \sqrt{3}x = 0$.

Hence, the correct answer is option (A).

4. A bird is sitting on the top of a vertical pole 20-m high and its elevation from a point *O* on the ground is 45°. It flies off horizontally straight away from the point *O*. After one second, the elevation of the bird from *O* is reduced to 30°. Then the speed (in m/s) of the bird is

(A) $20\sqrt{2}$ **(B)** $20(\sqrt{3}-1)$

(C) $40(\sqrt{2}-1)$ **(D)** $40(\sqrt{3}-\sqrt{2})$

[JEE MAIN 2014 (OFFLINE)]

Solution: See Fig. 2.23.

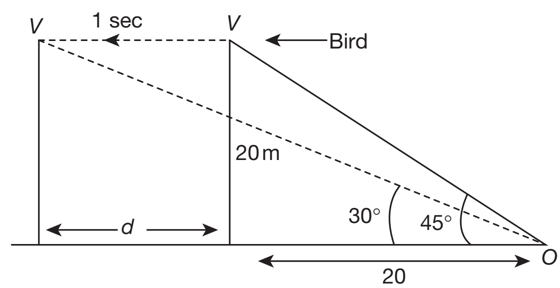


Figure 2.23

$$\frac{d+20}{20} = \cot 30° = \sqrt{3} \Rightarrow d = 20\sqrt{3} - 20 = 20(\sqrt{3}-1)$$

Hence, the correct answer is option (B).

5. If the angles of elevation of the top of a tower from three collinear points *A*, *B* and *C*, on a line leading to the foot of the tower, are 30°, 45° and 60°, respectively, then the ratio, *AB*: *BC*, is

(A) $\sqrt{3}:\sqrt{2}$ **(B)** $1:\sqrt{3}$

(C) 2 : 3 **(D)** $\sqrt{3}:1$

[JEE MAIN 2015 (OFFLINE)]

Solution: See Fig. 2.24.

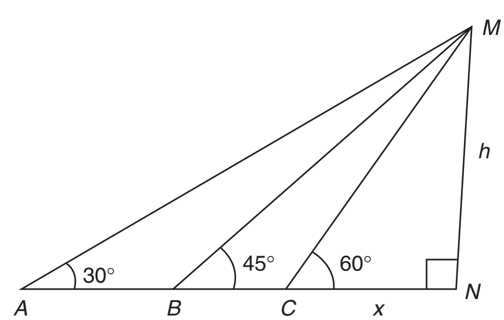


Figure 2.24

Now,

$$\tan 60° = \frac{h}{x} \Rightarrow x = h \cot 60° \quad (1)$$

$$\tan 45° = \frac{h}{BC + x} \Rightarrow BC = h - h \cot 60° \quad (2)$$

and $$\tan 30° = \frac{h}{AB + h} \Rightarrow AB = h \cot 30° - h \quad (3)$$

Therefore,

$$AB : BC = \frac{\cot 30° - 1}{1 - \cot 60°} = \frac{\sqrt{3} - 1}{1 - \frac{1}{\sqrt{3}}} = \frac{\sqrt{3}}{1}$$

Hence, the correct answer is option (D).

6. Let the tangents drawn to the circle $x^2 + y^2 = 16$ from the point $P(0, h)$ meet the x-axis at points A and B. If the area of ΔAPB is minimum, then h is equal to

(A) $4\sqrt{3}$ **(B)** $3\sqrt{3}$
(C) $3\sqrt{2}$ **(D)** $4\sqrt{2}$

[JEE MAIN 2015 (ONLINE SET-1)]

Solution: See Fig. 2.25.

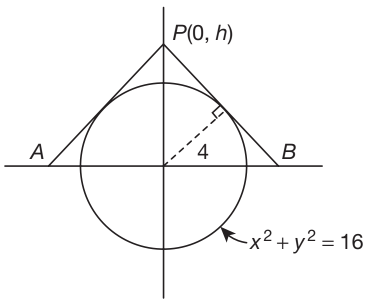


Figure 2.25

Equation of tangent to circle $x^2 + y^2 = 16$ is

$$y = mx \pm 4\sqrt{m^2 + 1}$$

It passes through $P(0, h)$; $h > 0 \Rightarrow h = 4\sqrt{m^2 + 1}$

Hence, equation of tangent PA or PB will be $y = mx \pm h$

They intersect at x-axis, where

$$0 = mx \pm h \Rightarrow mx \Rightarrow \mp h \Rightarrow x = \mp \frac{h}{m} \Rightarrow AB = \frac{2h}{|m|}$$

Therefore,

$$\text{Area of } \Delta PAB = \frac{1}{2}\left(\frac{2h}{|m|}\right) \cdot h = \frac{h^2}{|m|}$$

Also

$$\frac{|h|}{\sqrt{m^2 + 1}} = 4$$

$$\Rightarrow \sqrt{m^2 + 1} = \frac{|h|}{4} \Rightarrow m^2 + 1 = \frac{h^2}{16}$$

$$\Rightarrow m^2 = \frac{h^2 - 16}{16} \Rightarrow |m| = \frac{\sqrt{h^2 - 16}}{4}$$

Therefore, Area of $\Delta PAB = \dfrac{4h^2}{\sqrt{h^2 - 16}} = f(h)$ (say)

$$\Rightarrow f'(h) = \frac{4(h^3 - 32h)}{(h^2 - 16)^{3/2}} = 0 \Rightarrow h = 4\sqrt{2}$$

Hence, for minimum area, $h = 4\sqrt{2}$.

Hence, the correct answer is option (D).

7. In a ΔABC, $\frac{a}{b} = 2 + \sqrt{3}$ and $\angle C = 60°$. The ordered pair $(\angle A, \angle B)$ is equal to

(A) (15°, 105°) **(B)** (105°, 15°)
(C) (45°, 75°) **(D)** (75°, 45°)

[JEE MAIN 2015 (ONLINE SET-1)]

Solution:

$$\frac{a}{b} = 2 + \sqrt{3}, \ \angle C = 60°$$

$$\frac{a}{b} = \frac{\sin A}{\sin B} = (2 + \sqrt{3}) = \tan 75° = \frac{\sin 75°}{\cos 75°}$$

Now,

$$\frac{\sin A}{\sin B} = \frac{\sin(90° + 15°)}{\sin(90° - 15°)} = \frac{\sin(105°)}{\sin(75°)}$$

Also $\angle C = 60° \Rightarrow \angle A + \angle B = 120° \Rightarrow \angle A = 105°, \ \angle B = 15°$

Hence, the correct answer is option (B).

8. A man is walking towards a vertical pillar in a straight path, at a uniform speed. At a certain point A on the path, he observes that the angle of elevation of the top of the pillar is 30°. After walking for 10 min from A in the same direction, at a point B, he observes that the angle of elevation of the top of the pillar is now 60°. Then, the time taken (in minutes) by him from B to reach the pillar is

(A) 5 **(B)** 6
(C) 10 **(D)** 20

[JEE MAIN 2016 (OFFLINE)]

Solution:

The given situation is depicted in Fig. 2.26. We have

$$AB = x; BP = y; PQ = h$$

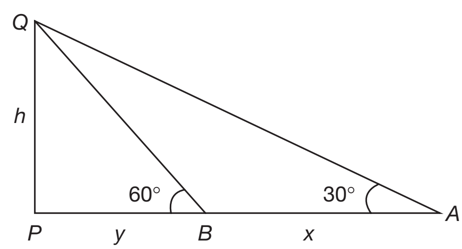


Figure 2.26

$$\tan 30° = \frac{h}{x + y} \Rightarrow \frac{1}{\sqrt{3}} = \frac{h}{x + y} \Rightarrow x + y = h\sqrt{3}$$

$$\tan 60° = \frac{h}{y} \Rightarrow h = y\sqrt{3}$$

Now,

$$x + y = \sqrt{3}y \times \sqrt{3} = 3y \Rightarrow x = 2y$$

Let the speed of man be u. Therefore,

$$\left.\begin{aligned} u &= \frac{x}{10} \Rightarrow 10 = \frac{x}{4} \\ u &= \frac{y}{t} \Rightarrow t = \frac{y}{4} \end{aligned}\right] \Rightarrow \frac{10}{t} = \frac{x}{y} = \frac{2y}{y} \Rightarrow t = 5 \text{ min}$$

Hence, the correct answer is option (A).

9. The angle of elevation of the top of a vertical tower from a point A due east of it is 45°. The angle of elevation of the top of the same tower from a point B due south of A is 30°. If the distance between A and B is $54\sqrt{2}$ m, then the height of the tower (in metres), is

(A) 108 **(B)** $36\sqrt{3}$
(C) $54\sqrt{3}$ **(D)** 54

[JEE MAIN 2016 (ONLINE SET-2)]

Solution:

Let the height of tower be h (see Fig. 2.27). Therefore,

$$OA = OC = h$$

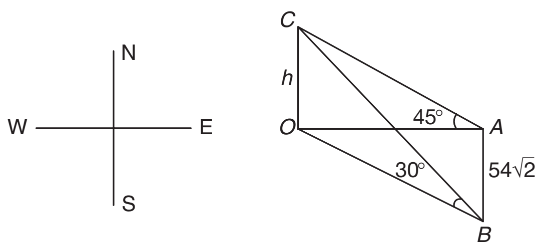


Figure 2.27

From ΔOBC, we get

$$\tan 30° = \frac{h}{OB} \Rightarrow OB = \sqrt{3}h$$

From ΔOAB, we get

$$OB^2 = OA^2 + AB^2$$
$$\Rightarrow 3h^2 = h^2 + (54^2 \times 2)$$
$$\Rightarrow 2h^2 = 54^2 \times 2$$
$$\Rightarrow h = 54$$

Hence, the correct answer is option (D).

Previous Years' Solved JEE Advanced/ IIT-JEE Questions

No questions appeared in JEE Advanced/IIT-JEE in the last 10 years (2007–2016) from this chapter.

Practice Exercise 1

1. If $\tan x = n \cdot \tan y$, $n \in R^+$, then maximum value of $\sec^2(x - y)$ is equal to

(A) $\frac{(n+1)^2}{2n}$ **(B)** $\frac{(n+1)^2}{n}$

(C) $\frac{(n+1)^2}{2}$ **(D)** $\frac{(n+1)^2}{4n}$

2. If $3\sin\theta + 5\cos\theta = 5$, then the value of $5\sin\theta - 3\cos\theta$ is equal to

(A) 5 **(B)** 3
(C) 4 **(D)** None of these

3. In ΔABC, if $\cot A \cdot \cot B \cdot \cot C > 0$, then the triangle is

(A) Acute angled **(B)** Right angled
(C) Obtuse angled **(D)** Does not exist

4. If $\pi < 2\theta < \frac{3\pi}{2}$, then $\sqrt{2+\sqrt{2+2\cos 4\theta}}$ equals

(A) $-2\cos\theta$ **(B)** $-2\sin\theta$
(C) $2\cos\theta$ **(D)** $2\sin\theta$

5. If $\tan\theta = \sqrt{n}$ for some non-square natural number n, then $\sec 2\theta$ is

(A) A rational number **(B)** An irrational number
(C) A positive number **(D)** None of these

(Non-square number is a number which is not a perfect square)

6. The minimum value of $\cos(\cos x)$ is

(A) 0 **(B)** $-\cos 1$
(C) $\cos 1$ **(D)** -1

7. If $\sin x \cos y = 1/4$ and $3 \tan x = 4 \tan y$, then find the value of $\sin(x + y)$.

(A) 1/16 **(B)** 7/16
(C) 5/16 **(D)** None of these

8. The maximum value of $4\sin^2 x + 3\cos^2 x + \sin\frac{x}{2} + \cos\frac{x}{2}$ is

(A) $4+\sqrt{2}$ **(B)** $3+\sqrt{2}$
(C) 9 **(D)** 4

9. If α and β are solutions of $\sin^2 x + a \sin x + b = 0$ as well as that of $\cos^2 x + c \cos x + d = 0$, then $\sin(\alpha + \beta)$ is equal to

(A) $\frac{2bd}{b^2+d^2}$ **(B)** $\frac{a^2+c^2}{2ac}$

(C) $\frac{b^2+d^2}{2bd}$ **(D)** $\frac{2ac}{a^2+c^2}$

10. If $\sin\alpha$, $\sin\beta$ and $\cos\alpha$ are in GP, then roots of the equation $x^2 + 2x \cot\beta + 1 = 0$ are always

(A) Equal **(B)** Real
(C) Imaginary **(D)** Greater than 1

11. If $S = \cos^2\frac{\pi}{n} + \cos^2\frac{2\pi}{n} + \cdots + \cos^2\frac{(n-1)\pi}{n}$, then S equals

(A) $\frac{n}{2}(n+1)$ **(B)** $\frac{1}{2}(n-1)$

(C) $\frac{1}{2}(n-2)$ **(D)** $\frac{n}{2}$

12. If in a ΔABC, $\angle C = 90°$, then the maximum value of $\sin A \sin B$ is

(A) $\frac{1}{2}$ **(B)** 1
(C) 2 **(D)** None of these

13. If in a ΔABC, $\sin^2 A + \sin^2 B + \sin^2 C = 2$, then the triangle is always

(A) Isosceles triangle **(B)** Right angled
(C) Acute angled **(D)** Obtuse angled

14. Maximum value of the expression $2\sin x + 4\cos x + 3$ is

(A) $2\sqrt{5}+3$ **(B)** $2\sqrt{5}-3$
(C) $\sqrt{5}+3$ **(D)** None of these

15. If $\cos\theta = \dfrac{\cos\alpha - \cos\beta}{1-\cos\alpha\cos\beta}$, then one of the values of $\tan\dfrac{\theta}{2}$ is

(A) $\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}$ **(B)** $\cot\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}$

(C) $\sin\dfrac{\alpha}{2}\sin\dfrac{\beta}{2}$ **(D)** None of these

16. If $0<\theta<\dfrac{\pi}{4}$ then $\sec 2\theta - \tan 2\theta$ is equal to

(A) $\tan\left(\dfrac{\pi}{4}+\theta\right)$ **(B)** $-\tan\left(\dfrac{\pi}{4}-\theta\right)$

(C) $\tan\left(\dfrac{\pi}{4}-\theta\right)$ **(D)** None of these

17. The equation $(\cos p - 1)x^2 + (\cos p)x + \sin p = 0$, where x is a variable, has real roots. Then the interval of p may be any one of the following:

(A) $(0, 2\pi)$ **(B)** $(-\pi, 0)$

(C) $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ **(D)** $(0, \pi)$

18. Let n be a positive integer such that $\sin\dfrac{\pi}{2^n}+\cos\dfrac{\pi}{2^n}=\dfrac{\sqrt{n}}{2}$ then

(A) $6 \le n \le 8$ **(B)** $4 \le n \le 8$
(C) $4 < n \le 8$ **(D)** $4 < n < 8$

19. If $\dfrac{x}{\cos\theta}=\dfrac{y}{\cos\left(\theta+\dfrac{2\pi}{3}\right)}=\dfrac{z}{\cos\left(\theta-\dfrac{2\pi}{3}\right)}$, then $x+y+z$ is equal to

(A) -1 **(B)** 1

(C) 0 **(D)** None of these

20. If $A+B+C=180°$, then the value of $\tan A + \tan B + \tan C$ is

(A) $\ge 3\sqrt{3}$ **(B)** $\ge 2\sqrt{3}$

(C) $> 3\sqrt{3}$ **(D)** $> 2\sqrt{3}$

21. Let $0 < A, B < \dfrac{\pi}{2}$ satisfying the equalities $3\sin^2 A + 2\sin^2 B = 1$ and $3\sin 2A - 2\sin 2B = 0$. Then $A + 2B =$

(A) $\dfrac{\pi}{4}$ **(B)** $\dfrac{\pi}{3}$

(C) $\dfrac{\pi}{2}$ **(D)** None of these

22. If $a\cos^3\theta + 3a\cos\theta\sin^2\theta = x$ and $a\sin^3\theta + 3a\cos^2\theta\sin\theta = y$, then $(x+y)^{2/3}+(x-y)^{2/3}=$
(A) $2a^{2/3}$ **(B)** $a^{2/3}$
(C) $3a^{2/3}$ **(D)** $2a^{1/3}$

23. If $(1+\sqrt{1+a})\tan\alpha = (1+\sqrt{1-a})$, then $\sin 4\alpha =$

(A) $a/2$ **(B)** a
(C) $a^{2/3}$ **(D)** $2a$

24. If $\cos^2\theta = \dfrac{1}{3}(a^2-1)$ and $\tan^2\dfrac{\theta}{2}=\tan^{2/3}\alpha$, then $\sin^{2/3}\alpha + \cos^{2/3}\alpha =$

(A) $2a^{2/3}$ **(B)** $\left(\dfrac{2}{a}\right)^{2/3}$

(C) $\left(\dfrac{2}{a}\right)^{1/3}$ **(D)** $2a^{1/3}$

25. The value of $\sum\limits_{r=1}^{5}\cos(2r-1)\dfrac{\pi}{11}$ is

(A) $\dfrac{1}{2}$ **(B)** $\dfrac{1}{3}$

(C) $\dfrac{1}{4}$ **(D)** $\dfrac{1}{6}$

26. $\sin nx = \sum\limits_{r=0}^{n} a_r \sin^r x$, where n is an odd natural number. Then

(A) $a_0 = 1, a_1 = 2n$ **(B)** $a_0 = 1, a_1 = n$

(C) $a_0 = 0, a_1 = n$ **(D)** $a_0 = 0, a_1 = -n$

27. $\tan\dfrac{5\pi}{12}-\tan\dfrac{\pi}{12}-\sqrt{3}\tan\dfrac{5\pi}{12}\tan\dfrac{\pi}{12}$ is equal to

(A) $-\sqrt{3}$ **(B)** $\dfrac{1}{\sqrt{3}}$

(C) 1 **(D)** $\sqrt{3}$

28. The maximum value of $27^{\cos 2x}81^{\sin 2x}$ is

(A) 3^2 **(B)** 3^5

(C) 3^7 **(D)** 3

29. If $\cos\theta + \sin\theta = a$, $\cos 2\theta = b$, then

(A) $a^2 = b^2(2-a^2)$ **(B)** $b^2 = a^2(2-a^2)$

(C) $a^2 = b^2(2-b^2)$ **(D)** $b^2 = a^2(2-b^2)$

30. If angle θ is divided into two parts A and B such that $A - B = x$ and $\tan A : \tan B = k:1$, then the value of $\sin x$ is

(A) $\dfrac{k-1}{k+1}\sin\theta$ **(B)** $\dfrac{k+1}{k}\sin\theta$

(C) $\dfrac{k+1}{k-1}\sin\theta$ **(D)** None of these

31. If α and β are solutions of $\sin^2 x + a\sin x + b = 0$ as well as of $\cos^2 x + c\cos x + d = 0$, then $\sin(\alpha+\beta)$ is equal to

(A) $\dfrac{2bd}{b^2+d^2}$ **(B)** $\dfrac{a^2+c^2}{2ac}$

(C) $\dfrac{b^2+d^2}{2bd}$ **(D)** $\dfrac{2ac}{a^2+c^2}$

32. If $\sin\theta$, $\cos\theta$ and $\tan\theta$ are in GP, then $\cot^6\theta - \cot^2\theta$ is equal to

(A) $\text{cosec}^2\theta$ **(B)** $\text{cosec}\,\theta$
(C) 1 **(D)** 0

33. If $3\cos x = 2\cos(x-2y)$, then $\tan(x-y)\tan y$ is equal to
(A) 5 **(B)** 6

(C) $\dfrac{1}{5}$ **(D)** $\dfrac{1}{6}$

34. If $A > 0$, $B > 0$ and $A+B=\frac{\pi}{3}$, then the maximum value of $\tan A \tan B$ is

(A) $\frac{1}{3}$ **(B)** 1

(C) $\sqrt{3}$ **(D)** $\frac{1}{\sqrt{3}}$

35. If in ΔABC, $\cos B + \cos C = 2 - 2\cos A$, then

(A) a, b, c are in AP **(B)** b, a, c are in AP

(C) c, a, b are in GP **(D)** c, b, a are in AP

36. The value of $\cos^2\frac{\pi}{9}+\cos^2\frac{2\pi}{9}+\cos^2\frac{4\pi}{9}$ is

(A) 0 **(B)** 3

(C) 9 **(D)** None of these

37. If $\frac{2\sin\alpha}{1+\cos\alpha+\sin\alpha}=y$, then $\frac{1-\cos\alpha+\sin\alpha}{1+\sin\alpha}$ is

(A) $\frac{1}{y}$ **(B)** y

(C) $1-y$ **(D)** $1+y$

38. If $4n\alpha=\pi$, then $\cot\alpha\cot 2\alpha\cot 3\alpha\cdots\cot(2n-1)\alpha$ is equal to

(A) 1 **(B)** –1

(C) ∞ **(D)** None of these

39. If $\sin\alpha+\sin\beta=a$ and $\cos\alpha-\cos\beta=b$, then $\tan\left(\frac{\alpha-\beta}{2}\right)$ is equal to

(A) $-\frac{a}{b}$ **(B)** $-\frac{b}{a}$

(C) $\sqrt{a^2+b^2}$ **(D)** None of these

40. If $\tan\theta=a\neq 0$, $\tan 2\theta=b\neq 0$ and $\tan\theta+\tan 2\theta=\tan 3\theta$, then

(A) $a=b$ **(B)** $ab=1$

(C) $a+b=0$ **(D)** $b=2a$

41. The greatest real number among sin1, sin2, sin3 and $\sin(\sqrt{10}-2)$ is

(A) sin 1 **(B)** sin 2

(C) sin 3 **(D)** $\sin(\sqrt{10}-2)$

42. If $\sin x+\sin^2 x=1$, then the value of $\cos^2 x+\cos^4 x+\cot^4 x-\cot^2 x$ is equal to

(A) 1 **(B)** 0

(C) 2 **(D)** None of these

43. $\sin\frac{\pi}{9}\sin\frac{2\pi}{9}\sin\frac{4\pi}{9}\sin\frac{\pi}{3}$ is equal to

(A) $\frac{3}{4}$ **(B)** $\frac{3}{8}$

(C) $\frac{3}{16}$ **(D)** None of these

44. If x, y, z are in AP, then $\frac{\sin x-\sin z}{\cos z-\cos x}$ is equal to

(A) $\tan y$ **(B)** $\cot y$

(C) $\sin y$ **(D)** $\cos y$

45. If $|\sin x+\cos x|=|\sin x|+|\cos x|$, then x belongs to the quadrant

(A) I or III **(B)** II or IV

(C) I or II **(D)** III or IV

46. Find the value of x if $x=\frac{1-4\sin 10^\circ\sin 70^\circ}{2\sin 10^\circ}$.

(A) 1 **(B)** 2

(C) 3 **(D)** None of these

47. Given $\frac{\pi}{2}<\alpha<\pi$, then the expression $\sqrt{\frac{1-\sin\alpha}{1+\sin\alpha}}+\sqrt{\frac{1+\sin\alpha}{1-\sin\alpha}}$ is equal to

(A) $\frac{1}{\cos\alpha}$ **(B)** $-\frac{2}{\cos\alpha}$

(C) $\frac{2}{\cos\alpha}$ **(D)** None of these

48. If $\cot\alpha\cot\beta=2$, then $\frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)}$ is equal to

(A) $\frac{1}{3}$ **(B)** $-\frac{1}{3}$

(C) $\frac{1}{2}$ **(D)** $-\frac{1}{2}$

49. If $\cos(x-y)$, $\cos x$ and $\cos(x+y)$ are in HP, then $\cos x\cdot\sec\frac{y}{2}$ is equal to

(A) $\sqrt{2}$ **(B)** $-\sqrt{2}$

(C) $\pm\sqrt{2}$ **(D)** None of these

50. If $\cot\theta=\frac{\sqrt{m}+\sqrt{n}}{\sqrt{m}-\sqrt{n}}$, then $(m+n)\cos 2\theta$ is equal to

(A) $2\sqrt{mn}$ **(B)** $\frac{m+n}{m-n}$

(C) $(\sqrt{m}+\sqrt{n})^2$ **(D)** None of these

51. If $\cos^2 x+\cos^4 x=1$, then the value of $\tan^4 x+\cot^4 x+\tan^2 x-\cot^2 x$ is equal to

(A) 0 **(B)** 2

(C) 1 **(D)** None of these

52. If $\cos 25^\circ+\sin 25^\circ=k$, then $\cos 20^\circ$ is equal to

(A) $\frac{k}{\sqrt{2}}$ **(B)** $-\frac{k}{\sqrt{2}}$

(C) $\pm\frac{k}{\sqrt{2}}$ **(D)** None of these

53. If $\tan\theta=n\tan\phi$, then the maximum value of $\tan^2(\theta-\phi)$ is equal to

(A) $\frac{(n-1)^2}{4n}$ **(B)** $\frac{(n+1)^2}{4n}$

(C) $\frac{(n+1)}{2n}$ **(D)** $\frac{(n-1)}{2n}$

54. If $a\le 16\sin x\cos x+12\cos^2 x-6\le b$ for all $x\in R$, then

(A) $a=-5, b=5$ **(B)** $a=-4, b=4$

(C) $a=-10, b=10$ **(D)** None of these

55. Let $f(\theta)=\frac{\cot\theta}{1+\cot\theta}$ and $\alpha+\beta=\frac{5\pi}{4}$. Then the value of $f(\alpha)\cdot f(\beta)$ is

(A) 2 **(B)** $-\frac{1}{2}$

(C) $\frac{1}{2}$ **(D)** None of these

56. If $\cos\alpha = \dfrac{2\cos\beta - 1}{2 - \cos\beta}$ $(0<\alpha<\pi,\ 0<\beta<\pi)$, then $\tan\dfrac{\alpha}{2}\cot\dfrac{\beta}{2}$ is equal to

(A) 1 **(B)** $\sqrt{2}$
(C) $\sqrt{3}$ **(D)** None of these

57. The value(s) of y for which the equation $4\sin x + 3\cos x = y^2 - 6y + 14$ has a real solution, is (are)

(A) 3 **(B)** 5
(C) –3 **(D)** None of these

58. Prove that $\sum_{\alpha,\beta,\gamma} \sin\beta\ \sin\gamma\ \cos^2\alpha\ \sin(\beta-\gamma) = -\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta)$.

59. If $\sin^2\phi = \dfrac{\cos 2\alpha\cos 2\beta}{\cos^2(\alpha+\beta)}$, prove that $\tan^2\dfrac{\phi}{2}$ is equal to either $\dfrac{\tan(\pi/4+\beta)}{\tan(\pi/4+\alpha)}$ or $\dfrac{\tan(\pi/4+\alpha)}{\tan(\pi/4+\beta)}$.

60. Find the sum of n terms $S_n = \tan x\tan 2x + \tan 2x\tan 3x + \cdots + \tan x\tan(n+1)x$.

61. Let A, B, C be three angles such that $A = \pi/4$ and $\tan B\tan C = p$. Find all possible values of p such that A, B, C are the angles of a triangle.

62. Let $A_1, A_2, A_3, \ldots, A_n$ be the vertices of an n-sided polygon such that $\dfrac{1}{A_1A_2} = \dfrac{1}{A_1A_3} + \dfrac{1}{A_1A_4}$. Find the value of n.

63. If $m\tan(\theta-30^\circ) = n\tan(\theta+120^\circ)$ show that the value of $\cos 2\theta$ in terms of m and n is $\dfrac{(m+n)}{2(m-n)}$.

Practice Exercise 2

Single/Multiple Correct Choice Type Questions

1. The number of solution of equation $8[x^2 - x] + 4[x] = 13 + 12[\sin x]$ is

(A) 0 **(B)** 2
(C) 4 **(D)** 6

2. Let $k = 1^\circ$. Then $2\sin 2k + 4\sin 4k + 6\sin 6k + \cdots + 180\sin 180k$ is equal to

(A) $90\cos k$ **(B)** $90\tan 89^\circ$
(C) $90\tan k$ **(D)** $90\cot 89^\circ$

3. If $(\sqrt{2}\cos x + \sqrt{2}\sin x + \sqrt{7})m = 1$ holds, then

(A) Greatest negative integral value of m is –1
(B) Least positive integral value of m is 5
(C) No such m exists
(D) $m \in [-7, -1) \cup (1, \infty)$

4. The equation $2x = (2n+1)\pi(1 - \cos x)$, where n is a positive integer, has

(A) Infinitely many real roots **(B)** Exactly one real root
(C) Exactly $2n+2$ real roots **(D)** Exactly $2n+3$ real roots

5. For any real θ, the maximum value of $\cos^2(\cos\theta) + \sin^2(\sin\theta)$

(A) Is 1 **(B)** Is $1 + \sin^2 1$
(C) Is $1 + \cos^2 1$ **(D)** Does not exist

6. Number of solutions of the equation $[y + [y]] = 2\cos x$ is, where $y = \dfrac{1}{3}[\sin x + [\sin x + [\sin x]]]$ and [.] denotes the greatest integer function

(A) 1 **(B)** 2
(C) 3 **(D)** None of these

7. If x and α are real, then the inequation $\log_2^x + \log_x^2 + 2\cos\alpha \le 0$

(A) Has no solution
(B) Has exactly two solutions
(C) Is satisfied for any real α and any real x in (0, 1)
(D) Is satisfied for any real α and any real x in $(1, \infty)$

8. If $y = \dfrac{\sin^4 x - \cos^4 x + \sin^2 x\cos^2 x}{\sin^4 x + \cos^4 x + \sin^2 x\cos^2 x}, x \in \left(0, \dfrac{\pi}{2}\right)$, then

(A) $-\dfrac{3}{2} \le y \le \dfrac{1}{2}$ **(B)** $1 \le y \le \dfrac{1}{2}$
(C) $-\dfrac{5}{3} \le y \le 1$ **(D)** None of these

9. If the value of expression $\dfrac{\operatorname{cosec}^4\theta - 3\cot^2\theta}{\operatorname{cosec}^4\theta - d\cot^2\theta}$ lies between $\dfrac{1}{3}$ and 3, then

(A) $d \in [-2, 2]$ **(B)** $d = -1$
(C) $d = 2$ **(D)** None of these

10. The number of solution of the equation $\log_{|x-1|}|x^2 - 1| = [|\sin x| + |\cos x|]$ (where [.] denotes the greatest integer function) is

(A) 3 **(B)** 2
(C) 1 **(D)** None of these

11. The minimum value of $\left(1 + \dfrac{1}{\sin^n\theta}\right)\left(1 + \dfrac{1}{\cos^n\theta}\right)$ is

(A) 1 **(B)** 4
(C) $(1 + 2^{n/2})^2$ **(D)** None of these

12. If A, B, C, D are the successive values of x satisfying the equation $\sin x = k$, $0 < k < 1$ where $A < B < C < D$, then $A + B$, $B + C$, $C + D$, … is in

(A) AP **(B)** GP
(C) HP **(D)** None of these

13. The range of $y = \sin^3 x - 6\sin^2 x + 11\sin x - 6$ is

(A) [–24, 2] **(B)** [–24, 0]
(C) [0, 24] **(D)** None of these

14. For $0 < x < \dfrac{\pi}{2}$, $(1 + 4\operatorname{cosec} x)(1 + 8\sec x)$ is

(A) ≥ 81 **(B)** > 81
(C) ≥ 83 **(D)** > 83

15. If $[\sin x] + [\cos x] + 2 = 0$, then range of $f(x) = \sin x - \cos x + 3$ corresponding to the solution set of the given equation is (where [.] denotes the greatest integer function)

(A) [2, 4) **(B)** (2, 4)
(C) (2, 4] **(D)** None of these

16. $\sin x + \cos x = y^2 - y + a$ has no value of x for any y if 'a' belongs to

(A) $(0, \sqrt{3})$ **(B)** $(-\sqrt{3}, 0)$
(C) $(-\infty, -\sqrt{3})$ **(D)** $(\sqrt{3}, \infty)$

17. If $x\cos\theta = y\cos\left(\theta + \dfrac{2\pi}{3}\right) = z\cos\left(\theta + \dfrac{4\pi}{3}\right)$, then which of the following is (are) true?

(A) $xy + yz + zx = 0$

(B) $xyz = \dfrac{1}{4}\cos(3\theta)$

(C) $x^2+y^2+z^2=\frac{xy^2}{z}+\frac{yz^2}{x}+\frac{zx^2}{y}$

(D) $x^2+y^2+z^2=xy+yz+zx$

18. Let x, y, z be elements from interval $[0, 2\pi]$ satisfying the inequality $(4+\sin 4x)(2+\cot^2 y)(1+\sin^4 z)\le 12\sin^2 z$ then

(A) The number of ordered pairs (x, y) is 4.

(B) The number of ordered pairs (y, z) is 8.

(C) The number of ordered pairs (z, x) is 8.

(D) The number of pairs (y, z) such that $z=y$ is 2.

Matrix Match Type Questions

19. Match the following:

Column I	Column II
(A) The number of solution of $\frac{x}{2}+\frac{\sin x}{\cos x}=\frac{\pi}{4}\ln[-\pi,\pi]$	**(i)** 1
(B) The number of solution of equation $\sin^{-1}(\lvert x^2-1\rvert)+\cos^{-1}(\lvert 2x^2-5\rvert)=\frac{\pi}{2}$	**(ii)** 0
(C) The number of solution of $x^4-2x^2\sin^2\frac{\pi}{2}x+1=0$	**(iii)** 3
(D) The number of solution of $x^2+2x+2\sec^2\pi x+\tan^2\pi x=0$	**(iv)** 2

20. Match the following:

Column I	Column II
(A) If $y=\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}(b)$ $(0<b<1)$ and $0<y\le\frac{\pi}{4}$, then the maximum value	**(i)** 3
(B) The number of solutions of $\sin^4 x+\cos^3 x \ge 1$ in $(0, 2\pi)$ will be	**(ii)** $\frac{1}{3}$
(C) If in ΔABC, $C>\frac{3\pi}{4}$, then value of $(1+\tan A)(1+\tan B)$ will lie in interval	**(iii)** $[1, \sqrt{2}]$
(D) $\left\lvert\sqrt{\sin^2 x+2a^2}-\sqrt{2a^2-1-\cos^2 x}\right\rvert=k$, then k lies in the interval	**(iv)** $(1, 2)$

Integer Type Questions

21. If θ_1, θ_2, θ_3 are three values lying in $[0, 2\pi]$ for which $\tan\theta=\lambda$, then $\left\lvert\tan\frac{\theta_1}{3}\tan\frac{\theta_2}{3}+\tan\frac{\theta_2}{3}\tan\frac{\theta_3}{3}+\tan\frac{\theta_1}{3}\tan\frac{\theta_3}{3}\right\rvert$ is equal to ________.

22. The number of solutions that the equation $\sin[\cos(\sin x)]=\cos[\sin(\cos x)]$ has in $\left[0,\frac{\pi}{2}\right]$ is ________.

Answer Key

Practice Exercise 1

1. (D)	**2.** (B)	**3.** (A)	**4.** (D)	**5.** (A)	**6.** (C)
7. (B)	**8.** (A)	**9.** (D)	**10.** (B)	**11.** (C)	**12.** (A)
13. (B)	**14.** (A)	**15.** (A)	**16.** (C)	**17.** (D)	**18.** (C)
19. (C)	**20.** (A)	**21.** (C)	**22.** (A)	**23.** (B)	**24.** (B)
25. (A)	**26.** (C)	**27.** (D)	**28.** (B)	**29.** (B)	**30.** (A)
31. (D)	**32.** (C)	**33.** (C)	**34.** (A)	**35.** (B)	**36.** (D)
37. (B)	**38.** (A)	**39.** (B)	**40.** (C)	**41.** (D)	**42.** (C)
43. (C)	**44.** (B)	**45.** (A)	**46.** (A)	**47.** (A)	**48.** (A)
49. (C)	**50.** (A)	**51.** (B)	**52.** (A)	**53.** (A)	**54.** (C)
55. (C)	**56.** (C)	**57.** (A)			

Practice Exercise 2

1. (A)	**2.** (B)	**3.** (A)	**4.** (C)	**5.** (B)	**6.** (D)
7. (C)	**8.** (D)	**9.** (D)	**10.** (C)	**11.** (C)	**12.** (A)
13. (B)	**14.** (B)	**15.** (B)	**16.** (D)	**17.** (A), (B)	**18.** (C), (D)

19. (A) → (iii); (B) → (iv); (C) → (iv); (D) → (ii)
20. (A) → (ii); (B) → (i); (C) → (iv); (D) → (iii)
21. –3
22. 1

Solutions

Practice Exercise 1

1. $\cos(x-y) = \cos x \cos y + \sin x \sin y = \cos x \cos y\,(1 + \tan x \tan y)$
Put $\tan x = n \tan y$

$$\cos(x-y) = \cos x \cos y(1 + n\tan^2 y) \Rightarrow \sec(x-y) = \frac{\sec x \sec y}{(1+n\tan^2 y)}$$

$$\sec^2(x-y) = \frac{\sec^2 x \cdot \sec^2 y}{(1+n\tan^2 y)^2}$$
$$= \frac{(1+\tan^2 x)(1+\tan^2 y)}{(1+n\tan^2 y)^2}$$
$$= \frac{(1+n^2\tan^2 y)(1+\tan^2 y)}{(1+n\tan^2 y)^2}$$
$$= 1 + \frac{(n-1)^2\tan^2 y}{(1+n\tan^2 y)^2}$$

(By division) using inequality

$$\frac{(1+n^2\tan^2 y + n^2\tan^2 y + n^4\tan^4 y)}{4} \geq (n^4\tan^8 y)^{1/4}$$

Now,

$$\frac{(1+n\tan^2 y)^2}{4} \geq n\tan^2 y$$
$$\Rightarrow \frac{\tan^2 y}{(1+n\tan^2 y)^2} \leq \frac{1}{4n}$$
$$\Rightarrow \sec^2(x-y) \leq 1 + \frac{(n-1)^2}{4n} = \frac{(n+1)^2}{4n}$$

2. $3\sin\theta = 5(1-\cos\theta) = 5 \times 2\sin^2(\theta/2) \Rightarrow \tan(\theta/2) = 3/5$

$$5\sin\theta - 3\cos\theta = 5 \times \frac{2\tan(\theta/2)}{1+\tan^2(\theta/2)} - 3\frac{[1-\tan^2(\theta/2)]}{1+\tan^2(\theta/2)}$$
$$= 5 \times \frac{2\times(3/5)}{1+(9/25)} - \frac{3\times[1-(9/25)]}{1+(9/25)} = 3$$

3. Since $\cot A \cot B \cot C > 0$, $\cot A$, $\cot B$, $\cot C$ are positive. So, the triangle is acute angled.

4. $\sqrt{2+\sqrt{2(1+\cos 4\theta)}} = \sqrt{2+2|\cos 2\theta|} = \sqrt{2(1-\cos 2\theta)}$

$= 2|\sin\theta| = 2\sin\theta$ as $\frac{\pi}{2} < \theta < \frac{3\pi}{4}$

5. $\sec 2\theta = \frac{1+\tan^2\theta}{1-\tan^2\theta} = \frac{1+n}{1-n}$

where n is a non-square natural number so $1-n \neq 0$. Hence $\sec 2\theta$ is a rational number.

6. $\cos x$ lies between –1 to 1 for all real x.
If $f(x) = \cos(\cos x)$, then $f'(x) = 0$ when either $\sin x = 0$ or $\sin\cos x = 0$, that is, at $x = 0$ or $x = \pi/2$.
At $x = 0$ we get minimum value of $f(x) = \cos 1$

7. $3\tan x = 4\tan y \Rightarrow 3\sin x\cos y = 4\cos x \sin y$
$\Rightarrow 3/4 = 4\cos x\sin y \Rightarrow \cos x \sin y = 3/16$
Therefore,
$$\sin(x+y) = \sin x\cos y + \cos x \sin y = \frac{1}{4} + \frac{3}{16} = \frac{7}{16}$$

8. Maximum value of $4\sin^2 x + 3\cos^2 x$, that is, $\sin^2 x + 3$ is 4 and that of $\sin\frac{x}{2} + \cos\frac{x}{2}$ is $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$, both attained at $x = \pi/2$.
Hence, the given function has the maximum value $4+\sqrt{2}$.

9. According to the given condition, $\sin\alpha + \sin\beta = -a$ and $\cos\alpha + \cos\beta = -c$. So

$$2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = -a \text{ and } 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = -c$$
$$\Rightarrow \tan\frac{\alpha+\beta}{2} = \frac{a}{c}$$

Now,
$$\sin(\alpha+\beta) = \frac{2\tan\frac{\alpha+\beta}{2}}{1+\tan^2\frac{\alpha+\beta}{2}} = \frac{2ac}{a^2+c^2}$$

10. $\sin\alpha$, $\sin\beta$, $\cos\alpha$ are in GP. Therefore,
$\sin^2\beta = \sin\alpha\cos\alpha \Rightarrow \cos 2\beta = 1 - \sin 2\beta \geq 0$
Now, the discriminant of the given equation is
$4\cot^2\beta - 4 = 4\cos 2\beta \cdot \text{cosec}^2\beta \geq 0 \Rightarrow$ roots are always real

11. $S = \cos^2\frac{\pi}{n} + \cos^2\frac{2\pi}{n} + \cdots + \cos^2(n-1)\frac{\pi}{n}$

$$= \frac{1}{2}\left[1+\cos\frac{2\pi}{n} + 1 + \cos\frac{4\pi}{n} + 1 + \cos\frac{6\pi}{n} + \cdots + 1 + \cos 2(n-1)\frac{\pi}{n}\right]$$
$$= \frac{1}{2}\left[n-1+\sum_{k=1}^{n-1}\cos\frac{2k\pi}{n}\right] = \frac{1}{2}[n-1-1] = \frac{1}{2}(n-2)$$

12. $\sin A\sin B = \frac{1}{2}\times 2\sin A\sin B$

$$= \frac{1}{2}[\cos(A-B) - \cos(A+B)] = \frac{1}{2}[\cos(A-B) - \cos 90°]$$
$$= \frac{1}{2}\cos(A-B) \leq \frac{1}{2}$$

$\Rightarrow$ Maximum value of $\sin A \sin B = \frac{1}{2}$

13. $\sin^2 A + \sin^2 B + \sin^2 C = 2 \Rightarrow 2\cos A\cos B\cos C = 0$
$\Rightarrow$ Either $A = 90°$ or $B = 90°$ or $C = 90°$

14. Maximum value of $2\sin x + 4\cos x = 2\sqrt{5}$
Hence, the maximum value of $2\sin x + 4\cos x + 3$ is $2\sqrt{5}+3$

15. $$\tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta} = \frac{1-\dfrac{\cos\alpha-\cos\beta}{1-\cos\alpha\cos\beta}}{1+\dfrac{\cos\alpha-\cos\beta}{1-\cos\alpha\cos\beta}}$$

$$= \frac{1-\cos\alpha\cos\beta-\cos\alpha+\cos\beta}{1-\cos\alpha\cos\beta+\cos\alpha-\cos\beta}$$

$$= \frac{(1-\cos\alpha)+\cos\beta(1-\cos\alpha)}{(1+\cos\alpha)-\cos\beta(1+\cos\alpha)} = \frac{(1-\cos\alpha)(1+\cos\beta)}{(1+\cos\alpha)(1-\cos\beta)}$$

$$= \tan^2\frac{\alpha}{2}\cot^2\frac{\beta}{2}$$

Therefore, $\tan\frac{\theta}{2} = \pm\tan\frac{\alpha}{2}\cot\frac{\beta}{2}$.

16. $\sec 2\theta - \tan 2\theta = \dfrac{1-\sin 2\theta}{\cos 2\theta} = \dfrac{1-\dfrac{2\tan\theta}{1+\tan^2\theta}}{\dfrac{1-\tan^2\theta}{1+\tan^2\theta}} = \dfrac{(1-\tan\theta)^2}{1-\tan^2\theta}$

$$= \frac{1-\tan\theta}{1+\tan\theta} = \tan\left(\frac{\pi}{4}-\theta\right)$$

17. Discriminant of the given equation = $(\cos p)^2 - 4(\cos p - 1)\sin p = \cos^2 p + 4(1-\cos p)\sin p \geq 0$, if $p \in (0, \pi)$

[$\because \cos^2 p \geq 0, 0 \leq 1-\cos p \leq 2$ and $\sin p > 0$ for all $p \in (0,\pi)$]

18. $\sin\frac{\pi}{2^n}+\cos\frac{\pi}{2^n} = \sqrt{2}\sin\left(\frac{\pi}{4}+\frac{\pi}{2^n}\right)$ lies in $[-\sqrt{2},\sqrt{2}]$.

Therefore, $\frac{\sqrt{n}}{2} \in [-\sqrt{2},\sqrt{2}] \Rightarrow \frac{\sqrt{n}}{2} \leq \sqrt{2} \Rightarrow \sqrt{n} \leq 2\sqrt{2} \Rightarrow n \leq 8$

Note that $n = 1$ does not satisfy the given equation and for $n > 1$

$$\frac{\pi}{2} \geq \frac{\pi}{4}+\frac{\pi}{2^n} > \frac{\pi}{4} \Rightarrow \sin\left(\frac{\pi}{4}+\frac{\pi}{2^n}\right) > \sin\frac{\pi}{4}$$

$$\Rightarrow \sqrt{2}\sin\left(\frac{\pi}{4}+\frac{\pi}{2^n}\right) > 1 \Rightarrow \frac{\sqrt{n}}{2} > 1 \Rightarrow n > 4$$

Hence, $4 < n \leq 8$.

19. Given $\dfrac{x}{\cos\theta} = \dfrac{y}{\cos\left(\theta+\frac{2\pi}{3}\right)} = \dfrac{z}{\cos\left(\theta-\frac{2\pi}{3}\right)} = \lambda$ (say)

$$\Rightarrow x+y+z = \lambda\left[\cos\theta+\cos\left(\theta+\frac{2\pi}{3}\right)+\cos\left(\theta-\frac{2\pi}{3}\right)\right]$$

$$= \lambda\left\{\cos\theta + 2\cos\theta\cos\frac{2\pi}{3}\right\} = 0$$

20. $\tan(A+B) = \tan(180° - C)$

$$\Rightarrow \frac{\tan A+\tan B}{1-\tan A\tan B} = -\tan C$$

$\Rightarrow \tan A + \tan B + \tan C = \tan A\tan B\tan C$

$\Rightarrow \dfrac{\tan A+\tan B+\tan C}{3} \geq \sqrt[3]{\tan A\tan B\tan C}$ [$\because$ A.M. ≥ G.M.]

$\Rightarrow \tan A\tan B\tan C \geq 3\sqrt[3]{\tan A\tan B\tan C}$

$\Rightarrow \tan^2 A\tan^2 B\tan^2 C \geq 27$ [cubing both sides]

$\Rightarrow \tan A\tan B\tan C \geq 3\sqrt{3}$

$\Rightarrow \tan A + \tan B + \tan C \geq 3\sqrt{3}$

21. From the second equation, we have

$$\sin 2B = \frac{3}{2}\sin 2A \quad (1)$$

and from the first equality

$$3\sin^2 A = 1 - 2\sin^2 B = \cos 2B \quad (2)$$

Now $\cos(A+2B) = \cos A\cdot\cos 2B - \sin A\cdot\sin 2B$

$$= 3\cos A\cdot\sin^2 A - \frac{3}{2}\cdot\sin A\cdot\sin 2A$$

$$= 3\cos A\cdot\sin^2 A - 3\sin^2 A\cdot\cos A = 0$$

$$\Rightarrow A+2B = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$

Given that $0 < A < \frac{\pi}{2}$ and $0 < B < \frac{\pi}{2} \Rightarrow 0 < A+2B < \pi+\frac{\pi}{2}$

Hence, $A+2B = \frac{\pi}{2}$.

22. $a\cos^3\theta + 3a\cos\theta\sin^2\theta = x$

$a\sin^3\theta + 3a\cos^2\theta\sin\theta = y$

$x+y = a[\sin^3\theta+\cos^3\theta+3\sin\theta\cos\theta(\sin\theta+\cos\theta)]$

$= a(\sin\theta+\cos\theta)^3$

$$\left(\frac{x+y}{a}\right)^{1/3} = \sin\theta+\cos\theta \quad (1)$$

$x-y = a[\cos^3\theta - \sin^3\theta + 3\cos\theta\sin^2\theta - 3\cos^2\theta\sin\theta]$

$= a[\cos\theta-\sin\theta]^3$

$$\left(\frac{x-y}{a}\right)^{1/3} = \cos\theta-\sin\theta \quad (2)$$

$$(\sin\theta+\cos\theta)^2 + (\cos\theta-\sin\theta)^2 = \frac{(x+y)^{2/3}+(x-y)^{2/3}}{a^{2/3}}$$

$$2(\sin^2\theta+\cos^2\theta) = \frac{(x+y)^{2/3}+(x-y)^{2/3}}{a^{2/3}}$$

$(x+y)^{2/3} + (x-y)^{2/3} = 2a^{2/3}$

23. Let $a = \sin 4\theta$. Then

$$\sqrt{1+a} = \cos 2\theta + \sin 2\theta$$

and $\sqrt{1-a} = \cos 2\theta - \sin 2\theta$ $(1+\sqrt{1+a})\tan\alpha = (1+\sqrt{1-a})$

$\Rightarrow (1+\cos 2\theta+\sin 2\theta)\tan\alpha = 1+\cos 2\theta - \sin 2\theta$

$$\Rightarrow \frac{2\cos\theta(\cos\theta+\sin\theta)}{2\cos\theta(\cos\theta-\sin\theta)} = \cot\alpha$$

$$\Rightarrow \frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta} = \cot\alpha \Rightarrow \frac{1+\tan\theta}{1-\tan\theta} = \cot\alpha$$

$$\Rightarrow \tan\left(\frac{\pi}{4}+\theta\right) = \tan\left(\frac{\pi}{2}-\alpha\right) \Rightarrow \theta = \left(\frac{\pi}{4}-\alpha\right)$$

So $a = \sin 4\theta = \sin(\pi - 4\alpha) = \sin 4\alpha$

24. $\cos^2\theta = \dfrac{a^2-1}{3}$, $\tan^2\dfrac{\theta}{2} = \tan^{2/3}\alpha$

Now,

$\tan^3\dfrac{\theta}{2} = \tan\alpha \Rightarrow \dfrac{\sin^3(\theta/2)}{\cos^3(\theta/2)} = \dfrac{\sin\alpha}{\cos\alpha}$

Let $\dfrac{\sin^3(\theta/2)}{\sin\alpha} = \dfrac{\cos^3(\theta/2)}{\cos\alpha} = k$. Then

$$\sin^3\frac{\theta}{2} = k\sin\alpha \quad (1)$$

$$\cos^3\frac{\theta}{2} = k\cos\alpha \quad (2)$$

Now,

$$k^{2/3}\sin^{2/3}\alpha + k^{2/3}\cos^{2/3}\alpha = 1$$

$$\Rightarrow \sin^{2/3}\alpha + \cos^{2/3}\alpha = \frac{1}{k^{2/3}}$$

Squaring and adding Eqs. (1) and (2), we get

$$k^2(\sin^2\alpha + \cos^2\alpha) = \sin^6\frac{\theta}{2} + \cos^6\frac{\theta}{2}$$

$$= \left(\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\right)^3 - 3\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}\left(\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\right)$$

$$\Rightarrow k^2 = 1 - \frac{3}{4}\sin^2\theta = 1 - \frac{3}{4} + \frac{3}{4}\cos^2\theta$$

$$\Rightarrow k^2 = \frac{a^2}{4} \Rightarrow k = \frac{a}{2}$$

Therefore,

$$\sin^{2/3}\alpha + \cos^{2/3}\alpha = \left(\frac{2}{a}\right)^{2/3}$$

25. $\displaystyle\sum_{r=1}^{5}\cos(2r-1)\frac{\pi}{11}$

$$= \cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \cos\frac{5\pi}{11} + \cos\frac{7\pi}{11} + \cos\frac{9\pi}{11}$$

$$= \frac{2\sin(\pi/11)}{2\sin(\pi/11)}\left(\cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \cos\frac{5\pi}{11} + \cos\frac{7\pi}{11} + \cos\frac{9\pi}{11}\right)$$

$$= \frac{\left(\sin\frac{2\pi}{11} + \sin\frac{4\pi}{11} - \sin\frac{2\pi}{11} + \sin\frac{6\pi}{11} - \sin\frac{4\pi}{11} + \sin\frac{8\pi}{11} - \sin\frac{6\pi}{11} + \sin\frac{10\pi}{11} - \sin\frac{8\pi}{11}\right)}{2\sin\frac{\pi}{11}}$$

$$= \frac{\sin\frac{10\pi}{11}}{2\sin\frac{\pi}{11}} = \frac{\sin\left(\pi - \frac{\pi}{11}\right)}{2\sin\frac{\pi}{11}} = \frac{1}{2}$$

26. $\sin nx = \operatorname{Im}(e^{inx}) = \operatorname{Im}[(\cos x + i\sin x)^n]$

$= {}^nC_1\cos^{n-1}x\cdot\sin x - {}^nC_3\cos^{n-3}\sin^3 x + {}^nC_5\cos^{n-5}x\cdot\sin^5 x + \cdots$

Since n is odd, let $n = 2\lambda + 1$. Then

$\sin nx = {}^nC_1(\cos^2 x)^{\lambda}\sin x - {}^nC_3(\cos^2 x)^{\lambda-1}\sin^3 x + \cdots$

$= {}^nC_1(1-\sin^2 x)^{\lambda}\sin x - {}^nC_3(1-\sin^2 x)^{\lambda-1}\cdot\sin^3 x + {}^nC_5(1-\sin^2 x)^{\lambda-2}\cdot\sin^5 x + \cdots$

$= {}^nC_1\sin x - ({}^nC_1\cdot{}^{\lambda}C_1 + {}^nC_3)\sin^3 x + \cdots$

$\Rightarrow a_0 = 0, a_1 = n$

27. We have $\tan\left(\dfrac{5\pi}{12} - \dfrac{\pi}{12}\right) = \dfrac{\tan\frac{5\pi}{12} - \tan\frac{\pi}{12}}{1 + \tan\frac{5\pi}{12}\cdot\tan\frac{\pi}{12}}$

$$\Rightarrow \tan\frac{\pi}{3} = \frac{\tan\frac{5\pi}{12} - \tan\frac{\pi}{12}}{1 + \tan\frac{5\pi}{12}\cdot\tan\frac{\pi}{12}}$$

$$\Rightarrow \tan\frac{5\pi}{12} - \tan\frac{\pi}{12} - \sqrt{3}\tan\frac{5\pi}{12}\cdot\tan\frac{\pi}{12} = \sqrt{3}$$

28. $27^{\cos 2x}\cdot 81^{\sin 2x} = 3^{3\cos 2x + 4\sin 2x}$

Maximum value of $3\cos 2x + 4\sin 2x = \sqrt{3^2 + 4^2} = 5$

Therefore, maximum value of $27^{\cos 2x}\cdot 81^{\sin 2x} = 3^5$

29. Given $\cos\theta + \sin\theta = a$

$\Rightarrow 1 + \sin 2\theta = a^2$ [squaring both sides]

$\Rightarrow \sin 2\theta = a^2 - 1$

$\Rightarrow \cos^2 2\theta = 1 - (a^2 - 1)^2$

$\Rightarrow b^2 = a^2(2 - a^2)$

30. Given $\dfrac{\tan A}{\tan B} = \dfrac{k}{1}$

$\Rightarrow \dfrac{\sin A}{\cos A}\cdot\dfrac{\cos B}{\sin B} = \dfrac{k}{1}$

$\Rightarrow \dfrac{\sin(A-B)}{\sin(A+B)} = \dfrac{k-1}{k+1}$ [By Componendo and Dividendo]

$\Rightarrow \sin x = \dfrac{k-1}{k+1}\sin\theta$ [$\because A - B = x$ and $A + B = \theta$]

31. Given $\sin\alpha + \sin\beta = -a$, $\sin\alpha\sin\beta = b$

$\cos\alpha + \cos\beta = -c$, $\cos\alpha\cos\beta = d$

So, $2\sin\dfrac{\alpha+\beta}{2}\cdot\cos\dfrac{\alpha-\beta}{2} = -a$

and $2\cos\dfrac{\alpha+\beta}{2}\cdot\cos\dfrac{\alpha-\beta}{2} = -c$

Therefore, $\tan\dfrac{\alpha+\beta}{2} = \dfrac{a}{c}$

$\Rightarrow \sin(\alpha+\beta) = \dfrac{2a/c}{1 + a^2/c^2} = \dfrac{2ac}{a^2 + c^2}$

32. Given $\sin\theta$, $\cos\theta$ and $\tan\theta$ are in GP. This means

$$\cos^2\theta = \frac{\sin^2\theta}{\cos\theta} \Rightarrow \sin^2\theta = \cos^3\theta \Rightarrow \operatorname{cosec}\theta = \cot^3\theta$$

Now,

$$\cot^6\theta-\cot^2\theta=\operatorname{cosec}^2\theta-\cot^2\theta=1$$

33. Given $\dfrac{\cos(x-2y)}{\cos x}=\dfrac{3}{2}$

$\Rightarrow \dfrac{\cos(x-2y)-\cos x}{\cos(x-2y)+\cos x}=\dfrac{1}{5}$ [By Componendo and Dividendo]

$\Rightarrow \dfrac{2\sin(x-y)\cdot\sin y}{2\cos(x-y)\cdot\cos y}=\dfrac{1}{5}$

$\Rightarrow \tan(x-y)\cdot\tan y=\dfrac{1}{5}$

34. Given $A+B=\dfrac{\pi}{3} \Rightarrow B=\dfrac{\pi}{3}-A$

Let $k=\tan A\tan B$

$=\tan A\cdot\tan\left(\dfrac{\pi}{3}-A\right)$

$=\tan A\cdot\dfrac{\sqrt{3}-\tan A}{1+\sqrt{3}\tan A}$

$\Rightarrow \tan^2 A+\sqrt{3}(k-1)\tan A+k=0$

Since $\tan A$ is real,

$3(k-1)^2-4k\ge 0$

$\Rightarrow (3k-1)(k-3)\ge 0$

$\Rightarrow k\le\dfrac{1}{3}$ or $k\ge 3$,

but k cannot be greater than 3, since $A+B=\pi/3$.

Therefore, maximum value of $\tan A\tan B$ is 1/3.

35. Given $\cos B+\cos C=2-2\cos A$

$\Rightarrow 2\cos\dfrac{B+C}{2}\cdot\cos\dfrac{B-C}{2}=4\sin^2\dfrac{A}{2}$

$\Rightarrow \cos\dfrac{B-C}{2}=2\sin\dfrac{A}{2}$

$\Rightarrow \sin\dfrac{B+C}{2}\cdot\cos\dfrac{B-C}{2}=\sin A$

$\Rightarrow \sin B+\sin C=2\sin A$

$\Rightarrow \sin B, \sin A, \sin C$ are in AP

$\Rightarrow B, A, C$ are in AP

36. We have $\cos^2\dfrac{\pi}{9}+\cos^2\dfrac{2\pi}{9}+\cos^2\dfrac{4\pi}{9}$

$=\dfrac{1}{2}\left[1+\cos\dfrac{2\pi}{9}+1+\cos\dfrac{4\pi}{9}+1+\cos\dfrac{8\pi}{9}\right]$

$=\dfrac{1}{2}\left[3+2\cos\dfrac{\pi}{3}\cdot\cos\dfrac{\pi}{9}+\cos\left(\pi-\dfrac{\pi}{9}\right)\right]$

$=\dfrac{1}{2}\left[3+\cos\dfrac{\pi}{9}-\cos\dfrac{\pi}{9}\right]$

$=\dfrac{3}{2}$

37. Given $\dfrac{2\sin\alpha}{1+\cos\alpha+\sin\alpha}=y$

$$\Rightarrow \frac{4\sin\frac{\alpha}{2}\cdot\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}+2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}=y$$

$$\Rightarrow \frac{2\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}=y$$

Now,

$$\frac{1-\cos\alpha+\sin\alpha}{1+\sin\alpha}=\frac{2\sin^2\frac{\alpha}{2}+2\sin\frac{\alpha}{2}\cdot\cos\frac{\alpha}{2}}{\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)^2}$$

$$=\frac{2\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}=y$$

38. Given $4n\alpha=\pi$

Now, $\cot\alpha\cdot\cot 2\alpha\cdot\cot 3\alpha\cdots\cot(2n-3)\alpha\cdot\cot(2n-2)\alpha\cdot\cot(2n-1)\alpha$

$=\cot\alpha\cdot\cot 2\alpha\cdot\cot 3\alpha\cdots\cot\left(\dfrac{\pi}{2}-3\alpha\right)\cdot\cot\left(\dfrac{\pi}{2}-2\alpha\right)\cdot\cot\left(\dfrac{\pi}{2}-\alpha\right)$

$=\cot\alpha\cdot\cot 2\alpha\cdot\cot 3\alpha\cdots\tan 3\alpha\cdot\tan 2\alpha\cdot\tan\alpha=1$

39. Given $\sin\alpha+\sin\beta=a$. Now

$$2\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}=a$$

and $\quad\cos\alpha-\cos\beta=b$

So we have

$$-2\sin\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}=b$$

Therefore, $\quad\tan\dfrac{\alpha-\beta}{2}=-\dfrac{b}{a}$

40. Given $\tan\theta+\tan 2\theta=\tan 3\theta$

$\Rightarrow \tan\theta+\tan 2\theta=\dfrac{\tan\theta+\tan 2\theta}{1-\tan\theta\cdot\tan 2\theta}$

$\Rightarrow (\tan\theta+\tan 2\theta)\left(1-\dfrac{1}{1-\tan\theta\tan 2\theta}\right)=0$

$\Rightarrow \tan\theta+\tan 2\theta=0$ or $\tan\theta\tan 2\theta=0$

$\Rightarrow a+b=0$ $[\because ab\ne 0]$

41. We know that $\sin x$ is an increasing function in $\left[0, \dfrac{\pi}{2}\right]$

Now,

$\sin 2=\sin(\pi-2)=\sin 1.14$

$\sin 3=\sin(\pi-3)=\sin 0.14$

$\sin(\sqrt{10}-2)=\sin 1.16$

Therefore, among sin 1, sin 2, sin 3 and $\sin(\sqrt{10}-2)$, $\sin(\sqrt{10}-2)$ is greatest.

42. Given $\sin x+\sin^2 x=1$. So

$$\sin x=\cos^2 x$$

$$\Rightarrow \sin^2 x=\cos^4 x$$

Now,

$$\cos^2 x+\cos^4 x+\cot^4 x-\cot^2 x$$

$$=\cos^2 x+\sin^2 x+\operatorname{cosec}^2 x-\cot^2 x \ =2$$

43. We have

$$\sin 20^\circ \sin 40^\circ \sin 80^\circ \sin 60^\circ$$

$$=\frac{\sqrt{3}}{2}[\sin 20^\circ\cdot\sin(60^\circ-20^\circ)\cdot\sin(60^\circ+20^\circ)]$$

$$=\frac{\sqrt{3}}{2}[\sin 20^\circ\cdot(\sin^2 60^\circ-\sin^2 20^\circ)]$$

$$=\frac{\sqrt{3}}{2}\left[\frac{3\sin 20^\circ-4\sin^3 20^\circ}{4}\right]$$

$$=\frac{\sqrt{3}}{8}\cdot\sin 60^\circ=\frac{3}{16}$$

44. Given x, y, z are in AP $\Rightarrow 2y=x+z$. Now

$$\frac{\sin x-\sin z}{\cos z-\cos x}=\frac{2\cos\frac{x+z}{2}\cdot\sin\frac{x-z}{2}}{2\sin\frac{x+z}{2}\cdot\sin\frac{x-z}{2}}=\cot y$$

45. Given $|\sin x+\cos x|=|\sin x|+|\cos x|$
Then $\sin x$ and $\cos x$ both will be positive or negative. Hence, x belongs to I quadrant or III quadrant.

46. We have

$$x=\frac{1-4\sin 10^\circ\cdot\sin 70^\circ}{2\sin 10^\circ}=\frac{1-2(\sin 30^\circ-\sin 10^\circ)}{2\sin 10^\circ}$$

$$=\frac{1-1+2\sin 10^\circ}{2\sin 10^\circ}=1$$

47. We have

$$\sqrt{\frac{1-\sin\alpha}{1+\sin\alpha}}+\sqrt{\frac{1+\sin\alpha}{1-\sin\alpha}}=\frac{1-\sin\alpha+1+\sin\alpha}{\sqrt{1-\sin^2\alpha}}$$

$$=\frac{2}{|\cos\alpha|}=-\frac{2}{\cos\alpha}\quad\left(\because \frac{\pi}{2}<\alpha<\pi\right)$$

48. Given $\cot\alpha\cot\beta=2$. So

$$\frac{\cos\alpha\cdot\cos\beta}{\sin\alpha\cdot\sin\beta}=\frac{2}{1}$$

$$\Rightarrow\frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)}=\frac{1}{3}$$

49. Given $\cos(x-y)$, $\cos x$ and $\cos(x+y)$ are in HP. So

$$\frac{2}{\cos x}=\frac{1}{\cos(x-y)}+\frac{1}{\cos(x+y)}$$

$$\Rightarrow\frac{2}{\cos x}=\frac{2\cos x\cdot\cos y}{\cos^2 x-\sin^2 y}$$

$$\Rightarrow\cos^2 x-\sin^2 y=\cos^2 x\cdot\cos y$$

$$\Rightarrow\cos^2 x\cdot 2\sin^2\frac{y}{2}=4\sin^2\frac{y}{2}\cdot\cos^2\frac{y}{2}$$

$$\Rightarrow\cos x\cdot\sec\frac{y}{2}=\pm\sqrt{2}$$

50. Given $\tan\theta=\dfrac{\sqrt{m}-\sqrt{n}}{\sqrt{m}+\sqrt{n}}$. So

$$\cos 2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}=\frac{1-\left(\frac{\sqrt{m}-\sqrt{n}}{\sqrt{m}+\sqrt{n}}\right)^2}{1+\left(\frac{\sqrt{m}-\sqrt{n}}{\sqrt{m}+\sqrt{n}}\right)^2}$$

$$=\frac{4\sqrt{mn}}{2(m+n)}$$

$$\Rightarrow (m+n)\cos 2\theta=2\sqrt{mn}$$

51. Given $\cos^2 x+\cos^4 x=1$

$$\Rightarrow\cos^4 x=\sin^2 x\Rightarrow\cot^4 x=\operatorname{cosec}^2 x$$

Now, $\tan^4 x+\cot^4 x+\tan^2 x-\cot^2 x$

$$=\tan^2 x(1+\tan^2 x)+\operatorname{cosec}^2 x-\cot^2 x$$

$$=\tan^2 x\cdot\sec^2 x+1$$

$$=\frac{\sin^2 x}{\cos^4 x}+1=2$$

52. Given $\cos 25^\circ+\sin 25^\circ=k$

$$\Rightarrow\cos(45^\circ-20^\circ)+\sin(45^\circ-20^\circ)=k$$

$$\Rightarrow\frac{1}{\sqrt{2}}\cos 20^\circ+\frac{1}{\sqrt{2}}\sin 20^\circ+\frac{1}{\sqrt{2}}\cos 20^\circ-\frac{1}{\sqrt{2}}\sin 20^\circ=k$$

$$\Rightarrow\cos 20^\circ=\frac{k}{\sqrt{2}}$$

53. We have

$$\tan(\theta-\phi)=\frac{\tan\theta-\tan\phi}{1+\tan\theta\cdot\tan\phi}=\frac{(n-1)\tan\phi}{1+n\tan^2\phi}$$

$$\Rightarrow\tan^2(\theta-\phi)=\frac{(n-1)^2}{(\cot\phi+n\tan\phi)^2}=\frac{(n-1)^2}{(\cot\phi-n\tan\phi)^2+4n}$$

$$\Rightarrow \tan^2(\theta-\phi) \le \frac{(n-1)^2}{4n}$$

Therefore, maximum value of $\tan^2(\theta-\phi)$ is $\frac{(n-1)^2}{4n}$.

54. We have

$$16\sin x \cdot \cos x + 12\cos^2 x - 6 = 8\sin 2x + 6\cos 2x$$

Now,

$$-\sqrt{8^2+6^2} \le 8\sin 2x + 6\cos 2x \le \sqrt{8^2+6^2}$$

$$\Rightarrow -10 \le 8\sin 2x + 6\cos 2x \le 10$$

Hence, $a = -10$, $b = 10$.

55. We have

$$f(\alpha)\cdot f(\beta) = f(\alpha)\cdot f\left(\frac{5\pi}{4}-\alpha\right)$$

$$= \frac{\cot\alpha}{1+\cot\alpha}\cdot\frac{\cot\left(\frac{5\pi}{4}-\alpha\right)}{1+\cot\left(\frac{5\pi}{4}-\alpha\right)}$$

$$= \frac{\cot\alpha}{1+\cot\alpha}\cdot\frac{\frac{\cot\alpha+1}{\cot\alpha-1}}{1+\frac{\cot\alpha+1}{\cot\alpha-1}}$$

$$= \frac{\cot\alpha}{1+\cot\alpha}\cdot\frac{\cot\alpha+1}{2\cot\alpha} = \frac{1}{2}$$

56. Given $\cos\alpha = \frac{2\cos\beta-1}{2-\cos\beta}$. So

$$\frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} = \frac{2\cdot\frac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}-1}{2-\frac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}}$$

$$\Rightarrow \frac{1-\tan^2(\alpha/2)}{1+\tan^2(\alpha/2)} = \frac{1-3\tan^2(\beta/2)}{1+3\tan^2(\beta/2)}$$

$$\Rightarrow \tan^2\frac{\alpha}{2} = 3\tan^2\frac{\beta}{2}$$ [By Componendo and Dividendo]

$$\Rightarrow \tan\frac{\alpha}{2}\cdot\cot\frac{\beta}{2} = \sqrt{3}$$

57. Given equation is

$$4\sin x + 3\cos x = y^2 - 6y + 14$$

Now, $-5 \le 4\sin x + 3\cos x \le 5$.

Hence, $-5 \le y^2 - 6y + 14 \le 5$. Now

$$y^2 - 6y + 14 \le 5$$

$$\Rightarrow y^2 - 6y + 9 \le 0 \Rightarrow (y-3)^2 \le 0$$

Hence, $y = 3$.

Again, if

$$y^2 - 6y + 14 \ge -5 \Rightarrow y^2 - 6y + 19 \ge 0$$

which is always true. Thus for $y = 3$, there exists one value of x for which equation is satisfied.

58. $\sin\beta\sin\gamma\cos^2\alpha\sin(\beta-\gamma)$

$$= \frac{1}{2}[\cos(\beta-\gamma)-\cos(\beta+\gamma)]\left[\frac{1+\cos 2\alpha}{2}\right]\sin(\beta-\gamma)$$

$$= \frac{1}{4}[\sin(\beta-\gamma)\cos(\beta-\gamma)-\cos(\beta+\gamma)\sin(\beta-\gamma)$$

$$+\cos 2\alpha\sin(\beta-\gamma)\cos(\beta-\gamma)-\cos 2\alpha\cos(\beta+\gamma)\sin(\beta-\gamma)] \quad (1)$$

$$\Rightarrow \sum\cos(\beta+\gamma)\sin(\beta-\gamma) = \frac{1}{2}\sum(\sin 2\beta-\sin 2\gamma)$$

$$= \frac{1}{2}(\sin 2\beta-\sin 2\gamma+\sin 2\gamma-\sin 2\alpha+\sin 2\alpha-\sin 2\beta) = 0$$

$$\sum\cos 2\alpha\sin(\beta-\gamma)\cos(\beta-\gamma)$$

$$= \frac{1}{2}\sum\cos 2\alpha(\sin 2\beta\cos 2\gamma-\cos 2\beta\sin 2\gamma)$$

$$= \frac{1}{2}[\cos 2\alpha\sin 2\beta\cos 2\gamma-\cos 2\alpha\cos 2\beta\sin 2\gamma$$

$$+\cos 2\beta\sin 2\gamma\cos 2\alpha-\cos 2\beta\cos 2\gamma\sin 2\alpha$$

$$+\cos 2\gamma\sin 2\alpha\cos 2\beta-\cos 2\gamma\cos 2\alpha\sin 2\beta] = 0$$

$$\sum\cos 2\alpha\cos(\beta+\gamma)\sin(\beta-\gamma)$$

$$= \frac{1}{2}\sum\cos 2\alpha(\sin 2\beta-\sin 2\gamma)$$

$$= \frac{1}{2}(\sin 2\beta\cos 2\alpha-\sin 2\gamma\cos 2\alpha+\sin 2\gamma\cos 2\beta$$

$$-\sin 2\alpha\cos 2\beta+\sin 2\alpha\cos 2\gamma-\sin 2\beta\cos 2\gamma)$$

$$= \frac{-1}{2}[\sin(2\beta-2\gamma)+\sin(2\gamma-2\alpha)+\sin(2\alpha-2\beta)]$$

Hence, substituting these values in Eq. (1), we get

$$\sum\sin\beta\sin\gamma\cos^2\alpha\sin(\beta-\gamma) = \frac{1}{4}\sum\sin(2\beta-2\gamma)$$

$$= \frac{1}{4}[\sin(2\beta-2\gamma)+\sin(2\gamma-2\alpha)+\sin(2\alpha-2\beta)]$$

$$= \frac{1}{4}[2\sin(\beta-\gamma)\cos(\beta-\gamma)+2\sin(\gamma-\beta)\cos(\gamma+\beta-2\alpha)]$$

$$= \frac{1}{2}\sin(\beta-\gamma)[\cos(\beta-\gamma)-\cos(\beta+\gamma-2\alpha)]$$

[$\because \sin(\gamma-\beta) = -\sin(\beta-\gamma)$]

$$= \frac{1}{2}\sin(\beta-\gamma)2\sin(\beta-\alpha)\sin(\gamma-\alpha)$$

$$= -\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta)$$

59. $$\sin^2\phi = \frac{\cos(\alpha+\beta+\alpha-\beta)\cos[\alpha+\beta-(\alpha-\beta)]}{\cos^2(\alpha+\beta)}$$

$$= \frac{[\cos^2(\alpha+\beta) - \sin^2(\alpha-\beta)]}{\cos^2(\alpha+\beta)} = 1 - \frac{\sin^2(\alpha-\beta)}{\cos^2(\alpha+\beta)}$$

Hence,

$$\cos^2\phi = \frac{\sin^2(\alpha-\beta)}{\cos^2(\alpha+\beta)}$$

Therefore,

$$\cos\phi = \frac{1-\tan^2(\phi/2)}{1+\tan^2(\phi/2)} = \pm\frac{\sin(\alpha-\beta)}{\cos(\alpha+\beta)}$$

Taking the positive sign, we get

$$\frac{1-\tan^2(\phi/2)}{1+\tan^2(\phi/2)} = \frac{\sin(\alpha-\beta)}{\cos(\alpha+\beta)}$$

$$\tan^2(\phi/2) = \frac{\cos(\alpha+\beta) - \sin(\alpha-\beta)}{\cos(\alpha-\beta) + \sin(\alpha-\beta)}$$

$$= \frac{\sin(\pi/2 + \overline{\alpha+\beta}) - \sin(\alpha-\beta)}{\sin(\pi/2 + \overline{\alpha+\beta}) + \sin(\alpha-\beta)}$$

$$= \frac{2\cos(\pi/4+\alpha)\sin(\pi/4+\beta)}{2\sin(\pi/4+\alpha)\cos(\pi/4+\beta)} = \frac{\tan(\pi/4+\beta)}{\tan(\pi/4+\alpha)}$$

By taking the negative sign the other value may be obtained.

60. Let T_r denote the r^{th} term. Hence

$$T_r = \tan rx \tan(r+1)x$$

$$\tan[(r+1)x - rx] = \frac{\tan(r+1)x - \tan rx}{1 + \tan(r+1)x \tan rx}$$

$$\Rightarrow \tan x + \tan x \tan(r+1)x \tan rx = \tan(r+1)x - \tan rx$$

$$\Rightarrow \tan(r+1)x \tan rx = \cot x[\tan(r+1)x - \tan rx] - 1$$

Putting $r = 1, 2, 3, \ldots, n$ and adding, we get

$$S_n = \cot x[\tan(n+1)x - \tan x] - n$$
$$= \cot x \tan(n+1)x - 1 - n$$
$$= \cot x \tan(n+1)x - (1+n)$$

61. $A + B + C = \pi \Rightarrow B + C = \frac{3\pi}{4} \Rightarrow 0 < B, C < \frac{3\pi}{4}$

$$\tan B \tan C = p$$

$$\Rightarrow \frac{\sin B \sin C}{\cos B \cos C} = \frac{p}{1}$$

$$\Rightarrow \frac{\cos B\cos C - \sin B \sin C}{\cos B \cos C + \sin B \sin C} = \frac{1-p}{1+p}$$

$$\Rightarrow \frac{\cos(B+C)}{\cos(B-C)} = \frac{1-p}{1+p}$$

$$\Rightarrow \frac{1+p}{\sqrt{2}(p-1)} = \cos(B-C)$$

Since B and C can vary from 0 to $\frac{3\pi}{4}$,

$$-\frac{1}{\sqrt{2}} < \frac{p+1}{\sqrt{2}(p-1)} \le 1$$

$$\Rightarrow 0 < 1 + \frac{p+1}{p-1}$$

$$\Rightarrow \frac{2p(p-1)}{(p-1)^2} > 0$$

$$\Rightarrow p < 0 \text{ or } p > 1 \quad (1)$$

Also $$\frac{p+1-\sqrt{2}(p-1)}{\sqrt{2}(p-1)} \le 0$$

$$\Rightarrow \frac{(p-1)[p(\sqrt{2}-1) - (\sqrt{2}+1)]}{\sqrt{2}(p-1)^2} \le 0$$

$$\Rightarrow p \le 1 \text{ or } p \ge (\sqrt{2}+1)^2 \qquad \left[\because \frac{\sqrt{2}+1}{\sqrt{2}-1} = (\sqrt{2}+1)^2\right] \quad (2)$$

Combining Eqs. (1) and (2), we get

$$p < 0 \text{ or } p \ge (\sqrt{2}+1)^2$$

62. A_1A_2, A_1A_3, A_1A_4 subtend angles $\frac{2\pi}{n}, \frac{4\pi}{n}, \frac{6\pi}{n}$, respectively, at the centre of polygon. Hence

$$A_1A_2 = 2r\sin\frac{\pi}{n};$$
$$A_1A_3 = 2r\sin\left(\frac{2\pi}{n}\right);$$
$$A_1A_4 = 2r\sin\left(\frac{3\pi}{n}\right)$$

(where r is the circumradius of polygon).

By problem,

$$\frac{1}{2r\sin(\pi/n)} = \frac{1}{2r\sin(2\pi/n)} + \frac{1}{2r\sin(3\pi/n)}$$

Hence,

$$2\sin\left(\frac{3\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right) = 2\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{3\pi}{n}\right) + 2\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)$$

$$\Rightarrow \cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{5\pi}{n}\right) = \cos\left(\frac{2\pi}{n}\right) - \cos\left(\frac{4\pi}{n}\right) + \cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{3\pi}{n}\right)$$

$$\Rightarrow \cos\left(\frac{3\pi}{n}\right) - \cos\left(\frac{5\pi}{n}\right) = \cos\left(\frac{2\pi}{n}\right) - \cos\left(\frac{4\pi}{n}\right)$$

$$\left[\cos\left(\frac{\pi}{n}\right) \text{ cancels out}\right]$$

$$2\sin\left(\frac{4\pi}{n}\right)\sin\left(\frac{\pi}{n}\right) = 2\sin\left(\frac{3\pi}{n}\right)\sin\left(\frac{\pi}{n}\right)$$

$$\Rightarrow \sin\frac{4\pi}{n} = \sin\left(\frac{3\pi}{n}\right) \text{ and } n \ge 3$$

Therefore, $$\frac{4\pi}{n} + \frac{3\pi}{n} = \pi \Rightarrow n = 7$$

63. Given $m\tan(\theta - 30°) = n\tan(\theta - 120°) = n\tan(90° + 30° + \theta)$ $= -n\cot(30° + \theta)$

$$\frac{m}{n} = -\frac{\cot(30°+\theta)}{\tan(\theta - 30°)}$$

$$\frac{m+n}{m-n}=-\frac{\cot(30°+\theta)+\tan(\theta-30°)}{-\cot(30°+\theta)-\tan(\theta-30°)}$$

(Componendo and Dividendo)

$$=\frac{-\cos(\theta+30°)\cos(\theta-30°)+\sin(\theta-30°)\sin(\theta+30°)}{-[\cos(\theta+30°)\cos(\theta-30°)+\sin(\theta+30°)\sin(\theta-30°)]}$$

$$=\frac{-\cos(\theta+30°+\theta-30°)}{-(1/2)}=2\cos 2\theta$$

Hence, $\cos 2\theta=\frac{(m+n)}{2(m-n)}$

Practice Exercise 2

1. LHS is an even number and RHS is an odd number always. Hence, number of solution is 0.
2. Let $y = 2\sin 2° + 4\sin 4° + \cdots + 178\sin 178°$
   $\Rightarrow y = 178\sin 178° + 176\sin 176° + \cdots + 2\sin 2°$
   $\Rightarrow 2y = 180(\sin 2° + \sin 4° + \cdots + \sin 178°)$
   $\Rightarrow y = 90\cdot\frac{\sin 89°}{\sin 1°}\sin(90°) = 90\tan 89°$
3. Suppose $\alpha=\sqrt{2}\cos x+\sqrt{2}\sin x+\sqrt{7}= 2\sin(x+\frac{\pi}{4})+\sqrt{7}$
   $-2+\sqrt{7}\le\alpha\le 2+\sqrt{7}$
   $\Rightarrow m=\frac{1}{\alpha}\Rightarrow m\in\left(-\infty,\ \frac{1}{-2+\sqrt{7}}\right]\cup\left[\frac{1}{2+\sqrt{7}},\ \infty\right)$
4. $\frac{x}{(2n+1)\pi}=\sin^2\left(\frac{x}{2}\right)$
   The graph of $\sin^2\left(\frac{x}{2}\right)$ will be above the axis of x and will be meeting it at 0, 2π, 4π, 6π, …, etc. It will attain maximum values at odd multiples of π, that is, π, 3π, …, $(2n + 1)\pi$.
   The last point after which graph of $y=\frac{x}{(2n+1)\pi}$ will stop cutting will be $(2n + 1)\pi$.
   Total intersection = $2(n + 1)$.
5. The maximum value of $\cos^2(\cos\theta)$ is 1 and that of $\sin^2(\sin\theta)$ is $\sin^2 1$, both exist for $\theta = \pi/2$. Hence, maximum value is $1 + \sin^2 1$.
6. $[y + [y]] = 2\cos x \Rightarrow [y] = \cos x$. So
   $$y=\frac{1}{3}[\sin x + [\sin x + (\sin x)] = (\sin x)$$
   $$\Rightarrow [\sin x] = \cos x$$
   Number of solution in $[0, 2\pi]$ is 0.
   Hence, the total solution is 0.
   Therefore, both are periodic with period 2π.
7. The equation has meaning if $x > 0, x \ne 1$
   Hence, domain = $(0, 1) \cup (1, \infty)$
   If $x \in (0, 1)$ then $\log_2 x < 0$ and $\log_2 x + \log_x 2 = \frac{\log x}{\log 2}+\frac{\log 2}{\log x}$
   = sum of a negative number ≤ -2.
   In this case any α will satisfy since $2\cos\alpha$ can never be more than 2.
   Thus, the inequation is satisfied for any x in (0, 1) and for any α.
   If $x \in (1, \infty)$ then $\log_2 x > 0$. So
   $$\frac{\log x}{\log 2}=\frac{\log 2}{\log x}>0$$
   The inequation cannot be satisfied unless
   $$\cos\alpha=-1 \text{ and } x=2, \text{ that is, } \log_2 x=1$$
   Option (D) is wrong since in the last case there are infinite solutions.
8. We have
   $$y=\frac{(\sin^2 x-\cos^2 x)(\sin^2 x+\cos^2 x)+\sin^2 x\cos^2 x}{(\sin^2 x+\cos^2 x)^2-\sin^2 x\cos^2 x}$$
   $$=\frac{-\cos 2x+\frac{1}{4}\sin^2 2x}{1-\frac{1}{4}\sin^2 2x}=\frac{1-4\cos 2x-\cos^2 2x}{3+\cos^2 2x}$$
   $$\Rightarrow (1 + y)\cos^2 2x + 4\cos 2x + 3y - 1 = 0$$
   Since, $\cos 2x$ is real,
   $16 - 4(3y - 1)(1 + y) \ge 0 \Rightarrow 3y^2 + 2y - 5 \le 0$
   or $(3y + 5)(y - 1) \le 0 \Rightarrow -\frac{5}{3}\le y\le 1$
   but $y = 1 \Rightarrow \cos 2x = -1$, that is, $x=\frac{\pi}{2}$ which is not permissible.
9. $y=\frac{t^2-t+1}{t^2+(2-d)t+1}$; $t=\cot^2\theta$
   $\Rightarrow (y - 1)t^2 + [(2 - d)y + 1]t + y - 1 = 0$
   $D \ge 0 \Rightarrow y^2[4 - (2 - d)^2] - 2(6 - d)y + 3 \le 0$
   also $3y^2 - 10y + 3 \le 0$
   $\Rightarrow \frac{4-(2-d)^2}{3}=\frac{-2(6-d)}{-10}=\frac{3}{3}\Rightarrow d=1$
10. $\log_{|x-1|}|x^2-1|=[|\sin x|+|\cos x|]$
    Period of $|\sin x|+|\cos x|$ is $\frac{\pi}{2}$ and the range of the function $|\sin x|+|\cos x|\in[1,\sqrt{2}]$ then the value $[|\sin x|+|\cos x|]=1$
    $$\log_{|x-1|}|x^2-1|=1$$
    According to the definition of log,
    $$|x^2-1|\ne 0\Rightarrow x\ne\pm 1$$
    $$|x-1|\ne 0,1\Rightarrow x-1\ne -1,0,1$$
    $$\Rightarrow x\ne 0,1,2$$
    So, $x\ne -1,0,1,2$
    By property of log,
    $$|x^2-1|=|x-1|$$
    $$|(x-1)(x+1)|=|x-1|$$
    $$|x+1|=1\ (|x-1|\ne 0)$$
    $$x+1=-1,1$$
    $$x=-2,0$$
    Value of x is -2.
    Number of solutions is 1.

11. $\left(1+\dfrac{1}{\sin^n\theta}\right)\left(1+\dfrac{1}{\cos^n\theta}\right)$

$$f(\theta)=1+\frac{1}{\sin^n\theta}+\frac{1}{\cos^n\theta}+\frac{1}{\sin^n\theta\cos^n\theta}$$

$$\frac{d}{d\theta}f(\theta)=-\frac{n}{\sin^{n+1}\theta}\cos\theta+\frac{n}{\cos^{n+1}\theta}\sin\theta-\frac{2^n n}{(\sin 2\theta)^{n+1}}2\cos 2\theta=0$$

will give $\theta=\dfrac{\pi}{4}$ $f(\theta)_{\min}$ will occur at $\theta=\dfrac{\pi}{4}$

12. Let $A=t$, then $B=\pi-t$, $C=2\pi+t$, $D=3\pi-t$

$$A+B=\pi, B+C=3\pi, C+D=5\pi \text{ are in AP.}$$

13. Put $\sin x=t$

$$y=t^3-6t^2+11t-6,\ -1\le t\le 1$$
$$f(-1)=-24,\ f(1)=0$$

14. $$\frac{4}{\sin x}+\frac{8}{\cos x}\ge 2\left(\frac{32}{\sin x\cos x}\right)^{1/2}=\frac{16}{\sqrt{\sin 2x}}\ge 16$$

But A.M. = G.M. and $\sin 2x=1$ cannot occur simultaneously

Hence, $\dfrac{4}{\sin x}+\dfrac{8}{\cos x}>16$

also $\dfrac{32}{\sin x\cos x}=\dfrac{64}{\sin 2x}\ge 64$

Therefore,

$$(1+4\operatorname{cosec}x)(1+8\sec x)=1+\left(\frac{4}{\sin x}+\frac{8}{\cos x}\right)+\frac{32}{\sin x\cos x}$$

$$>1+16+64=81$$

15. $[\sin x]+[\cos x]+2=0$ is possible only when

$[\sin x]=-1$ and $[\cos x]=-1$

or $-1\le\sin x<0$ and $-1\le\cos x<0\Rightarrow x\in\left(\pi,\dfrac{3\pi}{2}\right)$.

Now $f(x)=\sin x-\cos x+3=\sqrt{2}\sin\left(x-\dfrac{\pi}{4}\right)+3$

For $x\in\left(\pi,\dfrac{3\pi}{2}\right)$, $\sin\left(x-\dfrac{\pi}{4}\right)\in\left(-\dfrac{1}{\sqrt{2}},\dfrac{1}{\sqrt{2}}\right)$

$\Rightarrow y=f(x)=\sqrt{2}\sin\left(x-\dfrac{\pi}{4}\right)+3$, then $f(x)\in(2,4)$

16. $y^2-y+a=\left(y-\dfrac{1}{2}\right)^2+a-\dfrac{1}{4}$

Since $-\sqrt{2}\le\sin x+\cos x\le\sqrt{2}$, given equation will have no real value of x for any y if $a-\dfrac{1}{4}>\sqrt{2}$

17. Let $\cos\theta=\dfrac{k}{x}$, $\cos\left(\theta+\dfrac{2\pi}{3}\right)=\dfrac{k}{y}$, $\cos\left(\theta+\dfrac{4\pi}{3}\right)=\dfrac{k}{z}$

$\Rightarrow\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=0$ as $\cos\theta+\cos\left(\theta+\dfrac{2\pi}{3}\right)+\cos\left(\theta+\dfrac{4\pi}{3}\right)=0$

$\Rightarrow xy+yz+zx=0$

also, $xyz=\cos\theta\cos\left(\dfrac{\pi}{3}-\theta\right)\cos\left(\dfrac{\pi}{3}+\theta\right)=\dfrac{1}{4}\cos 3\theta$

18. $\sin^2 z+\operatorname{cosec}^2 z\ge 2$, $2+\cot^2 y\ge 2$, $4+\sin 4x\ge 3$

$\Rightarrow\sin^2 z=1$, $\cot^2 y=0$, $\sin 4x=-1$

$\Rightarrow z\in\left\{\dfrac{\pi}{2},\dfrac{3\pi}{2}\right\}$, $y\in\left\{\dfrac{\pi}{2},\dfrac{3\pi}{2}\right\}$, $x\in\left\{\dfrac{3\pi}{8},\dfrac{7\pi}{8},\dfrac{11\pi}{8},\dfrac{15\pi}{8}\right\}$.

19.

(A) $\dfrac{x}{2}+\tan x=\dfrac{\pi}{4}$

from graph, the equation has 3 solutions in $[-\pi,\pi]$.

(B) $\sin^{-1}|x^2-1|+\cos^{-1}|2x^2-5|=\dfrac{\pi}{2}\Rightarrow|x^2-1|=|2x^2-5|$

$\Rightarrow x^2=2\Rightarrow x=\pm\sqrt{2}$ (two solutions)

(C) $x^4-2x^2\sin\dfrac{\pi x}{2}+1=0\Rightarrow\left(x^2-\sin^2\dfrac{\pi x}{2}\right)^2+1-\sin^4\dfrac{\pi x}{2}=0$

$\Rightarrow x=(2n+1)$, $n\in$ I and $x^2=\sin^2\dfrac{\pi x}{2}=1$

$\Rightarrow x=\pm 1$ is the solution.

(D) Let $y=x^2+2x+2\sec^2\pi x+\tan^2\pi x$

$\Rightarrow y=(x+1)^2+(2\sec^2\pi x-1)+\tan^2\pi x>0\Rightarrow$ no solution.

20.

(A) $y=\tan^{-1}\dfrac{1}{2}+\tan^{-1}b$, $(0<b<1)$

$\Rightarrow y=\tan^{-1}\left(\dfrac{1/2+b}{1-b/2}\right)$, $\left(\because\ \dfrac{1}{2}b<1\right)$

$0<\tan^{-1}\left(\dfrac{1+2b}{2-b}\right)\le\dfrac{\pi}{4}\Rightarrow 0<\left(\dfrac{1+2b}{2-b}\right)\le 1$

$\Rightarrow 0<(1+2b)\le(2-b)$, $(1+2b>0)$

$\Rightarrow 3b\le 1\Rightarrow b\le\dfrac{1}{3}\Rightarrow b_{\max}=\dfrac{1}{3}$

(B) $\sin^4 x+\cos^3 x\ge 1$ (1)

Since $\sin^2 x+\cos^2 x=1$ and $-1\le\sin x,\cos x\le 1$, LHS of Eq. (1) cannot be >1. Therefore

$\sin^4 x+\cos^3 x=\pm 1$ (2)

Equation (2) is possible if either,

$\sin x=1$ and $\cos x=0$ or $\sin x=0$ and $\cos x=1$

$\Rightarrow x=(4n+1)\dfrac{\pi}{2}$, $x=(2n+1)\dfrac{\pi}{2}$ or $x=n\pi$, $x=2n\pi$

In $(0,2\pi)$, $x=\dfrac{\pi}{2}$, $x=\dfrac{\pi}{2},\dfrac{3\pi}{2}$ or $x=\pi$

Therefore, number of solutions will be 3.

(C) $C>\dfrac{3\pi}{4}$

$$0<A+B<\frac{\pi}{4}$$

$$\Rightarrow 0<\frac{\tan A+\tan B}{1-\tan A\tan B}<1$$

$$\Rightarrow\tan A+\tan B<1-\tan A\tan B$$

$$\Rightarrow(1+\tan A)(1+\tan B)<0$$

Also $\tan A > 0, \tan B > 0$

$(1+\tan A)(1+\tan B) > 1$

$$1 < (1+\tan A)(1+\tan B) < 2$$

(D) $\left|\sqrt{A}-\sqrt{B}\right| \le \sqrt{|A-B|}$

Given expression $\le \sqrt{\sin^2 x + 2a^2 - 2a^2 + 1 + \cos^2 x} \le \sqrt{2}$.

21. $\tan\theta = \dfrac{3\tan\frac{\theta}{3} - \tan^3\frac{\theta}{3}}{1-3\tan^2\frac{\theta}{3}} = \lambda$

$$\Rightarrow \tan^3\frac{\theta}{3} - 3\lambda\tan^2\frac{\theta}{3} - 3\tan\frac{\theta}{3} + \lambda = 0$$

$$\Rightarrow \sum \tan\frac{\theta_1}{3}\frac{\tan\theta_2}{3} = -3$$

22. Let $f(x) = \sin[\cos(\sin x)] - \cos[\sin(\cos x)]$. Then $f'(x) = \cos[\cos(\sin x)]\sin(\sin x)(-\cos x) - \sin[\sin(\cos x)]\cos(\cos x)\sin x$

$$\Rightarrow f'(x) < 0\ \forall\ x \in \left[0, \frac{\pi}{2}\right]$$

$$\Rightarrow f(x) \text{ is decreasing in } \left[0, \frac{\pi}{2}\right]$$

and $f(0) = \sin 1 - \cos(\sin 1)$ (1)

Now,

$$\sin 1 - \cos(\sin 1) = \cos\left(\frac{\pi}{2} - 1\right) - \cos(\sin 1)$$

$$\sin 1 > \sin\frac{\pi}{4} > \frac{1}{\sqrt{2}}$$

Hence,

$$\frac{\pi}{2} - 1 < \sin 1 \Rightarrow \sin 1 > \frac{\pi - 2}{2}$$

Therefore, $\sin 1 > \cos(\sin 1)$. From Eq. (1)

$$f(0) = \sin 1 - \cos(\sin 1) > 0$$

$$f\left(\frac{\pi}{2}\right) = \sin(\cos 1) - 1 < 0$$

Therefore, there will be only one root lying between $\left[0, \frac{\pi}{2}\right]$.

Solved JEE 2017 Questions

JEE Main 2017

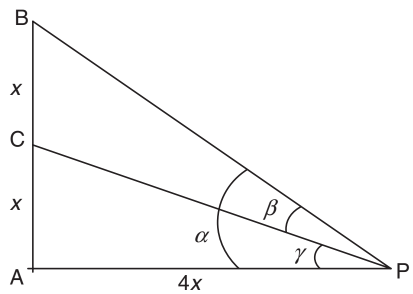
1. Let a vertical tower AB have its end a on the level ground. Let C be the mid-point of AB and P be a point on the ground such that AP = 2AB. If $\angle BPC = \beta$, then $\tan\beta$ is equal to:

(A) $\frac{1}{4}$ **(B)** $\frac{2}{9}$

(C) $\frac{4}{9}$ **(D)** $\frac{6}{7}$

(OFFLINE)

Solution: The given situation is depicted as shown in the following figure:



From the figure, we have

$$\beta = \alpha - \gamma$$

Therefore,

$$\tan\beta = \frac{\tan\alpha - \tan\gamma}{1+(\tan\alpha)(\tan\gamma)} = \frac{(1/2)-(1/4)}{1+(1/8)} = \frac{2}{9}$$

Hence, the correct answer is option (B).

3 Trigonometric Equation and Inequation

3.1 Introduction

An equation involving one or more trigonometrical ratios of unknown angle is called a trigonometric equation, for example, $\cos^2 x - 2\sin x = -1$.

It is to be noted that a trigonometrical identity is satisfied for every value of the unknown angle whereas trigonometric equation is satisfied only for some values (finite or infinite) of unknown angle.

For example, $\sec^2 x - \tan^2 x = 1$ is a trigonometrical identity as it is satisfied for every value of $x \in R$.

3.2 Solution of a Trigonometric Equation

A value of the unknown angle which satisfies the given equation is called a **solution of the equation**, for example, $\sin\theta = 1/2 \Rightarrow \theta = \pi/6$.

Since all trigonometrical ratios are periodic in nature, generally a trigonometrical equation has more than one solution or an infinite number of solutions. There are basically three types of solutions:

1. **Particular solution:** A specific value of unknown angle satisfying the equation.
2. **Principal solution:** Smallest numerical value of the unknown angle satisfying the equation (numerically smallest particular solution).
3. **General solution:** Complete set of values of the unknown angle satisfying the equation. It contains all particular solutions as well as principal solutions.

When we have two numerically equal smallest unknown angles, preference is given to the positive value in writing the principal solution. For example,

$$\sec\theta = \frac{2}{\sqrt{3}}$$

has

$$\frac{\pi}{6}, -\frac{\pi}{6}, -\frac{11\pi}{6}, \frac{23\pi}{6}, -\frac{23\pi}{6}\text{, etc.}$$

as its particular solutions. Out of these, the numerically smallest angles are $\pi/6$ and $-\pi/6$ but the principal solution is taken as $\theta = \pi/6$. To write the general solution, we notice that the position on P or P' can be obtained by the rotation of OP or OP' around O through a complete angle (2π) by any number of times and in any direction (clockwise or anticlockwise; see Fig. 3.1).

Therefore, the general solution is

$$\theta = 2k\pi \pm \frac{\pi}{6},\ k \in Z$$

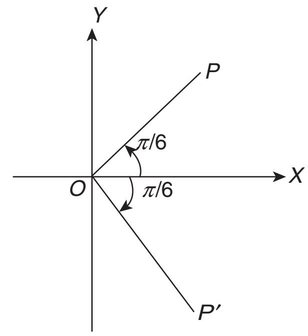


Figure 3.1

3.2.1 Method for Finding the Principal Value

Suppose we have to find the principal value of θ satisfying the equation $\sin\theta = -1/2$. Since $\sin\theta$ is negative, θ will be in the third or the fourth quadrant. We can approach the third or the fourth quadrant from two directions. If we take the anticlockwise direction, the numerical value of the angle will be greater than π. If we approach it in the clockwise direction, the angle will be numerically less than π. For the principal value, we have to take numerically smallest angle. So, for the principal value:

1. If the angle is in the first or the second quadrant, we must select the anticlockwise direction and if the angle is in the third or the fourth quadrant, we must select the clockwise direction (Fig. 3.2).

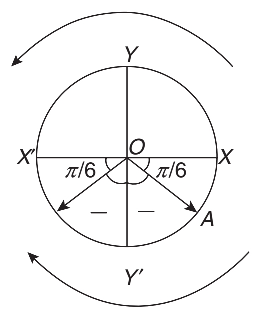


Figure 3.2

2. Principal value is never numerically greater than π.
3. Principal value always lies in the first circle (that is, in the first rotation).

   From the above criteria, θ will be $-\pi/6$ or $-5\pi/6$; between these two, $-\pi/6$ has the least numerical value. Hence, $-\pi/6$ is the principal value of θ satisfying the equation $\sin\theta = -1/2$.

3.2.2 Steps to Find Out the Principal Solution

Step 1: First draw a trigonometrical circle and mark the quadrant in which the angle may lie.
Step 2: Select the anticlockwise direction for the first and second quadrants and select the clockwise direction for the third and fourth quadrants.
Step 3: Find the angle in the first rotation.
Step 4: Select the numerically least angle. The angle thus found will be the principal value.
Step 5: In case two angles, one with positive sign and the other with negative sign, qualify for the numerically least angle, then it is the convention to select the angle with the positive sign as the principal value.

3.3 General Solution of the Standard Trigonometric Equation

1. General solution of the equation sin $\theta = \sin\alpha$:

$\sin\theta = \sin\alpha$ can be written as

$$\sin\theta - \sin\alpha = 0$$

$$\Rightarrow 2\sin\left(\frac{\theta-\alpha}{2}\right)\cos\left(\frac{\theta+\alpha}{2}\right) = 0$$

$$\left(\text{Apply}\ \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right)$$

$$\Rightarrow \sin\left(\frac{\theta-\alpha}{2}\right) = 0 \text{ or } \cos\left(\frac{\theta+\alpha}{2}\right) = 0$$

$$\Rightarrow \frac{\theta-\alpha}{2} = m\pi \text{ or}$$

$$\frac{\theta+\alpha}{2} = (2m+1)\frac{\pi}{2};\ m\in I$$

Hence,

$$\theta = 2m\pi + \alpha \text{ or } \theta = (2m+1)\pi - \alpha;\ m\in I$$

Hence,

$$\theta = n\pi + (-1)^n\alpha,\ n\in I$$

The same general solution works for the equation cosec θ = cosec α.

2. General solution of the equation $\cos\theta = \cos\alpha$:

$\cos\theta = \cos\alpha$ can be written as

$$\cos\theta - \cos\alpha = 0$$

$$\Rightarrow -2\sin\left(\frac{\theta+\alpha}{2}\right)\cdot\sin\left(\frac{\theta-\alpha}{2}\right) = 0$$

$$\left(\text{Apply}\ \cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right)$$

$$\Rightarrow \sin\left(\frac{\theta+\alpha}{2}\right) = 0 \text{ or } \sin\left(\frac{\theta-\alpha}{2}\right) = 0$$

$$\Rightarrow \frac{\theta+\alpha}{2} = n\pi \text{ or } \frac{\theta-\alpha}{2} = n\pi;\ n\in I$$

$$\Rightarrow \theta = 2n\pi - \alpha \text{ or } \theta = 2n\pi + \alpha;\ n\in I$$

Hence,

$$\theta = 2n\pi \pm \alpha,\ n\in I$$

The same general solution works for the equation $\sec\theta = \sec\alpha$.

3. General solution of the equation tan $\theta = \tan\alpha$:

tan θ = tan α can be written as

$$\frac{\sin\theta}{\cos\theta} = \frac{\sin\alpha}{\cos\alpha}$$

$$\Rightarrow \sin\theta\cos\alpha - \cos\theta\sin\alpha = 0$$

$$\Rightarrow \sin(\theta-\alpha) = 0 \Rightarrow \theta - \alpha = n\pi;\ n\in I$$

Hence,

$$\theta = n\pi + \alpha;\ n\in I$$

The same general solution works for the equation $\cot\theta = \cot\alpha$.

4. General solution of the equation $\tan^2\theta = \tan^2\alpha$:

$\tan^2\theta = \tan^2\alpha$ can be written as

$$\tan^2\theta - \tan^2\alpha = 0$$

$$\Rightarrow (\tan\theta - \tan\alpha)(\tan\theta + \tan\alpha) = 0$$

$$\Rightarrow \tan\theta - \tan\alpha = 0 \text{ or } \tan\theta + \tan\alpha = 0$$

$$\Rightarrow \tan\theta = \tan\alpha \text{ or } \tan\theta = \tan(-\alpha)$$

$$\Rightarrow \theta = n\pi + \alpha \text{ or } \theta = n\pi - \alpha,\ n\in I$$

Hence,

$$\theta = n\pi \pm \alpha,\ n\in I$$

The same general solution works for the equation $\sin^2\theta = \sin^2\alpha$ and $\cos^2\theta = \cos^2\alpha$.

3.3.1 General Solution of Some Particular Equation

1. sin $\theta = 0 \Rightarrow \theta = n\pi$
2. cos $\theta = 0 \Rightarrow \theta = (2n+1)\dfrac{\pi}{2}$
3. tan $\theta = 0 \Rightarrow \theta = n\pi$
4. sin θ = sin $\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha$, where $\alpha \in [-\pi/2, \pi/2]$
5. cos θ = cos $\alpha \Rightarrow \theta = 2n\pi \pm \alpha$, where $\alpha \in [0, \pi]$
6. tan θ = tan $\alpha \Rightarrow \theta = n\pi + \alpha$, where $\alpha \in (-\pi/2, \pi/2)$
7. $\sin^2\theta = \sin^2\alpha$, $\cos^2\theta = \cos^2\alpha$, $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha$
8. sin $\theta = 1 \Rightarrow \theta = (4n+1)\dfrac{\pi}{2}$
9. cos $\theta = 1 \Rightarrow \theta = 2n\pi$
10. cos $\theta = -1 \Rightarrow \theta = (2n+1)\pi$
11. sin θ = sin α and cos θ = cos $\alpha \Rightarrow \theta = 2n\pi + \alpha$

Note:

1. Everywhere in this chapter n is taken as an integer, if not stated otherwise.
2. The general solution should be given unless the solution is required in a specified interval.
3. α is taken as the principal value of the angle. The numerically least angle is called the principal value.

Illustration 3.1 Find the general solution:

1. $\sin\theta = \frac{\sqrt{3}}{2}$
2. $\tan 3x = 1$
3. $\sin 3\theta = \sin\theta$
4. $2\sin^2\theta - 3\sin\theta - 2 = 0$
5. $\tan\theta + \tan 2\theta + \sqrt{3}\tan\theta\tan 2\theta = \sqrt{3}$
6. $\tan^2\theta + \sec 2\theta = 1$
7. $\sec^2\theta = \frac{4}{3}$
8. $\frac{1-\cos 2\theta}{1+\cos 2\theta} = 3$

Solution:

1. $\sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \sin\theta = \sin\frac{\pi}{3} \Rightarrow \theta = n\pi + (-1)^n\frac{\pi}{3}$.

2. $\tan 3x = \tan\pi/4 \Rightarrow 3x = n\pi + \pi/4 \Rightarrow x = \frac{n\pi}{3} + \frac{\pi}{12}$.

3. $\sin 3\theta = \sin\theta \Rightarrow 3\theta = m\pi + (-1)^m\theta$

   For m even, that is, $m = 2n$, we have
$$\theta = \frac{2n\pi}{2} = n\pi$$
   For m odd, that is, $m = (2n+1)$, we have
$$\theta = (2n+1)\frac{\pi}{4}$$

4. $2\sin^2\theta - 3\sin\theta - 2 = 0$
$$\Rightarrow 2\sin^2\theta - 4\sin\theta + \sin\theta - 2 = 0$$
$$\Rightarrow 2\sin\theta(\sin\theta - 2) + (\sin\theta - 2) = 0$$
$$\Rightarrow (2\sin\theta + 1)(\sin\theta - 2) = 0$$
$$\Rightarrow \sin\theta = +2 \text{ (which is impossible)}$$
   Therefore,
$$\sin\theta = -\frac{1}{2} \Rightarrow \sin\theta = \sin\left(-\frac{\pi}{6}\right) \Rightarrow \theta = n\pi - (-1)^n\frac{\pi}{6}$$

5. $\tan\theta + \tan 2\theta + \sqrt{3}\tan\theta\tan 2\theta = \sqrt{3}$
$$\Rightarrow \tan\theta + \tan 2\theta = \sqrt{3}(1 - \tan\theta\tan 2\theta)$$
$$\Rightarrow \frac{\tan\theta + \tan 2\theta}{1 - \tan\theta\tan 2\theta} = \sqrt{3} \Rightarrow \tan 3\theta = \tan\left(\frac{\pi}{3}\right)$$
$$\Rightarrow 3\theta = n\pi + \frac{\pi}{3} \Rightarrow \theta = \frac{n\pi}{3} + \frac{\pi}{9} = (3n+1)\frac{\pi}{9}$$

6. $\tan^2\theta + \sec 2\theta = 1 \Rightarrow \tan^2\theta + \frac{1+\tan^2\theta}{1-\tan^2\theta} = 1$
$$\Rightarrow \tan^2\theta - \tan^4\theta + 1 + \tan^2\theta = 1 - \tan^2\theta$$
$$\Rightarrow \tan^4\theta - 3\tan^2\theta = 0$$
$$\Rightarrow \tan^2\theta(\tan^2\theta - 3) = 0$$
$$\Rightarrow \tan^2\theta = 0 \text{ and } \tan^2\theta = 3$$
   From this, we have
$$\tan^2\theta = \tan^2 0$$
   and
$$\tan^2\theta = \tan^2\frac{\pi}{3} \Rightarrow \theta = m\pi$$
   So,
$$\theta = n\pi \pm \frac{\pi}{3}$$

7. $\sec^2\theta = \frac{4}{3} \Rightarrow \cos^2\theta = \frac{3}{4} = \left(\frac{\sqrt{3}}{2}\right)^2 \Rightarrow \cos^2\theta = \cos^2\frac{\pi}{6}$

   Therefore,
$$\theta = n\pi \pm \frac{\pi}{6}$$

8. $$\frac{1-\cos 2\theta}{1+\cos 2\theta} = 3 \Rightarrow \frac{2\sin^2\theta}{2\cos^2\theta} = 3$$
$$\Rightarrow \tan^2\theta = 3 = (\sqrt{3})^2 \Rightarrow \tan^2\theta = \tan^2\frac{\pi}{3}$$
   Therefore,
$$\theta = n\pi \pm \frac{\pi}{3}$$

Illustration 3.2 Find the number of solutions of equation in the given interval:

1. $3\sin^2 x - 7\sin x + 2 = 0$ in $[0, 5\pi]$
2. $\tan x + \sec x = 2\cos x$ in $[0, 2\pi]$

Solution:

1. We have
$$3\sin^2 x - 7\sin x + 2 = 0$$
$$\Rightarrow 3\sin^2 x - 6\sin x - \sin x + 2 = 0$$
$$\Rightarrow (3\sin x - 1)(\sin x - 2) = 0$$
   But $\sin x \neq 2$. So $\sin x = 1/3$. Hence, from 0 to 2π we have 2 solutions (one in the first quadrant and the other in the second quadrant); from 2π to 4π we have 2 solutions and 4π to 5π we have 2 solutions. So, the total number of solutions = 6.

2. We have
$$\frac{\sin x}{\cos x} + \frac{1}{\cos x} = 2\cos x \Rightarrow \sin x + 1 = 2\cos^2 x$$
$$\Rightarrow 2\sin^2 x + \sin x - 1 = 0 \Rightarrow [2\sin x - 1][\sin x + 1] = 0$$
   So,
$$\sin x = -1 \text{ or } \sin x = \frac{1}{2} \Rightarrow x = \frac{3\pi}{2}$$
   Therefore, we have
$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$
   But $3\pi/2$ does not satisfy the equation, so the total number of solutions = 2.

Your Turn 1

1. Find the general value of θ in the equation $2\sqrt{3}\cos\theta = \tan\theta$.

Ans. $\theta = n\pi + (-1)^n \dfrac{\pi}{3}$

2. Find the general value of θ in the equation $\cos 2\theta = \sin\alpha$.

Ans. $\theta = n\pi \pm \left(\dfrac{\pi}{4} - \dfrac{\alpha}{2}\right)$

3. If $\begin{vmatrix} \cos(A+B) & -\sin(A+B) & \cos 2B \\ \sin A & \cos A & \sin B \\ -\cos A & \sin A & \cos B \end{vmatrix} = 0$, then find the value of B.

Ans. $B = (2n+1)\dfrac{\pi}{2}$

4. Find the general value of θ in the equation $\cos\theta + \cos 2\theta + \cos 3\theta = 0$.

Ans. $\theta = 2m\pi \pm \dfrac{2\pi}{3}$

5. Find the general value of θ in the equation $\sin 6\theta + \sin 4\theta + \sin 2\theta = 0$.

Ans. $\theta = n\pi \pm \dfrac{\pi}{3}$

3.4 System of Equations

1. **One equation with one unknown angle:**

(a) *Equation of the form $a\cos\theta + b\sin\theta = c$:*

In $a\cos\theta + b\sin\theta = c$, put

$$a = r\cos\alpha \text{ and } b = r\sin\alpha$$

where $r = \sqrt{a^2+b^2}$ and $|c| \le \sqrt{a^2+b^2}$.
Then,

$$r(\cos\alpha\cos\theta + \sin\alpha\sin\theta) = c$$

$$\Rightarrow \cos(\theta - \alpha) = \frac{c}{\sqrt{a^2+b^2}} = \cos\beta$$

$$\Rightarrow \theta - \alpha = 2n\pi \pm \beta \Rightarrow \theta = 2n\pi \pm \beta + \alpha$$

where $\tan\alpha = b/a$ is the general solution.

Alternatively, putting $a = r\sin\alpha$ and $b = r\cos\alpha$,

where $r = \sqrt{a^2+b^2}$, we get

$$\sin(\theta+\alpha) = \frac{c}{\sqrt{a^2+b^2}} = \sin\gamma \text{ (say)}$$

$$\Rightarrow \theta + \alpha = n\pi + (-1)^n\gamma \Rightarrow \theta = n\pi + (-1)^n\gamma - \alpha$$

where $\tan\alpha = b/a$ is the general solution.

Illustration 3.3 Find the number of the integral values of k for which the equation $7\cos x + 5\sin x = 2k+1$ has a solution.

Solution: We have

$$-\sqrt{7^2+5^2} \le (7\cos x + 5\sin x) \le \sqrt{7^2+5^2}$$

So, for solution

$$-\sqrt{74} \le (2k+1) \le \sqrt{74} \text{ or } -8.6 \le 2k+1 \le 8.6$$

$$\Rightarrow -9.6 \le 2k \le 7.6$$

$$\Rightarrow -4.8 \le k \le 3.8$$

So, the integral values of k are $-4, -3, -2, -1, 0, 1, 2, 3$ (eight values).

Illustration 3.4 Find the solution of equation $\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}$.

Solution:

$$\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}$$

$$\Rightarrow \frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \sin\frac{\pi}{3}\cos\theta + \cos\frac{\pi}{3}\sin\theta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \sin\left(\theta + \frac{\pi}{3}\right) = \sin\frac{\pi}{4}$$

$$\Rightarrow \theta = n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{3}$$

(b) *Equation of the form*

$$a_0\sin^n x + a_1\sin^{n-1}x\cos x + a_2\sin^{n-2}x\cos^2 x + \cdots + a_n\cos^n x = 0$$

Here, $a_0, a_1, \cdots, a_n$ are real numbers and the sum of the exponents in $\sin x$ and $\cos x$ in each term is equal to n. These numbers are said to be homogeneous with respect to $\sin x$ and $\cos x$. For $\cos x \ne 0$, the above equation can be written as $a_0\tan^n x + a_1\tan^{n-1}x + \cdots + a_n = 0$.

Illustration 3.5 Find the solution of equation $5\sin^2 x - 7\sin x\cos x + 16\cos^2 x = 4$.

Solution: To solve this kind of equation, we use the fundamental formula:

Trigonometrical identity, $\sin^2 x + \cos^2 x = 1$.
Writing the equation in the form

$$5\sin^2 x - 7\sin x\cos x + 16\cos^2 x = 4(\sin^2 x + \cos^2 x)$$

and simplifying, we get

$$\sin^2 x - 7\sin x\cos x + 12\cos^2 x = 0$$

Dividing by $\cos^2 x$ on both sides, we get

$$\tan^2 x - 7\tan x + 12 = 0$$

Now it can be factorized as

$$(\tan x - 3)(\tan x - 4) = 0 \Rightarrow \tan x = 3 \text{ or } 4$$

$$\tan x = \tan(\tan^{-1}3) \text{ or } \tan x = \tan(\tan^{-1}4)$$

$$\Rightarrow x = n\pi + \tan^{-1}3 \text{ or } x = n\pi + \tan^{-1}4$$

(c) Equation of the form $R(\sin kx, \cos nx, \tan mx, \cot lx) = 0$: Here, R is a rational function of the indicated arguments and k, l, m, n are the natural numbers, which can be reduced to a rational equation with respect to the arguments $\sin x$, $\cos x$, $\tan x$ and $\cot x$ by means of the formulas for trigonometric functions of the sum of angles (in particular, the formulas for double and triple angles) and then reduce equation of the given form to a rational equation with respect to the unknown $t = \tan(x/2)$ by means of the following formulas:

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}},$$

$$\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}},$$

$$\tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}},$$

$$\cot x = \frac{1-\tan^2\frac{x}{2}}{2\tan\frac{x}{2}}$$

Illustration 3.6 Find the general solution of the equation $(\cos x - \sin x)\left(2\tan x + \frac{1}{\cos x}\right) + 2 = 0.$

Solution: Using the formulas above the given equation can be rewritten as,

$$\left\{\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} - \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right\}\left\{\frac{4\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}} + \frac{1+\tan^2\frac{x}{2}}{1-\tan^2\frac{x}{2}}\right\} + 2 = 0$$

Let $t = \tan\frac{x}{2}$. Then

$$\left(\frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2}\right)\left(\frac{4t}{1-t^2} + \frac{1+t^2}{1-t^2}\right) + 2 = 0$$

$$\Rightarrow \frac{3t^4 + 6t^3 + 8t^2 - 2t - 3}{(t^2+1)(1-t^2)} = 0$$

Its roots are $t_1 = \frac{1}{\sqrt{3}}$ and $t_2 = -\frac{1}{\sqrt{3}}$.

Thus, the solution of the equation reduces to that of two elementary equations

$$\tan\frac{x}{2} = \frac{1}{\sqrt{3}}, \ \tan\frac{x}{2} = -\frac{1}{\sqrt{3}}$$

The solution is

$$\frac{x}{2} = n\pi \pm \frac{\pi}{6} \Rightarrow x = 2nx \pm \frac{\pi}{3}$$

(d) Equation of the form $R(\sin x + \cos x, \sin x \cdot \cos x) = 0$: Here, R is a rational function of the arguments in brackets. Put

$$\sin x + \cos x = t \quad (1)$$

and use the following identity:

$$(\sin x + \cos x)^2 = \sin^2 x + \cos^2 x + 2\sin x\cos x = 1 + 2\sin x \cos x$$

using Eq. (1) we thus have

$$\sin x \cos x = \frac{t^2-1}{2} \quad (2)$$

Taking Eqs. (1) and (2) into account, we can reduce the given equation to

$$R\left(t, \frac{t^2-1}{2}\right) = 0$$

Similarly, by the substitution $(\sin x - \cos x) = t$, we can reduce the equation of the form

$$R(\sin x - \cos x, \sin x \cos x) = 0$$

to an equation of the form

$$R\left(t, \frac{1-t^2}{2}\right) = 0$$

Illustration 3.7 Find the general solution of the equation $\sin^{10} x + \cos^{10} x = \frac{29}{16}\cos^4 2x$.

Solution: Using half-angle formulas, we can represent the given equation in the form

$$\left(\frac{1-\cos 2x}{2}\right)^5 + \left(\frac{1+\cos 2x}{2}\right)^5 = \frac{29}{16}\cos^4 2x$$

Put $\cos 2x = t$. We get

$$\left(\frac{1-t}{2}\right)^5 + \left(\frac{1+t}{2}\right)^5 = \frac{29}{16}t^4$$

$$\Rightarrow 24t^4 - 10t^2 - 1 = 0$$

whose only real root is $t^2 = 1/2$. Therefore,

$$\cos^2 2x = \frac{1}{2} \Rightarrow 1 + \cos 4x = 1$$

$$\Rightarrow \cos 4x = 0 \Rightarrow 4x = (2n+1)\frac{\pi}{2}$$

$$\Rightarrow x = \frac{n\pi}{4} + \frac{\pi}{8}; n \in I$$

2. **Two equations with one unknown angle:** Two equations are given and we have to find the values of variable θ which may satisfy the given equations.
   - **(a)** $\cos\theta = \cos\alpha$ and $\sin\theta = \sin\alpha$
     So the common solution is $\theta = 2n\pi + \alpha$.
   - **(b)** $\sin\theta = \sin\alpha$ and $\tan\theta = \tan\alpha$
     So the common solution is $\theta = 2n\pi + \alpha$.
   - **(c)** $\cos\theta = \cos\alpha$ and $\tan\theta = \tan\alpha$
     So the common solution is $\theta = 2n\pi + \alpha$.

Illustration 3.8 Find the most general value of θ satisfying the equation $\tan\theta = -1$ and $\cos\theta = \frac{1}{\sqrt{2}}$.

Solution:

$$\tan\theta = -1 = \tan\left(2\pi - \frac{\pi}{4}\right)$$

and

$$\cos\theta = \frac{1}{\sqrt{2}} = \cos\left(2\pi - \frac{\pi}{4}\right)$$

Hence, the general value is

$$\theta = 2n\pi + \left(2\pi - \frac{\pi}{4}\right) = 2n\pi + \frac{7\pi}{4}$$

3. **Two equations with two unknown angles:** Let $f(\theta,\phi)=0$, $g(\theta,\phi)=0$ be the system of two equations in two unknowns.

   Step (i): Eliminate any one variable, say ϕ. Let $\theta=\alpha$ be one solution.

   Step (ii): Then consider the system $f(\alpha,\phi)=0$, $g(\alpha,\phi)=0$ and use the method of two equations in one variable.

Illustration 3.9 Solve the system of equations $\sec\theta = \sqrt{2}\sec\phi$, $\tan\theta = \sqrt{3}\tan\phi$.

Solution: Usually, students proceed with such type of problems in the following way:

Squaring and subtracting the two equations, we get

$$\sec^2\theta - \tan^2\theta = 2\sec^2\phi - 3\tan^2\phi$$

$$\Rightarrow 2\tan^2\phi + 2 - 3\tan^2\phi = 1 \text{ or } \tan^2\phi = 1 \text{ or } \phi = n\pi \pm \frac{\pi}{4} \quad (1)$$

Also, we have

$$\sec^2\phi - \tan^2\phi = \frac{\sec^2\theta}{2} - \frac{\tan^2\theta}{3}$$

which gives

$$6 = 3\sec^2\theta - 2\tan^2\theta$$

$$\Rightarrow \tan^2\theta = 3$$

and so $\theta = m\pi \pm \frac{\pi}{3}$.

Thus, the solution of this system is

$$\theta = m\pi \pm \frac{\pi}{3} \text{ and } \phi = n\pi \pm \frac{\pi}{4}, m, n \in I \quad (2)$$

Now see the fallacies: $\theta = \frac{\pi}{3}$ and $\phi = -\frac{\pi}{4}$ (from the solution) give

$$\sec\left(\frac{\pi}{3}\right) = \sqrt{2}\sec\left(-\frac{\pi}{4}\right) \Rightarrow 2 = 2$$

But

$$\tan\left(\frac{\pi}{3}\right) = \sqrt{3}\tan\left(-\frac{\pi}{4}\right)$$

gives $\sqrt{3} = -\sqrt{3}$. Thus, the solution given in (2) consists of many extraneous (absurd) solutions. The simple reason for this is quite obvious. Equation (2) consists of solutions of the following four systems:

$$\sec\theta = \sqrt{2}\sec\phi,\ \tan\theta = \sqrt{3}\tan\phi \quad (3)$$

$$\sec\theta = \sqrt{2}\sec\phi,\ \tan\theta = -\sqrt{3}\tan\phi \quad (4)$$

$$\sec\theta = -\sqrt{2}\sec\phi,\ \tan\theta = \sqrt{3}\tan\phi \quad (5)$$

and

$$\sec\theta = -\sqrt{2}\sec\phi,\ \tan\theta = -\sqrt{3}\tan\phi \quad (6)$$

While we have to find the values which satisfy Eq. (3), we have to verify the solutions and should retain only the valid ones.

Alternative method: A better method for such type of equations is the following:

The given system is

$$\sec\theta = \sqrt{2}\sec\phi \quad (7)$$

$$\tan\theta = \sqrt{3}\tan\phi \quad (8)$$

Solving Eqs. $(7)^2 - (8)^2$ gives

$$\tan^2\phi = 1$$

Therefore,

$$\phi = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

Case 1: $\phi = \pi/4$. The system reduces to $\sec\theta = 2$, $\tan\theta = \sqrt{3}$ so $\theta = \frac{\pi}{3}$.

Therefore,

$$\theta = 2n\pi + \frac{\pi}{3},\ \phi = 2m\pi + \frac{\pi}{4} \quad (9)$$

Case 2: $\phi = 3\pi/4$. Then we have $\sec\theta = -2$, $\tan\theta = -\sqrt{3}$, so

$$\theta = \frac{2\pi}{3}$$

Thus, the general solution is

$$\theta = 2n\pi + \frac{2\pi}{3},\ \phi = 2m\pi + \frac{3\pi}{4} \quad (10)$$

Case 3: $\phi = \frac{5\pi}{4}\left(\text{or can be taken as} -\frac{3\pi}{4}\right)$. Then, $\sec\theta = -2$, $\tan\theta = \sqrt{3}$. Therefore, $\theta = \frac{4\pi}{3}$. Thus

$$\theta = 2n\pi + \frac{4\pi}{3},\ \phi = 2m\pi + \frac{5\pi}{4} \text{ or}$$

$$\theta = 2n\pi - \frac{2\pi}{3},\ \phi = 2m\pi - \frac{3\pi}{4} \quad (11)$$

Case 4: $\phi = \frac{7\pi}{4}\left(\text{or} -\frac{\pi}{4}\right)$. Then, $\sec\theta = 2$, $\tan\theta = -\sqrt{3}$, so $\theta = -\frac{\pi}{3}$.

Hence,

$$\theta = 2n\pi - \frac{\pi}{3},\ \phi = 2mx - \frac{\pi}{4} \quad (12)$$

Hence, the required solutions are given as

$$(\theta,\phi)=\left(2n\pi+\frac{\pi}{3},2m\pi+\frac{\pi}{4}\right);\ \left(2n\pi+\frac{2\pi}{3},2m\pi+\frac{3\pi}{4}\right);$$
$$\left(2n\pi-\frac{2\pi}{3},2m\pi-\frac{3\pi}{4}\right);\ \left(2n\pi-\frac{\pi}{3},2m\pi-\frac{\pi}{4}\right).$$

Your Turn 2

1. Find the solution of the equation $\sqrt{3}\sin x+\cos x=4$.

   Ans. No solution

2. Find the general solution of the equation $(\sqrt{3}-1)\sin\theta+(\sqrt{3}+1)\cos\theta=2$.

   Ans. $\theta=2n\pi\pm\frac{\pi}{4}+\frac{\pi}{12}$

3. Find the solution of the equation $\sin x+\cos x-2\sqrt{2}\sin x\cos x=0$.

   Ans. $x=n\pi+(-1)^n\frac{\pi}{6}-\left(\frac{\pi}{4}\right)$

4. Find the most general value of θ which will satisfy both the equations $\sin\theta=\frac{-1}{2}$ and $\tan\theta=\frac{1}{\sqrt{3}}$.

   Ans. $2n\pi+\frac{7\pi}{6}$

5. Find the value of θ and ϕ for the equation $\left(\frac{\sin\theta}{\sin\phi}\right)^2=\frac{\tan\theta}{\tan\phi}=3$.

   Ans. $\phi=n\pi\pm\frac{\pi}{6}$

3.5 Key Points to be Remembered for Solving the Trigonometric Equation

1. Check the validity of the equation before solving, if possible.
2. Squaring should be avoided as far as possible. If squaring is done, check for extraneous roots.
3. Do not cancel terms containing 'unknown' on two sides of the equation. It may cause root loss.
4. All solutions must come within the domain of the variable.
5. The problems of trigonometric equations can be solved either by the factorization method or by using the form $a\cos\theta+b\sin\theta$.
6. Given a choice of converting equation of a given problem into either the sine form or the cosine form, then one should prefer the cosine form.

3.6 Trigonometric Inequation

The basic method to solve trigonometric inequations is to find the angle that satisfies it, lying in the interval $[0, 2\pi]$ and then generalize it to accommodate all possible solutions. The methods can slightly change depending on the problem and the periods of trigonometric functions. Consider the following illustrations.

Illustration 3.10 Find the values of θ satisfying $\sin\theta>0$.

Solution: $\sin\theta$ is positive in quadrants 1 and 2. So $0<\theta<\pi$. But quadrants 1 and 2 can also be written as $2\pi<\theta<3\pi$ or $4\pi<\theta<5\pi$. So, the general solution can be obtained by adding $2n\pi$ to the first solution. That is,

$$2n\pi<\theta<(2n+1)\pi;\ n\in I$$

Illustration 3.11 Solve $\tan^2 x-(1+\sqrt{3})\tan x+\sqrt{3}<0$.

Solution: The given inequality is

$$(\tan x-1)(\tan x-\sqrt{3})<0$$
$$\Rightarrow 1<\tan x<\sqrt{3}\ \Rightarrow\ \frac{\pi}{4}<x<\frac{\pi}{3}$$

Since the tangent function repeats after an interval of length π, so the general solution is

$$n\pi+\frac{\pi}{4}<x<n\pi+\frac{\pi}{3};\quad n\in I$$

3.7 Equations Containing Combination of Trigonometric and Non-Trigonometric Expressions

Consider the equation $\sec^2 x=1-y^2$. There are two unknowns and one equation. We note that since $\sec^2 x\geq 1\ \forall x\in(-\infty,-1]\cup[1,\infty)$ and $1-y^2\leq 1\ \forall y\in R$, the given equation can hold only if $\sec^2 x=1$ and $1-y^2=1$, that is $x=n\pi$ and $y=0$.

So, the method is to find out the range of values that various expressions assume and then take their common portion.

Sometimes we are required to find the number of solutions and not the value of solutions. In such cases, we take the help of graphs. For example, consider the equation $\cos x=y$. If we sketch the graphs of $y=\cos x$ and $y=x$, we see that they intersect at exactly one point (Fig. 3.3). So, this equation has one solution lying in the interval $(0, \pi/2)$.

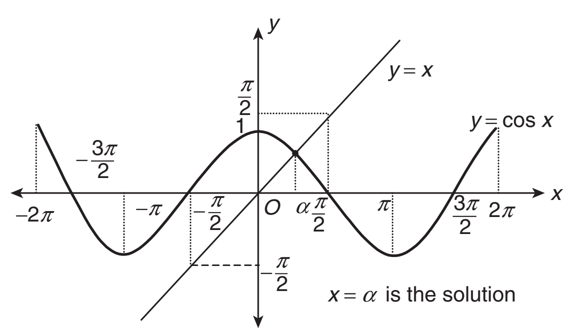


Figure 3.3

Illustration 3.12 By drawing graphs of $y=\tan x$ and $y=\frac{3(2-x)}{4}$, x measured in radians, find the number of solutions of the equation $4\tan x+3x=6$ for x between 0 and π.

Solution:

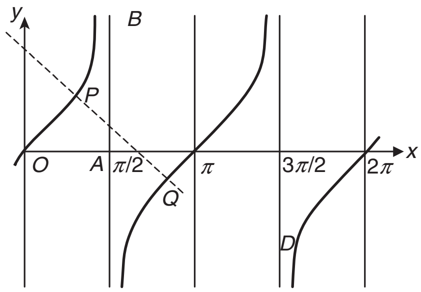


Figure 3.4

In Fig. 3.4, the graphs of $y = \tan x$ and $y = \frac{3(2-x)}{4}$ are drawn (dotted line) and these are found to intersect at two points P and Q for $0 \le x < \pi$. Hence the number of solutions is 2.

Additional Solved Examples

1. Solve cot (sin x + 3) = 1.

Solution:

$$\sin x + 3 = n\pi \pm \frac{\pi}{4} \Rightarrow 2 \le n\pi \pm \frac{\pi}{4} \le 4 \Rightarrow n = 1$$

$$\Rightarrow \sin x = \pi \pm \frac{\pi}{4} - 3$$

$$\Rightarrow x = n\pi + (-1)^n \sin^{-1}\left(\frac{5\pi}{4} - 3\right) \text{ or } n\pi + (-1)^n \sin^{-1}\left(\frac{3\pi}{4} - 3\right)$$

2. If sin 5x + sin 3x + sin x = 0, then find the value of x other than zero, lying between $0 \le x \le \pi/2$.

Solution:

$$\sin 5x + \sin 3x + \sin x = 0$$
$$\Rightarrow (\sin 5x + \sin x) + \sin 3x = 0$$

Using 2 sinA cosB = sin (A+B) + sin (A–B)

$$\Rightarrow 2 \sin 3x \cos 2x + \sin 3x = 0 \Rightarrow \sin 3x(2 \cos 2x + 1) = 0$$

$$\Rightarrow \sin 3x = 0; \cos 2x = -\frac{1}{2}$$

$$\Rightarrow 3x = n\pi, 2x = 2n\pi \pm \frac{2\pi}{3}$$

The required value of x is $\pi/3$.

3. Find all acute angles α such that $\cos \alpha \cos 2\alpha \cos 4\alpha = 1/8$.

Solution: It is given that

$$\cos\alpha \cos 2\alpha \cos 4\alpha = \frac{1}{8}$$

$$\Rightarrow 2\sin\alpha\cos\alpha\cos 2\alpha\cos 4\alpha = \frac{\sin\alpha}{4}$$

$$\Rightarrow 2\sin 2\alpha\cos 2\alpha\cos 4\alpha = \frac{\sin\alpha}{2}$$

$$\Rightarrow 2\sin 4\alpha\cos 4\alpha = \sin\alpha \Rightarrow \sin 8\alpha - \sin\alpha = 0$$

$$\Rightarrow 2\sin\frac{7\alpha}{2}\cos\frac{9\alpha}{2} = 0$$

Either $\sin\frac{7\alpha}{2} = 0$ or $\cos\frac{9\alpha}{2} = 0$.

Case I: $\sin\frac{7\alpha}{2} = 0$. Then

$$\frac{7\alpha}{2} = n\pi \Rightarrow \alpha = \frac{2n\pi}{7}$$

For $n = 0$, $\alpha = 0$ which is not a solution.

For $n = 1$, $\alpha = \frac{2\pi}{7}$.

Case II: $\cos\frac{9\alpha}{2} = 0$. Then

$$\frac{9\alpha}{2} = (2n+1)\frac{\pi}{2} \Rightarrow \alpha = (2n+1)\frac{\pi}{9} \Rightarrow \alpha = \frac{\pi}{9}, \frac{\pi}{3}$$

Hence,

$$\alpha = \frac{2\pi}{7}, \frac{\pi}{9}, \frac{\pi}{3}$$

4. Solve for x.

$$\log_{\sin^2 x}(2) + \log_{\cos^2 x}(2) + 2\log_{\sin^2 x}(2)\log_{\cos^2 x}(2) = 0$$

Solution:

$$\frac{1}{\log_2 \sin^2 x} + \frac{1}{\log_2 \cos^2 x} + \frac{2}{\log_2 \sin^2 x \times \log_2 \cos^2 x} = 0$$

$$\Rightarrow \frac{\log_2 \cos^2 x + \log_2 \sin^2 x + 2}{\log_2 \sin^2 x \times \log_2 \cos^2 x} = 0$$

$$\Rightarrow \log_2(\sin^2 x \cdot \cos^2 x) = -2$$

$$\Rightarrow \log_2\left(\frac{\sin 2x}{2}\right)^2 = -2$$

$$\Rightarrow \left(\frac{\sin 2x}{2}\right)^2 = \frac{1}{4} \Rightarrow \frac{\sin 2x}{2} = \pm\frac{1}{2}$$

$$\Rightarrow \sin 2x = \pm 1$$

$$\Rightarrow 2x = (2n+1)\frac{\pi}{2} \Rightarrow x = (2n+1)\frac{\pi}{4}, n \in I$$

5. Solve $\cos^2 x - \sin x - \frac{1}{4} = 0$.

Solution*:* Replacing $\cos^2 x$ by $1 - \sin^2 x$, we get a quadratic in sin of the form

$$4\sin^2 x + 4\sin x - 3 = 0$$
$$\Rightarrow (2\sin x + 3)(2\sin x - 1) = 0$$

Now

$$\sin x \ne -\frac{3}{2} \text{ since } |\sin x| \le 1$$

Therefore,

$$\sin x = \frac{1}{2}$$

The principal solution is $x = \frac{\pi}{6}$.

The general solution is $x = n\pi + (-1)^n \frac{\pi}{6}$.

6. Find the general solution of the equation $3\sin^2 x + 10\cos x - 6 = 0$.

Solution: The given equation can be written,

$$3(1-\cos^2 x) + 10\cos x - 6 = 0$$

On solving, we get

$$(\cos x - 3)(3\cos x - 1) = 0$$

Either $\cos x = 3$ (which is not possible) or

$$\cos x = \frac{1}{3} \Rightarrow x = 2n\pi \pm \cos^{-1}\left(\frac{1}{3}\right)$$

7. If the solutions for θ of $\cos p\theta + \cos q\theta = 0,\ p > 0,\ q > 0$ are in AP, then find the numerically smallest common difference of AP. **[Kerala (Engg.) 2001]**

Solution: Given

$$\cos p\theta = -\cos q\theta = \cos(\pi + q\theta) \Rightarrow p\theta = 2n\pi \pm (\pi + q\theta),\ n \in I$$

$$\Rightarrow \theta = \frac{(2n+1)\pi}{p-q} \text{ or } \frac{(2n-1)\pi}{p+q},\ n \in I$$

Both the solutions form an AP $\theta = \frac{(2n+1)\pi}{p-q}$ gives us an AP with common difference $= \frac{2\pi}{p-q}$.

$\theta = \frac{(2n-1)\pi}{p+q}$ gives us an AP with common difference $= \frac{2\pi}{p+q}$

Certainly, $\frac{2\pi}{p+q} < \left|\frac{2\pi}{p-q}\right|$.

8. Find the set of values of x for which

$$\frac{\tan 3x - \tan 2x}{1 + \tan 3x \tan 2x} = 1$$

Solution:

$$\tan(3x - 2x) = \tan x = 1 \Rightarrow x = n\pi + \frac{\pi}{4}$$

But this value does not satisfy the given equation.

9. Find the number of pairs (x, y) satisfying the equations $\sin x + \sin y = \sin(x+y)$ and $|x| + |y| = 1$.

Solution: The first equation can be written as

$$2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x+y)$$

when $\sin\frac{1}{2}(x+y) = 0$ we have that either $\sin\frac{1}{2}x = 0$ or $\sin\frac{1}{2}y = 0$.

Therefore, either $\sin\frac{1}{2}(x+y) = 0$ or $\cos\frac{1}{2}(x+y) = 0$.

$\Rightarrow x + y = 0$, or $x = 0$ or $y = 0$. As $|x| + |y| = 1$, therefore when $x + y = 0$, we have to reject $x + y = 1$, or $x + y = -1$ and solve it with $x - y = 1$ or $x - y = -1$ which gives $\left(\frac{1}{2}, \frac{-1}{2}\right)$ or $\left(\frac{-1}{2}, \frac{1}{2}\right)$ as the possible solution. Again solving with $x = 0$, we get $(0, \pm 1)$, and solving with $y = 0$, we get $(\pm 1, 0)$ as the other solution. Thus, we have six pairs of solution for x and y.

10. Find the value of $\cos\left(\theta - \frac{\pi}{4}\right)$ if $\tan(\pi\cos\theta) = \cot(\pi\sin\theta)$.

Solution: From $\tan(\pi\cos\theta) = \cot(\pi\sin\theta)$, we have

$$\tan(\pi\cos\theta) = \tan\left(\frac{\pi}{2} - \pi\sin\theta\right)$$

$$\pi\cos\theta = n\pi + \pi/2 - \pi\sin\theta, (n \in I)$$

$$\cos\theta + \sin\theta = n + \frac{1}{2} \Rightarrow \frac{\cos\theta + \sin\theta}{\sqrt{2}} = \frac{1}{2\sqrt{2}}(2n+1)$$

$$\Rightarrow \cos\left(\theta - \frac{\pi}{4}\right) = \frac{(2n+1)}{2\sqrt{2}} (n \in I)$$

11. Find the only value of x for which $2^{\sin x} + 2^{\cos x} > 2^{1-(1/\sqrt{2})}$ holds.

Solution: Since AM ≥ GM, we have

$$\frac{1}{2}(2^{\sin x} + 2^{\cos x}) \geq \sqrt{2^{\sin x} \cdot 2^{\cos x}}$$

$$\Rightarrow 2^{\sin x} + 2^{\cos x} \geq 2 \cdot 2^{\frac{\sin x + \cos x}{2}}$$

$$\Rightarrow 2^{\sin x} + 2^{\cos x} \geq 2^{1 + \frac{\sin x + \cos x}{2}}$$

And, we know that $\sin x + \cos x \geq -\sqrt{2}$. Therefore

$$2^{\sin x} + 2^{\cos x} > 2^{1-(1/\sqrt{2})} \text{ for } x = \frac{5\pi}{4}$$

Previous Years' Solved JEE Main/AIEEE Questions

1. Let A and B denote the statements

$$A: \cos\alpha + \cos\beta + \cos\gamma = 0$$

$$B: \sin\alpha + \sin\beta + \sin\gamma = 0$$

If $\cos(\beta-\gamma)+\cos(\gamma-\alpha)+\cos(\alpha-\beta)=-\frac{3}{2}$, then

(A) A is true and B is false
(B) A is false and B is true
(C) Both A and B are true
(D) Both A and B are false

[AIEEE 2009]

Solution: Given

$$\cos(\beta-\gamma)+\cos(\gamma-\alpha)+\cos(\alpha-\beta)=-\frac{3}{2}$$

$$\Rightarrow 2[\cos(\beta-\gamma)+\cos(\gamma-\alpha)+\cos(\alpha-\beta)]+3=0$$

$$\Rightarrow 2[\cos(\beta-\gamma)+\cos(\gamma-\alpha)+\cos(\alpha-\beta)]+\sin^2\alpha +\cos^2\alpha+\sin^2\beta+\cos^2\beta+\sin^2\gamma+\cos^2\gamma=0$$

$$\Rightarrow 2\cos\beta\cos\gamma+2\sin\beta\sin\gamma+2\cos\gamma\cos\alpha +2\sin\gamma\sin\alpha+2\cos\alpha\cos\beta+2\sin\alpha\sin\beta+\sin^2\alpha+\cos^2\alpha +\sin^2\beta+\cos^2\beta+\sin^2\gamma+\cos^2\gamma=0$$

$$\Rightarrow (\sin\alpha+\sin\beta+\sin\gamma)^2+(\cos\alpha+\cos\beta+\cos\gamma)^2=0$$

$$\Rightarrow \sin\alpha+\sin\beta+\sin\gamma=0 \text{ and } \cos\alpha+\cos\beta+\cos\gamma=0$$

Therefore, both A and B are true.

Hence, the correct answer is option (C).

2. Let $\cos(\alpha+\beta)=\frac{4}{5}$ and $\sin(\alpha-\beta)=\frac{5}{13}$, where $0\le\alpha,\beta\le\frac{\pi}{4}$. Then $\tan 2\alpha=$

(A) $\frac{56}{33}$ **(B)** $\frac{19}{12}$ **(C)** $\frac{20}{7}$ **(D)** $\frac{25}{16}$

[AIEEE 2010]

Solution:

$$\cos(\alpha+\beta)=\frac{4}{5}\Rightarrow\tan(\alpha+\beta)=\frac{3}{4}$$

$$\left(\text{by triangle method also } \because \alpha+\beta\in\left[0,\frac{\pi}{2}\right]\right)$$

$$\sin(\alpha-\beta)=\frac{5}{13}\Rightarrow\tan(\alpha-\beta)=\frac{5}{12}$$

$$\left(\begin{array}{l}\text{by triangle method also } \because \alpha-\beta\in\left[0,\frac{\pi}{4}\right]\\ \textbf{Note:}\ \alpha-\beta\notin\left[-\frac{\pi}{2},0\right]\because\sin(\alpha-\beta)>0\end{array}\right)$$

Therefore,

$$\tan2\alpha=\tan(\alpha+\beta+\alpha-\beta)=\frac{\tan(\alpha+\beta)+\tan(\alpha-\beta)}{1-\tan(\alpha+\beta)\tan(\alpha-\beta)}$$

$$=\frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}\cdot\frac{5}{12}}=\frac{56}{33}$$

Hence, the correct answer is option (A).

3. If $A=\sin^2x+\cos^4x$, then for all real x

(A) $\frac{13}{16}\le A\le1$ **(B)** $1\le A\le2$
(C) $\frac{3}{4}\le A\le\frac{13}{16}$ **(D)** $\frac{3}{4}\le A\le1$

[AIEEE 2011]

Solution:

$$A=\sin^2x+\cos^4x=\left(\frac{1-\cos2x}{2}\right)+\left(\frac{3+4\cos2x+\cos4x}{8}\right)$$

(since, $\cos2\theta=2\cos^2\theta-1$)

$$=\frac{3+\cancel{4\cos2x}+\cos4x+4-\cancel{4\cos2x}}{8}=\frac{7+\cos4x}{8}$$

Therefore,

$$\frac{7-1}{8}\le A\le\frac{7+1}{8}\Rightarrow\frac{3}{4}\le A\le1$$

Hence, the correct answer is option (D).

4. In a ΔPQR, if $3\sin P+4\cos Q=6$ and $4\sin Q+3\cos P=1$, then the angle R is equal to

(A) $\frac{5\pi}{6}$ **(B)** $\frac{\pi}{6}$ **(C)** $\frac{\pi}{4}$ **(D)** $\frac{3\pi}{4}$

[AIEEE 2012]

Solution: It is given that

$$3\sin P+4\cos Q=6 \quad (1)$$

$$4\sin Q+3\cos P=1 \quad (2)$$

From Eqs. (1) and (2), it is clear that $\angle P$ is obtuse. Therefore,

$$(3\sin P+4\cos Q)^2+(4\sin Q+3\cos P)^2=37$$

$$\Rightarrow 9+16+24(\sin P\cos Q+\cos P\sin Q)^2=37$$

$$\Rightarrow 24\sin(P+Q)=12\Rightarrow\sin(P+Q)=\frac{1}{2}$$

$$\Rightarrow P+Q=\frac{5\pi}{6}\Rightarrow R=\frac{\pi}{6}$$

Hence, the correct answer is option (B).

5. The expression $\frac{\tan A}{1-\cot A}+\frac{\cot A}{1-\tan A}$ can be written as

(A) $\sec A\operatorname{cosec}A+1$ **(B)** $\tan A+\cot A$
(C) $\sec A+\operatorname{cosec}A$ **(D)** $\sin A\cos A+1$

[JEE MAIN 2013]

Solution: We have

$$\frac{\tan A}{1-\cot A}+\frac{\cot A}{1-\tan A}=\frac{1}{\cot A(1-\cot A)}+\frac{\cot A\times\cot A}{\cot A\times(1-\tan A)}$$

$$=\frac{1}{\cot A(1-\cot A)}-\frac{\cot^2A}{(1-\cot A)}=\frac{1-\cot^3A}{\cot A(1-\cot A)}$$

$$=\frac{\operatorname{cosec}^2A+\cot A}{\cot A}=\sec A\operatorname{cosec}A+1$$

Hence, the correct answer is option (A).

6. Let $f_k(x)=\frac{1}{k}(\sin^k x+\cos^k x)$, where $x\in\mathbb{R}$ and $k \geq 1$. Then, $f_4(x)-f_6(x)$ equals

(A) $\frac{1}{4}$ **(B)** $\frac{1}{12}$ **(C)** $\frac{1}{6}$ **(D)** $\frac{1}{3}$

[JEE MAIN 2014 (OFFLINE)]

Solution:

$$f_4(x)-f_6(x)=\frac{1}{4}(\sin^4 x+\cos^4 x)-\frac{1}{6}(\cos^6 x+\sin^6 x)$$

$$=\frac{1}{4}[(\sin^2 x+\cos^2 x)^2-2\sin^2 x\cos^2 x]$$

$$-\frac{1}{6}[(\cos^2 x+\sin^2 x)^3-3\cos^2 x\sin^2 x(\cos^2 x+\sin^2 x)]$$

$$=\frac{1}{4}[1-2\sin^2 x\cos^2 x]-\frac{1}{6}[1-3\cos^2 x\sin^2 x]$$

$$=\frac{1}{4}-\frac{1}{2}\sin^2 x\cos^2 x-\frac{1}{6}+\frac{1}{2}\cos^2 x\sin^2 x=\frac{3-12}{12}=\frac{1}{12}$$

Hence, the correct answer is option (B).

7. The number of the values of α in $[0, 2\pi]$ for which $2\sin^3\alpha-7\sin^2\alpha+7\sin\alpha=2$ is

(A) 6 **(B)** 4 **(C)** 3 **(D)** 1

[JEE MAIN 2014 (ONLINE SET-1)]

Solution:

$$2\sin^3\alpha-7\sin^2\alpha+7\sin\alpha-2=0$$
$$\Rightarrow 2(\sin^3\alpha-1)-7\sin\alpha(\sin\alpha-1)=0$$
$$\Rightarrow 2(\sin\alpha-1)(\sin^2\alpha+\sin\alpha+1)-7\sin\alpha(\sin\alpha-1)=0$$
$$\Rightarrow (\sin\alpha-1)\{2\sin^2\alpha+2\sin\alpha+2-7\sin\alpha\}=0$$
$$\Rightarrow (\sin\alpha-1)\{2\sin^2\alpha-5\sin\alpha+2\}=0$$
$$\Rightarrow \sin\alpha=1, \sin\alpha=\frac{5\pm\sqrt{25-16}}{4}=\frac{5\pm 3}{4}=2,\frac{1}{2}$$

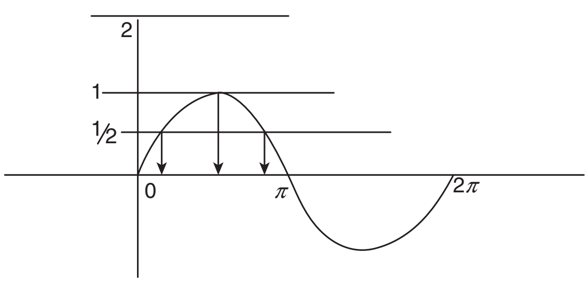


Figure 3.5

From Fig. 3.5, we find three solutions.
Hence, the correct answer is option (C).

8. If $\operatorname{cosec}\theta=\frac{p+q}{p-q}$ $(p\neq q\neq 0)$, then $\left|\cot\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right|$ is equal to

(A) $\sqrt{\frac{p}{q}}$ **(B)** $\sqrt{\frac{q}{p}}$ **(C)** $\sqrt{qp}$ **(D)** pq

[JEE MAIN 2014 (ONLINE SET-1)]

Solution: Given

$$\frac{\operatorname{cosec}\theta}{1}=\frac{p+q}{p-q}\ (p\neq q\neq 0)$$

By componendo and dividendo,

$$\frac{\operatorname{cosec}\theta+1}{\operatorname{cosec}\theta-1}=\frac{-p}{-q}\Rightarrow\frac{1+\sin\theta}{1-\sin\theta}=\frac{p}{q}$$

$$\frac{\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}=\frac{p}{q}$$

$$\Rightarrow\frac{\left(\sin^2\frac{\theta}{2}+\cos\frac{\theta}{2}\right)^2}{\left(\sin\frac{\theta}{2}-\cos\frac{\theta}{2}\right)^2}=\frac{p}{q}\Rightarrow\left|\frac{1+\cot\frac{\theta}{2}}{1-\cot\frac{\theta}{2}}\right|=\sqrt{\frac{p}{q}}$$

Now

$$\left|\cot\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right|=\left|\frac{1}{\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)}\right|=\left|\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}\right|=\left|\frac{\cot\frac{\theta}{2}-1}{\cot\frac{\theta}{2}+1}\right|$$

$$=\frac{1}{\left|\left(\cot\frac{\theta}{2}+1\right)\Big/\left(\cot\frac{\theta}{2}-1\right)\right|}$$

$$=\frac{1}{\left|1+\cot\frac{\theta}{2}\Big/1-\cot\frac{\theta}{2}\right|}=\frac{1}{\sqrt{\frac{p}{q}}}=\sqrt{\frac{q}{p}}$$

Hence, the correct answer is option (B).

9. Let f be a function defined on the set of the real numbers such that for $x\geq 0$, $f(x)=3\sin x+4\cos x$. Then $f(x)$ at $x=-\frac{11\pi}{6}$ is equal to

(A) $\frac{3}{2}+2\sqrt{3}$ **(B)** $-\frac{3}{2}+2\sqrt{3}$

(C) $\frac{3}{2}-2\sqrt{3}$ **(D)** $-\frac{3}{2}-2\sqrt{3}$

[JEE MAIN 2014 (ONLINE SET-2)]

Solution:

$$x=-\frac{11\pi}{6}=2\pi-\frac{\pi}{6}$$

$$\Rightarrow y\left(-\frac{11\pi}{6}\right)=y\left(2\pi-\frac{\pi}{6}\right)=3\sin\left(2\pi-\frac{\pi}{6}\right)+4\cos\left(2\pi-\frac{\pi}{6}\right)$$

$$=3\sin\left(-\frac{\pi}{6}\right)+4\cos\left(-\frac{\pi}{6}\right)\left[\begin{array}{l}\text{using, }\sin\left(-\frac{\pi}{6}\right)=-\sin\left(\frac{\pi}{6}\right)=\left(-\frac{1}{2}\right)\\ \text{and }\quad\cos\left(-\frac{\pi}{6}\right)=-\cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}\end{array}\right]$$

$$\Rightarrow y\left(-\frac{11\pi}{6}\right)=\frac{-3}{2}+2\sqrt{3}$$

Hence, the correct answer is option (B).

10. If $2\cos\theta + \sin\theta = 1 \left(\theta \neq \frac{\pi}{2}\right)$, then $7\cos\theta + 6\sin\theta$ is equal to

(A) $\frac{1}{2}$ **(B)** 2 **(C)** $\frac{11}{2}$ **(D)** $\frac{46}{5}$

[JEE MAIN 2014 (ONLINE SET-2)]

Solution:

$$2\cos\theta + \sin\theta = 1\left(\theta \neq \frac{\pi}{2}\right) \Rightarrow 2\cos\theta = 1 - \sin\theta$$

On squaring, we get

$$4\cos^2\theta = 1 + \sin^2\theta - 2\sin\theta$$
$$\Rightarrow 4(1 - \sin^2\theta) = 1 + \sin^2\theta - 2\sin\theta$$
$$\Rightarrow 4 - 4\sin^2\theta = 1 + \sin^2\theta - 2\sin\theta$$
$$\Rightarrow 5\sin^2\theta - 2\sin\theta - 3 = 0$$

Therefore,

$$\sin\theta = \frac{2 \pm \sqrt{4+60}}{10} = \frac{2 \pm 8}{10} = 1,\ -\frac{6}{10}\left(\text{or} -\frac{3}{5}\right),\ \text{since } \theta \neq \frac{\pi}{2}$$

Therefore,

$$\cos^2\theta = 1 - \left(\frac{-3}{5}\right)^2 = 1 - \frac{9}{25} = \frac{16}{25}$$

$$\Rightarrow \cos\theta = \pm\frac{4}{5}$$

Hence, two possibilities exist:

$$7\left(\frac{4}{5}\right) + 6\left(\frac{-3}{5}\right) = \frac{28}{5} - \frac{18}{5} = 2$$

and $$7\left(-\frac{4}{5}\right) + 6\left(\frac{-3}{5}\right) = \frac{-28}{5} - \frac{18}{5} = \frac{-46}{5}$$

Hence, the correct answer is option (B).

11. If $0 \leq x < 2\pi$, then the number of the real values of x, which satisfy the equation $\cos x + \cos 2x + \cos 3x + \cos 4x = 0$, is

(A) 9 **(B)** 3 **(C)** 5 **(D)** 7

[JEE MAIN 2016 (OFFLINE)]

Solution: We have

$$(\cos x + \cos 3x) + (\cos 2x + \cos 4x) = 0,\ x \in [0, 2\pi]$$
$$\Rightarrow 2\cos 2x \cos x + 2\cos 3x \cos x = 0$$
$$\Rightarrow 2\cos x(\cos 2x + \cos 3x) = 0$$

Therefore,

$$4\cos x \cos\frac{5x}{2}\cos\frac{x}{2} = 0$$

$\cos x = 0$	or	$\cos\frac{5x}{2} = 0$	or	$\cos\frac{x}{2} = 0$
$x = 90°, 270°$		$\frac{5x}{2} = (2n+1)\frac{\pi}{2}$		$\frac{x}{2} = (2m+1)\frac{\pi}{2}$
		$\Rightarrow x = (2n+1)\frac{\pi}{5}$		$\Rightarrow x = (2m+1)\pi$
		$\Rightarrow x = \frac{\pi}{5}, \frac{3\pi}{5}, \frac{7\pi}{5}, \frac{9\pi}{5}$		

Therefore, the total number of solutions is 7.

Hence, the correct answer is option (D).

12. If $A > 0$, $B > 0$ and $A + B = \pi/6$, then the minimum value of $\tan A + \tan B$ is

(A) $\sqrt{3} - \sqrt{2}$ **(B)** $4 - 2\sqrt{3}$

(C) $\frac{2}{\sqrt{3}}$ **(D)** $2 - \sqrt{3}$

[JEE MAIN 2016 (ONLINE SET-2)]

Solution: We have $A > 0$, $B > 0$ and $A + B = \pi/6$. Therefore,

$$\tan A + \tan B = \frac{\sin A}{\cos A} + \frac{\sin B}{\cos B} = \frac{\sin(A+B)}{\cos A \cos B}$$
$$= \frac{1}{2\cos A \cos B}$$
$$= \frac{1}{\cos(A+B) + \cos(A-B)}$$
$$= \frac{1}{(\sqrt{3}/2) + \cos(A-B)}$$

For the maximum value of $\tan A + \tan B$, $A = B$. Therefore, from the above equation, we get

$$\frac{1}{(\sqrt{3}/2) + 1} = \frac{2}{2 + \sqrt{3}} = 2(2 - \sqrt{3}) = 4 - 2\sqrt{3}$$

Hence, the correct answer is option (B).

Previous Years' Solved JEE Advanced/ IIT-JEE Questions

1. The number of solutions of the pair of equations $2\sin^2\theta - \cos 2\theta = 0$ and $2\cos^2\theta - 3\sin\theta = 0$ in the interval $[0, 2\pi]$ is

(A) zero **(B)** one **(C)** two **(D)** four

[IIT-JEE 2007]

Solution: The first equation is

$$2\sin^2\theta - \cos 2\theta = 0 \quad (1)$$

It can be written as

$$2\sin^2\theta - (1 - 2\sin^2\theta) = 0$$
$$\Rightarrow 4\sin^2\theta = 1$$
$$\Rightarrow \sin\theta = \frac{1}{2}, \frac{-1}{2}$$

The second equation given is,

$$2\cos^2\theta - 3\sin\theta = 0 \quad (2)$$

It can be written as

$$2 - 2\sin^2\theta - 3\sin\theta = 0$$
$$\Rightarrow 2\sin^2\theta + 3\sin\theta - 2 = 0$$
$$\Rightarrow \sin\theta = \frac{1}{2}, -2$$

Hence, $\sin\theta = \frac{1}{2}$ is a common solution.

Therefore, the number of solutions is two: $\theta=\frac{\pi}{6}$ and $\frac{5\pi}{6}$ {where $\theta\in[0,2\pi]$}.

Hence, the correct answer is option (C).

2. If $\frac{\sin^4 x}{2}+\frac{\cos^4 x}{3}=\frac{1}{5}$, then

(A) $\tan^2 x=\frac{2}{3}$ **(B)** $\frac{\sin^8 x}{8}+\frac{\cos^8 x}{27}=\frac{1}{125}$

(C) $\tan^2 x=\frac{1}{3}$ **(D)** $\frac{\sin^8 x}{8}+\frac{\cos^8 x}{27}=\frac{2}{125}$

[IIT-JEE 2009]

Solution: We have

$$\frac{\sin^4 x}{2}+\frac{\cos^4 x}{3}=\frac{1}{5}$$
$$3\sin^4 x+2(1-\sin^2 x)^2=\frac{6}{5}$$
$$\Rightarrow 25\sin^4 x-20\sin^2 x+4=0$$
$$\Rightarrow \sin^2 x=\frac{2}{5} \text{ and } \cos^2 x=\frac{3}{5}$$

Therefore, $\tan^2 x=\frac{2}{3}$ and $\frac{\sin^8 x}{8}+\frac{\cos^8 x}{27}=\frac{1}{125}$.

Hence, the correct answers are options (A) and (B).

3. For $0<\theta<\frac{\pi}{2}$, the solution(s) of

$$\sum_{m=1}^{6}\operatorname{cosec}\left(\theta+\frac{(m-1)\pi}{4}\right)\operatorname{cosec}\left(\theta+\frac{m\pi}{4}\right)=4\sqrt{2} \text{ is (are)}$$

(A) $\frac{\pi}{4}$ **(B)** $\frac{\pi}{6}$ **(C)** $\frac{\pi}{12}$ **(D)** $\frac{5\pi}{12}$

[IIT-JEE 2009]

Solution: Given solutions are

$$\frac{1}{\sin(\pi/4)}\left[\frac{\sin(\theta+\pi/4-\theta)}{\sin\theta\cdot\sin(\theta+\pi/4)}+\frac{\sin[\theta+\pi/2-(\theta+\pi/4)]}{\sin(\theta+\pi/4)\cdot(\theta+\pi/2)}+\cdots+\frac{\sin[(\theta+3\pi/2)-(\theta+5\pi/4)]}{\sin(\theta+3\pi/2)\cdot\sin(\theta+5\pi/4)}\right]$$
$$=4\sqrt{2}$$
$$\Rightarrow \sqrt{2}[\cos\theta-\cot(\theta+\pi/4)+\cot(\theta+\pi/4)-\cot(\theta+\pi/2)+\cdots$$
$$+\cot(\theta+5\pi/4)-\cot(\theta+3\pi/2)]=4\sqrt{2}$$
$$\Rightarrow \tan\theta+\cot\theta=4\Rightarrow\tan\theta=2\pm\sqrt{3}$$
$$\Rightarrow\theta=\frac{\pi}{12} \text{ or } \frac{5\pi}{12}$$

Hence, the correct answers are options (C) and (D).

4. If the angles A, B and C of a triangle are in an arithmetic progression and if a, b and c denote the lengths of the sides opposite to A, B and C, respectively, then the value of the expression $\frac{a}{c}\sin 2C+\frac{c}{a}\sin 2A$ is

(A) $\frac{1}{2}$ **(B)** $\frac{\sqrt{3}}{2}$ **(C)** 1 **(D)** $\sqrt{3}$

[IIT-JEE 2010]

Solution:

$$\frac{2R\sin A}{2R\sin C}\cdot 2\sin C\cos C+\frac{2R\sin C}{2R\sin A}2\sin A\cos A$$
$$=2(\sin A\cos C+\sin C\cos A)$$
$$=2\sin(A+C)=2\sin(A-B)=2\sin B$$

Given that angles of the triangle are in AP $\Rightarrow 2\angle B=\angle A+\angle C$ and $\angle A+\angle B+\angle C=\pi$

$$\Rightarrow\angle B=\frac{\pi}{3}$$

Hence, $2\sin B=2\sin\frac{\pi}{3}=\sqrt{3}.$

Hence, the correct answer is option (D).

5. The number of all possible values of θ where $0<\theta<\pi$, for which the system of equations

$$(y+z)\cos 3\theta=(xyz)\sin 3\theta$$
$$x\sin 3\theta=\frac{2\cos 3\theta}{y}+\frac{2\sin 3\theta}{z}$$
$$(xyz)\sin 3\theta=(y+2z)\cos 3\theta+y\sin 3\theta$$

has a solution (x_0, y_0, z_0) with $y_0z_0\neq 0$, is

[IIT-JEE 2010]

Solution: We have

$$xyz\sin 3\theta=(y+z)\cos 3\theta \quad (1)$$
$$xyz\sin 3\theta=2z\cos 3\theta+2y\sin 3\theta \quad (2)$$
$$xyz\sin 3\theta=(y+2z)\cos 3\theta+y\sin 3\theta \quad (3)$$

By Eqs. (1), (2), & (3)

$$(y+z)\cos 3\theta=2z\cos 3\theta+2y\sin 3\theta=(y+2z)\cos 3\theta+y\sin 3\theta \quad (4)$$

By Eq. (4)

$$y(\cos 3\theta-2\sin\theta)=z\cos 3\theta \text{ and}$$
$$y(\sin 3\theta-\cos 3\theta)=0$$
$$\Rightarrow\cos 3\theta=\sin 3\theta$$
$$\Rightarrow\tan 3\theta=1\Rightarrow 3\theta=n\pi+\frac{\pi}{4}$$
$$\Rightarrow\theta=\frac{n\pi}{3}+\frac{\pi}{12}$$

Hence, the correct answer is (3).

6. The number of values of θ in the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ such that $\theta\neq\frac{n\pi}{5}$ for $n=0,\pm1,\pm2$ and $\tan\theta=\cot 5\theta$ as well as $\sin 2\theta=\cos 4\theta$ is

[IIT-JEE 2010]

Solution: We have

$$\tan\theta=\cot 5\theta\Rightarrow\cos 6\theta=0$$
$$\Rightarrow 4\cos^3 2\theta-3\cos 2\theta=0\Rightarrow\cos 2\theta=0 \text{ or} \pm\frac{\sqrt{3}}{2} \quad (1)$$

Again, given that

$$\sin 2\theta = \cos 4\theta$$

$$\Rightarrow 2\sin^2 2\theta + \sin 2\theta - 1 = 0$$

$$\Rightarrow 2\sin^2 2\theta + 2\sin 2\theta - \sin 2\theta - 1 = 0$$

$$\Rightarrow \sin 2\theta = -1 \text{ or } \sin 2\theta = \frac{1}{2} \quad (2)$$

Now from Eqs. (1) and (2) we have

$$\cos 2\theta = 0 \text{ and } \sin 2\theta = -1$$

$$\Rightarrow 2\theta = -\frac{\pi}{2} \Rightarrow \theta = -\frac{\pi}{4}$$

Again from Eqs. (1) and (2) we have

$$\cos 2\theta = \pm\frac{\sqrt{3}}{2} \text{ and } \sin 2\theta = \frac{1}{2}$$

$$\Rightarrow 2\theta = \frac{\pi}{6}, \frac{5\pi}{6} \Rightarrow \theta = \frac{\pi}{12}, \frac{5\pi}{12}$$

Therefore,

$$\theta = -\frac{\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}$$

Hence, the correct answer is (3).

7. The maximum value of the expression $\dfrac{1}{\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta}$ is

[IIT-JEE 2010]

Solution:

$$\frac{1}{4\cos^2\theta + 1 + \frac{3}{2}\sin 2\theta}$$

$$\Rightarrow \frac{1}{2[1+\cos 2\theta] + 1 + \frac{3}{2}\sin 2\theta}$$

The minimum value of $1 + 4\cos^2\theta + 3\sin\theta\cos\theta$

$$1 + \frac{4(1+\cos 2\theta)}{2} + \frac{3}{2}\sin 2\theta$$

$$= 1 + 2 + 2\cos 2\theta + \frac{3}{2}\sin 2\theta$$

$$= 3 + 2\cos 2\theta + \frac{3}{2}\sin 2\theta$$

Therefore,

$$= 3 - \sqrt{4 + \frac{9}{4}} = 3 - \frac{5}{2} = \frac{1}{2}$$

So, the maximum value of $\dfrac{1}{4\cos^2\theta + 1 + \frac{3}{2}\sin 2\theta}$ is 2.

Hence, the correct answer is (2).

8. The positive integer value of $n > 3$ satisfying the equation $\dfrac{1}{\sin\left(\frac{\pi}{n}\right)} = \dfrac{1}{\sin\left(\frac{2\pi}{n}\right)} + \dfrac{1}{\sin\left(\frac{3\pi}{n}\right)}$ is

[IIT-JEE 2011]

Solution:

$$\frac{1}{\sin\frac{\pi}{n}} - \frac{1}{\sin\frac{3\pi}{n}} = \frac{1}{\sin\frac{2\pi}{n}}$$

$$\Rightarrow \frac{\sin\frac{3\pi}{n} - \sin\frac{\pi}{n}}{\sin\frac{\pi}{n}\sin\frac{3\pi}{n}} = \frac{1}{\sin\frac{2\pi}{n}} \quad \frac{\left(2\sin\frac{\pi}{n}\cos\frac{2\pi}{n}\right)\sin\frac{2\pi}{n}}{\sin\frac{\pi}{n}\sin\frac{3\pi}{n}} = 1$$

$$\Rightarrow \sin\frac{4\pi}{n} = \sin\frac{3\pi}{n} \Rightarrow \frac{4\pi}{n} + \frac{3\pi}{n} = \pi \Rightarrow n = 7$$

Hence, the correct answer is (7).

9. Let $\theta, \phi \in [0, 2\pi]$ be such that

$$2\cos\theta(1-\sin\phi) = \sin^2\theta\left(\tan\frac{\theta}{2} + \cot\frac{\theta}{2}\right)\cos\phi - 1,$$

$\tan(2\pi - \theta) > 0$ and $-1 < \sin\theta < -\dfrac{\sqrt{3}}{2}$. Then, ϕ **cannot** satisfy

(A) $0 < \phi < \dfrac{\pi}{2}$

(B) $\dfrac{\pi}{2} < \phi < \dfrac{4\pi}{3}$

(C) $\dfrac{4\pi}{3} < \phi < \dfrac{3\pi}{2}$

(D) $\dfrac{3\pi}{2} < \phi < 2\pi$

[IIT-JEE 2012]

Solution: The given equation can be written as

$$2\cos\theta\,(1-\sin\phi) = \frac{2\sin^2\theta}{\sin\theta}\cos\phi - 1$$

$$\Rightarrow 2\cos\theta - 2\cos\theta\sin\phi = 2\sin\theta\cos\phi - 1$$

$$\Rightarrow 2\cos\theta + 1 = 2\sin(\theta + \phi)$$

Now

$$\tan(2\pi - \theta) > 0 \Rightarrow \tan\theta < 0 \text{ and } -1 < \sin\theta < -\frac{\sqrt{3}}{2}$$

$$\Rightarrow \theta \in \left(\frac{3\pi}{2}, \frac{5\pi}{3}\right)$$

Also

$$\frac{1}{2} < \sin(\theta + \phi) < 1$$

$$\Rightarrow 2\pi + \frac{\pi}{6} < \theta + \phi < \frac{5\pi}{6} + 2\pi$$

$$\Rightarrow 2\pi + \frac{\pi}{6} - \theta_{\max} < \phi < 2\pi + \frac{5\pi}{6} - \theta_{\min}$$

$$\Rightarrow \frac{\pi}{2} < \phi < \frac{4\pi}{3}$$

Hence, the correct answers are options (A), (C) and (D).

10. Match List I with List II and select the correct answer using the code given below the lists:

	List I	List II
P.	$\left(\frac{1}{y^2}\left(\frac{\cos(\tan^{-1}y)+y\sin(\tan^{-1}y)^2}{\cot(\sin^{-1}y)+\tan(\sin^{-1}y)}\right)+y^4\right)^{1/2}$	**1.** $\frac{1}{2}\sqrt{\frac{5}{3}}$
Q.	If $\cos x+\cos y+\cos z=0=\sin x+\sin y+\sin z$ then possible value of $\cos\frac{x-y}{2}$ is	**2.** $\sqrt{2}$
R.	If $\cos\left(\frac{\pi}{4}-x\right)\cos 2x+\sin x\sin 2x\sec x$ $=\cos x\sin 2x\sec x+\cos\left(\frac{\pi}{4}+x\right)\cos 2x$ then possible value of $\sec x$ is	**3.** $\frac{1}{2}$
S.	If $\cot(\sin^{-1}\sqrt{1-x^2})=\sin(\tan^{-1}(x\sqrt{6})), x\neq 0,$ $x\neq 0,$ then possible value of x is	**4.** 1

Codes:

	P	Q	R	S
(A)	4	3	1	2
(B)	4	3	2	1
(C)	3	4	2	1
(D)	3	4	1	2

[JEE ADVANCED 2013]

Solution:

(P) $$\left(\frac{1}{y^2}\left(\frac{\cos(\tan^{-1}y)+y\sin(\tan^{-1}y)^2}{\cot(\sin^{-1}y)+\tan(\sin^{-1}y)}\right)+y^4\right)^{1/2}$$

$$=\left\{\frac{1}{y^2}\left(\frac{\frac{1}{\sqrt{1+y^2}}+\frac{y^2}{\sqrt{1+y^2}}}{\frac{\sqrt{1-y^2}}{y}+\frac{y}{\sqrt{1-y^2}}}\right)^2+y^4\right\}^{1/2}$$

$$=\left\{\frac{1}{y^2}\left(\frac{y\sqrt{1+y^2}\sqrt{1-y^2}}{1}\right)^2+y^4\right\}^{1/2}$$

$$=\{1-y^4+y^4\}^{1/2}=1$$

(Q) If $\cos x+\cos y+\cos z=\sin x+\sin y+\sin z=0$, then the possible value of $x-y$ is $\pm\frac{2\pi}{3}$. That is, $\cos\left(\frac{x-y}{2}\right)=\frac{1}{2}$.

(R) Given equation can be written as

$$\cos 2x\left\{\cos\left(\frac{\pi}{4}-x\right)-\cos\left(\frac{\pi}{4}+x\right)\right\}=\sin 2x(1-\tan x)$$

$$\Rightarrow \cos 2x\cdot 2\sin\frac{\pi}{4}\cdot\sin x=\sin 2x(1-\tan x)$$

$$\Rightarrow \frac{1}{\sqrt{2}}\cos 2x=\cos x-\sin x$$

$$\Rightarrow \frac{1}{\sqrt{2}}(\cos x+\sin x)=1$$

$$\Rightarrow x=\frac{\pi}{4}$$

So, $\sec x=\sqrt{2}$.

(S) Given equation is $\frac{x}{\sqrt{1-x^2}}=\frac{\sqrt{6}x}{\sqrt{1+6x^2}}$.

Either $x=0$ or $1+6x^2=6-6x^2 \Rightarrow x=\pm\frac{1}{2}\sqrt{\frac{5}{3}}$.

Hence, the correct answer is option (B).

11. For $x\in(0,\pi)$, the equation $\sin x+2\sin 2x-\sin 3x=3$ has

(A) Infinitely many solutions
(B) Three solutions
(C) One solution
(D) No solution

[JEE ADVANCED 2014]

Solution:

$$\sin x+2\sin 2x-\sin 3x=3$$
$$\sin x+4\sin x\cos x-3\sin x+4\sin^3 x=3$$
$$\sin x\,[-2+4\cos x+4(1-\cos^2 x)]=3$$
$$[4\cos x-4\cos^2 x+2]=3\operatorname{cosec} x$$
$$[3-(2\cos x-1)^2]=3\operatorname{cosec} x$$

Least value of R.H.S is 3 at $x=\pi/2$ while greatest value of L.H.S is 3 at $x=\pi/3$.
Hence, L.H.S and R.H.S are not equal at same value of x. so, no solution.

Hence, the correct answer is option (D).

12. The number of distinct solutions of the equation $\frac{5}{4}\cos^2 2x+\cos^4 x+\sin^4 x+\cos^6 x+\sin^6 x=2$ in the interval $[0, 2\pi]$ is ______.

[JEE ADVANCED 2015]

Solution: The given equation can be written as

$$f(x)=\frac{5}{4}\cos^2 2x+(\cos^2 x+\sin^2 x)^2-2\cos^2 x\sin^2 x$$
$$+(\cos^2 x+\sin^2 x)^3-3\cos^2 x\sin^2 x(\cos^2 x+\sin^2 x)=2$$
$$\Rightarrow f(x)=\frac{5}{4}\cos^2 2x+2-5\cos^2 x\sin^2 x=2$$

$$\Rightarrow \frac{5}{4}\cos^2 2x - \frac{5}{4}\sin^2 2x = 0$$

$$\Rightarrow \tan^2 2x = 1;\ x \in [0, 2\pi]$$

$$\Rightarrow \tan^2 2x = 1;\ 2x \in [0, 4\pi]$$

Hence, there will be 8 solutions, 2 in each interval $[0, \pi]$.

Hence, the correct answer is (8).

Practice Exercise 1

1. The general value of θ satisfying both $\sin\theta = \frac{-1}{2}$ and $\tan\theta = \frac{1}{\sqrt{3}}$ is
   (A) $2n\pi$ **(B)** $2n\pi + 7\pi/6$
   (C) $n\pi + \pi/4$ **(D)** $2n\pi + \pi/4$
2. The smallest positive value of x (in degrees) for which $\tan(x + 100°) = \tan(x + 50°) \tan x \tan(x - 50°)$ is
   (A) 30° **(B)** 45°
   (C) 60° **(D)** 90°
3. The most general value of θ satisfying $3 - 2\cos\theta - 4\sin\theta - \cos2\theta + \sin2\theta = 0$ is
   (A) $2n\pi$ **(B)** $2n\pi + \pi/2$
   (C) $4n\pi$ **(D)** $2n\pi + \pi/4$
4. The number of solutions of $\sin^3 x \cos x + \sin^2 x \cos^2 x + \sin x \cos^3 x = 1$ in $[0, 2\pi]$ is
   (A) 4 **(B)** 2
   (C) 1 **(D)** 0
5. The number of solutions of the equation $x^3 + 2x^2 + 5x + 2\cos x = 0$ in $[0, 2\pi]$ is
   (A) 0 **(B)** 1
   (C) 2 **(D)** 3
6. The equation $k\sin x + \cos 2x = 2k - 7$ possesses a solution if
   (A) $k > 6$ **(B)** $2 \le k \le 6$
   (C) $k > 2$ **(D)** None of these
7. The general solution of the equation $\tan 3x = \tan 5x$ is
   (A) $x = n\pi/2, n \in Z$ **(B)** $x = n\pi, n \in Z$
   (C) $x = (2n + 1)\pi, n \in Z$ **(D)** None of these
8. The equation $\sin^4 x - 2\cos^2 x + a^2 = 0$ is solvable if
   (A) $-\sqrt{3} \le a \le \sqrt{3}$ **(B)** $-\sqrt{2} \le a \le \sqrt{2}$
   (C) $-1 \le a \le 1$ **(D)** None of these
9. If $\tan m\theta + \cot n\theta = 0$, then the general value of θ is
   (A) $\frac{(2r+1)\pi}{2(m-n)}$ **(B)** $\frac{(2r+1)\pi}{2(m+n)}$
   (C) $\frac{r\pi}{m+n}$ **(D)** $\frac{r\pi}{m-n}$
10. If $\tan\theta + \sec\theta = \sqrt{3}, 0 < \theta < \pi$, then θ is equal to
    (A) $\pi/3$ **(B)** $2\pi/3$
    (C) $\pi/6$ **(D)** $5\pi/8$
11. If $\cos\theta + \sqrt{3}\sin\theta = 2$, then θ (only principal value) is
    (A) $\pi/3$ **(B)** $2\pi/3$
    (C) $4\pi/3$ **(D)** $5\pi/3$
12. The number of solutions of $5\cos^2\theta - 3\sin^2\theta + 6\sin\theta\cos\theta = 7$ in the interval $[0, 2\pi]$ is
    (A) 2 **(B)** 4
    (C) 0 **(D)** None of these
13. If $\sin\theta + \cos\theta = \sqrt{2}\cos\theta$, then the general solution for θ is
    (A) $2n\pi \pm \frac{\pi}{8}$ **(B)** $n\pi + \frac{\pi}{8}$
    (C) $n\pi + (-1)^n \frac{\pi}{8}$ **(D)** None of these
14. The number of solutions of $11 \sin x = x$ is
    (A) 4 **(B)** 6
    (C) 8 **(D)** None of these
15. If $\sqrt{3}\sin\pi x + \cos\pi x = x^2 - \frac{2}{3}x + \frac{19}{9}$, then x is equal to
    (A) $-\frac{1}{3}$ **(B)** $\frac{1}{3}$
    (C) $\frac{2}{3}$ **(D)** None of these
16. The general solution for θ if $\sin\left(2\theta + \frac{\pi}{6}\right) + \cos\left(\theta + \frac{5\pi}{6}\right) = 2$ is
    (A) $2n\pi + \frac{7\pi}{6}$ **(B)** $2n\pi + \frac{\pi}{6}$
    (C) $2n\pi - \frac{7\pi}{6}$ **(D)** None of these
17. The number of solutions of the equation $\tan x + \sec x = 2 \cos x$ lying in the interval $[0, 2\pi]$ is
    (A) 0 **(B)** 1
    (C) 2 **(D)** 3
18. One solution of the equation $4\cos^2\theta\sin\theta - 2\sin^2\theta = 3\sin\theta$ is
    (A) $\theta = n\pi + (-1)^n\left(\frac{-3\pi}{10}\right)$ **(B)** $\theta = n\pi + (-1)^n\left(\frac{3\pi}{10}\right)$
    (C) $\theta = 2n\pi \pm \frac{\pi}{6}$ **(D)** None of these
19. Solve the equations for x and y:
    $$x\cos^3 y + 3x \cos y \sin^2 y = 14$$
    $$x\sin^3 y + 3x \cos^2 y \sin y = 13$$
    (A) $y = \tan^{-1}\frac{1}{2}, x = 5\sqrt{5}$ where $2n\pi < y < 2n\pi + \frac{\pi}{2}$
    (B) $y = \tan^{-1}\frac{1}{2}, x = -5\sqrt{5}$ where $2n\pi + \pi < y < 2n\pi + \frac{3\pi}{2}$
    (C) Both
    (D) None of these
20. The solution of the equation $\tan\theta \tan 2\theta = 1$ is
    (A) $n\pi + \frac{5\pi}{12}$ **(B)** $n\pi - \frac{5\pi}{12}$
    (C) $2n\pi \pm \frac{\pi}{4}$ **(D)** $n\pi \pm \frac{\pi}{6}$
21. Find the general solution of the equation
    $$\sin x - 3 \sin 2x + \sin 3x = \cos x - 3 \cos 2x + \cos 3x$$
    (A) $\frac{n\pi}{2} + \frac{5\pi}{12}$ **(B)** $n\pi - \frac{5\pi}{12}$
    (C) $\frac{n\pi}{2} + \frac{\pi}{8}$ **(D)** $n\pi \pm \frac{\pi}{8}$

22. Solve for x the equation $\sin^3 x + \sin x \cos x + \cos^3 x = 1$:

(A) $2m\pi$ **(B)** $(4n+1)\dfrac{\pi}{2}$

(C) Both **(D)** None of these

23. The equation $e^{\sin x} - e^{-\sin x} - 4 = 0$ has

(A) No real solution **(B)** One real solution

(C) Two real solutions **(D)** Cannot be determined

24. If $\tan(\pi\cos x) = \cot(\pi\sin x)$, then $\cos\left(x - \dfrac{\pi}{4}\right)$ is

(A) $\dfrac{1}{\sqrt{2}}$ **(B)** $\dfrac{1}{2\sqrt{2}}$

(C) 0 **(D)** None of these

25. If $\cos\theta + \cos 7\theta + \cos 3\theta + \cos 5\theta = 0$, then $\theta =$

(A) $\dfrac{n\pi}{4}$ **(B)** $\dfrac{n\pi}{2}$

(C) $\dfrac{n\pi}{8}$ **(D)** None of these

26. The sum of all solutions of the equation $\cos x . \cos\left(\dfrac{\pi}{3} + x\right) . \cos\left(\dfrac{\pi}{3} - x\right) = \dfrac{1}{4},\ x \in [0, 6\pi]$ is

(A) $\sin(\theta/3) = 6\pi$ **(B)** 30π

(C) $\dfrac{110\pi}{3}$ **(D)** None of these

27. The equation $\sin x + \sin y + \sin z = -3$ for $0 \le x \le 2\pi, 0 \le y \le 2\pi, 0 \le z \le 2\pi$ has

(A) One solution **(B)** Two sets of solution

(C) Four sets of solution **(D)** No solution

28. The solution set of $(5 + 4\cos\theta)(2\cos\theta + 1) = 0$ in the interval $[0, 2\pi]$ is

(A) $\left\{\dfrac{\pi}{3}, \dfrac{2\pi}{3}\right\}$ **(B)** $\left\{\dfrac{\pi}{3}, \pi\right\}$

(C) $\left\{\dfrac{2\pi}{3}, \dfrac{4\pi}{3}\right\}$ **(D)** $\left\{\dfrac{2\pi}{3}, \dfrac{5\pi}{3}\right\}$

29. The equation $3\cos x + 4\sin x = 6$ has

(A) Finite solution **(B)** Infinite solutions

(C) One solution **(D)** No solution

30. The equation $|\sin x| + |\cos x| = 3/2$ has

(A) One solution **(B)** Two solutions

(C) Infinite number of solutions **(D)** No solution

31. If $2\cos x < \sqrt{3}$ and $x \in [-\pi, \pi]$, then the solution set for x is

(A) $\left[-\pi, \dfrac{-\pi}{6}\right) \cup \left(\dfrac{\pi}{6}, \pi\right]$ **(B)** $\dfrac{-\pi}{6}, \dfrac{\pi}{6}$

(C) $\left[-\pi, \dfrac{-\pi}{6}\right] \cup \left[\dfrac{\pi}{6}, \pi\right]$ **(D)** None of these

32. $\sin x + \cos x = y^2 - y + a$ has no value of x for any y if 'a' belongs to

(A) $(0, \sqrt{3})$ **(B)** $(-\sqrt{3}, 0)$

(C) $(-\infty, -\sqrt{3})$ **(D)** $(\sqrt{3}, \infty)$

33. The solution set of $(2\cos x - 1)(3 + 2\cos x) = 0$ in the interval $0 \le x \le 2\pi$ is

(A) $\left\{\dfrac{\pi}{3}\right\}$ **(B)** $\left\{\dfrac{\pi}{3}, \dfrac{5\pi}{3}\right\}$

(C) $\left\{\dfrac{\pi}{3}, \dfrac{5\pi}{3}, \cos^{-1}(-3/2)\right\}$ **(D)** None of these

34. The value of a for which the equation $4\operatorname{cosec}^2(\pi(a+x)) + a^2 - 4a = 0$ has a real solution is

(A) $a = 1$ **(B)** $a = 2$

(C) $a = 10$ **(D)** None of these

35. Let n be a positive integer such that $\sin\dfrac{\pi}{2^n} + \cos\dfrac{\pi}{2^n} = \dfrac{\sqrt{n}}{2}$, then

(A) $6 \le n \le 8$ **(B)** $4 < n \le 8$

(C) $4 \le n < 8$ **(D)** $4 < n < 8$

36. If $\sin^2 4x + \cos^2 x = 2\sin 4x \cos^2 x$, then

(A) $x = \dfrac{n\pi}{2}$ **(B)** $x = (2n+1)\dfrac{\pi}{2}$

(C) $c = (2n+1)\pi$ **(D)** None of these

37. The set of values of x for which the inequality $\sin^4\left(\dfrac{x}{3}\right) + \cos^4\left(\dfrac{x}{3}\right) \le \dfrac{1}{2}$ holds is

(A) R

(B) $\left\{x / x = \dfrac{3n\pi}{2} \pm \dfrac{3\pi}{4}; n \in I\right\}$

(C) $R - \left\{x / x = \dfrac{3n\pi}{2} \pm \dfrac{3\pi}{4};\ n \in I\right\}$

(D) ϕ

38. The solution set of the inequation $\cos^2 2x < \cos^2 x$ is

(A) $\left(2n\pi - \dfrac{2\pi}{3}, 2n\pi + \dfrac{2\pi}{3}\right)$ **(B)** $\left[2n\pi - \dfrac{2\pi}{3}, 2n\pi + \dfrac{2\pi}{3}\right]$

(C) $\left(n\pi - \dfrac{\pi}{3}, n\pi + \dfrac{\pi}{3}\right) - \{n\pi\}$ **(D)** $\left(n\pi - \dfrac{\pi}{3}, n\pi + \dfrac{\pi}{3}\right)$

39. Which of the following can be the solution of the equation $\sqrt{\sin(1-x)} = \sqrt{\cos x}$ for $x \in [0, 2\pi]$?

(A) $x \in \phi$ **(B)** $\dfrac{3\pi}{4} + \dfrac{1}{2}$

(C) $\dfrac{7\pi}{4} + \dfrac{1}{2}$ **(D)** None of these

40. The number of solutions of the equation $|\sin x| = |x - 1|$ in the interval $[0, \pi]$ is

(A) Three **(B)** One

(C) Two **(D)** None of these

41. The number of values of x in the interval $[0, 5\pi]$ satisfying the equation $3\sin^2 x - 7\sin x + 2 = 0$ is

(A) 0 **(B)** 5

(C) 6 **(D)** 10

42. A root of the equation $\sin x + x - 1 = 0$ lies in the interval

(A) $\left(0, \dfrac{\pi}{2}\right)$ **(B)** $\left(-\dfrac{\pi}{2}, 0\right)$

(C) $\left(\dfrac{\pi}{2}, \pi\right)$ **(D)** $\left(-\pi, -\dfrac{\pi}{2}\right)$

43. The general solution of the equation $\sin^2 x\cos^2 x+\sin x\cos x-1=0$ is

(A) $n\pi+\tan^{-1}\left(\frac{2}{1+\sqrt{7}}\right)$ **(B)** $n\pi+\tan^{-1}\left(\frac{1+\sqrt{7}}{2}\right)$

(C) $n\pi+\tan^{-1}\left(\frac{2}{1+\sqrt{7}}\right)$ **(D)** No solution

44. The total number of the integral values of n so that $\sin x(\sin x+\cos x)=n$ has at least one solution is

(A) 2 **(B)** 1

(C) 3 **(D)** Zero

45. If $3\sin x+4\cos ax=7$ has at least one solution, then a has to be necessarily

(A) An odd number **(B)** An even integer

(C) A rational number **(D)** An irrational number

46. If $\frac{\pi}{2}\le x\le\pi$ and $81^{\sin^2 x}+81^{\cos^2 x}=30$, then x is equal to

(A) $\frac{\pi}{3}$ or $\frac{\pi}{6}$ **(B)** $\frac{2\pi}{3}$ or $\frac{5\pi}{6}$

(C) $\frac{\pi}{6}$ or $\frac{5\pi}{6}$ **(D)** None of these

47. The number of pairs (x, y) satisfying $4x^2-4x+2=\sin^2 y$ and $x^2+y^2\le 3$ are

(A) 0 **(B)** 2

(C) 4 **(D)** None of these

48. The set of all x in $(-\pi, \pi)$ satisfying $|4\sin x-1|<\sqrt{5}$ is given by

(A) $\left(-\frac{\pi}{10}, \frac{3\pi}{10}\right)$ **(B)** $\left(\frac{\pi}{10}, \frac{3\pi}{10}\right)$

(C) $\left(\frac{\pi}{10}, -\frac{3\pi}{10}\right)$ **(D)** None of these

49. The most general values of θ which satisfy $\sin\theta=-\frac{1}{2}$, $\tan\theta=\frac{1}{\sqrt{3}}$ are

(A) $n\pi\pm\frac{\pi}{6}$ **(B)** $n\pi+(-1)^r\frac{\pi}{6}$

(C) $2n\pi+\frac{7\pi}{6}$ **(D)** $2n\pi+\frac{11\pi}{6}$

50. The solution of the equation $\cos^2\theta+\sin\theta+1=0$ lies in the interval

(A) $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ **(B)** $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$

(C) $\left(\frac{3\pi}{4}, \frac{5\pi}{7}\right)$ **(D)** None of these

51. If $x\ne\frac{n\pi}{2}$ and $(\cos x)^{\sin^2 x-3\sin x+2}=1$, then all solutions of x are given by

(A) $2n\pi+\frac{\pi}{2}$ **(B)** $(2n+1)\pi-\frac{\pi}{2}$

(C) $n\pi+(-1)\frac{\pi}{2}$ **(D)** None of these

52. The number of solutions of $16^{\sin^2 x}+16^{\cos^2 x}=10,\ 0\le x\le 2\pi$ is

(A) 8 **(B)** 6

(C) 4 **(D)** 2

53. If $\theta\in[0, 5\pi]$ and $r\in R$ such that $2\sin\theta=r^4-2r^2+3$, then the maximum number of values of the pair (r, θ) is

(A) 8 **(B)** 10

(C) 6 **(D)** None of these

54. The general solution of $\frac{1-\sin x+\sin^2 x+\cdots+\infty}{1+\sin x+\sin^2 x+\cdots+\infty}=\frac{1-\cos 2x}{1+\cos 2x}$ if $x=$

(A) $(-1)^n\frac{\pi}{3}+n\pi$ **(B)** $(-1)^n\frac{\pi}{6}+n\pi$

(C) $(-1)^{n+1}\frac{\pi}{6}+n\pi$ **(D)** $(-1)^{n-1}\frac{\pi}{3}+n\pi$ $(n\in I)$

55. The number of solutions of the equation $\cos 2x+\cos 2y+2\tan^2 x+2=0$ in the interval $[-2\pi, 2\pi]$ is

(A) 0 **(B)** 1

(C) 2 **(D)** None of these

56. If $4\sin^2 x-8\sin x+3\le 0,\ 0\le x\le 2\pi$, then the solution set for x is

(A) $\left[0, \frac{\pi}{6}\right]$ **(B)** $\left[0, \frac{5\pi}{6}\right]$

(C) $\left[\frac{5\pi}{6}, 2\pi\right]$ **(D)** $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$

57. The values of θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ satisfying the equation $(\sqrt{3})^{\sec^2\theta}=\tan^4\theta+2\tan^2\theta$ is

(A) $\frac{\pi}{4}$ **(B)** $\frac{\pi}{6}$

(C) π **(D)** None of these

58. The least value of a for which the equation $\frac{4}{\sin x}+\frac{1}{1-\sin x}=a$ has at least one solution for $x\in\left(0, \frac{\pi}{2}\right)$ is

(A) 9 **(B)** 4

(C) 8 **(D)** 1

59. If $\cot x\cot y=k$ and $x+y=\frac{\pi}{3}$, then $\tan x$, $\tan y$ satisfy the equation

(A) $kt^2-\sqrt{3}(k-1)t+1=0$ **(B)** $kt^2+\sqrt{3}(k-1)t+1=0$

(C) $kt^2-\sqrt{3}(k+1)t+1=0$ **(D)** $kt^2+\sqrt{3}(k+1)t+1=0$

60. The number of solutions for the equation $x^3+2x^2+5x+2\cos x=0$ in $[0, 2\pi]$ is

(A) 0 **(B)** 1

(C) 2 **(D)** 3

Practice Exercise 2

Single/Multiple Correct Choice Type Questions

1. If $0 < \alpha < \frac{\pi}{2}$ and $f(\alpha) = \pi^{\sec^2\alpha}\cos^2\alpha + \pi^{\operatorname{cosec}^2\alpha}\sin^2\alpha$, then

(A) $f(\alpha) < \pi^2$ **(B)** $f(\alpha) > \pi^2$
(C) $f(\alpha) < \pi$ **(D)** $f(\alpha) = \pi^2$

2. If $|\sin x + \cos x| = |\sin x| + |\cos x|$, then x belongs to the quadrant

(A) I or III **(B)** II or IV
(C) I or II **(D)** III or IV

3. $\sin x + \cos x = y^2 - y + a$ has no value of x for any y if 'a' belongs to

(A) $(0, \sqrt{3})$ **(B)** $(-\sqrt{3}, 0)$
(C) $(-\infty, -\sqrt{3})$ **(D)** $(\sqrt{3}, \infty)$

4. For any real θ, the maximum value of $\cos^2(\cos\theta) + \sin^2(\sin\theta)$ is

(A) 1 **(B)** $1 + \sin^2 1$
(C) $1 + \cos^2 1$ **(D)** Does not exist

5. In a ΔABC if $\cos A \cdot \cos B + \sin A \cdot \sin B \cdot \sin C = 1$, then the value of $\cos(A - B)$ is

(A) 1 **(B)** 0
(C) –1 **(D)** Does not exist

6. Find k if the equation $2\cos x + \cos 2kx = 3$ has only one solution.

(A) 0 **(B)** 2
(C) $\sqrt{2}$ **(D)** 1/2

7. The general solution of the equation $\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$ is

(A) $\frac{n\pi}{2} + \frac{\pi}{4}, n \in I$ **(B)** $\frac{n\pi}{2} + \frac{\pi}{8}, n \in I$
(C) $n\pi - \frac{\pi}{4}, n \in I$ **(D)** None of these

8. If the equation $x^2 + 12 + 3\sin(a + bx) + 6x = 0$ has atleast one real solution where $a, b \in [0, 2\pi]$, then value of $\cos\theta$ where θ is least positive value of $a + bx$ is

(A) π **(B)** 2π
(C) 0 **(D)** $\frac{\pi}{2}$

9. Total number of integral values of 'n' such that $\sin x(2\sin x + \cos x) = n$, has atleast one real solution is

(A) 3 **(B)** 1
(C) 2 **(D)** 0

10. For $0 < x < \frac{\pi}{2}$, $(1 + 4\operatorname{cosec} x)(1 + 8\sec x)$ is

(A) ≥81 **(B)** >81
(C) ≥83 **(D)** >83

11. The minimum value of $\left(1+\frac{1}{\sin^n\theta}\right)\left(1+\frac{1}{\cos^n\theta}\right)$ is

(A) 1 **(B)** 4
(C) $(1 + 2^{n/2})^2$ **(D)** None of these

12. If $\theta_1, \theta_2, \theta_3$ are three values lying in $[0, 2\pi]$ for which $\tan\theta = \lambda$, then $\left|\tan\frac{\theta_1}{3}\tan\frac{\theta_2}{3} + \tan\frac{\theta_2}{3}\tan\frac{\theta_3}{3} + \tan\frac{\theta_1}{3}\tan\frac{\theta_3}{3}\right| =$

(A) 1 **(B)** –2
(C) –3 **(D)** None of these

13. The solution of the equation $2\cos^4 x + \cos x - 2\cos x\sin^2 x - 3\sin^2 x + 1 = 0$ is

(A) $n\pi \pm \frac{\pi}{4}$ **(B)** $n\pi \pm \frac{\pi}{2}$
(C) $n\pi \pm \frac{\pi}{6}$ **(D)** None of these

14. If the value of expression $\frac{\operatorname{cosec}^4\theta - 3\cot^2\theta}{\operatorname{cosec}^4\theta - d\cot^2\theta}$ lies between $\frac{1}{3}$ and 3, then

(A) $d \in [-2, 2]$ **(B)** $d = -1$
(C) $d = 2$ **(D)** None of these

15. Let x, y, z be elements from interval $[0, 2\pi]$ satisfying the inequality $(4 + \sin 4x)(2 + \cot^2 y)(1 + \sin^4 z) \le 12\sin^2 z$. Then

(A) The number of ordered pairs (x, y) is 4.
(B) The number of ordered pairs (y, z) is 8.
(C) The number of ordered pairs (z, x) is 8.
(D) The number of pairs (y, z) such that $z = y$ is 2.

16. The expression $\cos 3\theta + \sin 3\theta + (2\sin 2\theta - 3)(\sin\theta - \cos\theta)$ is positive for all θ in

(A) $\left(2n\pi - \frac{3\pi}{4}, 2n\pi + \frac{\pi}{4}\right), n \in I$
(B) $\left(2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{6}\right), n \in I$
(C) $\left(2n\pi - \frac{\pi}{3}, 2n\pi + \frac{\pi}{3}\right), n \in I$
(D) $\left(2n\pi - \frac{\pi}{4}, 2n\pi + \frac{3\pi}{4}\right), n \in I$

17. If $\alpha, \beta, \gamma, \delta$ are four angles of a cyclic quadrilateral taken in clockwise direction then the value of $(2 + \sum\cos\alpha\cos\beta)$ will be

(A) $\sin^2\alpha + \sin^2\beta$ **(B)** $\cos^2\gamma + \cos^2\delta$
(C) $\sin^2\alpha + \sin^2\delta$ **(D)** $\cos^2\beta + \cos^2\gamma$

18. **Statement-1:** If $\sin x + A\cos x = B$, then $|A\sin x - \cos x| = \sqrt{A^2 - B^2 + 1}$.
Statement-2: The point $\left(\frac{1}{\sqrt{1+A^2}}, \frac{A}{\sqrt{1+A^2}}\right)$ lies on a circle of radius unity.

(A) Statement-1 is true, statement-2 is true; statement-2 is a correct explanation for statement-1.
(B) Statement-1 is true, statement-2 is true; statement-2 is not a correct explanation for statement-1.
(C) Statement-1 is true, statement-2 is false.
(D) Statement-1 is false, statement-2 is true.

19. If $0 < \alpha, \beta < 2\pi$, then the number of ordered pairs (α, β) satisfying $\sin^2(\alpha + \beta) - 2\sin\alpha\sin(\alpha + \beta) + \sin^2\alpha + \cos^2\beta = 0$ is

(A) 2 **(B)** 0
(C) 4 **(D)** None of these

20. If $A + B + C = \pi$, then the maximum value of $\cos A + \cos B + k\cos C$ (where $k > 1/2$) is

(A) $\frac{1}{k} + \frac{k}{2}$ **(B)** $\frac{2k^2 + 1}{3}$
(C) $\frac{k^2 + 2}{2}$ **(D)** $\frac{1}{2k} + k$

Answer Key

Practice Exercise 1

1. (B)	**2.** (A)	**3.** (A), (B)	**4.** (D)	**5.** (A)	**6.** (B)
7. (B)	**8.** (B)	**9.** (A)	**10.** (C)	**11.** (A)	**12.** (C)
13. (B)	**14.** (D)	**15.** (B)	**16.** (A)	**17.** (C)	**18.** (A)
19. (B)	**20.** (D)	**21.** (C)	**22.** (C)	**23.** (A)	**24.** (B)
25. (C)	**26.** (B)	**27.** (A)	**28.** (C)	**29.** (D)	**30.** (D)
31. (A)	**32.** (D)	**33.** (B)	**34.** (B)	**35.** (B)	**36.** (B)
37. (C)	**38.** (C)	**39.** (C)	**40.** (C)	**41.** (C)	**42.** (A)
43. (D)	**44.** (A)	**45.** (C)	**46.** (B)	**47.** (B)	**48.** (B)
49. (C)	**50.** (D)	**51.** (D)	**52.** (A)	**53.** (C)	**54.** (B)
55. (A)	**56.** (D)	**57.** (A)	**58.** (A)	**59.** (A)	**60.** (A)

Practice Exercise 2

1. (B)	**2.** (A)	**3.** (D)	**4.** (B)	**5.** (A)	**6.** (C)
7. (B)	**8.** (C)	**9.** (A)	**10.** (D)	**11.** (C)	**12.** (C)
13. (A)	**14.** (D)	**15.** (C), (D)	**16.** (A), (B)	**17.** (A), (C)	**18.** (B)
19. (C)	**20.** (D)				

Solutions

Practice Exercise 1

1. Let us first find out θ lying between 0° and 360°.
Since,

$$\sin\theta = \frac{-1}{2} \Rightarrow \theta = 210° \text{ or } 330°$$

and

$$\tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = 30° \text{ or } 210°$$

Hence, $\theta = 210°$ or $\frac{7\pi}{6}$ is the value satisfying both.

Therefore, the general value of $\theta = \left(2n\pi + \frac{7\pi}{6}\right), n \in \mathrm{I}$

2. The relation may be written as

$$\frac{\tan(x+100°)}{\tan(x-50°)} = \tan(x+50°)\tan x$$

$$\Rightarrow \frac{\sin(x+100°)\cos(x-50°)}{\sin(x-50°)\cos(x+100°)} = \frac{\sin(x+50°)\sin x}{\cos(x+50°)\cos x}$$

$$\Rightarrow \frac{\sin(2x+50°)+\sin(150°)}{\sin(2x+50°)-\sin(150°)} = \frac{\cos(50°)-\cos(2x+50°)}{\cos(50°)+\cos(2x+50°)}$$

$$\Rightarrow \frac{\sin(2x+50°)}{\sin 150°} = \frac{-\cos 50°}{\cos(2x+50°)}$$

$\Rightarrow \cos 50° + 2\sin(2x+50°)\cos(2x+50°) = 0$
$\Rightarrow \cos 50° + \sin(4x+100°) = 0$
$\Rightarrow \cos 50° + \cos(4x+10°) = 0$
$\Rightarrow \cos(2x+30°)\cos(2x-20°) = 0$
$\Rightarrow x = 30°, 55°$

So, the smallest value of $x = 30°$.

3. $3 - 2\cos\theta - 4\sin\theta - \cos 2\theta + \sin 2\theta = 0$
$\Rightarrow 3 - 2\cos\theta - 4\sin\theta - 1 + 2\sin^2\theta + 2\sin\theta\cos\theta = 0$
$\Rightarrow 2\sin^2\theta - 2\cos\theta - 4\sin\theta + 2\sin\theta\cos\theta + 2 = 0$
$\Rightarrow (\sin^2\theta - 2\sin\theta + 1) + \cos\theta(\sin\theta - 1) = 0$
$\Rightarrow (\sin\theta - 1)[\sin\theta - 1 + \cos\theta] = 0$

Case 1: Either $\sin\theta = 1$

$\Rightarrow \theta = 2n\pi + \pi/2$ where $n \in I$

Case 2: Or, $\sin\theta + \cos\theta = 1$

$\Rightarrow \cos(\theta - \pi/4) = \cos(\pi/4) \Rightarrow \theta - \pi/4 = 2n\pi \pm \pi/4$
$\Rightarrow \theta = 2n\pi, 2n\pi + \pi/2$ where $n \in I$

Hence, $\theta = 2n\pi, 2n\pi + \pi/2$.

4. $\sin x\cos x[\sin^2 x + \sin x\cos x + \cos^2 x] = 1$

$\Rightarrow \sin x\cos x + (\sin x\cos x)^2 = 1$
$\Rightarrow \sin^2 2x + 2\sin 2x - 4 = 0$

$$\sin 2x = \frac{-2 \pm \sqrt{4+16}}{2} = -1 \pm \sqrt{5}$$

which is not possible.

5. Let $f(x) = x^3 + 2x^2 + 5x + 2\cos x$. Differentiating w.r.t. x we get

$$f'(x) = 3x^2 + 4x + 5 - 2\sin x = 3\left(x + \frac{2}{3}\right)^2 + \frac{11}{3} - 2\sin x$$

Now

$$\frac{11}{3} - 2\sin x > 0 \;\forall\; x \text{ (as } -1 \le \sin x \le 1)$$

$\Rightarrow f'(x) > 0 \;\forall x$
$\Rightarrow f(x)$ is an increasing function

Now $f(0) = 2$. Hence, $f(x) = 0$ has no solution in $[0, 2\pi]$.

6. We have

$$k\sin x + (1 - 2\sin^2 x) = 2k - 7$$

$$\Rightarrow 2\sin^2 x - k\sin x + 2(k-4) = 0$$

$$\Rightarrow \sin x = \frac{k \pm \sqrt{k^2 - 16k + 64}}{4}$$

$$= \frac{k \pm (k-8)}{4} = \frac{1}{2}(k-4), 2$$

But $\sin \neq 2$. Therefore, $\sin x = \frac{1}{2}(k-4)$. Now

$$-1 \le \sin x \le 1 \Rightarrow -1 \le \frac{k-4}{2} \le 1 \Rightarrow 2 \le k \le 6$$

7. We have tan $3x$ = tan $5x$. So

$$5x = n\pi + 3x, n \in Z \Rightarrow x = n\pi/2, n \in Z$$

If n is odd, then $x = n\pi/2$ gives the extraneous solutions. Thus, the solution of the given equation can not given by $x = n\pi/2$, where n is even say $n = 2m$, $m \in Z$. Hence, the required solution is $x = n\pi$, $n \in Z$.

8. We have $\sin^4 x - 2\cos^2 x + a^2 = 0$. Let $y = \sin^2 x$. Then

$$y^2 - 2(1-y) + a^2 = 0$$

$$\Rightarrow y^2 + 2y + a^2 - 2 = 0$$

$$\Rightarrow y = -1 \pm \sqrt{3-a^2}$$

For y to be real, Discriminant ≥ 0.

So $\quad 4 - 4(a^2 - 2) \ge 0 \Rightarrow a^2 \le 3 \quad (1)$

But $\sin^2 x = y$. Therefore $0 \le y \le 1$. So

$$0 \le -1 + \sqrt{3-a^2} \le 1 \Rightarrow 1 \le \sqrt{3-a^2} \le 2$$

$$\Rightarrow 1 \le 3 - a^2 \le 4 \Rightarrow 2 - a^2 \ge 0 \Rightarrow a^2 \le 2 \quad (2)$$

From Eqs. (1) and (2), $a^2 \le 2 \Rightarrow -\sqrt{2} \le a \le \sqrt{2}$.

9. The given equation can be written as

$$\tan m\theta = -\cot n\theta = \tan(\pi/2 + n\theta)$$

Therefore,

$$m\theta = r\pi + \left(\frac{\pi}{2} + n\theta\right), r \in \text{I}$$

$$\Rightarrow (m-n)\theta = \frac{1}{2}(2r+1)\pi, r \in \text{I}$$

So,

$$\theta = \frac{(2r+1)\pi}{2(m-n)}, r \in \text{I}$$

10. $$\sqrt{3}\cos\theta - \sin\theta = 1$$

or $$\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta = \frac{1}{2}$$

Therefore,

$$\cos\left(\theta + \frac{\pi}{6}\right) = \cos\frac{\pi}{3}$$

$$\Rightarrow \theta + \frac{\pi}{6} = \frac{\pi}{3}$$

$$\Rightarrow \theta = \frac{\pi}{6}$$

11. $$\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta = 1$$

$$\cos\left(\theta - \frac{\pi}{3}\right) = 1$$

$$\cos\left(\theta - \frac{\pi}{3}\right) = \cos 0$$

Therefore,

$$\theta = \frac{\pi}{3}$$

12. $5\cos^2\theta - 3\sin^2\theta + 6\sin\theta\cos\theta = 7$

$$\Rightarrow 5\left(\frac{1+\cos 2\theta}{2}\right) - 3\left(\frac{1-\cos 2\theta}{2}\right) + 3\sin 2\theta = 7$$

$$\Rightarrow 4\cos 2\theta + 3\sin 2\theta = 6,$$

But $4\cos 2\theta + 3\sin 2\theta \le \sqrt{4^2 + 3^2} = 5$. So, solution does not exist.

13. $\sin\theta + \cos\theta = \sqrt{2}\cos\theta \Rightarrow \tan\theta = \sqrt{2} - 1$

$$\Rightarrow \theta = \frac{\pi}{8} \Rightarrow n\pi + \frac{\pi}{8}$$

14. We have

$$11 \sin x = x \quad (1)$$

On replacing + by −, we have

$$11\sin(-x) = -x \Rightarrow 11\sin x = x$$

So, for every positive solution, we have negative solution also and $x = 0$ satisfies Eq. (1), so the number of solution will always be odd. Therefore, (D) is appropriate choice.

15. $$\text{LHS} = \sqrt{3}\sin\pi x + \cos\pi x = 2\sin\left(\pi x + \frac{\pi}{6}\right) \le 2$$

So equality holds for $x = \frac{1}{3}$.

$$\text{RHS} = x^2 - \frac{2}{3}x + \frac{19}{9} = \left(x - \frac{1}{3}\right)^2 + 2 \ge 2$$

So equality holds if $x = \frac{1}{3}$.

Thus, LHS = RHS for $x = \frac{1}{3}$ only.

16. $$\sin\left(2\theta + \frac{\pi}{6}\right) + \cos\left(\theta + \frac{5\pi}{6}\right) = 2 \quad (1)$$

Since $\sin\left(2\theta + \frac{\pi}{6}\right) \le 1$ and $\cos\left(\theta + \frac{5\pi}{6}\right) \le 1$, therefore

Eq. (1) may hold true if $\sin\left(2\theta + \frac{\pi}{6}\right)$ and $\cos\left(\theta + \frac{5\pi}{6}\right)$.

Both are equal to 1 simultaneously. First, a common value of θ is $7\pi/6$ for which

$$\sin\left(2\theta+\frac{\pi}{6}\right)=\sin\frac{5\pi}{2}=\sin\frac{\pi}{2}=1$$

and

$$\cos\left(\theta+\frac{5\pi}{6}\right)=\cos\left(\frac{7\pi}{6}+\frac{5\pi}{6}\right)=\cos 2\pi=1$$

Since periodicity of $\sin\left(2\theta+\frac{\pi}{6}\right)$ is π and periodicity of $\cos\left(\theta+\frac{5\pi}{6}\right)$ is 2π, therefore, the periodicity of $\sin\left(2\theta+\frac{\pi}{6}\right)$ $\sin\left(2\theta+\frac{\pi}{6}\right)$ is 2π.

Therefore, the general solution is

$$\theta=2n\pi+\frac{7\pi}{6}$$

17. The given equation can be written as

$$\sin x+1=2\cos^2 x$$

$$\Rightarrow \sin x+1=2(1-\sin^2 x)$$

$$\Rightarrow 2\sin^2 x+\sin x-1=0$$

$$\Rightarrow \sin x=\frac{-1\pm\sqrt{1+8}}{4}=\frac{-1\pm 3}{4}=\frac{1}{2} \text{ or } -1$$

$$\Rightarrow x=\frac{\pi}{6},\frac{5\pi}{6} \text{ or } \frac{3\pi}{2}$$

but equation is not defined at $x=3\pi/2$.
Hence, the required number of solutions is 2.

18. The given equation can be written as

$$\sin\theta[4(1-\sin^2\theta)-2\sin\theta-3]=0$$

$$\Rightarrow \sin\theta[1-2\sin\theta-4\sin^2\theta]=0$$

$$\Rightarrow \sin\theta[4\sin^2\theta+2\sin\theta-1]=0$$

Therefore, either $\sin\theta=0$ which gives $\theta=n\pi$

or $4\sin^2\theta+2\sin\theta-1=0$ which gives

$$\sin\theta=\frac{-2\pm\sqrt{4+16}}{2\times 4}=\frac{-2\pm 2\sqrt{5}}{8}=\frac{-1\pm\sqrt{5}}{4}$$

$$=\frac{-1+\sqrt{5}}{4},\frac{-1-\sqrt{5}}{4}$$

Now

$$\sin\theta=\frac{1}{4}(\sqrt{5}-1)=\sin 18^\circ=\sin\left(\frac{\pi}{10}\right)$$

Therefore,

$$\theta=n\pi\pm(-1)^n\left(\frac{\pi}{10}\right)$$

Again

$$\sin\theta=-\frac{1}{4}(\sqrt{5}-1)=-\cos 36^\circ$$

$$=-\cos(90^\circ-54^\circ)=-\sin 54^\circ$$

$$=\sin(-54^\circ)=\sin\left(\frac{-3\pi}{10}\right)$$

Therefore,

$$\theta=n\pi+(-1)^n\left(-\frac{3\pi}{10}\right)$$

Thus, one solution of the given equation is

$$\theta=n\pi+(-1)^n\left(-\frac{3\pi}{10}\right)$$

19. Clearly, $x\neq 0$. On dividing the equations we get

$$\frac{\cos^3 y+3\cos y\sin^2 y}{\sin^3 y+3\cos^2 y\sin y}=\frac{14}{13}$$

By componendo and dividendo, we get

$$\frac{(\cos y+\sin y)^3}{(\cos y-\sin y)^3}=\frac{14+13}{14-13}$$

$$\Rightarrow \frac{(\cos y+\sin y)^3}{(\cos y-\sin y)^3}=27=(3)^3$$

$$\Rightarrow \frac{\cos y+\sin y}{\cos y-\sin y}=\frac{3}{1}$$

On dividing numerator and denominator by $\cos y$, we get

$$\frac{1+\tan y}{1-\tan y}=\frac{3}{1}$$

Again using componendo and dividendo, we get

$$\frac{2\tan y}{2}=\frac{2}{4}$$

Solving we get,

(a) $\sin y=\frac{1}{\sqrt{5}}, \cos y=\frac{2}{\sqrt{5}}$ (when y is in the first quadrant)

(b) $\sin y=-\frac{1}{\sqrt{5}}$ and $\cos y=-\frac{2}{\sqrt{5}}$ (when y is in the third quadrant)

When y is in the first quadrant

$$x\left[\frac{8}{5\sqrt{5}}+3\frac{2}{\sqrt{5}},\frac{1}{5}\right]=14\Rightarrow x=-5\sqrt{5}$$

When y is in the third quadrant

$$x\left[\frac{-8}{5\sqrt{5}}+3\left(\frac{-2}{\sqrt{5}}\right)\cdot\frac{1}{5}\right]=14\Rightarrow x=-5\sqrt{5}$$

Hence, $y=\tan^{-1}\frac{1}{2}, x=5\sqrt{5}$, where $2n\pi<y<2n\pi+\frac{\pi}{2}$

and $y=\tan^{-1}\frac{1}{2}, x=-5\sqrt{5}$, where $2n\pi+\pi<y<2n\pi+\frac{3\pi}{2}$

20. Given $\tan\theta\cdot\tan 2\theta=1\Rightarrow\frac{2\tan^2\theta}{1-\tan^2\theta}=1$

$$\Rightarrow 2\tan^2\theta=1-\tan^2\theta\Rightarrow 3\tan^2\theta=1$$

$$\Rightarrow \tan\theta=\pm\frac{1}{\sqrt{3}}\Rightarrow\theta=n\pi\pm\frac{\pi}{6}$$

21. Given $\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$
$\Rightarrow 2\sin 2x\cos x - 3\sin 2x = 2\cos x\cos 2x - 3\cos 2x$
$\Rightarrow \sin 2x\,(2\cos x - 3) = \cos 2x\,(2\cos x - 3) \Rightarrow \sin 2x = \cos 2x$
(since $\cos x \neq 3/2$)
$\Rightarrow \tan 2x = 1 \Rightarrow 2x = n\pi + \dfrac{\pi}{4} \Rightarrow x = \dfrac{n\pi}{2} + \dfrac{\pi}{8},\ n \in I$

22. The given equation is

$\sin^3 x + \cos^3 x + \sin x\cos x = 1$
$\Rightarrow (\sin x + \cos x)(\sin^2 x - \sin x\cos x + \cos^2 x) + \sin x\cos x - 1 = 0$
$\Rightarrow (1 - \sin x\cos x)[\sin x + \cos x - 1] = 0$

Either $1 - \sin x\cos x = 0 \Rightarrow \sin 2x = 2$ which is not possible

or

$\sin x + \cos x - 1 = 0 \Rightarrow \cos(x - \pi/4) = \dfrac{1}{\sqrt{2}} \Rightarrow x - \dfrac{\pi}{4} = 2m\pi \pm \dfrac{\pi}{4}$
$\Rightarrow x = 2m\pi$ and $x = (4n+1)\dfrac{\pi}{2}$

23. The given equation can be written as

$$e^{2\sin x} - 4e^{\sin x} - 1 = 0 \Rightarrow e^{\sin x} = \frac{4 \pm \sqrt{16+4}}{2} = 2 + \sqrt{5}$$

$\Rightarrow \sin x = \ln(2+\sqrt{5})$ [$\ln(2-\sqrt{5})$ not defined as $(2-\sqrt{5})$ is negative]

Now, $2 + \sqrt{5} > e \Rightarrow \ln(2+\sqrt{5}) > 1 \Rightarrow \sin x > 1$ which is not possible. Hence, there is no real solution.

24. Given that $\tan(\pi\cos x) = \cot(\pi\sin x)$. Now

$$\tan(\pi\cos x) = \tan\left(\frac{\pi}{2} - \pi\sin x\right)$$

$$\Rightarrow \pi\cos x = \frac{\pi}{2} - \pi\sin x$$

$$\Rightarrow \cos x + \sin x = \frac{1}{2}$$

$$\Rightarrow \frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x = \frac{1}{2\sqrt{2}}$$

$$\Rightarrow \cos\left(x - \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}$$

25. Combining θ and 7θ, 3θ and 5θ, we get

$$2\cos 4\theta(\cos 3\theta + \cos\theta) = 0$$

Therefore,

$$4\cos 4\theta\cdot\cos 2\theta\cdot\cos\theta = 0$$

$$\Rightarrow 4\frac{1}{2^3\sin\theta}(\sin 2^3\theta) = 0$$

$$\Rightarrow \sin 8\theta = 0$$

Hence, $\theta = \dfrac{n\pi}{8}$.

26. Here,

$$\cos x\left(\frac{1}{4}\cos^2 x - \frac{3}{4}\sin^2 x\right) = \frac{1}{4}$$

$$\frac{\cos x}{4}(4\cos^2 x - 3) = \frac{1}{4}$$

$$\Rightarrow \cos 3x = 1 \Rightarrow 3x = 2n\pi \Rightarrow x = \frac{2n\pi}{3} \quad \text{(where } n = 0 \text{ to } 9)$$

Therefore, the required sum $= \dfrac{2\pi}{3}\displaystyle\sum_{n=0}^{9} n = 30\pi.$

27. $\sin x + \sin y + \sin z = -3$ only possible when

$$\sin x = \sin y = \sin z = -1$$

Hence, $x = y = z = \dfrac{3\pi}{2}$

28. $$(5 + 4\cos\theta)(2\cos\theta + 1) = 0$$

$\cos\theta = \dfrac{-5}{4}$, which is not possible.

Therefore, $2\cos\theta + 1 = 0 \Rightarrow \cos\theta = -\dfrac{1}{2} \Rightarrow \theta = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$

The solution set is $\left\{\dfrac{2\pi}{3}, \dfrac{4\pi}{3}\right\} \in [0, 2\pi]$.

29. $3\cos x + 4\sin x = 6$

$$\Rightarrow \frac{3}{5}\cos x + \frac{4}{5}\sin x = \frac{6}{5} \Rightarrow \cos(x - \theta) = \frac{6}{5}$$

$\left[\text{Where, } \theta = \cos^{-1}\left(\dfrac{3}{5}\right)\right]$

So, the equation has no solution.

30. No solution as $|\sin x| \le 1, |\cos x| \le 1$ and both of them do not attain their maximum value for the same angle.

$$|\sin x| + |\cos x| \in [1, \sqrt{2}\,]$$

31. Here, $\cos x < \dfrac{\sqrt{3}}{2}$. The value scheme for this is shown below.

From Figure 3.7,

$$-\pi \le x < \frac{-\pi}{6} \text{ or } \frac{\pi}{6} < x \le \pi$$

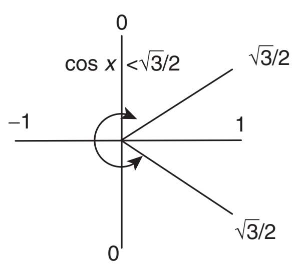


Figure 3.7

Therefore,

$$x \in \left[-\pi, \frac{-\pi}{6}\right) \cup \left(\frac{\pi}{6}, \pi\right]$$

32. Given equation is

$$\sin x + \cos x = y^2 - y + a$$

we know that $-\sqrt{2} \le \sin x + \cos x \le \sqrt{2}$.
Now, the given equation with root be satisfied for any x if $y^2 - y + a \ge \sqrt{2}$ or $y^2 - y + a \le -\sqrt{2}$, for all y, which is not possible
Therefore,

$$\left(y - \frac{1}{2}\right)^2 + a - \frac{1}{4} \ge \sqrt{2}$$

$$\Rightarrow a - \frac{1}{4} \ge \sqrt{2} \Rightarrow a \ge 1.65$$

That is,

$$a \in (\sqrt{3},\ \infty)$$

33. Given equation is

$$(2\cos x - 1)\ (3 + 2\cos x) = 0$$

$$\Rightarrow \cos x = \frac{1}{2}\ \left[\text{since } \cos x \ne -\frac{3}{2}\right]$$

Therefore,

$$x = \frac{\pi}{3},\ \frac{5\pi}{3}$$

$$\Rightarrow \text{solution set is } \left\{\frac{\pi}{3},\ \frac{5\pi}{3}\right\}$$

34. We have $4\operatorname{cosec}^2(\pi(a+x)) = -a^2 + 4a = 4 - (a-2)^2$.
The minimum value of $4\operatorname{cosec}^2(\pi(a+x)) = 4$
[since cosec $x \in [1,\ \infty)$]
and the maximum value of $-a^2 + 4a$ is 4 at $a = 2$.
Thus, the given relation is true for only $a = 2$.

35. Given $\sin\frac{\pi}{2^n} + \cos\frac{\pi}{2^n} = \frac{\sqrt{n}}{2}$

We have $\sin\frac{\pi}{2^n} + \cos\frac{\pi}{2^n} = \sqrt{2}\sin\left(\frac{\pi}{4} + \frac{\pi}{2^n}\right)$, which lies in $[-\sqrt{2},\ \sqrt{2}]$. Therefore,

$$\frac{\sqrt{n}}{2} \in [-\sqrt{2},\ \sqrt{2}] \Rightarrow \frac{\sqrt{n}}{2} \le \sqrt{2}$$

$$\Rightarrow \sqrt{n} \le 2\sqrt{2} \Rightarrow n \le 8$$

We note that $n = 1$ does not satisfy the given equation and for $n > 1$,

$$\frac{\pi}{2} \ge \frac{\pi}{4} + \frac{\pi}{2^n} > \frac{\pi}{4} \Rightarrow \sin\left(\frac{\pi}{4} + \frac{\pi}{2^n}\right) > \sin\frac{\pi}{4}$$

$$\Rightarrow \sqrt{2}\sin\left(\frac{\pi}{4} + \frac{\pi}{2^n}\right) > 1 \Rightarrow \frac{\sqrt{n}}{2} > 1 \Rightarrow n > 4$$

Hence, $4 < n \le 8$.

36. Given equation is

$$\sin^2 4x - 2\sin 4x \cdot \cos^2 x + \cos^2 x = 0$$

For $\sin x$ to be real, $D \ge 0$. So

$$4\cos^4 x - 4\cos^2 x \ge 0$$

$$\Rightarrow \cos^4 x \ge \cos^2 x$$

$$\Rightarrow \cos^4 x = \cos^2 x$$

$$\Rightarrow x = (2n+1)\frac{\pi}{2}$$

37. Given inequality is

$$\sin^4\left(\frac{x}{3}\right) + \cos^4\left(\frac{x}{3}\right) \le \frac{1}{2}$$

$$\Rightarrow 1 - \frac{1}{2}\sin^2\frac{2x}{3} \le \frac{1}{2}$$

$$\Rightarrow \sin^2\frac{2x}{3} \ge 1$$

But $\sin^2\frac{2x}{3} \le 1$ and $\sin^2\frac{2x}{3} = 1 \Rightarrow \frac{2x}{3} = n\pi \pm \frac{\pi}{2},\ n \in I$.

Therefore,

$$x \in R - \left\{x \,\middle|\, x = \frac{3n\pi}{2} \pm \frac{3\pi}{4},\ n \in I\right\}$$

38. **Hint:** $0 \le \cos^2 x,\ \cos^2 2x \le 1$

$$\cos^2 2x < \cos^2 x\ \forall\ x \in \left(\frac{-\pi}{3},\ \frac{\pi}{3}\right) - 0$$

Thus in general $\forall x \in \left(n\pi - \frac{\pi}{3},\ n\pi + \frac{\pi}{3}\right) - (n\pi)$.

39. **Hint:** $\cos x < 0,\quad \forall x \in \left(\frac{\pi}{2},\ \frac{3\pi}{2}\right)$

$$\sqrt{\sin(1-x)} = \sqrt{\cos x}$$

$$\Rightarrow \sin\ (1-x) = \cos x$$

$$\Rightarrow \cos\left(\frac{\pi}{2} - (1-x)\right) = \cos x$$

$$\Rightarrow x = 2n\pi \pm \left(\frac{\pi}{2} - (1-x)\right)$$

On simplifying, we get

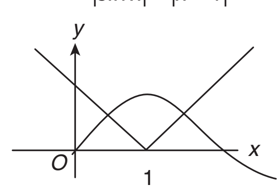
$$x = n\pi - \frac{\pi}{4} + \frac{1}{2} \Rightarrow x = \frac{3\pi}{4} + \frac{1}{2} \text{ and } \frac{7\pi}{4} + \frac{1}{2}$$

But at $x = \frac{3\pi}{4} + \frac{1}{2}$, $\cos x < 0$. Thus,

$$x = \frac{7\pi}{4} + \frac{1}{2}$$

40. Draw graph.

$$|\sin x| = |x - 1|$$



41. **Hint:** $\sin x \in [-1,\ 1]$

$3\sin^2 x - 6\sin x - \sin x + 2 = 0\ (3\sin x - 1)\ (\sin x - 2) = 0$

$\sin x = 2$ not possible

$\sin x = \frac{1}{3}$, 6 solutions

42. **Hint:** $\sin x = 1 - x$

One solution in $\left(0, \frac{\pi}{2}\right)$.

43. **Hint:** $\sin x \in [-1, 1]$

$$4\sin^2 x\cos^2 x + 4\sin x\cos x - 4 = 0$$

Let $t = \sin 2x \Rightarrow \sin^2 2x + 2\sin 2x - 4 = 0$

Then

$$t^2 + 2t - 4 = 0 \Rightarrow t = -1 \pm \sqrt{5} \Rightarrow t > 1, t < -1$$

Thus, $\sin 2x = t$ has no solution.

44. **Hint:** $-1 \le \sin x \le 1$

$$\frac{1}{2}\left[2\sin^2 x + 2\sin x\cos x\right] = n$$

$$\Rightarrow 1 - \cos 2x + \sin 2x = 2n$$

$$\Rightarrow 1 + \sqrt{2}\sin\left(2x - \frac{\pi}{4}\right) = 2n$$

$$\Rightarrow \sin\left(2x - \frac{\pi}{4}\right) = \frac{2n-1}{\sqrt{2}}$$

Hence, $-1 \le \sin y \le 1$. Only values of n which satisfy this are $n = 0$, $n = 1$. So, there are only two solutions.

45. **Hint:** $-1 \le \sin x \le 1$, $-1 \le \cos x \le 1$, only possible when

$\sin x = 1$, $\cos ax = 1$

$\Rightarrow ax = 2m\pi$

$\Rightarrow x = 2n\pi + \pi/2$

$\Rightarrow a(4n + 1)\pi/2 = 2m\pi$

$\Rightarrow a = \frac{4m}{4n+1}$

$a = \frac{4m}{2n+1}$ is a rational number.

46. **Hint:** $t + \frac{1}{t} = a$ can be expressed as quadratic.

See Fig. 3.8.

$$3^{4\sin^2 x} + 3^{4\cos^2 x} = 3\cdot(1+9)$$

$$\Rightarrow 81^{\sin^2 x} + 81^{(1-\cos^2 x)} = 30$$

$$\Rightarrow 81^{\sin^2 x} + \frac{81}{81^{\sin^2 x}} = 30$$

Let $81^{\sin^2 x} = t$. Then we have

$$t + \frac{81}{t} = 30$$

$$\Rightarrow t^2 - 30t + 81 = 0$$

$$\Rightarrow (t-3)(t-27) = 0$$

So, $81^{\sin^2 x} = 3$ or $81^{\sin^2 x} = 27$

$$\Rightarrow 4\sin^2 x = 1 \text{ or } 4\sin^2 x = 3$$

$$\Rightarrow \sin x = \pm\frac{1}{2} \text{ or } \sin x = \pm\frac{\sqrt{3}}{2}$$

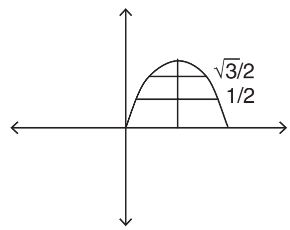


Figure 3.8

47. **Hint:** $0 \le \sin^2 y \le 1$

$$\Rightarrow \sin^2 y = 4x^2 - 4x + 2.$$

Let $f(x) = 4x^2 - 4x + 2 = (2x-1)^2 + 1$

$f(x)$ has minimum value

$$\Rightarrow \text{at } x = \frac{1}{2},\ \sin^2 y = 1$$

$f(x)$ has $D < 0$

$$\Rightarrow f(x) \ge 0\ (2x-1)^2 + 1 = \sin^2 y$$

$$\Rightarrow x = \frac{1}{2},\ \cos y = 0$$

Therefore,

$y = (2n+1)\frac{\pi}{2}$ minimum value at $x = \frac{1}{2}$

$$x^2 + y^2 \le 3 \Rightarrow \frac{1}{4} + \frac{(2n+1)^2\pi^2}{4} \le 3$$

$$1 + (2n+1)^2\pi^2 \le 12$$

$$(2n+1)^2\pi^2 \le 11$$

Only satisfies when $n = 0, -1$

That is, only two solutions satisfy $\left.\begin{matrix}\left(\frac{1}{2}, \frac{\pi}{2}\right)\\ \left(\frac{1}{2}, \frac{-\pi}{2}\right)\end{matrix}\right\}$

48. $|4\sin x - 1| < \sqrt{5} \Rightarrow -\sqrt{5} < 4\sin x - 1 < \sqrt{5}$

$$\Rightarrow \frac{-\sqrt{5}+1}{4} < \sin x < \frac{\sqrt{5}+1}{4} \Rightarrow \frac{\pi}{10} < x < \frac{3\pi}{10}$$

49. θ is in the third quadrant. So

$$\sin\theta = -\frac{1}{2} = \sin\frac{7\pi}{6}$$

$$\tan\theta = \frac{1}{\sqrt{3}} = \tan\frac{7\pi}{6}$$

$$\theta = 2n\pi + \frac{7\pi}{6}$$

50. $1-\sin^2\theta+\sin\theta+1=0 \Rightarrow \sin^2\theta-\sin\theta-2=0$

$$\Rightarrow \sin\theta=\frac{1\pm\sqrt{1+8}}{2}=\frac{1\pm 3}{2}\Rightarrow \sin\theta=-1$$

Therefore,

$$\theta=2n\pi-\frac{\pi}{2}=(4n-1)\frac{\pi}{2}$$

For n = 1, 2, we have

$$\theta=\frac{3\pi}{2},\ \frac{7\pi}{2}\Rightarrow\quad \theta\in\left(\frac{-5\pi}{4},\ \frac{7\pi}{4}\right)$$

51. Hint: Equate powers.

$\sin^2 x-3\sin x+2=0$. When $\cos x=1$, we have $x=2n\pi$
When $\sin x=1$, we have

$$\sin x=\sin\frac{\pi}{2}\Rightarrow x=n\pi+(-1)^n\frac{\pi}{2}$$

But x cannot be any multiple of $\pi/2$. Thus there is no solution.

52. Hint: $ax^2+bx+c=a(x-\alpha)(x-\beta)$

$$16^{\sin^2 x}+16^{1-\sin^2 x}=10$$
$$\Rightarrow 16^{\sin^2 x}+16\cdot 16^{-\sin^2 x}=10$$
$$\Rightarrow 16^{\sin^2 x}+\frac{16}{16^{\sin^2 x}}=10$$

Let $16^{\sin^2 x}=t$. Then

$$\Rightarrow t+\frac{16}{t}=10\Rightarrow t^2+16-10t=0$$
$$\Rightarrow (t-2)(t-8)=0\Rightarrow t=2,\ 8$$

When $16^{\sin^2 x}=2\Rightarrow 2\Rightarrow 4\sin^2 x=2$, $4\sin^2 x=1\Rightarrow \sin x=\pm\frac{1}{2}$.

When $16^{\sin^2 x}=8\Rightarrow 2^{4\sin^2 x}=2^3\Rightarrow \sin x=\pm\frac{\sqrt{3}}{2}$ interval is $[0, 2\pi]$.

From the graph of sin x in the interval $[0, 2\pi]$, it is clear that there are 8 solutions.

53. Hint: For real values of x, the discriminant of a quadratic equation is greater than or equal to zero.
Given equation is $r^4-2r^2+3-2\sin\theta=0$.
The above equation is quadratic in r^2. For the real values of r^2, discriminant > 0. Therefore,

$$4-4(3-2\sin\theta)\geq 0$$
$$\Rightarrow 1-3+2\sin\theta\geq 0$$
$$\Rightarrow -2+2\sin\theta\geq 0\Rightarrow \sin\theta\geq 1 \qquad (1)$$

That is, the above inequality holds only at $\sin\theta=1$.

Possible value of $\theta=\frac{\pi}{2}, 2\pi+\frac{\pi}{2}, 4\pi+\frac{\pi}{2}$

And $r^4-2r^2+3-2=0\Rightarrow (r^2-1)^2=0\Rightarrow r=-1, 1$

Hence, number of ordered pair of (r,θ) is 6.

54. Hint: Sum to infinite terms of a GP $=\frac{a}{1-r}, |r|<1$

$$\frac{\frac{1}{1+\sin x}}{\frac{1}{1-\sin x}}=\frac{1-\cos 2x}{1+\cos 2x}$$

$$\Rightarrow \frac{1-\sin x}{1+\sin x}=\frac{1-\cos 2x}{1+\cos 2x}$$

Applying componendo and dividendo, we have

$$\sin x=\cos 2x\Rightarrow \sin x=\sin\left(\frac{\pi}{2}-2x\right)$$

$$\Rightarrow 3x=\frac{\pi}{2}\Rightarrow x=\frac{\pi}{6}$$

Multiplying by sin on both sides, we get

$$\sin x=\sin\frac{\pi}{6}$$

$$\Rightarrow x=n\pi+(-1)^n\frac{\pi}{6}$$

55. Hint: $a^2+b^2\neq -c^2$

$$\cos 2x+\cos 2y+2\tan^2 x+2=0$$
$$\Rightarrow 2\cos^2 x-1+2\cos^2 y-1+2\tan^2 x+2=0$$
$$\Rightarrow \cos^2 x+\cos^2 y+\tan^2 x=0$$
$$\Rightarrow \cos^2 x+\tan^2 x=-\cos^2 y$$

which is never possible.

56. See Fig. 3.9.

$$4\sin^2 x-6\sin x-2\sin x+3\leq 0$$
$$\Rightarrow 2\sin x(2\sin x-3)-1(2\sin x-3)\leq 0$$

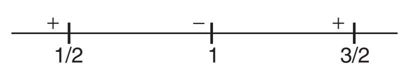


Figure 3.9

$\Rightarrow(2\sin x-1)(2\sin x-3)\leq 0\Rightarrow 2\sin x-1\leq 0$ and $2\sin x-3\geq 0$ or $2\sin x-1\geq 0$ and $2\sin x-3\leq 0$

$$\sin x\geq\frac{1}{2}\ \text{or}\ \sin x\leq\frac{3}{2}$$

$$\Rightarrow x\in\left[\frac{\pi}{6},\frac{5\pi}{6}\right]$$

Hence, the required solution is $\left[\frac{\pi}{6},\frac{5\pi}{6}\right]$.

57. Hint: $\sec\frac{\pi}{4}=\sqrt{2}$

Only $\theta=\frac{\pi}{4}$ satisfies the given equation.

58. Hint: $ax^2+bx+c=0$, for real $b^2-4ac\ge 0$

$$\frac{4}{\sin x}+\frac{1}{1-\sin x}=a$$
$$\Rightarrow 4-4\sin x+\sin x=a\sin x(1-\sin x)$$
$$\Rightarrow 4-3\sin x=a\sin x-a\sin^2 x$$
$$\Rightarrow a\sin^2 x-(a+3)\sin x+4=0$$

The above is a quadratic equation for sin x. For the real values of sin x, discriminant ≥ 0. So

$$\Rightarrow (a+3)^2-4\cdot 4\cdot a\ge 0$$
$$\Rightarrow a^2+6a+9-16a\ge 0$$
$$\Rightarrow a^2-10a+a\ge 0$$
$$\Rightarrow (a-9)(a-1)\ge 0$$
$$\Rightarrow a\ge 9, a\ge 1$$

But $a=1$ does not satisfy. Therefore, $a=9$.

59. Hint: $ax^2+bx+c=x^2-x(\alpha+\beta)+\alpha\beta,\ \alpha,\beta$ are the roots of the equation.

$$\cot x\cot y=k \ \Rightarrow\ \frac{1}{k}=\tan x\cdot\tan y$$
$$\tan(x+y)=\tan\frac{\pi}{3}\ \Rightarrow\ \frac{\tan x+\tan y}{1-\tan x\cdot\tan y}=\sqrt{3}$$
$$\Rightarrow \frac{\tan x+\tan y}{1-\frac{1}{k}}=\sqrt{3}\Rightarrow \tan x+\tan y=\frac{\sqrt{3}(k-1)}{k}$$

Therefore, the quadratic equation whose roots are tan x and tan y is given by

$$t^2-\frac{\sqrt{3}(k-1)}{k}t+\frac{1}{k}=0$$
$$\Rightarrow\ kt^2-\sqrt{3}(k-1)t+1=0$$

60. Hint: For the increasing function $f'(x)>0$.

$$x^3+2x^2+5x=-2\cos x$$

Let
$$f(x)=x^3+2x^2+5x=x(x^2+2x+5)$$

The term in the bracket has no root.
Also, $f'(x)>0$ and so $f(x)$ is always increasing.
From the graph (Fig. 3.10), it is clear that in $(0, 2\pi)$ equations do not have any solution.

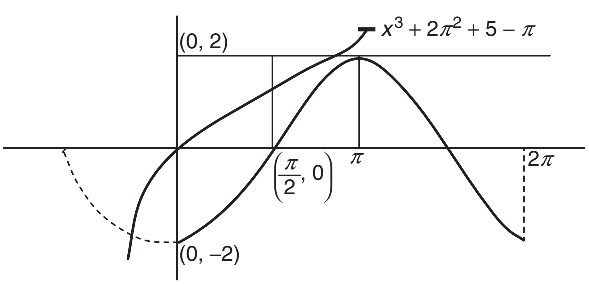


Figure 3.10

Practice Exercise 2

1. AM $\ge$ GM. Also

$$f(x)=2\ (\text{AM of } \pi^{\sec^2\alpha}\cos^2\alpha \text{ and } \pi^{\operatorname{cosec}^2\alpha}\sin^2\alpha)$$

Using AM > GM we see that
$$f(x)\ge 2\ (\text{GM of } \pi^{\sec^2\alpha}\cos^2\alpha \text{ and } \pi^{\operatorname{cosec}^2\alpha}\sin^2\alpha)$$
$$=2\sqrt{(\pi^{\sec^2\alpha}\cos^2\alpha)(\pi^{\operatorname{cosec}^2\alpha}\sin^2\alpha)}=\sin 2\alpha\ (\pi^{2/\sin^2(2\alpha)})$$

Now let $g(x)=x\,\pi^{2/x^2}$ is a decreasing function for $x\in(0,1)$.
Also for $0<\alpha<\pi/2$, $\sin 2\alpha\in(0,1)$.
Therefore, $g(\sin 2\alpha)>g(1)>\pi^2$

2. The given equality is possible if and only if sin$x$ and cos$x$ are of same sign, which is true only in I^st^ and III^rd^ quadrants.

3. $$y^2-y+a=\left(y-\frac{1}{2}\right)^2+a-\frac{1}{4}$$

Since $-\sqrt{2}\le\sin x+\cos x\le\sqrt{2}$, given equation will have no real value of x for any y if $a-\frac{1}{4}>\sqrt{2}$, that is $a\in(\sqrt{3},\infty)$ (as $\sqrt{2}+1/4<\sqrt{3}$).

4. The maximum value of $\cos^2(\cos\theta)$ is 1 and that of $\sin^2(\sin\theta)$ is $\sin^2 1$. Both exist for $\theta=\pi/2$. Hence maximum value is $1+\sin^2 1$.
5. $\cos A\cdot\cos B+\sin A\cdot\sin B\cdot\sin C\le\cos(A-B)=1$.
6. $2\cos x+\cos 2kx=3$

$$\Rightarrow 2\left(1-2\sin^2\frac{x}{2}\right)+1-2\sin^2 kx=3$$
$$\Rightarrow 2\sin^2\frac{x}{2}+\sin^2 kx=0 \qquad (1)$$

Since $\sin^2\frac{x}{2}$ and $\sin^2 kx$ are both positive, Eq. (1) is possible only if $\sin^2\frac{x}{2}=0$ and $\sin^2 kx=0$.

Therefore, $x=0,\pm 2\pi,\pm 4\pi,\dots$. But for Eq. (1) to have unique solution, the possible value of k must be irrational. Therefore, $k=\sqrt{2}$ is the possible option.

7. $\sin x+\sin 3x-3\sin 2x=\cos x+\cos 3x-3\cos 2x$
$\Rightarrow 2\sin 2x\cos x-3\sin 2x=2\cos x\cos 2x-3\cos 2x$
$\Rightarrow \sin 2x(2\cos x-3)=\cos 2x(2\cos x-3)$
$\Rightarrow (2\cos x-3)(\sin 2x-\cos 2x)=0$ (Since $\cos x\ne 3/2$)
Therefore, $\sin 2x=\cos 2x$

$$\tan 2x=1\Rightarrow 2x=n\pi+\frac{\pi}{4}\Rightarrow x=\frac{n\pi}{2}+\frac{\pi}{8}, n\in I$$

8. $$(x+3)^2+3+3\sin(a+bx)=0$$
Now, $x=-3$, so
$$\sin(a+bx)=-1$$
$$\Rightarrow \sin(a-3b)=-1$$
$$\Rightarrow a-3b=(4n-1)\frac{\pi}{2}, n\in I$$
As $n=1$, Therefore, $a-3b=\frac{3\pi}{2}\Rightarrow \cos(a-3b)=0$.

9. We have

$$2\sin^2 x+\frac{2\sin x\cos x}{2}=n$$

$$\Rightarrow 1-\cos 2x+\frac{\sin 2x}{2}=n$$
$$\Rightarrow \sin 2x-2\cos 2x=2n-2$$
$$\Rightarrow -\sqrt{5}\le 2n-2\le \sqrt{5}$$
$$\Rightarrow \frac{-\sqrt{5}}{2}\le n-1\le \frac{\sqrt{5}}{2}$$
$$\Rightarrow 1-\frac{\sqrt{5}}{2}\le n\le 1+\frac{\sqrt{5}}{2}$$
$$\Rightarrow n=0, 1, 2$$

10. $\frac{4}{\sin x}+\frac{8}{\cos x}\ge 2\left(\frac{32}{\sin x\cos x}\right)^{1/2}=\frac{16}{\sqrt{\sin 2x}}\ge 16$

But AM = GM and $\sin 2x = 1$ cannot occur simultaneously

Therefore, $\frac{4}{\sin x}+\frac{8}{\cos x}>16$

Also $\frac{32}{\sin x\cos x}=\frac{64}{\sin 2x}\ge 64$

Therefore,

$$(1+4\operatorname{cosec}x)(1+8\sec x)=1+\left(\frac{4}{\sin x}+\frac{8}{\cos x}\right)+\frac{32}{\sin x\cos x}$$
$$>1+16+64=81$$

11. $\left(1+\frac{1}{\sin^n\theta}\right)\left(1+\frac{1}{\cos^n\theta}\right)$

Now,

$$f(\theta)=1+\frac{1}{\sin^n\theta}+\frac{1}{\cos^n\theta}+\frac{1}{\sin^n\theta\cos^n\theta}$$

and

$$\frac{d}{d\theta}f(\theta)=-\frac{n}{\sin^{n+1}\theta}\cos\theta+\frac{n}{\cos^{n+1}\theta}\sin\theta-\frac{2^n n}{(\sin 2\theta)^{n+1}}2\cos 2\theta=0$$

will give $\theta=\frac{\pi}{4}$

Hence, $f(\theta)_{\min}$ will occur at $\theta=\frac{\pi}{4}$.

12. $\tan\theta=\dfrac{3\tan\frac{\theta}{3}-\tan^3\frac{\theta}{3}}{1-3\tan^2\frac{\theta}{3}}=\lambda$

$$\Rightarrow \tan^3\frac{\theta}{3}-3\lambda\tan^2\frac{\theta}{3}-3\tan\frac{\theta}{3}+\lambda=0$$
$$\Rightarrow \sum\tan\frac{\theta_1}{3}\tan\frac{\theta_2}{3}=-3.$$

13. $2\cos^4x+\cos x-2\cos x\sin^2x-3\sin^2x+1=0$

$\Rightarrow 2\cos^4x+2\cos x-2\cos x\sin^2x-3\sin^2x-\cos x+1=0$

$\Rightarrow 2\cos^4x+2\cos^3x+3\cos^2x-\cos x-2=0$

$\Rightarrow (2\cos^2x-1)(\cos^2x+\cos x+2)=0$

$\Rightarrow \cos x=\frac{1}{\sqrt{2}}\Rightarrow x=n\pi\pm\frac{\pi}{4}.$

14. $y=\dfrac{t^2-t+1}{t^2+(2-d)t+1}$; $t=\cot^2\theta$

$$\Rightarrow (y-1)t^2+((2-d)y+1)t+y-1=0$$
$$D\ge 0\Rightarrow y^2(4-(2-d)^2)-2(6-d)y+3\le 0$$

Also, $3y^2-10y+3\le 0$

$$\Rightarrow \frac{4-(2-d)^2}{3}=\frac{-2(6-d)}{-10}=\frac{3}{3}$$
$$\Rightarrow \frac{4-(2-d)^2}{3}=\frac{-2(6-d)}{-10}=\frac{3}{3}\Rightarrow d=1$$

15. $(4+\sin 4x)(2+\cot^2y)(1+\sin^4z)\le 12\sin^2z$

$$\Rightarrow (4+\sin 4x)(2+\cot^2y)\left(\frac{1+\sin^4z}{\sin^2z}\right)\le 12$$
$$\Rightarrow (4+\sin 4x)(2+\cot^2y)(\sin^2z+\operatorname{cosec}^2z)\le 12$$

If $(4+\sin 4x)\ge 3$, $(2+\cot^2y)\ge 2$ and $(\sin^2z+\operatorname{cosec}^2z)\ge 2$, then least value of

$$(4+\sin 4x)(2+\cot^2y)(\sin^2z+\operatorname{cosec}^2z)\ge 12$$

From the question, we can see that only below equality holds

$$4+\sin 4x=3, 2+\cot^2y=2, \sin^2z+\operatorname{cosec}^2z=2$$
$$\sin 4x=-1, \cot^2y=0, \sin^2z=1$$
$$\sin 4x=-1, x\in\left\{\frac{3\pi}{8},\frac{7\pi}{8},\frac{11\pi}{8},\frac{15\pi}{8}\right\}$$
$$\cot^2y=0, x\in\left\{\frac{\pi}{2},\frac{3\pi}{2}\right\}$$
$$\sin^2z=1, x\in\left\{\frac{\pi}{2},\frac{3\pi}{2}\right\}$$

16. $(\cos\theta-\sin\theta)(\cos^2\theta+\sin\theta\cos\theta+\sin^2\theta-\sin\theta\cos\theta)$

$$=-4\sqrt{2}\sin\left(\theta-\frac{\pi}{4}\right)>0$$

Now, $\sin\left(\theta-\frac{\pi}{4}\right)$ is negative. Hence,

$$(2n-1)\pi<\theta-\frac{\pi}{4}<2n\pi$$

17. See Fig. 3.11.

$\alpha+\gamma=\pi$ and $\beta+\gamma=\pi$,

$\Rightarrow \cos\alpha+\cos\beta+\cos\gamma+\cos\delta=0$

$\Rightarrow \cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta+2\sum\cos\alpha\cos\beta=0$

$\Rightarrow 2\sum\cos a\cos\beta=-[2\cos^2\alpha+2\cos^2\beta]$

$\Rightarrow 2+\sum\cos\alpha\cos\beta=[\sin^2\alpha+\sin^2\beta]$

$=\sin^2\alpha+\sin^2\delta$ (therefore, $\delta=\pi-\beta$)

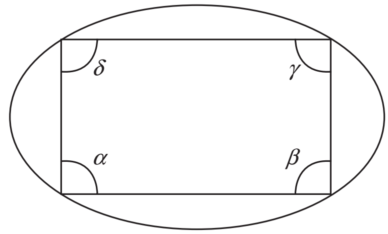


Figure 3.11

18. $$1 + A^2 = (\sin x + A\cos x)^2 + (A\sin x - \cos x)^2$$

$$\Rightarrow \sqrt{1 + A^2 - B^2} = |A\sin x - \cos x|$$

So Statement–1 is true.
Statement–2 is obviously true but Statement–2 is not a correct explanation of Statement–1.
$y = ax^2 + bx + c = 0$ is a quadratic equation which has real roots if and only if $b^2 - 4ac \geq 0$. If $f(x, y) = 0$ is a second-degree equation, then using above fact we can get the range of x and y by treating it as quadratic equation in y or x. Similarly, $ax^2 + bx + c \geq 0 \ \forall \ x \in R$ if $a > 0$ and $b^2 - 4ac \leq 0$.

19. By solving, we get

$$\sin(\alpha + \beta) = \sin\alpha \pm \sqrt{-\cos^2\beta} \Rightarrow \cos\beta = 0$$

$$\Rightarrow \beta = \frac{\pi}{2}, \frac{3\pi}{2}$$

(i) $\beta = \frac{\pi}{2} \Rightarrow \tan\alpha = 1, \alpha \in \left\{\frac{\pi}{4}, \frac{5\pi}{4}\right\}$

(ii) $\beta = \frac{3\pi}{2} \Rightarrow \tan\alpha = -1, \alpha \in \left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$

20. Let $\cos A + \cos B + k\cos C = y$.

$$\Rightarrow 2k\sin^2\frac{C}{2} - 2\cos\left(\frac{A-B}{2}\right)\sin\frac{C}{2} + y - k = 0. \text{ As } D \geq 0$$

$$\Rightarrow k(y-k) \leq \frac{1}{2}\cos^2\left(\frac{A-B}{2}\right) \leq \frac{1}{2} \Rightarrow y \leq \frac{1}{2k} + k$$

Solved JEE 2017 Questions

JEE Main 2017

1. If $5(\tan^2 x - \cos^2 x) = 2\cos 2x + 9$, then the value of $\cos 4x$ is:

(A) $\frac{1}{3}$ **(B)** $\frac{2}{9}$

(C) $-\frac{7}{9}$ **(D)** $-\frac{3}{5}$

(OFFLINE)

Solution: Let $\cos^2 x = t$. Therefore, from the given equation, we get

$$5\left[\frac{1-t}{t} - t\right] = 2(2t-1)+9$$
$$5(1-t-t^2) = t(4t+7)$$
$$9t^2 + 12t - 5 = 0$$
$$9t^2 + 15t - 3t - 5 = 0$$
$$(3t-1)(3t+5) = 0$$

Thus, we consider $t = \frac{1}{3}$ since $t \neq \frac{-5}{3}$. Therefore,

$$\cos 2x = 2\left(\frac{1}{3}\right) - 1 = \frac{-1}{3}$$

and $$\cos 4x = 2\left(\frac{-1}{3}\right)^2 - 1 = \frac{-7}{9}$$

Hence, the correct answer is option (C).

JEE Advanced 2017

1. Let α and β be non-zero real numbers such that $2(\cos\beta - \cos\alpha) + \cos\alpha\cos\beta = 1$. Then, which of the following is/are true?

(A) $\tan\left(\frac{\alpha}{2}\right) + \sqrt{3}\tan\left(\frac{\beta}{2}\right) = 0$

(B) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right) = 0$

(C) $\tan\left(\frac{\alpha}{2}\right) - \sqrt{3}\tan\left(\frac{\beta}{2}\right) = 0$

(D) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right) - \tan\left(\frac{\beta}{2}\right) = 0$

Solution: It is given that

$$2(\cos\beta - \cos\alpha) + \cos\alpha\cos\beta = 1$$

Using

$$\cos\alpha = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$$

and $$\cos\beta = \frac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}$$

we get

$$2\left(\frac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}} - \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}\right) + \left[\frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} \times \frac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}\right] = 1$$

$$\Rightarrow 2\left[\frac{\left(1-\tan^2\frac{\beta}{2}\right)\left(1+\tan^2\frac{\alpha}{2}\right) - \left(1+\tan^2\frac{\beta}{2}\right)\left(1-\tan^2\frac{\alpha}{2}\right)}{\left(1+\tan^2\frac{\alpha}{2}\right)\left(1+\tan^2\frac{\beta}{2}\right)}\right] + \left[\frac{\left(1-\tan^2\frac{\alpha}{2}\right)\left(1-\tan^2\frac{\beta}{2}\right)}{\left(1+\tan^2\frac{\alpha}{2}\right)\left(1+\tan^2\frac{\beta}{2}\right)}\right] = 1$$

$$\Rightarrow 2\left(\frac{1-\tan^2\frac{\beta}{2}+\tan^2\frac{\alpha}{2}-\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}-1+\tan^2\frac{\alpha}{2}-\tan^2\frac{\beta}{2}+\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}}{\left(1+\tan^2\frac{\alpha}{2}\right)\left(1+\tan^2\frac{\beta}{2}\right)}\right) + \left[\frac{1-\tan^2\frac{\alpha}{2}-\tan^2\frac{\beta}{2}+\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}}{\left(1+\tan^2\frac{\alpha}{2}\right)\left(1+\tan^2\frac{\beta}{2}\right)}\right] = 1$$

$$\Rightarrow 2\left[\frac{\left(2\tan^2\frac{\alpha}{2}\right)-\left(2\tan^2\frac{\beta}{2}\right)}{\left(1+\tan^2\frac{\alpha}{2}\right)\left(1+\tan^2\frac{\beta}{2}\right)}\right] + \left[\frac{\left(1-\tan^2\frac{\alpha}{2}\right)-\left(\tan^2\frac{\beta}{2}\right)+\left(\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}\right)}{\left(1+\tan^2\frac{\alpha}{2}\right)\left(1+\tan^2\frac{\beta}{2}\right)}\right] = 1$$

$$\Rightarrow \left(4\tan^2\frac{\alpha}{2} - 4\tan^2\frac{\beta}{2} + 1 - \tan^2\frac{\alpha}{2} - \tan^2\frac{\beta}{2} + \tan^3\frac{\alpha}{2}\tan^2\frac{\beta}{2}\right)$$
$$= \left(1+\tan^2\frac{\alpha}{2}\right)\left(1+\tan^2\frac{\beta}{2}\right)$$

$$\Rightarrow \left(3\tan^2\frac{\alpha}{2} - 5\tan^2\frac{\beta}{2} + \tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2} + 1\right)$$
$$= \left(1+\tan^2\frac{\alpha}{2}+\tan^2\frac{\beta}{2}+\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}\right)$$

$$\Rightarrow \left(2\tan^2\frac{\alpha}{2} - 6\tan^2\frac{\beta}{2}\right) = 0$$

$$\Rightarrow \left(\tan^2\frac{\alpha}{2}\right) = \left(3\tan^2\frac{\beta}{2}\right)$$

$$\Rightarrow \left(\tan\frac{\alpha}{2}\right) = \pm\left(\sqrt{3}\tan\frac{\beta}{2}\right)$$

$$\Rightarrow \left(\tan\frac{\alpha}{2}\right) \pm \left(\sqrt{3}\tan\frac{\beta}{2}\right) = 0$$

Hence, the correct answers are options (A) and (C).

4 Properties of Triangle

4.1 Introduction

In a triangle *ABC*, the angles are denoted by capital letters *A*, *B* and *C* and the lengths of the sides opposite to these angles are denoted by small letters *a*, *b* and *c*, respectively (Fig. 4.1). Semi-perimeter of the triangle is defined as $s = \frac{a+b+c}{2}$ and its area is denoted by *S* or Δ.

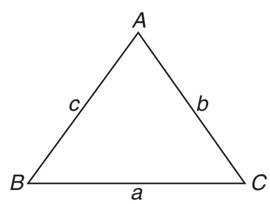


Figure 4.1

Some properties

1. $A + B + C = 180°$ (or π)
2. $a + b > c,\ b + c > a,\ c + a > b$
3. $|a - b| < c, |b - c| < a, |c - a| < b$

Generally, the relations involving the sides and angles of a triangle are cyclic in nature, e.g. to obtain the second similar relation to $a + b > c$, we simply replace *a* by *b*, *b* by *c* and *c* by *a*. So, to write all the relations follow the cycles given.

4.2 Relation Between Sides and Angles of a Triangle

4.2.1 Sine Rule

See Fig. 4.2. In ΔABC, the sides of a triangle are proportional to the sine of the angles opposite to them

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

If *S* is the centre of the circumcircle and *R* the circumradius, then in ΔBDC, right-angled at *B*, with $\angle BDC = \angle BAC = A$,

$$\sin A = \sin\angle BDC = \frac{BC}{DC} = \frac{a}{2R} \Rightarrow \frac{a}{\sin A} = 2R;$$

Similarly,

$$\frac{b}{\sin B} = 2R \text{ and } \frac{c}{\sin C} = 2R$$

Hence,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

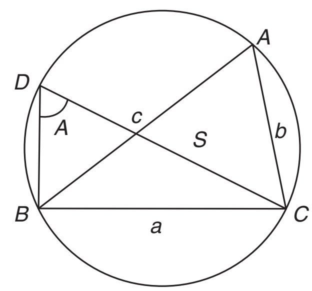


Figure 4.2

Note:

1. The above rule may also be expressed as $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.
2. The sine rule is a very useful tool to express sides of a triangle in terms of sines of angle and vice versa in the following manner:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = K \text{ (say)}$$

$$\Rightarrow a = K \sin A,\ b = K \sin B,\ c = K \sin c$$

Similarly,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \lambda \text{ (say)}$$

$$\Rightarrow \sin A = \lambda a,\ \sin B = \lambda b,\ \sin C = \lambda c$$

Illustration 4.1 If the angles of a triangle are in the ratio 4:1:1, then find the ratio of the longest side to the perimeter.

Solution: Let *x* be the angle of a triangle. Then

$$4x + x + x = 180 \Rightarrow 6x = 180 \Rightarrow x = 30°$$

$$\frac{\sin 120°}{a} = \frac{\sin 30°}{b} = \frac{\sin 30°}{c}$$

Therefore,

$$a : (a + b + c) = (\sin 120°) : (\sin 120° + \sin 30° + \sin 30°)$$

$$= \frac{\sqrt{3}}{2} : \frac{\sqrt{3}+2}{2} = \sqrt{3} : \sqrt{3} + 2$$

Illustration 4.2 In a triangle *ABC*, $\angle B = \pi/3$ and $\angle C = \pi/4$ and *D* divides *BC* internally in the ratio 1:3. Then, find the value of $\frac{\sin\angle BAD}{\sin\angle CAD}$.

Solution: See Fig. 4.3. Let $\angle BAD = \alpha$, $\angle CAD = \beta$.

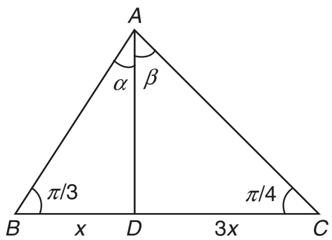


Figure 4.3

In ΔADB, applying sine formulae, we get

$$\frac{x}{\sin\alpha} = \frac{AD}{\sin\left(\frac{\pi}{3}\right)} \quad (1)$$

In ΔADC, applying sine formulae, we get

$$\frac{3x}{\sin\beta} = \frac{AD}{\sin(\pi/4)} \quad (2)$$

Dividing Eq. (1) by Eq. (2), we get

$$\frac{x}{\sin\alpha} \times \frac{\sin\beta}{3x} = \frac{AD}{\sin\left(\frac{\pi}{3}\right)} \times \frac{\sin\left(\frac{\pi}{4}\right)}{AD}$$

$$\Rightarrow \frac{\sin\beta}{3\sin\alpha} = \frac{\frac{1}{\sqrt{2}}}{\frac{\sqrt{3}}{2}} = \sqrt{\frac{2}{3}}$$

$$\Rightarrow \frac{\sin\beta}{\sin\alpha} = 3\sqrt{\frac{2}{3}} = \sqrt{6}$$

Therefore,

$$\frac{\sin\angle BAD}{\sin\angle CAD} = \frac{\sin\alpha}{\sin\beta} = \frac{1}{\sqrt{6}}$$

Illustration 4.3 In any triangle ABC if $2\cos B = \frac{a}{c}$, then the triangle is

(A) Right angled **(B)** Equilateral
(C) Isosceles **(D)** None of these

Solution:

$$2\cos B = \frac{a}{c} = \frac{k\sin A}{k\sin C} = \frac{\sin A}{\sin C}$$

$$\Rightarrow 2\cos B\sin C = \sin A$$

$$\Rightarrow \sin(B+C) - \sin(B-C) = \sin A$$

$$\Rightarrow \sin(180^\circ - A) - \sin(B-C) = \sin A$$

$$\Rightarrow \sin A - \sin(B-C) = \sin A$$

$$\Rightarrow \sin(B-C) = 0$$

$$\Rightarrow B - C = 0 \Rightarrow B = C$$

Therefore, the triangle is isosceles.
Hence, the correct answer is option (C).

Illustration 4.4 Prove that $a\cos A + b\cos B - c\cos C = 2c\cos A\cos B$.

Solution:

$$\begin{aligned}
\text{LHS} &= a\cos A + b\cos B - c\cos C \\
&= 2R\{\sin A\cos A + \sin B\cos B - \sin C\cos C\} \\
&= R\{\sin 2A + \sin 2B - \sin 2C\} \\
&= R\{2\sin(A+B)\cos(A-B) - 2\sin C\cos C\} \\
&= R\{2\sin C\cos(A-B) + 2\sin C\cos(A+B)\} \\
&\qquad (\text{since } A+B = \pi - C) \\
&= 2R\sin C\{\cos(A-B) + \cos(A+B)\} \\
&= 4R\sin C\cos A\cos B \\
&= 2c\cos A\cos B
\end{aligned}$$

(since $c = 2R\sin C$)

4.2.2 Cosine Rule

See Fig. 4.4. In a ΔABC,

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = c^2 + a^2 - 2ca\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

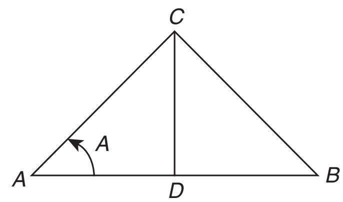


Figure 4.4

We shall prove the first one: We have

$$\begin{aligned}
BC^2 &= DC^2 + DB^2 \\
&= DC^2 + (AB - AD)^2 \\
&= (DC^2 + AD^2) + AB^2 - 2\,AB\cdot AD \\
&= AC^2 + AB^2 - 2ABAC\cos A
\end{aligned}$$

That is,

$$a^2 = b^2 + c^2 - 2bc\cos A$$

From these formulas, we also have the following:

1. $a^2 = b^2 + c^2 - 2bc\cos A \Rightarrow \cos A = \frac{b^2 + c^2 - a^2}{2bc}$
2. $b^2 = c^2 + a^2 - 2ca\cos B \Rightarrow \cos B = \frac{c^2 + a^2 - b^2}{2ca}$
3. $c^2 = a^2 + b^2 - 2ab\cos C \Rightarrow \cos C = \frac{a^2 + b^2 - c^2}{2ab}$

Combining with $\sin A = \frac{a}{2R}$, $\sin B = \frac{b}{2R}$, $\sin C = \frac{c}{2R}$, we have by division,

$$\tan A = \frac{abc}{R(b^2 + c^2 - a^2)},\ \tan B = \frac{abc}{R(c^2 + a^2 - b^2)},$$

$$\tan C = \frac{abc}{R(a^2 + b^2 - c^2)}$$

where R is the radius of the circumcircle of the triangle ABC.

Illustration 4.5 Find the smallest angle of the ΔABC, when $a=7,\ b=4\sqrt{3}$ and $c=\sqrt{13}$.

Solution: Smallest angle is opposite to smallest side. Therefore,

$$\cos C=\frac{b^2+a^2-c^2}{2ab}=\frac{49+48-13}{2\times 7\times 4\sqrt{3}}=\frac{3}{2\sqrt{3}}=\frac{\sqrt{3}}{2}\Rightarrow \angle C=30^\circ$$

Illustration 4.6 If $\frac{b+c}{11}=\frac{c+a}{12}=\frac{a+b}{13}$, prove that

$$\frac{\cos A}{7}=\frac{\cos B}{19}=\frac{\cos C}{25}$$

Solution:

$$\frac{b+c}{11}=\frac{c+a}{12}=\frac{a+b}{13}=\frac{b+c+c+a+a+b}{11+12+13}=\frac{a+b+c}{18}$$

By ratio of proportion, that is,

$$\frac{A}{B}=\frac{C}{D}=\frac{A+C}{B+D}$$

$$\frac{b+c}{11}=\frac{a}{7},\ \frac{c+a}{12}=\frac{b}{6},\ \frac{a+b}{13}=\frac{c}{5}$$

$$\Rightarrow \frac{a}{7}=\frac{b}{6}=\frac{c}{5}=k\,(\text{say})$$

$$\cos A=\frac{b^2+c^2-a^2}{2bc}=k^2\frac{(6^2+5^2-7^2)}{k^2 2(6)(5)}=\frac{1}{5}=\frac{7}{35}$$

$$\cos B=\frac{c^2+a^2-b^2}{2ca}=\frac{k^2(5^2+7^2-6^2)}{k^2 2(5)(7)}=\frac{19}{35}$$

$$\cos C=\frac{a^2+b^2-c^2}{2}=\frac{k^2(7^2+6^2-5^2)}{k^2 2(7)(6)}=\frac{5}{7}=\frac{25}{35}$$

Therefore,

$$\frac{\cos A}{7}=\frac{\cos B}{19}=\frac{\cos C}{25}$$

Illustration 4.7 In a triangle ABC, AD is altitude from A. Given $b>c$, $\angle C=23^\circ$ and $AD=\frac{abc}{b^2-c^2}$. Find $\angle B$.

Solution: We know that,

$$\cos B=\frac{a^2+c^2-b^2}{2ac}=\frac{a^2-(b^2-c^2)}{2ac}$$

Now, $AD=\frac{abc}{b^2-c^2}$. Therefore,

$$\cos B=\frac{a^2-\frac{abc}{AD}}{2ac}$$

Also, $AD=b\sin 23^\circ$. Therefore,

$$\cos B=\frac{a-\frac{c}{\sin 23^\circ}}{2c}$$

By sine formulae,

$$\frac{a}{c}=\frac{\sin(B+23^\circ)}{\sin 23^\circ}$$

Therefore,

$$\cos B=\frac{\left[\frac{\sin(B+23^\circ)}{\sin 23^\circ}-\frac{1}{\sin 23^\circ}\right]}{2}$$

$$\Rightarrow \sin(23^\circ-B)=-1=\sin(-90^\circ)$$

Therefore, $23^\circ-B=-90^\circ$ or $B=113^\circ$.

4.2.3 Projection Rule

In a ΔABC,

$$a=b\cos C+c\cos B$$
$$b=c\cos A+a\cos C$$
$$c=a\cos B+b\cos A$$

With reference to the figure drawn for the cosine formula

$$c=AB=AD+DB=\frac{AD}{AC}+BC\frac{DB}{BC}=b\cos A+a\cos B$$

Illustration 4.8 In any triangle ABC, prove that

$$(b+c)\cos A+(c+a)\cos B+(a+b)\cos C=a+b+c$$

Solution:

$$\begin{aligned}\text{LHS}&=(b+c)\cos A+(c+a)\cos B+(a+b)\cos C\\&=b\cos A+c\cos A+c\cos B+a\cos B+a\cos C+b\cos C\\&=(b\cos A+a\cos B)+(c\cos A+a\cos C)+(c\cos B+b\cos C)\\&=c+b+a=\text{RHS}\quad[\text{By using projection rule}]\end{aligned}$$

Illustration 4.9 In a ΔABC, find the value of $\frac{\cos C+\cos A}{c+a}+\frac{\cos B}{b}$.

[EAMCET 2001]

Solution:

$$\frac{\cos C+\cos A}{c+a}+\frac{\cos B}{b}=\frac{(b\cos C+b\cos A)+(c\cos B+a\cos B)}{b(c+a)}$$

$$=\frac{(b\cos C+c\cos B)+(b\cos A+a\cos B)}{b(c+a)}$$

$$=\frac{a+c}{b(c+a)}\ \text{(Using projection formulae)}$$

$$=\frac{1}{b}$$

Illustration 4.10 If k is the perimeter of ΔABC, then find the value of $b\cos^2\frac{C}{2}+c\cos^2\frac{B}{2}$.

Solution:

$$b\cos^2\frac{C}{2}+c\cos^2\frac{B}{2}=\frac{b}{2}(1+\cos C)+\frac{c}{2}(1+\cos B)$$

$$=\frac{b}{2}+\frac{c}{2}+\frac{1}{2}(b\cos C+c\cos B)$$

$$=\frac{a+b+c}{2}=\frac{k}{2}$$

4.2.4 Tangent Rule or Napier Analogy

In a ΔABC,

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cdot\cot\frac{A}{2}$$

From the sine formula, we have

$$\frac{b}{\sin B}=\frac{c}{\sin C}\Rightarrow\frac{b}{c}=\frac{\sin B}{\sin C}$$

Using the componendo-dividendo principle, we get

$$\frac{b-c}{b+c}=\frac{\sin B-\sin C}{\sin B+\sin C}$$

$$=\frac{2\cos\frac{B+C}{2}\cdot\sin\frac{B-C}{2}}{2\sin\frac{B+C}{2}\cdot\cos\frac{B-C}{2}}$$

$$=\cot\frac{B+C}{2}\cdot\tan\frac{B-C}{2}$$

$$=\cot\left(90^\circ-\frac{A}{2}\right)\tan\frac{B-C}{2}$$

$$=\tan\frac{A}{2}\cdot\tan\frac{B-C}{2}$$

Therefore,

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$$

Similarly,

$$\tan\frac{C-A}{2}=\frac{c-a}{c+a}\cot\frac{B}{2}$$

and

$$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}$$

Illustration 4.11 In a ΔABC, $A=\frac{\pi}{3}$ and $b:c=2:3$. If $\tan\alpha=\frac{\sqrt{3}}{5}$, $0<\alpha<\frac{\pi}{2}$ then find the value of angles B and C.

Solution:

$$\tan\frac{C-B}{2}=\frac{c-b}{c+b}\cot\frac{A}{2}$$

$$\Rightarrow\ \tan\frac{C-B}{2}=\frac{1}{5}\cot 30^\circ=\frac{\sqrt{3}}{5}=\tan\alpha$$

Therefore,

$$C-B=2\alpha \text{ and } C+B=180^\circ-60^\circ=120^\circ$$

That is, $B=60^\circ-\alpha$, $C=60^\circ+\alpha$.

Illustration 4.12 In a ΔABC if $a=2b$ and $|A-B|=\pi/3$, then the measure of $\angle C$ is ______.

Solution: Clearly, $A>B$ $(a>b)$. Now

$$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}$$

$$\Rightarrow\ \tan 30^\circ=\frac{1}{3}\cot\frac{C}{2}$$

Therefore,

$$\sqrt{3}=\cot\frac{C}{2}\text{ or }\frac{C}{2}=\frac{\pi}{6}\Rightarrow C=\pi/3$$

Your Turn 1

1. In a ΔABC, $A:B:C=3:5:4$. Then $[a+b+c\sqrt{2}]$ is equal to ______. **[DCE 2001]**
Ans. 3b
2. Prove that $(b+c)\sin\frac{A}{2}=a\cos\frac{B-C}{2}$.
3. In a ΔABC find the value of $2ac\sin\left(\frac{A-B+C}{2}\right)$.
Ans. $c^2+a^2-b^2$
4. In a ΔABC if $\frac{\sin A}{\sin C}=\frac{\sin(A-B)}{\sin(B-C)}$, then a^2, b^2, c^2 are in which progression.
Ans. AP
5. In a ΔABC, find the value of $(a+b+c)(\cos A+\cos B+\cos C)$.
Ans. $2\sum a\cos^2\frac{A}{2}$
6. In a triangle ABC, $a=6$, $b=3$ and $\cos(A-B)=\frac{4}{5}$. Find angle C.
Ans. 90°

4.3 Theorem of the Medians (Apollonius Theorem)

In every triangle, the sum of the squares of any two sides is equal to twice the square of half the third side together with twice the square of the median that bisects the third side.

See Fig. 4.5. For any triangle ABC, using cosine rule we have

$$b^2+c^2=2(h^2+m^2)=2\{m^2+(a/2)^2\}$$

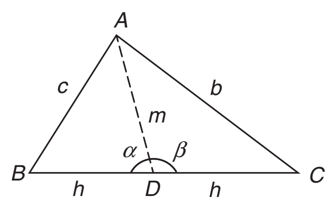


Figure 4.5

If the triangle is right angled, the mid-point of hypotenuse is equidistant from the three vertices, so that $DA=DB=DC$. Therefore, $b^2+c^2=a^2$, which is the **Pythagoras theorem**. This theorem is very useful for solving problems of height and distance.

Illustration 4.13 AD is a median of the ΔABC if AE and AF are medians of the triangles ABD and ADC, respectively, and $AD=m_1$, $AE=m_2$, $AF=m_3$. Find the value of $\frac{a^2}{8}$.

Solution: See Fig. 4.6. In ΔABC,

$$AD^2=m_1^2=\frac{c^2+b^2}{2}-\frac{a^2}{4}$$

In ΔABD,

$$AE^2=m_1^2=\frac{c^2+AD^2}{2}-\frac{\left(\frac{a}{2}\right)^2}{4}$$

In ΔADC,

$$AF^2 = m_3^2 = \frac{AD^2+b^2}{2} - \frac{\left(\frac{a}{2}\right)^2}{4}$$

Therefore,

$$m_2^2 + m_3^2 = AD^2 + \frac{b^2+c^2}{2} - \frac{a^2}{8}$$

$$\Rightarrow m_2^2 + m_3^2 = 2m_1^2 + \frac{a^2}{8}$$

$$\Rightarrow \frac{a^2}{8} = m_2^2 + m_3^2 - 2m_1^2$$

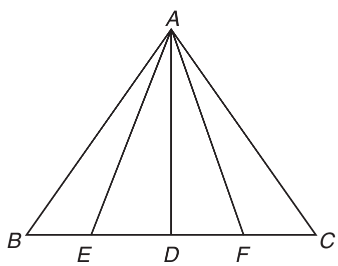


Figure 4.6

4.4 Half-Angle Formulae

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}},\ \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} \text{ and } \tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\sin\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}},\ \cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}} \text{ and } \tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$$

$$\sin\frac{C}{2} = \sqrt{\frac{(s-b)(s-a)}{ba}},\ \cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ba}} \text{ and } \tan\frac{C}{2} = \sqrt{\frac{(s-b)(s-a)}{s(s-c)}}$$

We shall prove the first of these

$$2\sin^2\frac{A}{2} = 1-\cos A = 1-\left(\frac{b^2+c^2-a^2}{2bc}\right)$$

$$= \frac{a^2-(b^2+c^2-2bc)}{2bc}$$

$$= \frac{a^2-(b-c)^2}{2bc} = \frac{(a+b-c)(a-b+c)}{2bc}$$

$$= \frac{(2s-2c)(2s-2b)}{2bc}$$

where $2s = a+b+c =$ Perimeter of the Δ

Therefore,

$$\sin^2\frac{A}{2} = \frac{(s-b)(s-c)}{2bc}$$

and

$$\sin\frac{A}{2} = \frac{\sqrt{(s-b)(s-c)}}{\sqrt{bc}}$$

Note:

$\sin\frac{A}{2} > 0$ since $\frac{A}{2}$ is an acute angle in a triangle.

Similarly, writing $2\cos^2\frac{A}{2} = 1+\cos A$ we get

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

and

$$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \frac{\sqrt{(s-b)(s-c)}}{\sqrt{s(s-a)}}$$

Illustration 4.14 If in a triangle ABC, $\tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}$ are in harmonic progression, then show that the sides a, b, c are in arithmetic progression.

Solution: Given $\tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}$ are in HP. Then

$$\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2} \text{ are in HP}$$

$$\Rightarrow 2\cot\frac{B}{2} = \cot\frac{A}{2} + \cot\frac{C}{2}$$

$$\Rightarrow 2\sqrt{\frac{s(s-b)}{(s-a)(s-c)}} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} + \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

Multiply throughout by $\sqrt{(s-a)(s-b)(s-c)}$ we get

$$2(s-b) = (s-a) + (s-c)$$

$$\Rightarrow 2b = a+c$$

$$\Rightarrow a, b, c \text{ are in AP}$$

Illustration 4.15 In a ΔABC, if $3a = b+c$, then find the value of $\cot\frac{B}{2}\cot\frac{C}{2}$.

Solution:

$$\cot\frac{B}{2}\cdot\cot\frac{C}{2} = \sqrt{\frac{s(s-b)}{(s-a)(s-c)}}\cdot\sqrt{\frac{s(s-c)}{(s-a)(a-b)}} = \frac{s}{s-a}$$

Given $3a = b+c \Rightarrow a+b+c = 4a$. Therefore,

$$\cot\frac{B}{2}\cot\frac{C}{2} = \frac{s}{s-a} = \frac{2a}{a} = 2$$

Illustration 4.16 If the sides of triangle a, b, c be in AP, then find the value of $\tan\frac{A}{2}+\tan\frac{C}{2}$.

Solution:

$$\tan\frac{A}{2}+\tan\frac{C}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} + \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

$$= \frac{b}{s}\cot\frac{B}{2} = \frac{2b}{2s}\cot\frac{B}{2}$$

Now a, b, c are in AP. Therefore,

$$a+c = 2b \Rightarrow 2s = 3b$$

Hence,

$$\tan\frac{A}{2}+\tan\frac{C}{2} = \frac{2b}{3b}\cot\frac{B}{2} = \frac{2}{3}\cot\frac{B}{2}$$

4.5 Area of a Triangle

Let three angles of ΔABC be denoted by A, B, C and the sides opposite to these angles by letters a, b, c, respectively.

1. **When two sides and the included angle are given:** See Fig. 4.7.

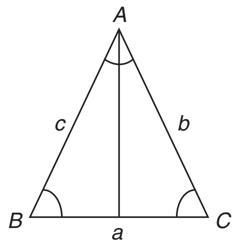


Figure 4.7

The area of triangle ABC is given by

$$\Delta = \frac{1}{2}bc\sin A = \frac{1}{2}ca\ \sin B = \frac{1}{2}ab\sin C$$

That is, $\Delta = \frac{1}{2}$ (Product of two sides) $\times$ (Sine of included angle)

2. **When three sides are given:**

Area of ΔABC is

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

where semi-perimeters of triangle is defined by $s = \frac{a+b+c}{2}$

3. **When three sides and the circumradius are given:**

$$\text{Area of triangle } \Delta = \frac{abc}{4R}$$

where R is the circumradius of the triangle.

4. **When two angles and the included sides are given:**

$$\Delta = \frac{1}{2}a^2\frac{\sin B\sin C}{\sin(B+C)} = \frac{1}{2}b^2\frac{\sin A\sin C}{\sin(A+C)} = \frac{1}{2}c^2\frac{\sin A\sin B}{\sin(A+B)}$$

Illustration 4.17 Prove that in any ΔABC

$$\frac{(a+b+c)^2}{a^2+b^2+c^2} = \frac{\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}}{\cot A+\cot B+\cot C}$$

Solution:

$$\cot A = \frac{\cos A}{\sin A} = \frac{2bc\cos A}{2bc\sin A} = \frac{b^2+c^2+a^2}{4\Delta}$$

Therefore,

$$\sum\cot a \sum\frac{b^2+c^2-a^2}{4\Delta} = \frac{a^2+b^2+c^2}{4\Delta}$$

Also,

$$\frac{1+\cos A}{\sin A} = \frac{2\cos^2\frac{A}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}} = \cot\frac{A}{2}$$

$$\frac{2bc(1+\cos A)}{2bc\sin A} = \cot\frac{A}{2}$$

Therefore,

$$\sum\cot\frac{A}{2} = \frac{\sum 2bc + \sum 2bc\cos A}{4\Delta}$$

$$= \frac{\sum 2bc + \sum(b^2+c^2-a^2)}{4\Delta}$$

$$= \frac{a^2+b^2+c^2+2ab+2bc+2ca}{4\Delta} = \frac{(a+b+c)^2}{4\Delta}$$

So,

$$\frac{\sum\cot\frac{A}{2}}{\sum\cot A} = \frac{(a+b+c)^2}{a^2+b^2+c^2}$$

Hence proved.

Illustration 4.18 In a ΔABC, $A = \frac{2\pi}{3}$, $b-c = 3\sqrt{3}$ and area $(\Delta ABC) = \frac{9\sqrt{3}}{2}\text{cm}^2$. Find a.

Solution: Since area (ΔABC) = $9\sqrt{3}/2$ we have

$$\frac{1}{2}bc\sin\frac{2\pi}{3} = \frac{9\sqrt{3}}{2} \Rightarrow \frac{1}{2}\cdot\frac{\sqrt{3}}{2}bc = \frac{9\sqrt{3}}{2} \Rightarrow bc = 18$$

Also,

$$\cos\frac{2\pi}{3} = \frac{b^2+c^2-a^2}{2bc}$$

$$\Rightarrow -\frac{1}{2} = \frac{(b-c)^2+2bc-a^2}{2bc}$$

$$\Rightarrow (b-c)^2+3bc-a^2 = 0$$

$$\Rightarrow 27+54 = a^2$$

$$\Rightarrow a = 9$$

Illustration 4.19 If p_1, p_2, p_3 are the length of the altitudes of a triangle ABC from the vertices A, B, C and Δ is the area of the triangle, then find the value of $p_1^{-2}+p_2^{-2}+p_3^{-2}$.

Solution: We have

$$\frac{1}{2}ap_1 = \Delta,\ \frac{1}{2}bp_2 = \Delta,\ \frac{1}{2}cp_3 = \Delta$$

$$\Rightarrow p_1 = \frac{2\Delta}{a},\ p_2 = \frac{2\Delta}{b},\ p_3 = \frac{2\Delta}{c}$$

Therefore,

$$\frac{1}{p_1^2}+\frac{1}{p_2^2}+\frac{1}{p_3^2} = \frac{a^2+b^2+c^2}{4\Delta^2}$$

Your Turn 2

1. If AD, BE and CF are the medians of a ΔABC then find the value of $(AD^2+BE^2+CF^2):(a^2+b^2+c^2)$.

Ans. 3:4

2. In ΔABC, find the value of $\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)\left(a\sin^2\frac{B}{2}+b\sin^2\frac{A}{2}\right)$.

[Roorkee 1988]

Ans. $c\sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = c\cot\frac{C}{2}$

3. If p_1, p_2, p_3 are the length of the altitudes of a triangle ABC, prove that $p_1^{-2}+p_2^{-2}+p_3^{-2} = \frac{(\cot A+\cot B+\cot C)}{\Delta}$.

4. Find the area of a triangle ΔABC, if $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$ $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$ and $a = 2$.

Ans. $\sqrt{3}$

5. Find the value of $a^2 \sin 2B + b^2 \sin 2A$ $a^2 \sin 2B + b^2 \sin 2A =$ ____ if Δ stands for the area of a triangle ABC.

Ans. 4Δ

4.6 Circle Connected with the Triangle

4.6.1 Circumcircle of a Triangle and its Radius

See Fig. 4.8. The circle passing through the vertices of the triangle ABC is called the circumcircle. Its radius R is called the circumradius. In the triangle ABC,

$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}$$

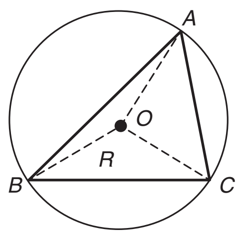


Figure 4.8

The centre of this circle is the point of intersection of perpendicular bisectors of the sides and is called the circumcentre.

4.6.2 In-circle of a Triangle and Its Radius

The circle touching the three sides of the triangle internally is called the inscribed or the in-circle of the triangle. Its radius r is called the in-radius of the triangle. Its centre is known as the in-centre.

The in-centre I is the point of concurrence of internal angle bisectors of the angles A, B, C. The circle with centre I and radius $r = IL = IM = IN$ touches the sides of the triangle.

$$\Delta = \Delta IBC + \Delta ICA + \Delta IAB$$

$$= \frac{1}{2}r(BC) + \frac{1}{2}r(CA) + \frac{1}{2}r(AB)$$

$$= \frac{r(a+b+c)}{2} = \frac{r \cdot 2s}{2} = rs$$

Therefore,

$$r = \frac{\Delta}{s}$$

$$r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$$

In Fig. 4.9, $AN = AM$ are tangents and similarly we have for other pairs. So

$$AN + BL + LC = \frac{2s}{2} = s$$

$$AN + a = s \Rightarrow AN = s - a$$

In $\Delta\, ANI$,

$$\frac{r}{AN} = \tan\frac{A}{2}$$

Therefore,

$$r = (s-a)\tan\frac{A}{2}$$

The other two values are obtained in a similar way. Also

$$r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

From Fig. 4.9, we have

$$a = BC = BL + LC = r\cot\frac{B}{2} + r\cot\frac{C}{2}$$

$$= r\left\{\frac{\cos\frac{B}{2}}{\sin\frac{B}{2}} + \frac{\cos\frac{C}{2}}{\sin\frac{C}{2}}\right\} = \frac{r\sin\left(\frac{B+C}{2}\right)}{\sin\frac{B}{2}\sin\frac{C}{2}} = \frac{r\cos\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}$$

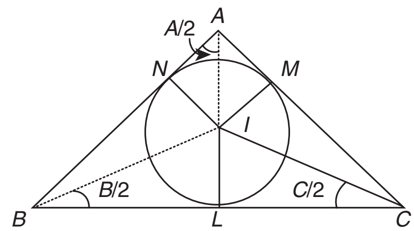


Figure 4.9

Therefore,

$$r = \frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} = \frac{2R\sin A\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}$$

$$= 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

4.6.3 Ex-Circles (Escribed Circle) of a Triangle and Their Radius

The circle touching BC and the two sides AB and AC produced of ΔABC externally is called the escribed circle opposite A. Its radius is denoted by r_1. Similarly, r_2 and r_3 denote the radii of the escribed circles opposite to angles B and C, respectively. r_1, r_2, r_3 are called the ex-radii of the ΔABC.

There are three ex-circles. The ex-circle opposite A is drawn in Fig. 4.10. I_1 is the point of intersection of internal bisector of angle A and external bisectors of angles B and C. The perpendiculars I_1L, I_1M, I_1N to the three sides of the Δ are equal and the radius r_1 of the ex-circle opposite is to A.

$$AN = AM = \frac{1}{2}(AN + AM)$$

$$= \frac{1}{2}[AB + BL + AC + CL]$$

$$= \frac{1}{2}(a+b+c) = s$$

Hence, from $\Delta\, ANI_1$

$$r_1 = I_1N = AN\tan\frac{A}{2} = s\tan\frac{A}{2}$$

$$\Rightarrow r_1 = s\tan\frac{A}{2}$$

Similarly,

$$r_2 = s\tan\frac{B}{2} \quad \text{and} \quad r_3 = s\tan\frac{C}{2}$$

Also,

$$r_1 = s\tan\frac{A}{2} = s\frac{\sqrt{(s-b)(s-c)}}{\sqrt{s(s-a)}}$$

$$= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s-a} = \frac{\Delta}{s-a}$$

Hence,

$$r_1 = \frac{\Delta}{s-a}$$

Similarly,

$$r_2 = \frac{\Delta}{s-b} \text{ and } r_3 = \frac{\Delta}{s-c}$$

Another formula for $r_1 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$ can be derived as follows:

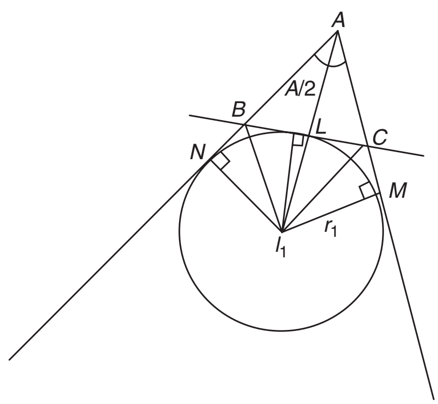


Figure 4.10

$$r_1 = s\tan\frac{A}{2} = \frac{s\sin\frac{A}{2}}{\cos\frac{A}{2}} = \frac{s\sqrt{\frac{(s-b)(s-c)}{bc}}}{\cos\frac{A}{2}}$$

$$= \frac{a\sqrt{\frac{s(s-b)}{ac}}\sqrt{\frac{s(s-c)}{ab}}}{\cos\frac{A}{2}} = \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$$

$$= \frac{4R\sin\frac{A}{2}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$$

$$r_1 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

Similarly,

$$r_1 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

$$r_2 = \cos\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\cos\frac{C}{2}$$

$$r_3 = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

Few more results: In any ΔABC, we have

1. $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3} = \frac{1}{r}$
2. $\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab} = \frac{1}{2Rr}$
3. $r_1r_2 + r_2r_3 + r_3r_1 = s^2$
4. $\Delta = 2R^2\sin A\cdot\sin b\cdot\sin C = 4Rr\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}$

Illustration 4.20 Prove that $r_1 + r_2 + r_3 - r = 4R$.

Solution: Taking the LHS, we have

$$(r_1+r_2)+(r_3-r) = \left(\frac{\Delta}{s-a}+\frac{\Delta}{s-b}\right)+\left(\frac{\Delta}{s-c}-\frac{\Delta}{s}\right)$$

$$= \Delta\left\{\frac{2s-a-b}{(s-a)(s-b)}\right\}+\Delta\left\{\frac{s-(s-c)}{s(s-c)}\right\}$$

$$= \Delta\left\{\frac{c}{(s-a)(s-b)}+\frac{c}{s(s-c)}\right\}$$

$$= \frac{\Delta c}{s(s-a)(s-b)(s-c)}\{s(s-c)+(s-a)(s-b)\}$$

$$= \frac{\Delta cab}{s(s-a)(s-b)(s-c)} = \frac{\Delta abc}{\Delta^2} = \frac{abc}{\Delta} = 4R$$

Illustration 4.21 Prove that

$$\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r^2} = \frac{a^2+b^2+c^2}{\Delta^2}$$

Solution: Taking the LHS, we have

$$\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r^2} = \frac{1}{\Delta^2}\{(s-a)^2+(s-b)^2+(s-c)^2+s^2\}$$

$$= \frac{1}{\Delta^2}\{4s^2-2s(a+b+c)+a^2+b^2+c^2\}$$

$$= \frac{1}{\Delta^2}\{4s^2-2s(2s)+a^2+b^2+c^2\}$$

$$= \frac{a^2+b^2+c^2}{\Delta^2}$$

4.7 Orthocentre of a Triangle

Let ABC be any triangle, and let AK, BL and CM be the perpendiculars for A, B and C upon the opposite sides of the triangle. Three perpendiculars meet at a common point H. This point H is the orthocentre of the triangle. The triangle formed by joining the feet of the three perpendiculars is called the pedal triangle of ABC.

4.7.1 Lengths of Altitudes

The distance of the orthocentre H from the vertices and sides of the triangle ABC (Fig. 4.11).

$$HK = KB\tan\angle HBK$$

$$= KB\tan(90^\circ - C) = AB\cos B\cot C$$

$$= \frac{c}{\sin C}\cos B\cos C = 2R\cos B\cos C$$

Similarly,

$$HL = 2R\cos A\cos C \text{ and } HM = 2R\cos A\cos B$$

Also,

$$AH = AL\sec\angle KAC = c\cos A\,\text{cosec}\, C$$

$$= \frac{c}{\sin C}\cos A = 2R\cos A$$

Similarly,

$$BH = 2R\cos B \text{ and } CH = 2R\cos C$$

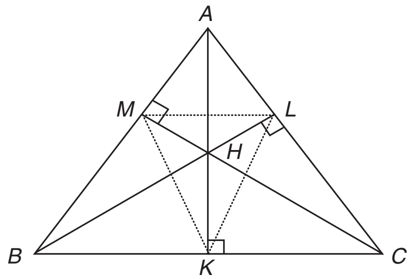


Figure 4.11

4.8 Centroid of a Triangle

If ABC is a triangle, and D, E and F are, respectively, the middle points of BC, CA and AB, the lines AD, BE and CF are called the medians of the triangle. The point where these medians are concurrent is called centroid, G, of the triangle (Fig. 4.12). So

$$AG = \frac{2}{3}AD,\ BG = \frac{2}{3}BE \text{ and } CG = \frac{2}{3}CF$$

That is, the centroid divides every median in the ratio 2:1.

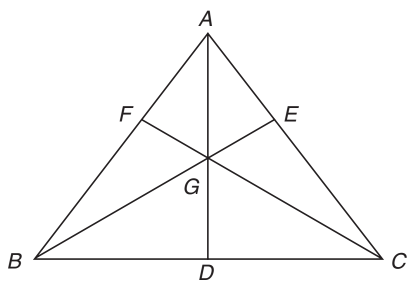


Figure 4.12

4.9 Pedal Triangle

Let the perpendiculars AD, BE and CF from the vertices A, B and C on the opposite sides BC, CA and AB of ΔABC, respectively, meet at O. Then O is the orthocentre of ΔABC. Triangle DEF is called the **pedal triangle** of the ΔABC.

Orthocentre of the triangle is the in-centre of the pedal triangle (Fig. 4.13).

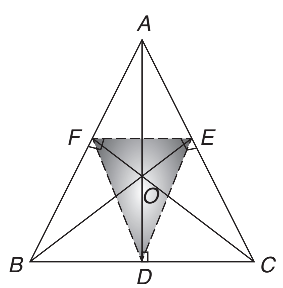


Figure 4.13

If O is the orthocentre and DEF the pedal triangle of ΔABC, where AD, BE, CF are the perpendiculars drawn from A, B, C on the opposite sides BC, CA, AB, respectively, then

$$OA = 2R\cos A,\ OB = 2R\cos B \text{ and } OC = 2R\cos C$$
$$OD = 2R\cos B\cos C,\ OE = 2R\cos C\cos A \text{ and}$$
$$OF = 2R\cos A\cos B$$

4.9.1 Sides and Angles of a Pedal Triangle

See Fig. 4.14. The angles of pedal triangle DEF are: $180° - 2A$, $180° - 2B$, $180° - 2C$ and sides of pedal triangle are

$$EF = a\cos A \text{ or } R\sin 2A;\ FD = b\cos B \text{ or } R\sin 2B;$$
$$DE = c\cos C \text{ or } R\sin 2C$$

If given ΔABC is obtuse, then angles are been represented by $2A, 2B, 2C - 180°$ and the sides are $a\cos A, b\cos B, -c\cos C$.

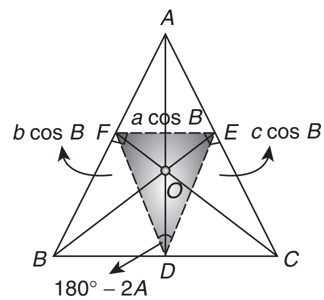


Figure 4.14

4.9.2 Area and Circumradius and In-radius of Pedal Triangle

Area of pedal triangle

$$= \frac{1}{2}(\text{Product of the sides}) \times (\text{sine of included angle})$$

$$\Delta = \frac{1}{2}R^2 \cdot \sin 2A \cdot \sin 2B \cdot \sin 2C$$

$$\text{Circumradius of pedal triangle} = \frac{EF}{2\sin\angle FDE} = \frac{R\sin 2A}{2\sin(180° - 2A)} = \frac{R}{2}$$

$$\text{In-radius of pedal triangle} = \frac{\text{area of } \Delta DEF}{\text{semi-perimeter of } \Delta DEF}$$

$$= \frac{\frac{1}{2}R^2\sin 2A \cdot \sin 2B \cdot \sin 2C}{2R\sin A \cdot \sin B \cdot \sin C}$$

$$= 2R\cos A \cdot \cos B \cdot \cos C$$

4.10 Ex-Central Triangle

Let ABC be a triangle and I be the centre of incircle. Let I_1, I_2 and I_3 be the centres of the escribed circles which are opposite to A, B, C, respectively. Then $I_1I_2I_3$ is called the ex-central triangle of ΔABC (Fig. 4.15).

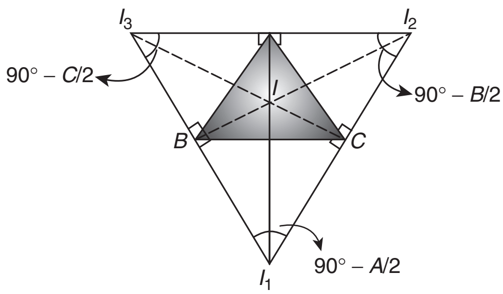


Figure 4.15

$I_1I_2I_3$ is a triangle. Thus, triangle ABC is the pedal triangle of its ex-central triangle $I_1I_2I_3$.
The angles of ex-central triangle $I_1I_2I_3$ are

$$90° - \frac{A}{2},\ 90° - \frac{B}{2},\ 90° - \frac{C}{2}$$

and sides are

$$I_1I_3 = 4R\cos\frac{B}{2};\ I_1I_2 = 4R\cos\frac{C}{2};\ I_2I_3 = 4R\cos\frac{A}{2}$$

4.10.1 Area and Circumradius of the Ex-Central Triangle

Area of triangle $= \frac{1}{2}$ (Product of two sides) $\times$ (Sine of included angles)

$$\Delta = \frac{1}{2}\cdot\left(4R\cos\frac{B}{2}\right)\cdot\left(4R\cos\frac{C}{2}\right)\times\left(90° - \frac{A}{2}\right)$$

$$\Delta = 8R^2\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}$$

$$\text{Circumradius } = \frac{I_2I_3}{2\sin\angle I_1I_2I_3} = \frac{4R\cos\frac{A}{2}}{2\sin\left(90° - \frac{A}{2}\right)} = 2R$$

Remarks:

1. Orthocentre of the triangle is the in-centre of the pedal triangle.
2. If I_1, I_2 and I_3 are the centres of escribed circles which are opposite to A, B and C, respectively, and I the centre of in-circle then triangle ABC is the pedal triangle of the triangle $I_1I_2I_3$ and I is the orthocentre of the triangle $I_1I_2I_3$.
3. The centroid of the triangle lies on the line joining the circumcentre to the orthocentre and divides it in the ratio 1: 2.
4. Circle circumscribing the pedal triangle of a given triangle bisects the sides of the given triangle and also the lines joining the vertices of the given triangle to the orthocentre of the given triangle. This circle is known as **nine-point circle**.
5. Circumcentre of the pedal triangle of a given triangle bisects the line joining the circumcentre of the triangle to the orthocentre.

Illustration 4.22 In a ΔABC, $r_1 < r_2 < r_3$. Then

(A) $a < b < c$ **(B)** $a > b > c$
(C) $b < a < c$ **(D)** $a < c < b$

Solution: In a ΔABC, $r_1 < r_2 < r_3$. Then

$$\frac{1}{r_1} > \frac{1}{r_2} > \frac{1}{r_3} \Rightarrow \frac{s-a}{\Delta} > \frac{s-b}{\Delta} > \frac{s-c}{\Delta}$$

$$\Rightarrow (s-a) > (s-b) > (s-c) \Rightarrow -a > -b > -c \Rightarrow a < b < c$$

Hence, the correct answer is option (A).

Illustration 4.23 In a triangle ABC, let $\angle C = \frac{\pi}{2}$. If r is the in-radius and R is the circumradius of the triangle, then $2(r+R)$ is equal to ____.

[IIT Screening 2000]

(A) $a+b$ **(B)** $b+c$
(C) $c+a$ **(D)** $a+b+c$

Solution: We have

$$\frac{c}{\sin C} = 2R$$

Therefore,

$$c = 2R\sin 90° = 2R$$

Also,

$$r = (s-c)\tan\frac{C}{2} = (s-c)(\tan 45°)$$

$$2r = 2s - 2c = a+b-c = a+b-2R,$$

$$\Rightarrow 2(r+R) = a+b$$

Hence, the correct answer is option (A).

Illustration 4.24 In an equilateral triangle the in-radius and the circumradius are connected by

(A) $r = 4R$ **(B)** $r = R/2$
(C) $r = R/3$ **(D)** None of these

Solution:

$$r = 4R\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}$$

For an equilateral triangle, $A = B = C = 60°$. Therefore

$$r = 4R\sin 30°\cdot\sin 30°\cdot\sin 30° = 4R\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{R}{2}$$

Hence, the correct answer is option (B).

Illustration 4.25 In a triangle ABC, the vertices A, B, C are at distance of p, q, r from the orthocentre, respectively. Show that $aqr + brp + cpq = abc$.

Solution: Let H be the orthocentre of triangle ABC (Fig. 4.16). From question, $HA = p$, $HB = q$, $HC = r$

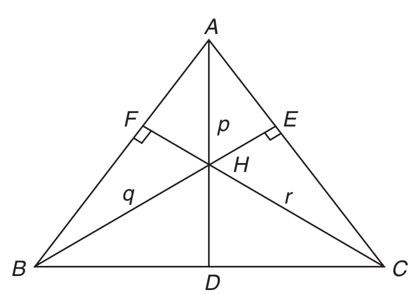


Figure 4.16

From Fig. 4.16,

$$\angle HBD = \angle EBC = 90^\circ - C$$
$$\angle HCD = \angle FCB = 90^\circ - B$$

Therefore,

$$\angle BHC = 180^\circ - (\angle HBD + \angle HCD)$$
$$= 180^\circ - [90^\circ - C + 90^\circ - B]$$
$$= B + C = 180^\circ - A$$

Similarly,

$$\angle AHC = 180^\circ - B \text{ and } \angle AHB = 180^\circ - C$$

Now,

Area of ΔBHC + Area of ΔCHA + Area of ΔAHB
= Area of ΔABC

$$\Rightarrow \frac{1}{2}q\cdot r\cdot \sin\angle BHC + \frac{1}{2}\cdot r\cdot p\sin\angle CHA + \frac{1}{2}p\cdot q\sin\angle AHB = \Delta$$

Therefore,

$$\Delta = \frac{1}{2}bc\cdot \sin A$$

$$\Rightarrow \frac{1}{2}qr\cdot\sin(180^\circ - A) + \frac{1}{2}rp\sin(180^\circ - B) + \frac{1}{2}pq\sin(180^\circ - C) = \Delta$$

$$\Rightarrow \frac{1}{2}qr\cdot\sin A + \frac{1}{2}rp\sin B + \frac{1}{2}pq\cdot\sin C = \Delta$$

$$\Rightarrow \frac{1}{2}qr\cdot\frac{a}{2R} + \frac{1}{2}pr\cdot\frac{b}{2R} + \frac{1}{2}pq\cdot\frac{c}{2R} = \Delta$$

$$\Rightarrow aqr + brp + cpq = 4R\cdot\Delta = 4\cdot\frac{abc}{4\Delta} = abc \qquad \left[\text{Since } \Delta = \frac{abc}{4R}\right]$$

Your Turn 3

1. If R is the radius of the circumcircle of the ΔABC and Δ is its area, then

   (A) $R = \frac{as+b+c}{\Delta}$ **(B)** $R = \frac{as+b+c}{4\Delta}$

   (C) $R = \frac{abc}{4\Delta}$ **(D)** $R = \frac{abc}{\Delta}$

   Ans. (C)

2. If the lengths of the sides of a triangle are 3, 4 and 5 units, then R (the circumradius) is

   (A) 2.0 unit **(B)** 2.5 unit
   (C) 3.0 unit **(D)** 3.5 unit

   Ans. (B)

3. If x, y, z are perpendiculars drawn from the vertices of triangle having sides a, b and c, then the value of $\frac{bx}{c} + \frac{cy}{a} + \frac{az}{b}$ will be

   (A) $\frac{a^2+b^2+c^2}{2R}$ **(B)** $\frac{a^2+b^2+c^2}{R^2}$

   (C) $\frac{a^2+b^2+c^2}{R^2}$ **(D)** $\frac{2(a^2+b^2+c^2)}{R}$

   Ans. (A)

4. If r_1, r_2, r_3 in a triangle be in HP, then the sides are in

   (A) AP **(B)** GP
   (C) HP **(D)** None of these

   Ans. (A)

5. If the sides of a triangle are 13, 14, 15 then the radius of its in-circle is

   (A) $\frac{67}{8}$ **(B)** $\frac{65}{4}$

   (C) 4 **(D)** 24

   Ans. (C)

6. If the bisector of the angle C of a triangle ABC cuts AB in D and the circumcircle in E, prove that $CE{:}DE = (a+b)^2{:}c^2$.

4.11 Cyclic Quadrilateral

A quadrilateral $PQRS$ is said to be cyclic quadrilateral if there exists a circle passing through all its four vertices P, Q, R and S (Fig. 4.17). Let a cyclic quadrilateral be such that

$$PQ = a,\ QR = b,\ RS = c \text{ and } SP = d$$

Then $\angle Q + \angle S = 180^\circ$ and $\angle P + \angle S = 180^\circ$. Let $2s = a+b+c+d$. Now

Area of cyclic quadrilateral $PQRS$ = Area of ΔPQR + Area of ΔPRS

$$= \frac{1}{2}ab\sin Q + \frac{1}{2}cd\sin S$$
$$= \frac{1}{2}ab\sin Q + \frac{1}{2}cd\sin(\pi - Q)$$
$$= \frac{1}{2}(ab+cd)\sin Q \qquad (1)$$

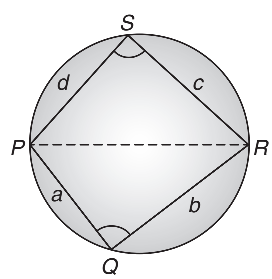


Figure 4.17

In ΔPQR and ΔPRS, from cosine rule we have, respectively,

$$PR^2 = PQ^2 + QR^2 - 2PQ{\cdot}QR\cos Q = a^2 + b^2 - 2ab\cos Q \qquad (2)$$

and
$$PR^2 = PS^2 + RS^2 - 2PS{\cdot}RS\cos S$$
$$= d^2 + c^2 - 2cd\cos(\pi - Q)$$
$$= d^2 + c^2 + 2cd\cos Q \qquad (3)$$

Therefore,

1. Area of cyclic quadrilateral $= \frac{1}{2}(ab+cd)\sin Q$
2. Area of cyclic quadrilateral $= \sqrt{(s-a)(s-b)(s-c)(s-d)}$, where $2s = a+b+c+d$
3. $\cos Q = \frac{a^2+b^2-c^2-d^2}{2(ab+cd)}$

4.11.1 Circumradius of Cyclic Quadrilateral

Circumcircle of quadrilateral PQRS is also the circumcircle of ΔPQR. Hence, circumradius of cyclic quadrilateral $PQRS = R$

$$\text{Circumradius of } \Delta PQR = \frac{PR}{2\sin B} = \frac{PR(ab+cd)}{4\Delta}$$

But

$$PR = \sqrt{\frac{(ac+bd)(ad+bc)}{(ab+cd)}}$$

Hence,

$$R = \frac{1}{4\Delta}\sqrt{(ac+bd)(ad+bc)(ab+cd)}$$
$$= \frac{1}{4}\sqrt{\frac{(ac+bd)(ad+bc)(ab+cd)}{(s-a)(s-b)(s-c)(s-d)}}$$

4.11.2 Ptolemy's Theorem

See Fig. 4.18. In a cyclic quadrilateral PQRS, the product of diagonals is equal to the sum of the products of the length of the opposite sides, i.e. according to Ptolemy's theorem, for a cyclic quadrilateral *PQRS*.

$$PR\cdot QS = PQ\cdot RS + RQ\cdot PS$$

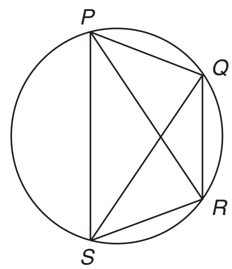


Figure 4.18

Illustration 4.26 In a cyclic quadrilateral *ABCD*, prove that

$$\tan^2\frac{B}{2} = \left(\frac{(s-a)(s-b)}{(s-c)(s-d)}\right)$$

a, *b*, *c* and *d* being the lengths of sides *AB*, *BC*, *CD* and *DA*, respectively, and '*s*' is semi-perimeter of quadrilateral.

Solution: See Fig. 4.19. In ΔABC

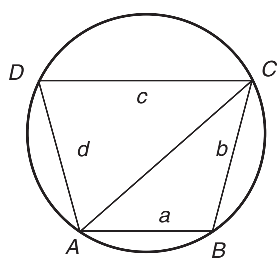


Figure 4.19

$$AC^2 = a^2 + b^2 - 2ab\cos B \quad (1)$$

In ΔADC

$$AC^2 = c^2 + d^2 - 2cd\cos D$$
$$= c^2 + d^2 - 2cd\cos(180^\circ - B)$$
$$= c^2 + d^2 + 2cd\cos B \quad (2)$$

From Eqs. (1) and (2), we get

$$\cos B = \frac{a^2+b^2-c^2-d^2}{2(ab+cd)} \quad (3)$$

Since, $\tan^2\left(\frac{B}{2}\right) = \frac{1-\cos B}{1+\cos B}$, we get using Eq. (3)

$$\tan^2\left(\frac{B}{2}\right) = \frac{2(ab+cd)-(a^2+b^2-c^2-d^2)}{2(ab+cd)+(a^2+b^2-c^2-d^2)}$$
$$= \frac{(c+d)^2-(a-b)^2}{(a+b)^2-(c-d)^2}$$
$$= \frac{(c+d+a-b)(c+d-a+b)}{(a+b+c-d)(a+b-c+d)}$$
$$= \frac{(s-a)(s-b)}{(s-d)(s-c)}, \text{ where } s = a+b+c+d$$

Illustration 4.27 A cyclic quadrilateral *ABCD* of area $\frac{3\sqrt{3}}{4}$ is inscribed in a unit circle. If one of its sides $AB = 1$ and the diagonal $BD = \sqrt{3}$ then find the lengths of the other sides.

Solution:

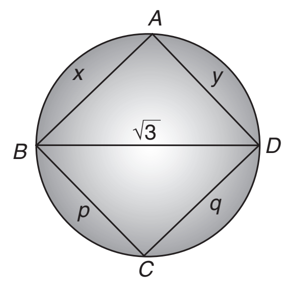


Figure 4.20

See Fig. 4.20. By sine formula in ΔABC,

$$\frac{\sqrt{3}}{\sin A} = 2R \Rightarrow \frac{\sqrt{3}}{\sin A} = 2$$
$$\Rightarrow \sin A = \frac{\sqrt{3}}{2} \Rightarrow A = \frac{\pi}{3}$$

Now, $AB = x = 1$.

By cosine formula in ΔABD

$$\cos\frac{\pi}{3} = \frac{x^2+y^2-3}{2xy} \Rightarrow \frac{1}{2} = \frac{1+y^2-3}{2y} \Rightarrow y = y^2 - 2$$
$$\Rightarrow y^2 - y - 2 = 0 \Rightarrow (y-2)(y+1) = 0$$
$$\Rightarrow y = 2 \ [\because y \neq -1]$$

Since $\angle A = 60^\circ$, therefore $\angle C = 120^\circ$

In ΔBDC,

$$3 = p^2 + q^2 - 2pq\cos 120^\circ \Rightarrow 3 = p^2 + q^2 + pq \quad (1)$$

Also, area of quadrilateral $ABCD = \frac{3\sqrt{3}}{4}$. Therefore,

$$\frac{3\sqrt{3}}{4} = \Delta ABD + \Delta BCD$$

$$= \frac{1}{2}\cdot 1\cdot 2\sin 60° + \frac{1}{2}p\cdot q\cdot \sin 120°$$

$$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4}pq$$

$$\Rightarrow \frac{\sqrt{3}}{4}pq = \frac{3\sqrt{3}}{4} - \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}-2\sqrt{3}}{4} = \frac{\sqrt{3}}{4}$$

$$\Rightarrow pq = 1$$

Therefore, Eq. (1) gives,

$$3 = p^2 + q^2 + 1 \Rightarrow p^2 + q^2 = 2 \ [p, q > 0]$$

Thus,

$$p^2 + \frac{1}{p^2} = 2 \Rightarrow p^4 - 2p^2 + 1 = 0$$

$$\Rightarrow (p^2 - 1)^2 = 0 \Rightarrow p^2 = 1$$

So, $p^2 = 1, q = 1$. Therefore, $AB = 1, AD = 2, BC = CD = 1$.

4.12 Regular Polygon

A regular polygon is a polygon that has all its sides equal and all its angles equal.

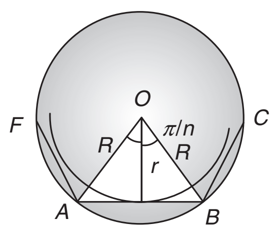


Figure 4.21

1. See Fig. 4.21. Each interior angle of a regular polygon of n sides is

$$\left(\frac{2n-4}{n}\right)\times \text{Right angles} = \left[\frac{2n-4}{n}\right]\times\frac{\pi}{2} \text{ radians}$$

2. The circle passing through all the vertices of a regular polygon is called its *circumscribed* circle.

   If a is the length of each side of a regular polygon of n sides, then the radius R of the circumscribed circle, is given by

$$R = \frac{a}{2}\cdot \text{cosec}\left(\frac{\pi}{n}\right)$$

3. The circle which can be inscribed within the regular polygon so as to touch all its sides, is called its inscribed circle.

   Again if a is the length of each side of a regular polygon of n sides, then the radius r of the inscribed circle is given by

$$R = \frac{a}{2}\cdot \cot\left(\frac{\pi}{n}\right)$$

4. The area of a regular polygon is given by

$$\Delta = n \times \text{area of triangle } OAB$$

$$= \frac{1}{2}na^2\cdot\cot\left(\frac{\pi}{n}\right) \text{ (In terms of side)}$$

$$= nr^2\cdot\tan\left(\frac{\pi}{n}\right) \text{ (In terms of in-radius)}$$

$$= \frac{n}{2}\cdot R^2\sin\left(\frac{2\pi}{n}\right) \text{ (In terms of circumradius)}$$

4.12.1 Area of Sector

Area included between two radius and circumference (Fig. 4.22) is given by

$$\text{Area} = \frac{R^2\theta}{2}, \text{ where } \theta \text{ is in radians}$$

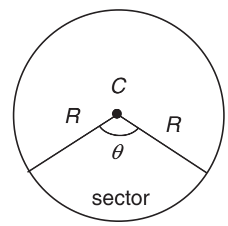


Figure 4.22

4.12.2 Area of Segment

Area between a circumference and a chord (Fig. 4.23) is given by

$$\text{Area} = \frac{R^2}{2}(\theta - \sin\theta)$$

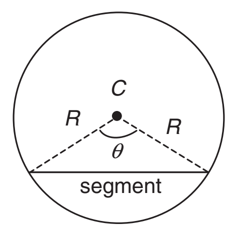


Figure 4.23

Illustration 4.28 The area of the circle and the area of a regular polygon of n sides and its perimeter equal to that of the circle are in the ratio of **[Roorkee 1992]**

(A) $\tan\left(\frac{\pi}{n}\right):\frac{\pi}{n}$ **(B)** $\cos\left(\frac{\pi}{n}\right):\frac{\pi}{n}$

(C) $\sin\left(\frac{\pi}{n}\right):\frac{\pi}{n}$ **(D)** $\cot\left(\frac{\pi}{n}\right):\frac{\pi}{n}$

Solution: Let r be the radius of the circle and A_1 be its area, therefore, $A_1 = \pi r^2$. Since the perimeter of the circle is the same as the perimeter of a regular polygon of n sides, therefore, $2\pi r = na$, where 'a' is the length of one side of the regular polygon. This gives $a = \frac{2\pi r}{n}$.

Let A_2 be the area of the polygon. Then

$$A_2 = \frac{1}{4}na^2\cdot\cot\frac{\pi}{n} = \frac{1}{4}n\cdot\frac{4\pi^2 r^2}{n^2}\cot\frac{\pi}{n} = \pi r^2\cdot\frac{\pi}{n}\cdot\cot\frac{\pi}{n}$$

Therefore,

$$A_1 : A_2 = \pi r^2 : \pi r^2 \cdot \frac{\pi}{n} \cdot \cot\frac{\pi}{n} = 1 : \frac{\pi}{n}\cot\frac{\pi}{n} = \tan\frac{\pi}{n} : \frac{\pi}{n}$$

Hence, the correct answer is option (A).

Illustration 4.29 If the number of sides of two regular polygons having the same perimeter be n and $2n$, respectively, their areas are in the ratio

(A) $\dfrac{2\cos\left(\frac{\pi}{n}\right)}{\cos\left(\frac{\pi}{2n}\right)}$ **(B)** $\dfrac{2\cos\left(\frac{\pi}{n}\right)}{1+\cos\left(\frac{\pi}{n}\right)}$

(C) $\dfrac{\cos\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)}$ **(D)** None of these

Solution: Let s be the perimeter of both the polygons. Then the length of each side of the first polygon is s/n and that of second polygon is $s/2n$.

If A_1, A_2 denote their areas, then

$$A_1 = \frac{n}{4}\left[\frac{s}{n}\right]^2 \cot\frac{\pi}{n} \quad (1)$$

and

$$A_2 = \frac{1}{4}\cdot(2n)\left(\frac{s}{2n}\right)^2 \cdot \cot\left(\frac{\pi}{2n}\right) \quad (2)$$

Ratio of Eqs. (1) and (2) is

$$\frac{A_1}{A_2} = \frac{2\cot\left(\frac{\pi}{n}\right)}{\cot\left(\frac{\pi}{2n}\right)} = \frac{2\cos\left(\frac{\pi}{n}\right)\sin\left(\frac{\pi}{2n}\right)}{\sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{2n}\right)}$$

$$= \frac{2\cos\left(\frac{\pi}{n}\right)\sin\left(\frac{\pi}{2n}\right)}{2\sin\left(\frac{\pi}{2n}\right)\cos\left(\frac{\pi}{2n}\right)\cos\left(\frac{\pi}{2n}\right)}$$

$$= \frac{2\cos\left(\frac{\pi}{n}\right)}{1+\cos\left(\frac{\pi}{n}\right)}$$

Hence, the correct answer is option (B).

Illustration 4.30 The ratio of the area of the regular polygon of n sides circumscribed about a circle to the area of the regular polygon of equal number of sides inscribed in the circle is 4:3. Find the value of n.

Solution: Area of circle inscribed about a regular polygon of n sides is

$$\pi R^2 = \pi\left(\frac{a}{2}\operatorname{cosec}\frac{\pi}{n}\right)^2 = \frac{\pi a^2}{4}\operatorname{cosec}^2\frac{\pi}{n}$$

Area of circle inscribed about the same regular polygon is

$$\pi r^2 = \pi\left(\frac{a}{2}\cot\frac{\pi}{n}\right)^2 = \frac{\pi a^2}{4}\cot^2\frac{\pi}{n}$$

$$\text{Given ratio } = \frac{\pi R^2}{\pi r^2} = \frac{4}{3} \Rightarrow \frac{\operatorname{cosec}^2\frac{\pi}{n}}{\cot^2\frac{\pi}{n}} = \frac{4}{3}$$

$$\Rightarrow \cos\frac{\pi}{n} = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}$$

$$\Rightarrow \text{Either } \frac{\pi}{n} = \frac{\pi}{6} \text{ or } \frac{\pi}{n} = \frac{5\pi}{6}$$

As n is a natural number, therefore $n = 6$.

4.13 Solution of a Triangle

The three sides a, b, c and the three angles A, B, C are called the elements of the triangle ABC. When any three of these six elements (except all the three angles) of a triangle are given, the triangle is known completely; that is, the other three elements can be expressed in terms of the given elements and can be evaluated. This process is called the **solution of a triangle**.

In this section, we will discuss the solution of oblique triangles only.

4.13.1 Type I

Problems based on finding the angles when three sides are given.

If the data given is in sine we use the following formula, whichever is applicable:

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}},\ \sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}},\ \sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

If the given data are in cosine, first of all try the following formula, whichever is needed:

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}},\ \cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}},\ \cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

and see whether logarithm of the number on RHS can be determined from the given data. If s proceed further, if not then try the following formula, whichever is needed:

$$\cos A = \frac{b^2+c^2-a^2}{2bc},\ \cos B = \frac{a^2+c^2-b^2}{2ac},\ \cos C = \frac{a^2+b^2-c^2}{2ab}$$

If the given data are in tangent, use the following formula, whichever is applicable:

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}},\ \tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}},\ \tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

4.13.2 Type II

Problem based on finding the angles when any two sides and the angles between them are given or any two sides and the difference of the angles opposite to them are given.

Working Rule: Use the following formula, whichever is needed:

1. $\tan\dfrac{B-C}{2} = \dfrac{b-c}{b+c}\cot\dfrac{A}{2}$
2. $\tan\dfrac{C-A}{2} = \dfrac{c-a}{c+a}\cot\dfrac{B}{2}$
3. $\tan\dfrac{A-B}{2} = \dfrac{a-b}{a+b}\cot\dfrac{C}{2}$

4.13.3 Type III

Problems based on finding the sides and angles when any two angles and side opposite to one of them are given.

Working Rule: Use the following formula, whichever is needed:

1. $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$
2. $A+B+C=180°$

4.13.4 Type IV

When all the three angles are given, then unique solution of triangle is not possible. In this case only the ratio of the sides can be determined.

For this the following formula can be used:

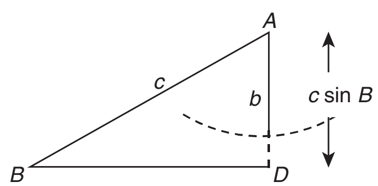
$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

4.13.5 Type V

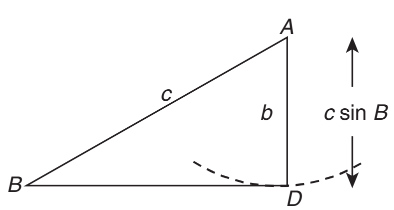
If two sides b and c and the angle B (opposite to side b) are given, then $\sin C=\frac{c}{b}\sin B$, $A = 180° - (B + C)$ and $a=\frac{b\sin A}{\sin B}$ give the remaining elements.

1. If $b < c \sin B$, there is no triangle possible [Fig. 4.24(i)].
2. If $b = c \sin B$ and B is an acute angle, then there is only one triangle possible [Fig. 4.24(ii)].
3. If $c \sin B < b < c$ and B is an acute angle, then there are two values of angle C [Fig. 4.24(iii)].
4. If $c < b$ and B is an acute angle, then there is only one triangle [Fig. 4.24(iv)].

This is, sometimes, called an **ambiguous case**.



(i)



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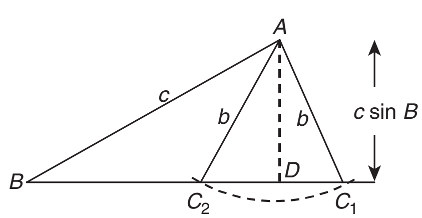

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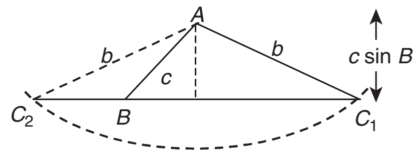


(iv)

Figure 4.24

4.13.6 Alternative Method

By applying cosine rule, we have

$$\cos B=\frac{a^2+c^2-b^2}{2ac}$$

$$\Rightarrow a^2-(2c\cos B)\,a+(c^2-b^2)=0$$

$$\Rightarrow\ a=c\cos B\pm\sqrt{(c\cos B)^2-(c^2-b^2)}$$

$$\Rightarrow\ a=c\cos B\pm\sqrt{b^2-(c\sin B)^2}$$

This equation leads to the following cases:

Case 1.
If $b < c \sin B$, no such triangle is possible.

Case 2.
Let $b = c \sin B$. There are further following cases:

(a) B is an obtuse angle
$\Rightarrow$ cos B is negative. There exists no such triangle.
(b) B is an acute angle
$\Rightarrow$ cos B is positive. There exists only one such triangle.

Case 3.
Let $b > c \sin B$. There are further following cases:

(**a**) B is an acute angle
$\Rightarrow$ cos B is positive. In this case two values of a will exist if and only if

$$c\cos B>\sqrt{b^2-(c\sin B)^2}\ \text{or}\ c>b$$

$\Rightarrow$ Two such triangles are possible. If $c < b$, only one such triangle is possible.

(**b**) B is an obtuse angle
$\Rightarrow$ cos B is negative. In this case triangle will exist if and only if

$$\sqrt{b^2-(c\sin B)^2}>|c\cos B|$$

If $b > c$. In this case only one such triangle is possible.
If $b < c$ there exists no such triangle.

1. If one side a and angles B and C are given, then $A = 180° - (B + C)$, and
$$b=\frac{a\sin B}{\sin A},\ c=\frac{a\sin C}{\sin A}$$
2. If the three angles A, B, C are given, we can only find the ratios of the sides a, b, c by using sine rule (since there are infinite similar triangles possible).

Illustration 4.31 If in a right-angled triangle the hypotenuse is four times as long as the perpendicular drawn to it from opposite vertex, then find one of its acute angle.

Solution: See Fig. 4.25. If x is length of perpendicular drawn to it from opposite vertex of a right-angled triangle, so, length of the diagonal is

$$AB = y_1 + y_2 \quad (1)$$

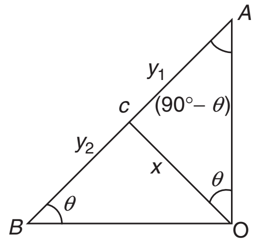


Figure 4.25

From $\Delta OCB, y_2 = x\cot\theta$ and from ΔOCA, $y_1 = x\tan\theta$. Putting the values in Eq. (1), we get

$$AB = x(\tan\theta + \cot\theta) \quad (2)$$

Since, length of hypotenuse = 4 (Length of perpendicular) therefore,

$$x(\tan\theta + \cot\theta) = 4x \Rightarrow \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} = 4$$

$$\Rightarrow \sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = 30° \Rightarrow \theta = 15°$$

Illustration 4.32 Find the number of triangles ABC that can be formed with $a = 3$, $b = 8$ and $\sin A = \frac{5}{3}$.

Solution: Given $a = 3$, $b = 8$ and $\sin A = \frac{5}{13}$. Therefore

$$b\sin A = 8 \times \left(\frac{5}{13}\right) = \frac{40}{13} > a\ (= 3)$$

Thus, in this case no triangle is possible.

Illustration 4.33 If two sides of a triangle are $2\sqrt{3}$ and $2\sqrt{2}$ and the angle opposite the shorter side is $45°$, then find the maximum value of the third side.

Solution: Let $a = 2\sqrt{3}$, $b = 2\sqrt{2}$. Therefore, $B = 45°$. So,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{2\sqrt{3}}{\sin A} = \frac{2\sqrt{2}}{\sin 45°} = \frac{c}{\sin C} \quad (1)$$

$$\sin A = \frac{\sqrt{3}}{2} \Rightarrow A = 60°$$

Therefore,

$$C = 180° - A - B = 75°$$

From Eq. (1), we have

$$c = 4\sin C = 4\sin(45° + 30°) = \sqrt{2} + \sqrt{6}$$

Illustration 4.34 In an ambiguous case, if the remaining angles of the triangles formed with a, b and A be B_1, C_1 and B_2, C_2 then $\frac{\sin C_1}{\sin B_1} + \frac{\sin C_2}{\sin B_2} =$ ____.

(A) $2\cos A$ **(B)** $\cos A$

(C) $2\sin A$ **(D)** $\sin A$

Solution: See Fig. 4.26. In Δ's ACB_1 and ACB_2

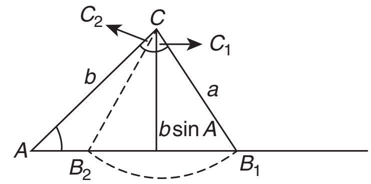


Figure 4.26

$$\frac{\sin C_1}{\sin B_1} = \frac{AB_1}{AC} = \frac{c_1}{b} \text{ and } \frac{\sin C_2}{\sin B_2} = \frac{c_2}{b}$$

Therefore,

$$\frac{\sin C_1}{\sin B_1} + \frac{\sin C_2}{\sin B_2} = \frac{c_1 + c_2}{b}$$

$$\left[\begin{array}{l}\text{since, } \cos A = \dfrac{b^2 + c^2 - a^2}{2bc} \Rightarrow c^2 - (2b\cos A)c + (b^2 - a^2) = 0 \\ \text{therefore, } c_1 + c_2 = 2b\cos A\end{array}\right]$$

$$\frac{\sin C_1}{\sin B_1} + \frac{\sin C_2}{\sin B_2} = \frac{2b\cos A}{b} = 2\cos A$$

Hence, the correct answer is option (A).

Your Turn 4

1. The two adjacent sides of a cyclic quadrilateral are 2 and 5 and the angle between them is 60°. If the third side is 3, the remaining fourth side is

(A) 2 **(B)** 3

(C) 4 **(D)** 5

Ans. (A)

2. Two adjacent sides of a cyclic quadrilateral are 2 and 5 and the angle between them is 60°. If the area of the quadrilateral is $4\sqrt{3}$, then the remaining two sides are

(A) 2, 3 **(B)** 3, 4

(C) 4, 5 **(D)** 5, 6

Ans. (A)

3. The sum of the radii of inscribed and circumscribed circles for an n-sided regular polygon of side a, is

(A) $a\cot\left(\frac{\pi}{n}\right)$ **(B)** $\frac{a}{2}\cot\left(\frac{\pi}{2n}\right)$

(C) $a\cot\left(\frac{\pi}{2n}\right)$ **(D)** $\frac{a}{4}\cot\left(\frac{\pi}{2n}\right)$

Ans. (C)

4. A regular polygon of nine sides, each of length 2, is inscribed in a circle. The radius of the circle is

(A) $\operatorname{cosec}\frac{\pi}{9}$ **(B)** $\operatorname{cosec}\frac{\pi}{3}$

(C) $\cot\frac{\pi}{9}$ **(D)** $\tan\frac{\pi}{9}$

Ans. (A)

5. In a triangle ABC, $AB=2, BC=4, CA=3$ and D is mid-point of BC. Then

(A) $\cos B=\frac{11}{6}$ (B) $\cos B=\frac{7}{8}$

(C) $AD=2.4$ (D) $AD^2=2.5$

Ans. (D)

6. If $b=3, c=4$ and $B=\frac{\pi}{3}$, then find the number of triangles that can be constructed.

Ans. 0

4.13.7 *m – n* Theorem

See Fig. 4.27. If a point D divides the side BC of ΔABC internally in the ratio $m{:}n$ and $\angle BAD=\alpha$, $\angle DAC=\beta$ and $\angle ADC=\theta$ then

$$(m+n)\cot\theta = m\cot\alpha - n\cot\beta = n\cot B - m\cot C$$

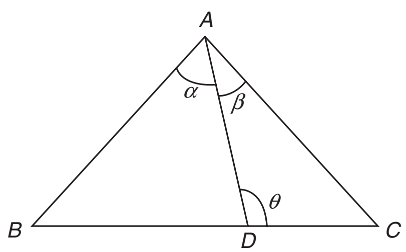


Figure 4.27

The result can be derived using the sine rule in ΔABD and ΔADC.

Illustration 4.35 In Fig. 4.28, ABC is a triangle in which angle $C=90°$ and $AB=5$ cm. D is a point on AB such that $AD=3$ cm and $\angle ACD=60°$. Find the length of side AC.

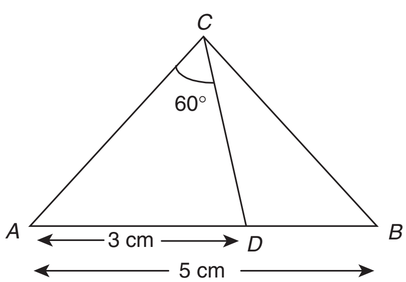


Figure 4.28

Solution: Using $m-n$ theorem,

$$(3+2)\cot\angle CDA = 2\cot 30° - 3\cot 60°$$

$$\Rightarrow \cot\angle CDA = \frac{\sqrt{3}}{5}$$

Now, using sine rule in ΔCDA,

$$\frac{AC}{\sin\angle CDA} = \frac{AD}{\sin\angle ACD}$$

$$\Rightarrow AC = \frac{3}{\sin 60°}\cdot\frac{5}{\sqrt{28}} = 5\sqrt{\frac{3}{7}}\text{ cm}$$

Additional Solved Examples

1. If in a ΔABC, $\cos A+2\cos B+\cos C=2$, then a, b, c are in

(A) HP (B) GP

(C) AP (D) None of these

Solution: From the given condition, we have

$$\cos A+\cos C = 2(1-\cos B)$$

$$\Rightarrow 2\cos\frac{A+C}{2}\cos\frac{A-C}{2} = 2\ x\ 2\sin^2\frac{B}{2}$$

$$\Rightarrow \cos\frac{A-C}{2} = 2\sin\frac{B}{2}\left(\because \cos\frac{A+C}{2}=\sin\frac{B}{2}\neq 0\right)$$

$$\Rightarrow 2\cos\frac{A-C}{2}\cos\frac{B}{2} = 4\sin\frac{B}{2}\cos\frac{B}{2}$$

$$\Rightarrow 2\sin\frac{A+C}{2}\cos\frac{A-C}{2} = 2\sin B\left(\because \cos\frac{B}{2}=\sin\frac{A+C}{2}\right)$$

$$\Rightarrow \sin A+\sin C = 2\sin B$$

$$\Rightarrow \frac{a}{k}+\frac{c}{k} = 2\frac{b}{k} \Rightarrow a+c=2b$$

$$\Rightarrow a, b, c \text{ are in AP}$$

Hence, the correct answer is option (C).

2. Points D, E are taken on the side BC of a triangle ABC such that $BD=DE=EC$. If $\angle BAD=x$, $\angle DAE=y$, $\angle EAC=z$, then the value of $\frac{\sin(x+y)\sin(y+z)}{\sin x\sin z}$ is equal to

(A) 4 (B) 1

(C) 2 (D) None of these

Solution: See Fig. 4.29. From ΔADC

$$\frac{\sin(y+z)}{DC} = \frac{\sin C}{AD} \quad (1)$$

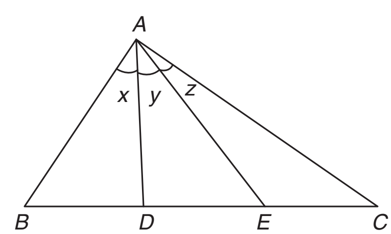


Figure 4.29

From ΔABD

$$\frac{\sin x}{BD} = \frac{\sin B}{AD} \quad (2)$$

and from ΔAEC,

$$\frac{\sin z}{EC} = \frac{\sin C}{AE} \quad (3)$$

Also, from ΔABE,

$$\frac{\sin(x+y)}{BE} = \frac{\sin B}{AE} \quad (4)$$

From Eqs. (1), (2), (3) and (4), we get

$$\frac{\sin(x+y)\sin(y+z)}{\sin x \sin z} = \frac{BE}{AE}\times\frac{DC}{AD}\times\frac{AD}{BD}\times\frac{AE}{EC}$$
$$= \frac{BE}{BD}\times\frac{DC}{EC} = 2\times 2 = 4$$

($\because BE = 2\,BD$ and $DC = 2\,EC$)

Hence, the correct answer is option (A).

3. In any ΔABC, if $C = 90°$, then $\tan\frac{B}{2}$ is equal to

(A) $\sqrt{\frac{c-a}{c+a}}$ **(B)** $\sqrt{\frac{a-c}{a+c}}$

(C) $\sqrt{\frac{c+a}{c-a}}$ **(D)** None of these

Solution: Using Napier's analogy, we get

$$\tan\frac{C-A}{2} = \frac{c-a}{c+a}\cot\frac{B}{2}$$

$$\Rightarrow \tan\frac{90°-(90°-B)}{2} = \frac{c-a}{c+a}\cdot\frac{1}{\tan\frac{B}{2}}$$

(since $C = 90°$, therefore $A + B = 90°$)

$$\Rightarrow \left(\tan\frac{B}{2}\right)^2 = \frac{c-a}{c+a}$$

$$\Rightarrow \tan\frac{B}{2} = \sqrt{\frac{c-a}{c+a}}$$

(since, $0 < B < 180°$, $\tan\frac{B}{2}$ cannot be negative)

Hence, the correct answer is option (A).

4. If x, y, z are perpendiculars drawn from the vertices of a triangle having sides a, b and c, then $\frac{bx}{c}+\frac{cy}{a}+\frac{az}{b}$ is equal to

(A) $\frac{a^2+b^2+c^2}{2R}$ **(B)** $\frac{a^2+b^2+c^2}{R}$

(C) $\frac{a^2+b^2+c^2}{4R}$ **(D)** $\frac{2(a^2+b^2+c^2)}{R}$

Solution:

$$\frac{1}{2}ax = \Delta,\ \frac{1}{2}by = \Delta,\ \frac{1}{2}cz = \Delta$$

$$\Rightarrow x = \frac{2\Delta}{a},\ y = \frac{2\Delta}{b},\ z = \frac{2\Delta}{c}$$

Hence,

$$\frac{bx}{c}+\frac{cy}{a}+\frac{az}{b} = \frac{2b\Delta}{ac}+\frac{2c\Delta}{ab}+\frac{2a\Delta}{bc}$$

$$\frac{2\Delta}{abc}\{b^2+c^2+a^2\} = \frac{4\Delta}{abc}\left(\frac{a^2+b^2+c^2}{2}\right)$$
$$= \frac{1}{R}\left(\frac{a^2+b^2+c^2}{2}\right)$$

$$\left(\text{since } R = \frac{abc}{4\Delta}\right)$$

Hence, the correct answer is option (A).

5. In a ΔABC, $\frac{\cos C+\cos A}{c+a}+\frac{\cos B}{b}$ is equal to

(A) $\frac{1}{a}$ **(B)** $\frac{1}{b}$

(C) $\frac{1}{c}$ **(D)** $\frac{c+a}{b}$

Solution: We have

$$\frac{\cos C+\cos A}{c+a}+\frac{\cos B}{b} = \frac{b\cos C+b\cos A+c\cos B+a\cos B}{b(c+a)}$$
$$= \frac{(b\cos C+c\cos B)+(b\cos A+a\cos B)}{b(c+a)}$$
$$= \frac{a+c}{b(c+a)} \text{ (using projection formulae)}$$
$$= \frac{1}{b}$$

Hence, the correct answer is option (B).

6. In a ΔABC, prove that $\cos A + \cos B + \cos C \le \frac{3}{2}$.

Solution:

$$\cos A+\cos B+\cos C = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+1-2\sin^2\frac{C}{2}$$
$$= 2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\right]+1$$
$$\left[\frac{A+B}{2} = \frac{\pi}{2}-\frac{C}{2}, \text{ therefore, } \cos\left(\frac{A+B}{2}\right) = \sin\frac{C}{2}\right]$$
$$= 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}+1$$
$$\le 1+\frac{1}{8}\cdot 4\left[\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\le\frac{1}{8}\right]\le\frac{3}{2}$$

7. For a triangle, it is given that $\cos A + \cos B + \cos C = \frac{3}{2}$. Prove that the triangle is equilateral.

Solution: If a, b, c are the sides of ΔABC then given that

$$\cos A+\cos B+\cos C = \frac{3}{2}$$
$$\Rightarrow \frac{b^2+c^2-a^2}{2bc}+\frac{c^2+a^2-b^2}{2ca}+\frac{a^2+b^2-c^2}{2ab} = \frac{3}{2}$$
$$\Rightarrow ab^2+ac^2-a^3+bc^2+ba^2-b^3+ca^2+cb^2-c^3 = 3abc$$
$$\Rightarrow ab^2+ac^2+bc^2+ba^2+ca^2+cb^2-6abc = a^3+b^3+c^3-3abc$$
$$\Rightarrow a(b-c)^2+b(c-a)^2+c(a-b)^2 = \frac{(a+b+c)}{2}[(a-b)^2+(b-c)^2+(c-a)^2]$$
$$\Rightarrow (a+b-c)(a-b)^2+[b+c-a)(b-c)^2+(c+a-b)(c-a^2) = 0$$

Now $a + b > c$, $b + c > a$, $c + a > b$. Since each term on the left side has positive coefficient multiplied by perfect square, each must be separately zero. So $a = b = c$. Hence, the triangle is equilateral.

8. In a triangle ABC, prove that $\dfrac{a(a+c-b)}{b(b+c-a)} = \dfrac{1-\cos A}{1+\cos A}$.

Solution:

$$\text{LHS} = \frac{a(a+c-b)}{b(b+c-a)} = \frac{a[(a+c)^2 - b^2]}{b[(b+c)^2 - a^2]}$$

$$= \frac{a}{b}\frac{(a^2+c^2-b^2+2ac)}{(b^2+c^2-a^2+2bc)}$$

$$= \frac{a}{b}\frac{(2ac\ \cos B + 2ac)}{2bc\cos A + 2bc}$$

$$= \frac{a}{b}\frac{2ac\ (\cos B + 1)}{2bc\ \ (\cos A + 1)}$$

$$= \frac{a^2}{b^2}\frac{(1+\cos B)}{(1+\cos A)}$$

$$= \frac{\sin^2 A}{\sin^2 B}\frac{(1+\cos B)}{(1+\cos A)}$$

$$= \frac{(1-\cos^2 A)}{(1-\cos^2 B)}\frac{(1+\cos B)}{(1+\cos A)}$$

$$= \frac{1-\cos A}{1-\cos B}$$

9. If the median AD of a triangle ABC divides the angle $\angle BAC$ in ratio 1:2, then show that $\dfrac{\sin B}{\sin C} = \dfrac{1}{2}\sec\dfrac{A}{3}$.

Solution: See Fig. 4.30.

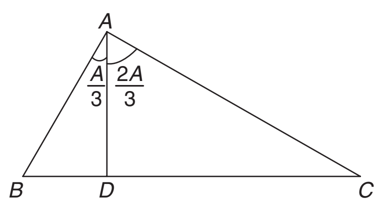


Figure 4.30

$$\angle BAD = \frac{A}{3},\ \angle DAC = \frac{2A}{3}$$

$$\frac{BD}{\sin\frac{A}{3}} = \frac{AD}{\sin B}$$

Therefore,

$$AD = \frac{BD\ \sin B}{\sin\frac{A}{3}}$$

From ΔADC, we have

$$\frac{DC}{\sin\frac{2A}{3}} = \frac{AD}{\sin C}$$

Therefore,

$$AD = \frac{DC\ \sin C}{\sin\frac{2A}{3}}$$

But $BD = CD$. Therefore,

$$\frac{\sin B}{\sin\frac{A}{3}} = \frac{\sin C}{\sin\frac{2A}{3}} = \frac{\sin C}{2\sin\frac{A}{3}\cdot\cos\frac{A}{3}}$$

$$\Rightarrow \frac{\sin B}{\sin C} = \frac{1}{2\cos\frac{A}{3}} = \frac{1}{2}\sec\frac{A}{3}$$

10. If I is the in-centre of ΔABC and R_1, R_2, R_3 are the radii of the circumcircles of the triangles IBC, ICA and IAB, respectively, then show that $R_1R_2R_3 \le R_3$.

Solution: See Fig. 4.31. In ΔIBC apply sine rule. We get

$$\frac{a}{\sin\left(\pi - \frac{B+C}{2}\right)} = 2R_1\ \frac{2R\sin A}{\cos A/2} = 2R_1$$

$$\Rightarrow R_1 = 2R\sin A/2$$

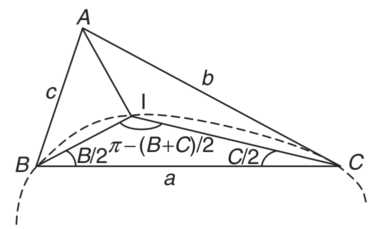


Figure 4.31

Similarly, $R_2 = 2R\sin B/2$ and $R_3 = 2R\sin C/2$
Now, $R_1R_2R_3 = R^3\ 8\sin A/2\sin B/2\sin C/2 \le R^3$

11. In any ΔABC, if $a = 2$, $b = \sqrt{3}+1$ and $C = 60°$. Find the other two angles and the remaining side.

Solution: Two sides and included angle are given.

$$\tan\frac{B-A}{2} = \frac{b-a}{b+a}\cot\frac{C}{2} = \left(\frac{\sqrt{3}+1-2}{\sqrt{3}+1+2}\right)\cot 30°$$

$$= \left(\frac{\sqrt{3}-1}{\sqrt{3}+3}\right)\sqrt{3} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

$$= \frac{\tan 60° - \tan 45}{1+\tan 60°\ \tan 45}$$

$$= \tan(60° - 45°) = \tan 15°$$

Therefore,

$$\frac{B-A}{2} = 15° \Rightarrow B - A = 30° \quad (1)$$

We know that

$$A + B + C = 180° \Rightarrow A + B = 120° \quad (2)$$

Solving Eqs. (1) and (2), we get $B = 75°$ and $A = 45°$.

To find side c, we use the sine rule

$$\frac{a}{\sin A} = \frac{c}{\sin 60^\circ} \Rightarrow c = \sqrt{2}\left(\frac{\sqrt{3}}{2}\right)\left(\frac{2}{1}\right) = \sqrt{6}$$

Thus, $A = 45^\circ$, $B = 75^\circ$ and $c = \sqrt{6}$.

12. If $A = 30^\circ$, $a = 100$, $c = 100\sqrt{2}$, find the number of triangles that can be formed.

Solution: Here a, c and A are given. Therefore we will have to examine whether two triangles are possible or not. For two triangles

$$a > c\sin A \quad (1)$$
$$a < c \quad (2)$$

Now

$$100 > 100\sqrt{2}\sin 30^\circ$$
$$\Rightarrow 100 > 50\sqrt{2} \text{ and } a < c$$

That is, $100 < 100\sqrt{2}$. So two triangles can be formed.

13. In a ΔABC if $\angle C = 90^\circ$ prove that $\dfrac{1}{a+c} + \dfrac{1}{b+c} = \dfrac{3}{a+b+c}$.

Solution: Using cosine rule we have

$$c^2 = a^2 + b^2 - 2ab\cos 60^\circ$$
$$= a^2 + b^2 - 2ab\left(\frac{1}{2}\right) = a^2 + b^2 - ab \quad (1)$$

Therefore,

$$ab - b^2 = a^2 - c^2$$
$$\Rightarrow b(a-b) = (a-c)(a+c)$$
$$\Rightarrow \frac{1}{a+c} = \frac{a-c}{b(a-b)}$$

From Eq. (1),

$$ab - a^2 = b^2 - c^2$$
$$\Rightarrow a(b-a) = (b-c)(b+c)$$
$$\Rightarrow \frac{1}{b+c} = \frac{b-c}{a(b-a)}$$

Therefore,

$$\frac{1}{a+c} + \frac{1}{b+c} = \frac{a-c}{b(a-b)} + \frac{b-c}{a(b-a)}$$
$$= \frac{a(a-c) - b(b-c)}{ab(a-b)} = \frac{(a^2-b^2) - c(a-b)}{ab(a-b)}$$
$$= \frac{(a-b)(a+b-c)}{ab(a-b)} = \frac{a+b-c}{ab}$$
$$= \frac{(a+b-c)(a+b+c)}{ab(a+b+c)} = \frac{(a+b)^2 - c^2}{ab(a+b+c)}$$
$$= \frac{(a+b)^2 - (a^2+b^2-ab)}{ab(a+b+c)} \text{ (using Eq. (1) to replace } c^2\text{)}$$
$$= \frac{3ab}{ab(a+b+c)} = \frac{3}{a+b+c}$$

14. Prove that $\sum a^3\cos(B-C) = 3abc$.

Solution:

$$a^3\cos(B-C) = a^2 \cdot a\cos(B-C)$$
$$= a^2 \cdot 2R\sin A\cos(B-C)$$
$$= Ra^2 2\sin(B+C)\cdot\cos(B-C)$$
$$(\text{since } B+C = 180^\circ - A)$$
$$= Ra^2\{\sin 2B + \sin 2C\}$$
$$= a^2(b\cos B + c\cos C)$$
$$\sum a^3\cos(B-C) = \sum a^2(b\cos B + c\cos C)$$
$$= ab(a\cos B + b\cos A) + bc(b\cos C + c\cos B) + ca(c\cos A + a\cos C)$$
$$= ab\cdot c + bc\cdot a + ca\cdot b = 3abc$$

15. The sides of a triangle are in AP. If the angles A and C are the greatest and smallest angles, respectively, prove that $4(1-\cos A)(1-\cos C) = \cos A + \cos C$.

Solution: We have $2b = a + c$. Therefore,

$$2\sin B = \sin A + \sin C$$
$$\Rightarrow 4\sin\frac{B}{2}\cos\frac{B}{2} = 2\sin\frac{A+C}{2}\cos\frac{A-C}{2} = 2\cos\frac{B}{2}\cdot\cos\frac{A-C}{2}$$
$$\Rightarrow 2\sin\frac{B}{2} = \cos\frac{A-C}{2}$$
$$\Rightarrow 2\cos\frac{A+C}{2} = \cos\frac{A-C}{2} \quad (1)$$

Now,

$$\cos A + \cos C = 2\cos\frac{A+C}{2}\cdot\cos\frac{A-C}{2}$$
$$= 2\cos\frac{A+C}{2}\left(2\cos\frac{A+C}{2}\right) \text{ [using Eq. (1)]}$$
$$= 4\cos^2\frac{A+C}{2} \quad (2)$$
$$4(1-\cos A)(1-\cos C) = 4\cdot 2\sin^2\frac{A}{2}\cdot 2\sin^2\frac{C}{2}$$
$$= 4\left(2\sin\frac{A}{2}\sin\frac{C}{2}\right)^2$$
$$= 4\left\{\cos\frac{A-C}{2} - \cos\frac{A+C}{2}\right\}^2$$
$$= 4\left\{2\cos\frac{A+C}{2} - \cos\frac{A+C}{2}\right\}^2$$
$$= 4\cos^2\frac{A+C}{2} \quad (3)$$

From Eqs. (2) and (3), we get

$$\cos A + \cos C = 4(1-\cos A)(1-\cos C)$$

16. A triangle has base 6 cm and an area of 12 sq. cm. The difference of the base angles is 60°. Prove that the angle opposite is given by the equation

$$8\cos A - 6\cos A = 3$$

Solution: We have $B+C=180°-A$; $B-C=60°$. Therefore,

$$B=120-\frac{A}{2} \text{ and } C=60°-\frac{A}{2}$$

$$\sin B=\sin(180°-B)=\sin\left(60°+\frac{A}{2}\right)$$

$$\sin C=\sin\left(60°-\frac{A}{2}\right)$$

Also

$$\text{Area of the } \Delta=\frac{1}{2}ca\sin B=\frac{1}{2}a2R\sin C\sin B$$

$$=\frac{1}{2}a\left(\frac{a}{\sin A}\right)\sin B\sin C=\frac{a^2\sin B\sin C}{2\sin A}$$

Therefore,

$$12=\frac{1}{2}\frac{(36)\sin\left(60°-\frac{A}{2}\right)\sin\left(60°+\frac{A}{2}\right)}{\sin A}$$

$$\Rightarrow 4\cos A=3(\cos A-\cos 120^\circ)=3\cos A+\frac{3}{2}$$

$$\Rightarrow 8\cos A-6\cos A=3$$

17. If $\cos A\cos B+\sin A\sin B\sin C=1$, show that $a:b:c=1:1:\sqrt{2}$.

Solution:

$$\cos A\cos B+\sin A\sin B\sin C=1$$

$$\Rightarrow \sin C=\frac{1-\cos A\cos B}{\sin A\sin B}\le 1 \text{ (since } \sin C\le 1)$$

$$\Rightarrow 1-\cos A\cos B\le \sin A\sin B$$

$$\Rightarrow 1\le \cos A\cos B+\sin A\sin B$$

$$\Rightarrow 1\le \cos(A-B) \quad (1)$$

But,

$$\cos(A-B)\le 1 \quad (2)$$

Equations (1) and (2) can hold, only if $\cos(A-B)=1$.
This implies $A=B$. Hence,

$$\sin C=\frac{1-\cos A\cos B}{\sin A\sin B}=\frac{1-\cos^2 A}{\sin^2 A}=1$$

Therefore, $C=90°$ and hence $A=B=45°$. So

$$a:b:c=\sin 45°:\sin 45°:\sin 90°=1:1:\sqrt{2}$$

18. If $rr_1=r_2r_3$, prove that the triangle is right-angled.

Solution:

$$rr_1=r_2r_3\Rightarrow \frac{\Delta}{s}\cdot\frac{\Delta}{s-a}=\frac{\Delta}{s-b}\cdot\frac{\Delta}{s-c}$$

$$\Rightarrow (s-b)(s-c)=s(s-a) \text{ [since } \Delta^2=s(s-a)(s-b)(s-c)]$$

$$\Rightarrow s^2-s(b+c)+bc=s^2-sa$$

$$\Rightarrow s(b+c-a)-bc=0$$

$$\Rightarrow (a+b+c)(b+c-a)-2bc=0$$

$$\Rightarrow (b+c)^2-a^2-2bc=0$$

$$\Rightarrow b^2+c^2=a^2$$

So, the triangle is right-angled by the converse of Pythagoras theorem.

19. Prove that the distance of the middle point of the side BC from the foot of the altitude from A to BC is $\frac{b^2-c^2}{2a}$ (assuming $b>c$).

Solution: See Fig. 4.32.

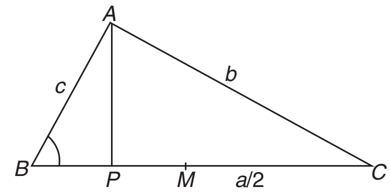


Figure 4.32

The required distance $=MP$

$$=\frac{a}{2}-BP=\frac{a}{2}-c\cos B$$

$$=\frac{a^2-2ac\cos B}{2a}$$

$$=\frac{a^2-(a^2+c^2-b^2)}{2a}=\frac{b^2-c^2}{2a}$$

Note: If $b<c$, the same distance $=\frac{c^2-b^2}{2a}$

20. If O is a point inside a triangle ABC such that $\angle OAB=\angle OBC=\angle OCA=\omega$, then show that

(i) $\cot\omega=\cot A+\cot B+\cot C$

(ii) $\operatorname{cosec}^2\omega=\operatorname{cosec}^2 A+\operatorname{cosec}^2 B+\operatorname{cosec}^2 C$

Solution: See Fig. 4.33. ABC is the triangle and O is so taken (inside the triangle) such that

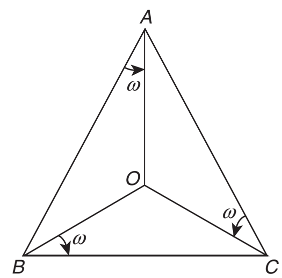


Figure 4.33

$$\angle OAB=\angle OBC=\angle OCA=\omega$$

$$\text{Area of triangle } OBC=\frac{1}{2}OB\cdot BC\cdot\sin\omega$$

$$=\frac{1}{2}OB\cdot OC\cdot\sin\angle BOC$$

and

$$\angle BOC=180^\circ-\{\omega+C-\omega\}=180°-C$$

Therefore,

$$OC=\frac{a\sin\omega}{\sin C} \quad (1)$$

Also,

$$\text{Area of triangle } OAC = \frac{1}{2}OA \cdot OC \cdot \sin A$$
$$= \frac{1}{2}OA \cdot AC \sin(A-\omega)$$

Therefore,

$$OC = \frac{b\sin(A-\omega)}{\sin A} \quad (2)$$

From Eqs. (1) and (2), we get

$$a\frac{\sin\omega}{\sin C} = \frac{b\sin(A-\omega)}{\sin A}$$
$$\Rightarrow \; a\sin\omega\sin A = b\sin C\sin(A-\omega)$$

Since $a = 2R\sin A, b = 2R\sin B$ we get

$$\sin A\sin\omega\sin(B+C) = \sin B\sin C(\sin A\cos\omega - \cos A\sin\omega)$$
$$\Rightarrow \; \cos\omega\sin A\sin B\sin C = \sin\omega\begin{Bmatrix}\sin B\sin C\cos A + \sin C\sin A\cos B\\ +\sin A\sin B\cos C\end{Bmatrix}$$

Dividing by $\sin A\sin B\sin C\sin\omega$, we get

$$\cot\omega = \cot A + \cot B + \cot C \quad (3)$$

This is the result of (i).

Squaring Eq. (3), we get

$$\cot^2\omega = \cot^2 A + \cot^2 B + \cot^2 C + 2\sum\cot A\cot B$$

and $\sum\cot A\cot B = 1$

Using $\text{cosec}^2\theta - 1 = \cot^2\theta$ we get

$$\text{cosec}^2\omega = \text{cosec}^2 A + \text{cosec}^2 B + \text{cosec}^2 C$$

21. If the area of ΔABC is $a^2-(b-c)^2$, then find the value of tan A.

Solution: Given

$$\sqrt{s(s-a)(s-b)(s-c)} = a^2-(b-c)^2$$
$$\Rightarrow \sqrt{s(s-a)(s-b)(s-c)} = (a+b-c)(a-b+c)$$
$$\Rightarrow \sqrt{s(s-a)(s-b)(s-c)} = (2s-2c)(2s-2b)$$
$$\Rightarrow \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} = 4 \Rightarrow \quad \cot\frac{A}{2} = 4$$
$$\Rightarrow \tan\frac{A}{2} = \frac{1}{4}$$

Therefore,

$$\tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}} = \frac{2\left(\frac{1}{4}\right)}{1-\frac{1}{16}} = \frac{8}{15}$$

Previous Years' Solved JEE Main/AIEEE Questions

1. Three distinct points A, B and C are given in the two-dimensional coordinate plane such that the ratio of the distance of any one of them from the point (1, 0) to the distance from the point (–1, 0) is equal to $\frac{1}{3}$. Then the circumcentre of the triangle ABC is at the point

(A) (0, 0)
(B) $\left(\frac{5}{4}, 0\right)$
(C) $\left(\frac{5}{2}, 0\right)$
(D) $\left(\frac{5}{3}, 0\right)$

[AIEEE 2009]

Solution: Let $P = (1, 0)$, $Q(-1, 0)$ and $A = (x, y)$. Then

$$\frac{AP}{AQ} = \frac{BP}{BQ} = \frac{CP}{CQ} = \frac{1}{3}$$
$$\Rightarrow 3AP = AQ \Rightarrow 9AP^2 = AQ^2$$
$$\Rightarrow 9(x-1)^2 + 9y^2 = (x+1)^2 + y^2$$
$$\Rightarrow 9x^2 - 18x + 9 + 9y^2 = x^2 + 2x + 1 + y^2$$
$$\Rightarrow 8x^2 - 20x + 8y^2 + 8 = 0$$
$$\Rightarrow x^2 + y^2 - \frac{5}{2}x + 1 = 0 \quad (1)$$

Therefore, A lies on the circle and similarly, B and C also lie on the same circle. Therefore, the circumcentre of ABC is the centre of the circle 1, which is given by $\left(\frac{-\left(-\frac{5}{2}\right)}{2}, 0\right) = \left(\frac{5}{4}, 0\right)$

Hence, the correct answer is option (B).

2. For a regular polygon, let r and R be the radii of the inscribed and the circumscribed circles. A false statement among the following is

(A) There is a regular polygon with $\frac{r}{R} = \frac{1}{\sqrt{2}}$

(B) There is a regular polygon with $\frac{r}{R} = \frac{2}{3}$

(C) There is a regular polygon with $\frac{r}{R} = \frac{\sqrt{3}}{2}$

(D) There is a regular polygon with $\frac{r}{R} = \frac{1}{2}$

[AIEEE 2010]

Solution: Consider a regular polygon of n sides. Draw a line segment from its centre to each of its n sides to get n number of similar triangles which will look as shown in Fig. 4.34.

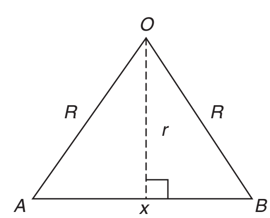


Figure 4.34

Let θ be angle AOB. Therefore,

$$\theta = \frac{360°}{n}$$

Then from trigonometry of right triangles, we have

$$\tan\frac{\theta}{2} = \frac{x/2}{r} \text{ and } \sin\frac{\theta}{2} = \frac{x/2}{R}$$

So we have,

$$r = \frac{a}{2}\cot\frac{\pi}{n}$$

where a is the side of polygon. Therefore,

$$R = \frac{a}{2}\operatorname{cosec}\frac{\pi}{n}$$

$$\Rightarrow \frac{r}{R} = \frac{\cot\frac{\pi}{n}}{\operatorname{cosec}\frac{\pi}{n}} = \cos\frac{\pi}{n} \neq \frac{2}{3} \text{ for any } n \in N$$

Hence, the correct answer is option (B).

3. $ABCD$ is a trapezium such that AB and CD are parallel and $BC \perp CD$. If $\angle ADB = \theta$, $BC = p$ and $CD = q$, then AB is equal to

(A) $\dfrac{p^2 + q^2\cos\theta}{p\cos\theta + q\sin\theta}$ **(B)** $\dfrac{p^2 + q^2}{p^2\cos\theta + q^2\sin\theta}$

(C) $\dfrac{(p^2 + q^2)\sin\theta}{(p\cos\theta + q\sin\theta)^2}$ **(D)** $\dfrac{(p^2 + q^2)\sin\theta}{p\cos\theta + q\sin\theta}$

[JEE MAIN 2013]

Solution: Using sine rule in the triangle ABD, as shown in Fig. 4.35, we get

$$\frac{AB}{\sin\theta} = \frac{BD}{\sin(\theta+\alpha)} = \frac{\sqrt{p^2+q^2}}{\sin(\theta+\alpha)}$$

$$\Rightarrow AB = \frac{\sqrt{p^2+q^2}\,\sin\theta}{\sin\theta\cos\alpha + \cos\theta\sin\alpha}$$

$$= \frac{\sqrt{p^2+q^2}\,\sin\theta}{[(\sin\theta\cdot q)/(\sqrt{p^2+q^2})] + [(\cos\theta\cdot p)/\sqrt{p^2+q^2}]}$$

$$= \frac{(p^2+q^2)\sin\theta}{p\cos\theta + q\sin\theta}$$

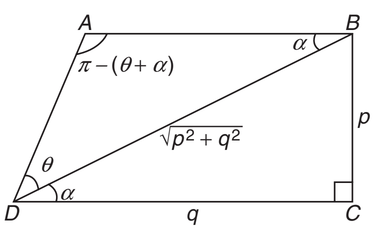


Figure 4.35

Hence, the correct answer is option (D).

4. The angle of elevation of the top of a vertical tower from a point P on the horizontal ground was observed to be α. After moving a distance 2 m from P towards the foot of the tower, the angle of elevation changes to β. Then the height (in metres) of the tower is

(A) $\dfrac{2\sin\alpha\sin\beta}{\sin(\beta-\alpha)}$ **(B)** $\dfrac{\sin\alpha\sin\beta}{\cos(\beta-\alpha)}$

(C) $\dfrac{2\sin(\beta-\alpha)}{\sin\alpha\sin\beta}$ **(D)** $\dfrac{\cos(\beta-\alpha)}{\sin\alpha\sin\beta}$

[JEE MAIN 2014 (ONLINE SET-2)]

Solution: See Fig. 4.36.

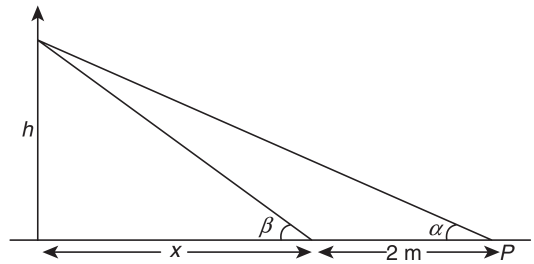


Figure 4.36

$$\frac{x+2}{h} = \cos t\alpha \Rightarrow x = h\cot\alpha - 2$$

Also $\dfrac{x}{h} = \cot\beta \Rightarrow x = h\cot\beta$

Therefore,

$$h\cot\alpha - 2 = h\cot\beta$$

$$\Rightarrow h\left(\frac{\cos\alpha}{\sin\alpha} - \frac{\cos\beta}{\sin\beta}\right) = 2$$

$$\Rightarrow h\left(\frac{\sin\beta\cos\alpha - \cos\beta\sin\alpha}{\sin\alpha\sin\beta}\right) = 2$$

$$\Rightarrow h = \frac{2\sin\alpha\sin\beta}{\sin(\beta-\alpha)}$$

Hence, the correct answer is option (A).

5. From the top of a 64 m high tower, a stone is thrown upwards vertically with the velocity of 48 m/s. The greatest height (in meters) attained by the stone, assuming the value of the gravitational acceleration, $g = 32$ m/s^2 is

(A) 100 **(B)** 88

(C) 128 **(D)** 112

[JEE MAIN 2015 (ONLINE SET-2)]

Solution: See Fig. 4.37.

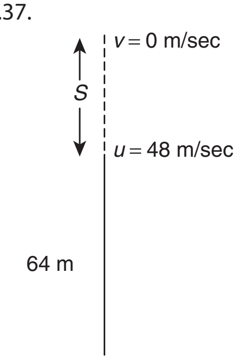


Figure 4.37

$$g = 32 \text{ m/s}^2 \Rightarrow a = -32 \text{ m/sec}^2$$

By laws of motion

$$v^2 - u^2 = 2as$$

$$\Rightarrow 0 - (48)^2 = 2(-32)(s)$$

$$\Rightarrow s = \frac{48\times 48}{2\times 32} = 36 \text{ m}$$

Therefore, greatest height attained by stone from ground $= (64 + 36)$ m $= 100$ m.

Hence, the correct answer is option (A).

6. ABC is a triangle in a plane with vertices $A(2, 3, 5)$, $B(-1, 3, 2)$ and $C(\lambda, 5, \mu)$. If the median through A is equally inclined to the coordinate axes, then the value of $(\lambda^3 + \mu^3 + 5)$ is
   (A) 1130 **(B)** 1348
   (C) 1077 **(D)** 676

[JEE MAIN 2016 (ONLINE SET-2)]

Solution: The specified triangle is shown in Fig. 4.38.

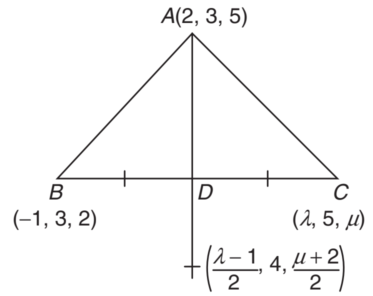


Figure 4.38

The direction ratios (DRs) of AD is

$$\left(\frac{\lambda-1}{2}-2, 4-3, \frac{\mu+2}{2}-5\right)$$
$$\left(\frac{\lambda-5}{2}, 1, \frac{\mu-8}{2}\right)$$

That is,

$$\frac{\lambda-5}{2}=1, \quad \frac{\mu-8}{2}=1$$
$$\lambda=7, \mu=10$$

Therefore,

$$\lambda^3+\mu^3+5=343+1000+5=1348$$

Hence, the correct answer is option (B).

Previous Years' Solved JEE Advanced/ IIT-JEE Questions

1. In a triangle PQR, P is the largest angle and $\cos P = \frac{1}{3}$. Further the in-circle of the triangle touches the sides PQ, QR and RP at N, L and M, respectively, such that the lengths of PN, QL and RM are consecutive even integers. Then possible length(s) of the side(s) of the triangle is (are)
   (A) 16 **(B)** 18
   (C) 24 **(D)** 22

[JEE ADVANCED 2013]

Solution: From Fig. 4.39, we see that QR is the largest side.

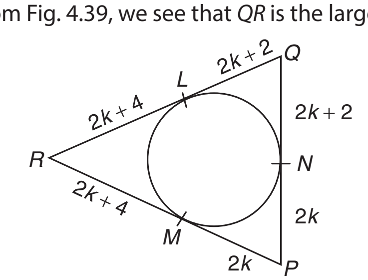


Figure 4.39

Therefore,

$$PM = PN = 2k$$
$$RM = RL = 2k + 4$$
$$QL = QN = 2k + 2$$

Therefore,

$$\left.\begin{aligned} QR &= 4k+6 \\ RP &= 4k+4 \\ PQ &= 4k+2 \end{aligned}\right\} \quad (1)$$

Hence,

$$\cos P = \frac{(PQ)^2+(PR)^2-(QR)^2}{2(PQ)(PR)}$$
$$\Rightarrow \frac{1}{3} = \frac{(4k+2)^2+(4k+4)^2-(4k+6)^2}{2(4k+2)(4k+4)}$$
$$\Rightarrow \frac{1}{3} = \frac{(2k+1)^2+4(k+1)^2-(2k+3)^2}{4(2k+1)(k+1)}$$
$$\Rightarrow \frac{1}{3} = \frac{4(k+1)^2-(4k+4)(2)}{4(2k+1)(k+1)}$$
$$\Rightarrow \frac{1}{3} = \frac{(k+1)^2-2(k+1)}{(k+1)(2k+1)}$$
$$\Rightarrow (k+1)(2k+1) = 3(k-1)(k+1)$$
$$\Rightarrow k=-1 \text{ or } 2k+1=3k-3$$
$$\Rightarrow 4=k$$

Substituting the values in the set of Eq. (1), we get $PQ = 18$; $QR = 22$; and $RP = 22$.

Hence, the correct options are (B) and (D).

2. In a triangle, the sum of two sides is x and the product of the same two sides is y. If $x^2 - c^2 = y$, where c is the third side of the triangle, then the ratio of the in-radius to the circumradius of the triangle is
   (A) $\frac{3y}{2x(x+c)}$ **(B)** $\frac{3y}{2c(x+c)}$
   (C) $\frac{3y}{4y(x+c)}$ **(D)** $\frac{3y}{4c(x+c)}$

[JEE ADVANCED 2014]

Solution: See Fig. 4.40. Let

$$a+b=x \quad (1)$$
$$ab=y \quad (2)$$
$$x^2-c^2=y \quad (3)$$

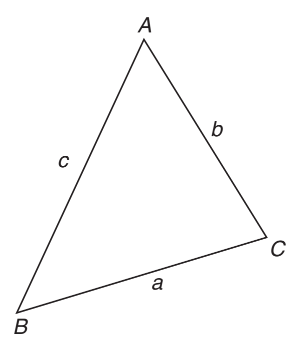


Figure 4.40

From Eq. (1),

$$a+b+c=c+x \tag{4}$$

From Eq. (2),

$$abc=cy \tag{5}$$

From Eq. (4) we have

$$2s=c+x \Rightarrow 2\frac{\Delta}{r}=c+x \Rightarrow r=\frac{2\Delta}{c+x}$$

$$\left(\text{since } r=\frac{\Delta}{s}\right)$$

From Eq. (5) we have

$$4\,\Delta R=cy \Rightarrow R=\frac{cy}{4\Delta}\left(\text{since } R=\frac{abc}{4\Delta}\right)$$

Therefore,

$$\frac{r}{R}=\frac{2\Delta}{c+x}\times\frac{4\Delta}{abc}=\frac{8\Delta^2}{(c+x)abc}$$

$$=\frac{8\times\left(\frac{1}{2}ab\sin C\right)^2}{(c+x)(abc)}=\frac{2a^{\not 2}b^{\not 2}\sin^2 C}{(c+x)\not a\,\not b c} \tag{6}$$

Now from Eq. (3),

$$(a+b)^2-c^2=y$$
$$\Rightarrow 2ab\cos c+2ab=y$$
$$\Rightarrow 2\not{ab}(1+\cos c)=\not{ab}$$

Therefore,

$$\cos c=-\frac{1}{2}$$

Now from Eq. (6),

$$\frac{r}{R}=\frac{2ab(1-\cos^2 c)}{(c+x)c}=\frac{2ab\left(1-\frac{1}{4}\right)}{(c+x)c}$$

$$=\frac{2ab\times\frac{3}{4}}{c(c+x)}=\frac{3ab}{2x(x+c)}=\frac{3y}{2c(x+c)}$$

Hence, the correct answer is option (B).

3. In a triangle XYZ, let x, y, z be the lengths of sides opposite to the angles X, Y, Z, respectively, and $2s=x+y+z$. If $\frac{s-x}{4}=\frac{s-y}{3}=\frac{s-z}{2}$ and the area of in-circle of the triangle XYZ is $\frac{8\pi}{3}$, then

(A) the area of the triangle XYZ is $6\sqrt{6}$.

(B) the radius of circumcircle of the triangle XYZ is $\frac{35}{6}\sqrt{6}$.

(C) $\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2}=\frac{4}{35}$

(D) $\sin^2\left(\frac{X+Y}{2}\right)=\frac{3}{5}$

[JEE ADVANCED 2016]

Solution: See Fig. 4.41. It is given that

$$2s=x+y+z$$

Let us consider

$$\frac{s-x}{4}=\frac{s-y}{3}=\frac{s-z}{2}=k \text{ (say)}$$

That is,

$$s=4k+x;\ s=3k+y;\ s=2k+z$$

Adding the three, we get

$$3s=9k+(x+y+z)=9k+2s$$
$$\Rightarrow s=9k$$

Hence

$$x=9k-4k=5k,\ y=6k,\ z=7k$$

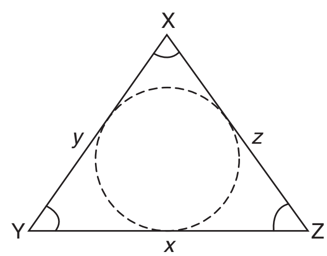


Figure 4.41

Area of in-circle of the triangle XYZ is

$$\pi r^2=\pi\left(\frac{\Delta}{s}\right)^2=\frac{\pi}{s^2}\Delta^2$$

$$=\frac{\pi}{s^2}s\,(s-x)(s-y)(s-z)$$

$$=\frac{\pi}{81k^2}\times 9k\times 4k\times 3k\times 2k$$

$$=\frac{\pi}{9k}\times 24k^3=\frac{\pi}{9}24k^2$$

Therefore,

$$\frac{\pi}{9}24k^2=\frac{8\pi}{3}$$

$$\Rightarrow k^2=\frac{8\pi}{3}\times\frac{9}{24\pi}=1\Rightarrow k=1$$

The sides of the triangle are given by $x=5$, $y=6$, $z=7$. Now, the area of ΔXYZ is

$$\sqrt{s(s-x)(s-y)(s-z)}=\sqrt{9\times 4\times 3\times 2}=6\sqrt{6}$$

Hence, option (A) is correct.

Now,

$$R=\frac{xyz}{4\Delta}=\frac{5\times 6\times 7}{4\times 6\sqrt{6}}=\frac{35}{4\sqrt{6}}$$

$$\sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}=\sqrt{\left[\frac{(s-y)(s-z)}{yz}\right]\left[\frac{(s-z)(s-z)}{xz}\right]\left[\frac{(s-y)(s-x)}{xy}\right]}$$

$$=\frac{(s-z)(s-y)(s-z)}{xyz}$$

So,

$$\frac{(s-z)(s-y)(s-z)}{xyz}=\frac{4\times 3\times 2}{5\times 6\times 7}=\frac{4}{25}$$

Hence, option (C) is correct.
Again,

$$\sin^2\left(\frac{X+Y}{2}\right)=\sin^2\left(\frac{\pi-Z}{2}\right)=\cos^2\frac{Z}{2}=\frac{s(s-Z)}{XY}=\frac{9\times 2}{5\times 6}=\frac{3}{5}$$

Hence, option (D) is correct.
Hence, the correct options are (A), (C) and (D).

Practice Exercise 1

1. If the lengths of arcs AB, BC and CA of a circle are 3, 4 and 5, respectively, then the area of triangle ABC is
   (A) $\frac{9\sqrt{3}(\sqrt{3}+1)}{\pi^2}$ **(B)** $\frac{9\sqrt{3}(\sqrt{3}-1)}{\pi^2}$
   (C) $\frac{9\sqrt{3}(\sqrt{3}-1)}{\pi}$ **(D)** None of these
2. The area of right-angled triangle in terms of r and r_1, if $\angle A=90°$ (where r, r_1 have their usual meanings), is
   (A) $r+r_1$ **(B)** rr_1
   (C) $r-r_1$ **(D)** r_1-r
3. If in a ΔABC (whose circumcentre is origin), $a\le \sin A$, then for any point (x, y) inside the circumcircle of ΔABC
   (A) $|xy|<1/8$ **(B)** $|xy|>1/8$
   (C) $1/8<xy<1/2$ **(D)** None of these
4. If the sine of the angles of a triangle ABC satisfy the equation $c^3x^3-c^2(a+b+c)x^2+\lambda x+\mu=0$ (where a, b, c are the sides of ΔABC), then triangle ABC is
   (A) always right-angled for any λ, μ
   (B) right-angled only when $\lambda=c(ab+bc+ca)$, $\mu=-abc$
   (C) right-angled only when $\lambda=\frac{c(ab+bc+ca)}{4}$, $\mu=\frac{-abc}{8}$
   (D) never right-angled
5. If $\sin A$ and $\sin B$ of a triangle ABC satisfy $c^2x^2-c(a+b)x+ab=0$, then the triangle is
   (A) equilateral **(B)** isosceles
   (C) right-angled **(D)** acute angled
6. $ABCD$ is a quadrilateral circumscribed about a circle of unit radius. Then
   (A) $AB\sin\frac{C}{2}\cdot\sin\frac{A}{2}=CD\sin\frac{B}{2}\cdot\sin\frac{D}{2}$
   (B) $AB\sin\frac{A}{2}\cdot\sin=\frac{B}{2}=CD\sin\frac{C}{2}\cdot\sin\frac{D}{2}$
   (C) $AB\sin\frac{A}{2}\cdot\sin\frac{D}{2}=CD\sin\frac{C}{2}\cdot\sin\frac{B}{2}$
   (D) $AB\sin\frac{A}{2}\cdot\cos=\frac{B}{2}=CD\sin\frac{C}{2}\cdot\cos\frac{D}{2}$
7. If in triangle ABC, line joining the circumcentre and orthocentre is parallel to side AC, then value of $\tan A\cdot\tan C$ is equal to
   (A) $\sqrt{3}$ **(B)** 3
   (C) $3\sqrt{3}$ **(D)** None of these
8. AA_1, BB_1, CC_1 are the medians of triangle ABC whose centroid is G. If the points A, C_1, and B_1 are concyclic, then
   (A) $2b^2=a^2+c^2$ **(B)** $2c^2=a^2+b^2$
   (C) $2a^2=b^2+c^2$ **(D)** None of these
9. The area of a triangle ABC, where $a=2(\sqrt{3}+1)$, $B=45°$, $C=60°$ is
   (A) $\sqrt{3}(\sqrt{3}+1)$ square unit **(B)** $2(\sqrt{3}+1)$ square unit
   (C) $2\sqrt{3}(\sqrt{3}+1)$ square unit **(D)** $\sqrt{3}(2\sqrt{3}+1)$ square unit
10. In a triangle ABC, the value of
    $$\frac{\cos^2 B-\cos^2 C}{b+c}+\frac{\cos^2 C-\cos^2 A}{c+a}+\frac{\cos^2 A-\cos^2 B}{a+b}\text{ is}$$
    (A) 0 **(B)** 1
    (C) 2 **(D)** 3
11. In a triangle ABC if $\cos A+2\cos B+\cos C=2$, the sides of the triangle are in
    (A) HP **(B)** GP
    (C) AP **(D)** None of these
12. In a triangle, $1-\tan\frac{A}{2}\tan\frac{B}{2}=$
    (A) $\frac{2}{a+b+c}$ **(B)** $\frac{2c}{a+b+c}$
    (C) $\frac{c}{a+b+c}$ **(D)** None of these
13. In a triangle ABC $a^2b^2c^2(\sin 2A+\sin 2B+\sin 2C)=$
    (A) Δ^3 **(B)** $8\Delta^3$
    (C) $16\Delta^3$ **(D)** $32\Delta^3$
14. If ex-radii r_1, r_2, r_3 of a triangle are in HP then its sides a, b, c are in
    (A) AP **(B)** GP
    (C) HP **(D)** None of these
15. In a right-angled ΔABC, $\sin^2 A+\sin^2 B+\sin^2 C$ is equal to
    (A) 0 **(B)** 1
    (C) –1 **(D)** None of these
16. In any ΔABC, $b^2\sin 2C+c^2\sin 2B$ is equal to
    (A) Δ **(B)** 2Δ
    (C) 3Δ **(D)** 4Δ
17. If p_1, p_2, p_3 are, respectively, the perpendiculars from the vertices of a triangle to the opposite sides, then $p_1p_2p_3$ is equal to
    (A) $\frac{a^2b^2c^2}{R^2}$ **(B)** $\frac{a^2b^2c^2}{4R^2}$
    (C) $\frac{4a^2b^2c^2}{R^2}$ **(D)** $\frac{a^2b^2c^2}{8R^2}$
18. If p_1, p_2, p_3 are, respectively, the perpendiculars from the vertices of a triangle to the opposite sides, then $\frac{\cos A}{p_1}+\frac{\cos B}{p_2}+\frac{\cos C}{p_3}$ is equal to
    (A) $1/r$ **(B)** $1/R$
    (C) $1/\Delta$ **(D)** None of these

19. If $\Delta = a^2 - (b-c)^2$, where Δ is the area of triangle ABC, then $\tan A$ is equal to

(A) $\frac{15}{16}$ **(B)** $\frac{8}{15}$ **(C)** $\frac{8}{17}$ **(D)** $\frac{1}{2}$

20. If the angles of ΔABC are in the ratio 1:2:3, then the corresponding sides are in the ratio

(A) 2:3:1 **(B)** $\sqrt{3}:2:1$

(C) $2:\sqrt{3}:1$ **(D)** $1:\sqrt{3}:\sqrt{2}$

21. If $c^2 = a^2 + b^2$, then $4s(s-a)(s-b)(s-c)$ is equal to

(A) s^4 **(B)** b^2c^2

(C) c^2a^2 **(D)** a^2b^2

22. In a triangle ABC, O is a point inside the triangle such that $\angle OBC = \angle OCA = \angle OAB = 15°$. Then value of $\cot A + \cot B + \cot C$ is

(A) $2-\sqrt{3}$ **(B)** $\sqrt{2}-1$

(C) $\sqrt{2}+1$ **(D)** $2+\sqrt{3}$

23. In a ΔABC if $a^2\sin(B-C) + b^2\sin(C-A) + c^2\sin(A-B) = 0$, then triangle is

(A) right-angled **(B)** obtuse angled

(C) isosceles **(D)** None of these

24. In any ΔABC, the least value of $\pi\left(\frac{\sin^2 A + \sin A + 1}{\sin A}\right)$ is

(A) 27 **(B)** $\sqrt{3}$

(C) 9 **(D)** None of these

25. If A is the area and $2s$ the sum of the sides of a triangle, then

(A) $A \le \frac{s^2}{4}$ **(B)** $A \ge \frac{s^2}{3\sqrt{3}}$

(C) $A > \frac{s^2}{\sqrt{3}}$ **(D)** None of these

26. If a, b, c and d are the sides of a quadrilateral, then the value of $\frac{a^2+b^2+c^2}{d^2}$ is always greater than

(A) 1 **(B)** $\frac{1}{2}$

(C) $\frac{1}{3}$ **(D)** $\frac{1}{4}$

27. If $A+B+C+D=\pi$, then the value of $\Sigma \cos A \cos C - \Sigma \sin A \sin C =$

(A) –1 **(B)** 1

(C) 2 **(D)** 0

28. In a triangle ABC, D is the mid-point of BC and $AD \perp AC$. Then which of the following is true (a, b, c are sides of ΔBAC as usual):

(A) $3b^2 = a^2 + c^2$ **(B)** $2b^2 = a^2 + c^2$

(C) $3b^2 = a^2 - c^2$ **(D)** $2b^2 = a^2 - c^2$

29. In a triangle ABC, $\sqrt{a}+\sqrt{b}-\sqrt{c}$ is

(A) always positive

(B) always negative

(C) positive only when c is smallest

(D) None of these

30. Let a, b and c be the sides of a triangle and $\frac{a^2+b^2+c^2}{ab+bc+ca} = P$. Then

(A) $1 \le P \le 2$ **(B)** $1 < P \le 2$

(C) $1 < P < 2$ **(D)** $1 \le P < 2$

31. The area of the triangle inscribed in a circle of radius of 4 and the measures of whose angles are in the ratio 5:4:3 is

(A) $4(3+\sqrt{3})$ **(B)** $4(\sqrt{3}+\sqrt{2})$

(C) $4(3-\sqrt{3})$ **(D)** $4(\sqrt{3}-\sqrt{2})$

32. In a triangle ABC if $a^4+b^4+c^4 = 2c^2(a^2+b^2)$, then which of the following does not hold?

(A) $A = \frac{\pi}{4} \Rightarrow r_1 = r_3$ **(B)** $C = \frac{\pi}{2} \Rightarrow r_1 = r_2$

(C) $A = \frac{\pi}{2} \Rightarrow r_2 = r_3$ **(D)** $B = \frac{\pi}{2} \Rightarrow r_1 = r_3$

33. If in a ΔABC, $\Sigma\cos 3A = 1$, then ABC is

(A) an equilateral triangle

(B) an acute-angled scalene triangle

(C) an obtuse angled triangle

(D) a right-angled triangle

34. In an equilateral triangle $r:R:r_1$ is

(A) 2:1:3 **(B)** 1:3:2

(C) 1:2:3 **(D)** 3:2:1

35. In a ΔABC if $2R + r = r_1$, then

(A) $\angle C = \frac{\pi}{2}$ **(B)** $\angle B = \frac{\pi}{2}$

(C) $\angle A = \frac{\pi}{2}$ **(D)** None of these

36. If the sines of the angles of a triangle are in the ratio 4:5:6, then their cosines are in the ratio

(A) 12:2:9 **(B)** 12:9:2

(C) 9:12:2 **(D)** None of these

37. The perimeter of a ΔABC is six times the arithmetic mean of the sine of its angles. If the side a is 1, then the angle $\angle A$ is

(A) $\frac{\pi}{6}$ **(B)** $\frac{\pi}{3}$

(C) $\frac{\pi}{2}$ **(D)** π

38. If $\left(1-\frac{r_1}{r_2}\right)\left(1-\frac{r_1}{r_3}\right) = 2$, then the Δ is

(A) equilateral **(B)** isosceles

(C) right-angled **(D)** None of these

39. In a triangle, $\tan A + \tan B + \tan C = 6$ and $\tan A \tan B = 2$. Then the values of $\tan A$, $\tan B$ and $\tan C$ are

(A) 1, 2, 3 **(B)** 2, 1, 4

(C) 1, 2, 0 **(D)** None of these

40. If in a ΔABC, $c = 3b$ and $C - B = 90°$, then $\tan B$ equals

(A) $\sqrt{3}-2$ **(B)** $\frac{1}{3}$

(C) –1 **(D)** None of these

41. If sides of a triangle are 18, 24, 30 cm, then radius of circumcircle is

(A) 2 **(B)** 4
(C) 6 **(D)** None of these

42. If P is a point on the altitude AD of the triangle ABC such that $\angle CBP = \frac{B}{3}$, then AP is equal to

(A) $2a\sin\frac{C}{3}$ **(B)** $2b\sin\frac{C}{3}$
(C) $2c\sin\frac{B}{3}$ **(D)** $2c\sin\frac{C}{3}$

43. Given $b = 2$, $c = \sqrt{3}$, $\angle A = 30°$. Then the in-radius of ΔABC is

(A) $\frac{\sqrt{3}-1}{2}$ **(B)** $\frac{\sqrt{3}+1}{2}$
(C) $\frac{\sqrt{3}-1}{4}$ **(D)** None of these

44. In a ΔABC, $\sin A + \sin B + \sin C = 1 + \sqrt{2}$ and $\cos A + \cos B + \cos B + \cos C = \sqrt{2}$ if the triangle is

(A) equilateral **(B)** isosceles
(C) right-angled **(D)** right-angled isosceles

45. In a triangle ABC, $\tan\frac{A-B}{2}\cot\frac{A+B}{2}$ is equal to

(A) $\frac{a+b}{c}$ **(B)** $\frac{a+b}{a-b}$
(C) $\frac{a-b}{a+b}$ **(D)** $\frac{a+b}{2R}$

46. In any ΔABC, the expression
$\frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4b^2c^2}$ is equal to

(A) $\cos^2 A$ **(B)** $\sin^2 A$
(C) $1 + \cos A$ **(D)** $1 - \cos A$

47. If $r_1 = 2r_2 = 3r_3$ then $a + b + c$ is equal to

(A) $3b$ **(B)** $2b$
(C) $2a$ **(D)** $3c$

48. In a triangle ABC, if $\left(1+\frac{a}{b}+\frac{c}{b}\right)\left(1+\frac{b}{c}-\frac{a}{c}\right) = 3$, then the angle A is equal to

(A) $\frac{\pi}{3}$ **(B)** $\frac{\pi}{4}$
(C) $\frac{\pi}{6}$ **(D)** None of these

49. If twice the square of the diameter of a circle is equal to the sum of the squares of the sides of the inscribed triangle ABC, then $\sin^2 A + \sin^2 B + \sin^2 C$ is equal to

(A) 2 **(B)** 3
(C) 4 **(D)** 1

50. If H is orthocentre of triangle PQR then $PH + QH + RH$ is

(A) $QR\cot P + PR\cot Q + PQ\cot R$
(B) $(PQ + QR + RP)(\cot P + \cot Q + \cot R)$
(C) $\frac{1}{2r}(\cot P + \cot Q + \cot R)$
(D) None of these

51. The maximum value of $\left(\frac{b+c-a}{2R}\right)\sin A \cdot \sec^2\frac{A}{2}$ is

(A) 1 **(B)** $\frac{3}{2}$
(C) $\frac{2}{8}$ **(D)** None of these

52. Two triangles are possible if

(A) $A < \frac{\pi}{2}$, $a > c\sin A$, $a > c$ **(B)** $A < \frac{\pi}{2}$, $a > c\sin A$, $a < c$
(C) $A < \frac{\pi}{2}$, $a < c\sin A$, $a < c$ **(D)** None of these

53. If r_1, r_2, r_3 are ex-radii of the encircles of triangle ABC then $(s-a)r_1 + (s-b)r_2 + (s-c)r_3$ is

(A) rs **(B)** $3rs$
(C) $\frac{rs}{3}$ **(D)** None of these

54. In a triangle ABC, angles are in AP and $b : c = \sqrt{3} : \sqrt{2}$. Then the angle A is

(A) 60° **(B)** 75°
(C) 120° **(D)** 135°

55. In the ambiguous case, if a, b and A are given and c_1, c_2 are the two values of the third sides, then $(c_1 - c_2)^2 + (c_1 + c_2)^2 \cdot \tan^2 A$ is equal to

(A) 4 **(B)** $4a^2$
(C) $4b^2$ **(D)** $4c^2$

56. If d_1, d_2, d_3 are the diameters of the three escribed circles of a triangle, then $d_1d_2 + d_2d_3 + d_3d_1$ is equal to

(A) Δ^2 **(B)** $4s^2$
(C) $2\Delta^2$ **(D)** $4\Delta^2$

57. In a triangle ABC, $\angle B = \frac{\pi}{3}$ and $\angle C = \frac{\pi}{4}$. Let D divide BC internally in the ratio 1:3. Then $\frac{\sin\angle BAD}{\sin\angle CAD}$ equals

(A) $\frac{1}{\sqrt{6}}$ **(B)** $\frac{1}{3}$
(C) $\frac{1}{\sqrt{3}}$ **(D)** $\sqrt{\frac{2}{3}}$

58. The two adjacent sides of a cyclic quadrilateral are 2 and 5 and the angle between them is 60°. If the third side is 3, the fourth side is

(A) 2 **(B)** 3
(C) 4 **(D)** 5

59. In a triangle ABC, angle A is greater than angle B. If the measure of angles A and B satisfy the equation $3\sin x - 4\sin^3 x - k = 0$, $0 < k < 1$, then the measure of angle C is

(A) $\frac{\pi}{3}$ **(B)** $\frac{\pi}{2}$
(C) $\frac{2\pi}{3}$ **(D)** $\frac{5\pi}{6}$

60. Consider a triangle *ABC*, with given $\angle A$ and side '*a*'. If $bc = x^2$, then such a triangle would exist if (*x* is a given positive real number)

(A) $a < x\sin\frac{A}{2}$ **(B)** $a > 2x\sin\frac{A}{2}$
(C) $a < 2x\sin\frac{A}{2}$ **(D)** None of these

Practice Exercise 2

Single/Multiple Correct Choice Type Questions

1. If a right-angled triangle has integer sides then which of the following is necessarily an integer?
(A) Area **(B)** Circumradius
(C) In-radius **(D)** None of these

2. In a triangle *ABC*, $\angle C = 120°$. If *h* is the harmonic mean of the lengths of the sides *BC* and *CA*, then the length of the bisector of $\angle BCA$ is
(A) h **(B)** $\frac{h}{2}$
(C) $\frac{h}{\sqrt{2}}$ **(D)** $\frac{\sqrt{3}}{2}h$

3. A triangle is inscribed in a circle of radius 1. The distance between the orthocentre and the circumcentre of the triangle cannot be
(A) 1 **(B)** 2
(C) $\frac{3}{2}$ **(D)** 4

4. Two of the altitudes of a scalar triangle *ABC* have lengths 4 and 12. If the length of the third altitude is also an integer, then its largest possible value is
(A) 3 **(B)** 4
(C) 5 **(D)** 6

5. In a triangle *ABC* the altitude from *A* is not less than *BC* and altitude from *B* is not less than *AC*. The triangle is
(A) right-angled **(B)** isosceles
(C) obtuse angled **(D)** equilateral

6. If $\alpha, \beta, \gamma, \delta$ are the smallest positive angles in ascending order of magnitude which have their sines equal to the positive quantity *k*, then the value of $4\sin\frac{\alpha}{2} + 3\sin\frac{\beta}{2} + 2\sin\frac{\gamma}{2} + \sin\frac{\delta}{2}$ is equal to
(A) $2\sqrt{1-k}$ **(B)** $2\sqrt{1+k}$
(C) $2\sqrt{k}$ **(D)** None of these

7. If *D*, *E*, *F* are the feet of perpendiculars from the vertices *A*, *B*, *C* to the opposite sides of ΔABC and the semi-perimeter of ΔDEF is equal to the in-radius of ΔABC then $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$ is equal to
(A) $\frac{1}{4}$ **(B)** $\frac{1}{2}$
(C) $\frac{1}{8}$ **(D)** $\frac{\sqrt{3}}{8}$

8. The line ℓ is tangent to the circle *S* at the point *A*. *B* and *C* are points on ℓ on opposite sides of *A* and other tangents from *B*. *C* to *S* intersect at a point *P*. If *B*, *C* vary along ℓ in such a way that the product $|AB| \cdot |AC|$ is constant, then locus of *P* is
(A) circle **(B)** a line parallel to *BC*
(C) a set of points **(D)** None of these

9. Three straight lines are drawn through a point *M*, lying in the interior of triangle ABC, parallel to its sides. The areas of the resulting three triangles (see Fig. 4.42) are S_1, S_2 and S_3. The area of triangle *ABC* is

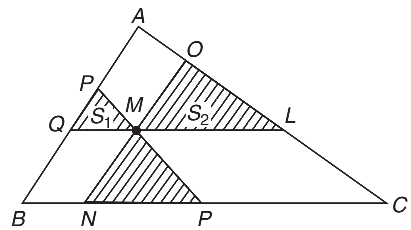


Figure 4.42

(A) $S_1 + S_2 + S_3$ **(B)** $(\sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3})^2$
(C) $\frac{(S_1 + S_2 + S_3)^{3/2}}{S_1 + S_2 + S_3}$ **(D)** None of these

10. A point *P* moves inside the square $A_1 A_2 A_3 A_4$ of side length *b* such that distance of point *P* from *O* is less than its distance from each of the four vertices of square. The area moved by the point *P* is
(A) b^2 sq. units **(B)** $\frac{b^2}{2}$ sq. units
(C) $2b^2$ **(D)** None of these

11. In Fig. 4.43, *AB* is tangent at *A* to the circle with centre *O*; point *D* is interior to the circle and *DB* intersects the circle at *C*. If $BC = DC = 3$, $OD = 2$ and $AB = 6$, then the radius of the circle is

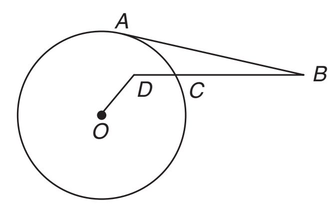


Figure 4.43

(A) $3+\sqrt{3}$ **(B)** $\sqrt{22}$
(C) $2\sqrt{6}$ **(D)** $\frac{9}{2}$

12. In any ΔABC, which is not right angled, $\sum \cos A \operatorname{cosec} B \operatorname{cosec} C$ is
(A) constant **(B)** less than 1
(C) greater than 2 **(D)** None of these

13. If the sides of a triangle *ABC* are *a*, *b* and *c* such that $2b = a + c$, then exhaustive range of $\frac{b}{c}$ is
(A) $\left(\frac{2}{3}, 2\right)$ **(B)** $(0, 1)$
(C) $\left(-\frac{1}{2}, 2\right)$ **(D)** $(4, 6)$

14. Let $ABCD$ be a parallelogram and let $\overrightarrow{AA'}$, $\overrightarrow{BB'}$, $\overrightarrow{CC'}$ and $\overrightarrow{DD'}$ be parallel rays in space on the same side of the plane determined by $ABCD$. If $AA' = 10$, $BB' = 8$, $CC' = 18$, $DD' = 22$ and M and N are the mid-points of $\overline{A'C'}$ and $\overline{B'D'}$ respectively, then $MN =$

(A) 1 **(B)** 2
(C) 3 **(D)** 4

15. The sides of a triangle have length 11, 15 and k, where k is an integer. Then the number of values of k for which the triangle is obtuse is

(A) 5 **(B)** 12
(C) 13 **(D)** 17

16. A pentagon is formed by cutting a triangular corner from a rectangular piece of paper. The five sides of the pentagon have lengths 13, 19, 20, 25 and 31, not necessarily in that order. The area of the pentagon is

(A) 459 sq. units **(B)** 600 sq. units
(C) 680 sq. units **(D)** 745 sq. units

17. For an acute angle ΔABC, if $p = \dfrac{\sqrt{3} + \sin A + \sin B + \sin C}{2\sin A \sin B \sin C}$ then $p \in$

(A) $\left[\frac{5}{3}, \frac{10}{3}\right]$ **(B)** $\left[\frac{10}{3}, \infty\right)$
(C) $[2, \infty)$ **(D)** $[-1, 1]$

18. If $\alpha, \beta, \gamma, \delta$ be four angles of a cyclic quadrilateral taken in clockwise direction then the value of $(2 + \Sigma \cos\alpha \cos\beta)$ will be

(A) $\sin^2\alpha + \sin^2\beta$ **(B)** $\cos^2\gamma + \cos^2\delta$
(C) $\sin^2\alpha + \sin^2\delta$ **(D)** $\cos^2\beta + \cos^2\gamma$

19. If in a non-right-angled triangle ABC, $\tan A$ and $\tan B$ are rational and the vertices A and B are also rational points, then the correct statements is/are

(A) $\tan C$ must be rational **(B)** C must be a rational point
(C) $\tan C$ may be irrational **(D)** C may be an irrational point

20. If in ΔABC, $\sec A$, $\sec B$, $\sec C$ are in harmonic progression, then

(A) a, b, c, are in harmonic progression.
(B) $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in harmonic progression
(C) r_1, r_2, r_3 are in arithmetic progression
(D) $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in arithmetic progression

21. Two circles C_1 and C_2 intersect at two distinct points P and Q in a plane. Let a line passing through P meet the circles C_1 and C_2 in A and B, respectively. Let Y be the mid-point of AB and QY meet the circles C_1 and C_2 in X and Z, respectively. Then

(A) Y is the mid-point of XZ **(B)** $\frac{XY}{YZ} = \frac{2}{1}$
(C) $YX = YZ$ **(D)** $XY + YZ = 3YZ$

22. If in a ΔABC, a, b, c are in AP, then it is necessary that

(A) $\frac{2}{3} < \frac{b}{c} < 2$ **(B)** $\frac{1}{3} < \frac{b}{c} < \frac{2}{3}$
(C) $\frac{2}{3} < \frac{b}{a} < 2$ **(D)** $\frac{1}{3} < \frac{b}{a} < \frac{2}{3}$

23. If a, b, c are the sides of a triangle then $\dfrac{a}{c+a-b} + \dfrac{b}{a+b-c} + \dfrac{c}{b+c-a}$ can take value(s)

(A) 1 **(B)** 2
(C) 3 **(D)** 4

24. If l_1, l_2, l_3 are lengths of altitudes of a triangle and $\frac{R}{l_1} + \frac{R}{l_2} + \frac{R}{l_3} = 2$ R is circumradius, then

(A) $l_1 = 3r$ **(B)** $l_2 = 3r$
(C) $l_3 = 4r$ **(D)** $l_2 = 4r$

25. If r_1, r_2, r_3 are the radii of the escribed circles of a triangle ABC and r is the radius of its in-circle then the root(s) of the equation $x^2 - r(r_1r_2 + r_2r_3 + r_3r_1)x + r_1r_2r_3 - 1 = 0$ is/are

(A) 1 **(B)** $r_1 + r_2 + r_3$
(C) r **(D)** $r_1r_2r_3 - 1$

Comprehension Type Questions

Paragraph for Questions 26–28: Vertices of a variable acute-angled triangle ABC lie on a circle of radius R such that $\frac{da}{dA} + \frac{db}{dB} + \frac{dc}{dC} = 6$. Distance of orthocentre of triangle ABC from vertices A, B and C is x_1, x_2 and x_3, respectively.

26. In-radius of triangle ABC is

(A) 1 **(B)** 2
(C) 3 **(D)** 4

27. Maximum value of $x_1x_2x_3$ is

(A) 4 **(B)** 6
(C) 8 **(D)** 10

28. $\frac{dx_1}{da} + \frac{dx_2}{db} + \frac{dx_3}{dc}$ is always less than equal to

(A) $-3\sqrt{3}$ **(B)** $3\sqrt{3}$
(C) 1 **(D)** 6

Paragraph for Questions 29–31: In a ΔABC, the equation of the side BC is $2x - y = 3$ and its circumcentre and orthocentre are at (2, 4) and (1, 2), respectively.

29. Circumradius of ΔABC is

(A) $\sqrt{\frac{61}{5}}$ **(B)** $\sqrt{\frac{51}{5}}$
(C) $\sqrt{\frac{41}{5}}$ **(D)** $\sqrt{\frac{43}{5}}$

30. $\sin B \sin C$ is equal to

(A) $\frac{9}{2\sqrt{61}}$ **(B)** $\frac{9}{4\sqrt{61}}$
(C) $\frac{9}{\sqrt{61}}$ **(D)** $\frac{9}{3\sqrt{61}}$

31. The distance of orthocentre of vertex A is

(A) $\frac{1}{\sqrt{5}}$ **(B)** $\frac{6}{\sqrt{5}}$
(C) $\frac{3}{\sqrt{5}}$ **(D)** $\frac{2}{\sqrt{5}}$

Paragraph for Questions 32–34: Let I be the in-centre and I_1, I_2, I_3 be the ex-centre opposite to angle A, B, C, respectively, in ΔABC. If α, β, γ be the circumradius of ΔBIC, ΔAIC and ΔAIB, respectively, and R, r, r_1, r_2, r_3 have their usual meaning, then

32. $II_1 + II_2 + II_3$ is equal to

(A) $2R\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\right)$ **(B)** $4R\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\right)$

(C) $4R\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)$ **(D)** $4R\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}$

33. α, β, γ is equal to

(A) $2R^2r$ **(B)** $4R^2r$
(C) $8R^2r$ **(D)** $16Rr^2$

34. $\frac{II_1}{\alpha}+\frac{II_2}{\beta}+\frac{II_3}{\gamma}$ is equal to

(A) $\frac{3}{2}$ **(B)** $\frac{3}{4}$
(C) 3 **(D)** 6

Paragraph for Questions 35–37: The area of any cyclic quadrilateral $ABCD$ is given by $A^2 = (s - a)(s - b)(s - c)(s - d)$, where $2s = a + b + c + d$, a, b, c and d are the sides of the quadrilateral.
For a cyclic quadrilateral $ABCD$ of area 1 sq. unit answer the following questions:

35. The minimum perimeter of the quadrilateral is

(A) 4 **(B)** 2
(C) 1 **(D)** None of these

36. The minimum value of the sum of the lengths of diagonals is

(A) $2\sqrt{2}$ **(B)** 2
(C) $\sqrt{2}$ **(D)** None of these

37. When the perimeter is minimum the quadrilateral is necessarily

(A) a square
(B) a rectangle but not a square
(C) a rhombus but not a square
(D) None of these

Paragraph for Questions 38–40: Let ABC be any triangle and P be a point inside it such that $\angle PAB = \frac{\pi}{18}$, $\angle PBA = \frac{\pi}{9}$, $\angle PCA = \frac{\pi}{6}$, $\angle PAC = \frac{2\pi}{9}$. Let $\angle PCB = x$.

38. $\angle PBC$ is equal to

(A) $\frac{\pi}{9}$ **(B)** $\frac{2\pi}{9}$
(C) $\frac{\pi}{3}$ **(D)** None of these

39. Triangle ABC is an

(A) equilateral triangle
(B) isosceles triangle
(C) right-angled triangle
(D) None of these

40. Which of the following is true?

(A) $BC > AC$ **(B)** $BC < AB$
(C) $AC > AB$ **(D)** $BC = AC$

Matrix Match Type Questions

41. Match the following:

Column-I	Column-II
(A) If in a triangle ABC, $\sin^2 A + \sin^2 B = \sin(A + B)$, then the triangle must be	**(i)** Right-angled
(B) If in a triangle ABC $\frac{bc}{2\cos A} = b^2 + c^2 - 2bc\cos A$, then the triangle must be	**(ii)** Equilateral
(C) If in a triangle ABC, $\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} = \sqrt{3}$, then the triangle must be	**(iii)** Isosceles
(D) If in a triangle the sides and the altitudes are in AP, then the triangle must be	**(iv)** Obtuse-angled

42. Let ABC be a triangle with G_1, G_2, G_3 the mid-points of BC, AC and AB, respectively. Also let M be the centroid of the triangle. It is given that the circumcircle of ΔMAC touches the side AB of the triangle at point A.

Column-I	Column-II
(A) $\frac{AG_1}{b} =$	**(p)** $\frac{2}{\sqrt{3}}$
(B) Maximum value of $S_n\angle CAM + S_n\angle CBM =$	**(q)** $\frac{\sqrt{3}}{2}$
(C) $\frac{a^2 + b^2}{c^2} =$	**(r)** $\sqrt{2}$
(D) If $(\sin\angle CAM + \sin\angle CBM)$ is maximum then $\frac{c^2}{ab} =$	**(s)** 2

Integer Type Questions

43. Two circles are circumscribed and inscribed about a square $ABCD$ of side 2 units. If P and Q are two points on respective circles, $\sum(PA)^2 - \sum(QA)^2 =$ ____________.

44. The base AB of a triangle is 1 and height h of C from AB is less than or equal to $\frac{1}{2}$. The maximum value of 4 times the product of the altitudes of triangle is ____________.

Answer Key

Practice Exercise 1

1. (A)	**2.** (B)	**3.** (A)	**4.** (B)	**5.** (C)	**6.** (B)
7. (B)	**8.** (C)	**9.** (C)	**10.** (A)	**11.** (C)	**12.** (B)
13. (D)	**14.** (A)	**15.** (D)	**16.** (D)	**17.** (D)	**18.** (B)
19. (B)	**20.** (D)	**21.** (D)	**22.** (D)	**23.** (C)	**24.** (A)
25. (A)	**26.** (C)	**27.** (D)	**28.** (C)	**29.** (A)	**30.** (D)
31. (A)	**32.** (B)	**33.** (C)	**34.** (C)	**35.** (C)	**36.** (B)
37. (A)	**38.** (C)	**39.** (A)	**40.** (B)	**41.** (D)	**42.** (C)
43. (A)	**44.** (D)	**45.** (C)	**46.** (B)	**47.** (A)	**48.** (A)
49. (A)	**50.** (A)	**51.** (A)	**52.** (B)	**53.** (B)	**54.** (B)
55. (B)	**56.** (B)	**57.** (A)	**58.** (A)	**59.** (C)	**60.** (B)

Practice Exercise 2

1. (C)	**2.** (B)	**3.** (D)	**4.** (C)	**5.** (A)	**6.** (B)
7. (A)	**8.** (B)	**9.** (B)	**10.** (B)	**11.** (B)	**12.** (A)
13. (A)	**14.** (A)	**15.** (C)	**16.** (D)	**17.** (B)	**18.** (A), (C)
19. (A), (B)	**20.** (B), (C)	**21.** (A), (C)	**22.** (A), (C)	**23.** (C), (D)	**24.** (A), (B)
25. (A), (D)	**26.** (C)	**27.** (C)	**28.** (A)	**29.** (A)	**30.** (A)
31. (B)	**32.** (B)	**33.** (A)	**34.** (D)	**35.** (A)	**36.** (A)
37. (A)	**38.** (C)	**39.** (B)	**40.** (C)	**41.** (A) → (i); (B) → (iii); (C) → (ii); (D) → (ii)	
42. (A) → (q); (B) → (p); (C) → (s); (D) → (r)		**43.** 12	**44.** 2		

Solutions

Practice Exercise 1

1. See Fig. 4.44. Angle subtended by the chord AB, BC and CA at centre of circle is in ratio 3:4:5, that is, 90°, 120°, 150°. So,

$$\angle B = 75°, \angle C = 45°, \angle A = 60°.$$

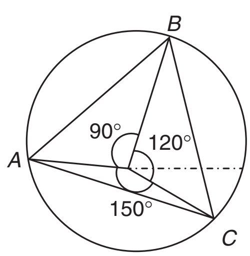


Figure 4.44

Now,

$$\text{Perimeter of circle, } 2\pi r = 12$$

$$r = \frac{12}{2\pi} = \frac{6}{\pi}$$

$$\text{Area of triangle} = 2r^2 \sin A \sin B \sin C$$

$$= 2\left(\frac{6}{\pi}\right)^2 \sin 60° \sin 45° \sin 75°$$

$$= \frac{72}{\pi^2} \times \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$= \frac{9\sqrt{3}(\sqrt{3}+1)}{\pi^2}$$

2.
$$r_1 = s \tan\frac{A}{2} = s$$

$$r = (s-a)\tan\frac{A}{2} = s-a$$

Also,

$$a^2 = b^2 + c^2$$
$$r_1 + r = b + c$$
$$r_1 - r = a$$

So,

$$(r_1+r)^2 - (r_1-r)^2 = 4r_1 r = 2bc$$

or

$$\Delta = \frac{1}{2}bc = rr_1$$

3.
$$a \le \sin A \Rightarrow \frac{a}{\sin A} \le 1 \Rightarrow R \le 1/2$$

So for any point (x, y) inside the circumcircle,

$$x^2 + y^2 < 1/4 \Rightarrow |xy| < 1/8$$

4. $\sin A + \sin B + \sin C = \dfrac{c^2(a+b+c)}{c^3} = \dfrac{a+b+c}{c}$

But $\sin A + \sin B + \sin C = \dfrac{a}{2R} + \dfrac{b}{2R} + \dfrac{c}{2R}$

Comparing both we get $c = 2R$. So, the triangle is a right-angled triangle.

Putting the same value of c, we get $\lambda = ab + bc + ca$, $\mu = -abc$

5. $\sin A+\sin B=\dfrac{c(a+b)}{c^2}=\dfrac{a+b}{c}=\dfrac{k\sin A+k\sin B}{k\sin C}$

Thus, $C=\pi/2\Rightarrow \Delta ABC$ is right-angled.

6. Let 'O' be the centre of circle and 'P' be its point of contact with side AB (Fig. 4.45).

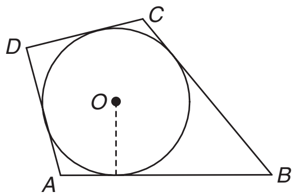


Figure 4.45

Thus,

$$AP=OP\cdot\cot\frac{A}{2}=\cot\frac{A}{2} \quad (1)$$

and $$PB=OC\cdot\cot\frac{B}{2}=\cot\frac{B}{2} \quad (2)$$

Adding Eqs. (1) and (2) we get

$$AP+PB=\cot\frac{A}{2}+\cot\frac{B}{2}=\frac{\sin\left(\frac{A+B}{2}\right)}{\sin\frac{A}{2}\cdot\sin\frac{B}{2}}$$

Similarly,

$$CD=\frac{\sin\left(\frac{C+D}{2}\right)}{\sin\frac{C}{2}\cdot\sin\frac{D}{2}}$$

Since,

$$A+B+C+D=2\pi$$

Therefore,

$$\frac{A+B}{2}=\pi-\frac{C+D}{2}$$

$$\Rightarrow \sin\left(\frac{A+B}{2}\right)=\sin\left(\frac{C+D}{2}\right)$$

$$\Rightarrow AB\cdot\sin\frac{A}{2}\cdot\sin\frac{B}{2}=CD\sin\frac{C}{2}\cdot\sin\frac{D}{2}$$

7. Distance of circumcentre from side $AC=R\cos B$ and distance of orthocentre from side $AC=2R\cos A\cdot\cos C$. So,

$$R\cos B=2R\cos A\cdot\cos C$$
$$\Rightarrow -\cos(A+C)=2\cos A\cdot\cos C$$
$$\Rightarrow \sin A\cdot\sin C=3\cos A\cdot\cos C$$
$$\Rightarrow \tan A\cdot\tan C=3$$

8. Since A, C_1, G and B_1 are concyclic, therefore

$BG\cdot BB_1=BC_1\cdot BA$

$$\Rightarrow \frac{2}{3}(BB_1)^2=\frac{c}{2}\cdot c$$

$$\Rightarrow \frac{2}{3}\cdot\frac{1}{4}(2a^2+2c^2-b^2)=\frac{c^2}{2}$$

$$\Rightarrow 2a^2+2c^2-b^2=3c^2$$

$$\Rightarrow b^2+c^2=2a^2$$

9. $$\Delta=\frac{1}{2}bc\sin A,\ \angle A=75^\circ$$

Hence,

$$\frac{b}{\sin 45^\circ}=\frac{a}{\sin 75^\circ}\text{ or } b=\frac{a/\sqrt{2}}{\frac{\sqrt{3}+1}{2\sqrt{2}}}=\frac{2a}{\sqrt{3}+1}$$

Therefore,

$$\Delta=\frac{1}{2}ab\sin C=\frac{1}{2}\frac{2a^2}{\sqrt{3}+1}\cdot\sin 60$$
$$=4(\sqrt{3}+1)\cdot\frac{\sqrt{3}}{2}=2\sqrt{3}(\sqrt{3}+1)\text{ sq. unit}$$

10. $\dfrac{\cos^2 B-\cos^2 C}{b+c}=\dfrac{\sin^2 C-\sin^2 B}{k(\sin B+\sin C)}=\dfrac{\sin C-\sin B}{k}$

Therefore,

$$\sum\frac{\cos^2 B-\cos^2 C}{b+c}=\frac{1}{k}\sum(\sin C-\sin B)=0$$

11. $\cos A+2\cos B+\cos C=2$

$$\Rightarrow \cos A+\cos C=2(1-\cos B)$$
$$\Rightarrow 2\cos\left(\frac{A+C}{2}\right)\cos\left(\frac{A-C}{2}\right)=4\sin^2\frac{B}{2}$$
$$\Rightarrow \cos\frac{A-C}{2}=2\sin\frac{B}{2}$$
$$\Rightarrow \cos\frac{A-C}{2}=2\cos\frac{A+C}{2}$$
$$\Rightarrow \cos\frac{A}{2}\cos\frac{C}{2}+\sin\frac{A}{2}\sin\frac{C}{2}=2\cos\frac{A}{2}\cos\frac{C}{2}-2\sin\frac{A}{2}\sin\frac{C}{2}$$
$$\Rightarrow \cot\frac{A}{2}\cot\frac{C}{2}=3$$
$$\Rightarrow \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}\cdot\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}=3$$
$$\Rightarrow \frac{s}{s-b}=3\Rightarrow s=3s-3b\Rightarrow 2s=3b$$
$\Rightarrow a+c=2b\Rightarrow a, b, c$ are in AP.

12. $1-\tan\dfrac{A}{2}\tan\dfrac{B}{2}=1-\sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}\sqrt{\dfrac{(s-a)(s-c)}{s(s-b)}}$

$$=1-\frac{s-c}{s}=\frac{c}{s}=\frac{2c}{a+b+c}$$

13. $a^2b^2c^2(\sin 2A+\sin 2B+\sin 2C)$

$$=a^2b^2c^2(2\sin(A+B)\cos(A-B)+2\sin C\cos C)$$
$$=a^2b^2c^2\,2\sin C[\cos(A-B)-\cos(A+B)]$$
$$=a^2b^2c^2[4\sin A\sin B\sin C]$$
$$=\frac{4a^2b^2c^2abc}{8R^3}=32\left(\frac{abc}{4R}\right)^3=32\Delta^3$$

14. $$\frac{1}{r_1}-\frac{1}{r_2}=\frac{1}{r_2}-\frac{1}{r_3}$$

$$\Rightarrow \frac{\Delta}{s-a}-\frac{\Delta}{s-b}=\frac{\Delta}{s-b}-\frac{\Delta}{s-c}$$

$$\Rightarrow (s-b)-(s-a)=(s-c)-(s-b)$$

$$\Rightarrow a-b=b-c$$

So a, b, c are in AP.

15. Let $C=90°$, then

$$\sin^2 A+\sin^2 B+\sin^2 C=\sin^2 A+\sin^2 B+1$$

$$=\sin^2 A+\sin^2\left(\frac{\pi}{2}-A\right)+1$$

$$=\sin^2 A+\cos^2 A+1=2$$

16. $b^2\sin 2C+c^2\sin 2B=b^2\cdot 2\sin C\cos C+c^2\cdot 2\sin B\cos B$

$$=2(b\sin C)(b\cos C)+2(c\sin B)(c\cos B)$$

$$=2c\sin B(b\cos C+c\cos B)=2ac\sin B=4\Delta$$

17. We have

$$\Delta=\frac{1}{2}ap_1=\frac{1}{2}bp_2=\frac{1}{2}cp_3$$

Therefore,

$$p_1=\frac{2\Delta}{a};\ p_2=\frac{2\Delta}{b};\ p_3=\frac{2\Delta}{c}$$

So,

$$p_1p_2p_3=\frac{8\Delta^3}{abc}=\frac{8}{abc}\left(\frac{abc}{4R}\right)^3=\frac{a^2b^2c^2}{8R^2}$$

18. We have

$$\frac{\cos A}{p_1}+\frac{\cos B}{p_2}+\frac{\cos C}{p_3}$$

$$=\frac{1}{2\Delta}(a\cos A+b\cos B+c\cos C)$$

$$=\frac{R}{\Delta}(\sin A\cos A+\sin B\cos B+\sin C\cos C)$$

$$=\frac{R}{2\Delta}(\sin 2A+\sin 2B+\sin 2C)$$

$$=R\frac{4\sin A\sin B\sin C}{2\Delta}=\frac{2R\sin A\sin B\sin C}{\Delta}$$

$$=\frac{2R}{\Delta}\cdot\frac{2\Delta}{bc}\cdot\frac{2\Delta}{ca}\cdot\frac{2\Delta}{ab}=\frac{16R\Delta^2}{a^2b^2c^2}=\frac{16R\Delta^2}{(4R\Delta)^2}=\frac{1}{R}$$

19. $\Delta=2bc-(b^2+c^2-a^2)=2bc(1-\cos A)$

$$=2bc\cdot 2\sin^2\frac{A}{2} \quad (1)$$

$$\Delta=\frac{1}{2}bc\sin A=\frac{1}{2}(bc)\ 2\sin\frac{A}{2}\cos\frac{A}{2}$$

$$=bc\sin\frac{A}{2}\cos\frac{A}{2} \quad (2)$$

Therefore, by Eqs. (1) and (2)

$$\tan\frac{A}{2}=\frac{1}{4}=t$$

So,

$$\tan A=\frac{2t}{1-t^2}=\frac{8}{15}$$

20. $(1+2+3)k=180° \Rightarrow k=30°$

Therefore, $A=30°$, $B=60°$, $C=90°$. Now

$$a{:}b{:}c=\sin A{:}\sin B{:}\sin C=\frac{1}{2}:\frac{\sqrt{3}}{2}:1 \text{ or } 1:\sqrt{3}:\sqrt{2}$$

21. Δ is right-angled, $\angle C=90°$. Therefore

$$4\Delta^2=4\cdot\left(\frac{1}{2}ab\right)^2=a^2b^2$$

22. $\cot A+\cot B+\cot C=\cot 15°=2+\sqrt{3}$

23. Consider

$$a^2\sin(B-C)=2aR\sin(B+C)\sin(B-C)$$

$$=2aR(\sin^2 B-\sin^2 C)=\frac{a}{2R}(b^2-c^2)$$

Similarly,

$$b^2\sin(C-A)=\frac{b}{2R}(c^2-a^2)$$

and

$$c^2\sin(A-B)=\frac{c}{2R}(a^2-b^2)$$

Therefore,

$$a^2\sin(B-C)+b^2\sin(C-A)+c^2\sin(A-B)=0$$

$$\Rightarrow a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)=0$$

$$\Rightarrow a(b-c)(b+c)+bc^2-b^2c-a^2b+a^2c=0$$

$$\Rightarrow a(b-c)(b+c)-bc(b-c)-a^2(b-c)=0$$

$$\Rightarrow (b-c)(ab+ac-bc-a^2)=0$$

$$\Rightarrow (b-c)(c-a)(a-b)=0$$

$\Rightarrow$ Either $a=b$ or $b=c$ or $c=a$

24. Lower bound value of $\pi\left(\frac{\sin^2 A+\sin A+1}{\sin A}\right)$

Hint: $\left(\sin A+\frac{1}{\sin A}\right)\geq 2$

Now,

$$\left(\sin A+\frac{1}{\sin A}+1\right)\geq 3$$

$$\left(\sin B+\frac{1}{\sin B}+1\right)\geq 3$$

$$\left(\sin C+\frac{1}{\sin C}+1\right)\geq 3$$

$$\Rightarrow 3\times 3\times 3=27$$

25. $A=$ Area, $2s=a+b+c$

Hint: AM $\geq$ GM

$$\frac{s+(s-a)+(s-b)+(s-c)}{4}\geq\left[s(s-a)(s-b)(s-c)\right]^{1/4}$$

$$\Rightarrow\ A\leq\frac{s^2}{4}$$

26. See Fig. 4.46.

Hint:

$$(a-b)^2+(b-c)^2+(c-a)^2 \ge 0$$

$$\Rightarrow 2(a^2+b^2+c^2) \ge 2(ab+bc+ca)$$

$$\Rightarrow 3(a^2+b^2+c^2) \ge a^2+b^2+c^2+2(ab+bc+ca)$$

$$\Rightarrow 3(a^2+b^2+c^2) \ge (a+b+c)^2$$

$$\Rightarrow 3(a^2+b^2+c^2) \ge (a+b+c)^2 \ge d^2$$

(as, $a+b+c>d$)

$$\Rightarrow \frac{(a^2+b^2+c^2)}{d^2} \ge \frac{1}{3}$$

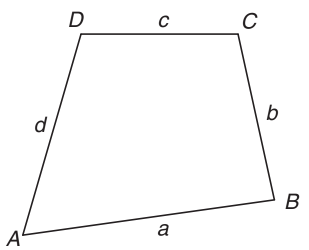


Figure 4.46

27. $A+B+C+D=\pi$

Hint: $A+B=\pi-(C+D)$

$\cos(A+B)+\cos(C+D)=0$

$\sum \cos A\cos C - \sum \sin A \sin C = \cos(A+C)+\cos(C+D)$
$+\cos(A+D)+\cos(B+C)+\cos(B+D)+\cos(A+B)=0$

28. See Fig. 4.47.

Hint: sine rule for a Δ.

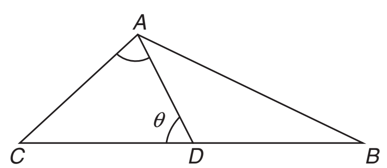


Figure 4.47

$$\Delta ACD: \frac{\sin 90}{\frac{a}{2}} = \frac{\sin\theta}{b} : \sin\theta = \frac{2b}{a}$$

In ΔABD

$$\frac{\sin(A-90)}{\frac{a}{2}} = \frac{\sin(\pi-\theta)}{c}$$

$$\Rightarrow \frac{-2\cos A}{a} = \frac{\sin\theta}{c}$$

$$\frac{2b}{a} = \frac{-2c\cos A}{a} = b + c\cos A = 0$$

$$\Rightarrow 3b^2 = a^2 - c^2$$

29. Hint: $a+b>c$

$$(\sqrt{a}+\sqrt{b})^2 > a+b > c \Rightarrow \sqrt{a}+\sqrt{b} > \sqrt{c}$$

Hence, $\sqrt{a}+\sqrt{b} > \sqrt{c}$, where a, b, c always positive.

30. $$\frac{a^2+b^2+c^2}{ab+bc+ca} = P$$

Hint: Use $(a-b)^2+(b-c)^2+(c-a)^2 \ge 0$ and $(a+b+c)>0$

$$(a-b)^2+(b-c)^2+(c-a)^2 \ge 0$$

$$\Rightarrow \frac{a^2+b^2+c^2}{ab+bc+ca} \ge 1 \quad (1)$$

$$a > c-b \Rightarrow a^2 > c^2+b^2-2bc \quad (2)$$

Similarly

$$b^2 > a^2+c^2-2ac \quad (3)$$

$$c^2 > a^2+b^2-2ab \quad (4)$$

Adding Eqs. (2)–(4) we get

$$\frac{a^2+b^2+c^2}{ab+bc+ca} < 2 \quad (5)$$

From Eqs. (1) and (5), we get $1 \le p < 2$.

31. See Fig. 4.48.

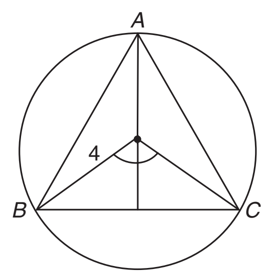


Figure 4.48

Hint: Angle at circumference = Half of centre

$$\Delta = \frac{1}{2}\times(2\sqrt{3}+6)\times 4 = 4(\sqrt{3}+3)$$

Angles of Δ: 75, 60, 45. Now,

$$\text{Area} = \frac{1}{2}ab\sin C = \frac{1}{2}\cdot 2R\cdot 2R\cdot \sin A\cdot \sin B\cdot \sin C$$

$$\Rightarrow \frac{1}{2}\,\frac{\sqrt{3}+1}{2\sqrt{2}}\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}\cdot 4R^2 = 4(3+\sqrt{3})$$

32. Hint: $(a^2+b^2-c^2)^2 = 2a^2b^2$

Now, $\dfrac{a^2+b^2-c^2}{2ab} = \pm\dfrac{1}{\sqrt{2}} = \cos C$

Hence, c can never be $\dfrac{\pi}{2}$.

33. Hint: Expand $\cos 3A + \cos 3B + \cos 3C = 1$

Now,

$$2\cos\frac{3}{2}(A+B)\cos\frac{3}{2}(A-B) - 2\sin^2\frac{3C}{2} = 0$$

$$\Rightarrow 4\sin\frac{3A}{2}\sin\frac{3B}{2}\sin\frac{3C}{2}=0$$

It means $\frac{3A}{2}$ or $\frac{3B}{2}$ or $\frac{3C}{2}=\pi$

Therefore, A or B or C is $\frac{2\pi}{3}$ obtuse angle triangle.

34. Hint: $\frac{\Delta}{s}:\frac{abc}{4\Delta}:\frac{\Delta}{s-a}$

Now,

$$\frac{1}{s}:\frac{abc}{4\Delta^2}:\frac{2}{a}=\frac{2}{3a}:\frac{a^3\cdot 4}{4\cdot\frac{3}{4}a^4}:\frac{2}{a} = \frac{1}{3}:\frac{2}{3}:1=1:2:3$$

35. Hint: $\frac{abc}{2\Delta}+\frac{\Delta}{s}=\frac{\Delta}{s-a}$

Now,

$$\frac{abc}{2\Delta}=\Delta\left(\frac{1}{s-a}-\frac{1}{s}\right)\Rightarrow bc=2(s-b)+(s-c)$$

$$\Rightarrow a^2=b^2+c^2\Rightarrow\Delta=\angle A=\frac{\pi}{2}$$

36. Hint: Sides are in ratio: 4:5:6.

Now, apply $\cos A:\cos B:\cos C=12:9:2$

37. Hint: Convert $\sin B$ and $\sin C$ to $\sin A$

$$\left(\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}\Rightarrow b\sin A=\sin B,\ c\sin A=\sin C\right)$$

Now,

$$(a+b+c)=\frac{6(\sin A+\sin B+\sin C)}{3}$$

$$\Rightarrow\frac{1+b+c}{2}=\sin A(1+b+c)$$

$$\Rightarrow\sin A=\frac{1}{2}\quad A=\frac{\pi}{6}$$

38. Hint: $r_1=\frac{\Delta}{(s-a)}$

Now,

$$\left(1-\frac{(s-b)}{s-a}\right)\left(1-\frac{(s-c)}{s-a}\right)=2$$

$$\Rightarrow(b-a)(c-a)=2(s-a)^2$$

$$\Rightarrow a^2=b^2+c^2:\angle A=\frac{\pi}{2}\text{ right-angled}$$

39. Hint: $A+B=\pi-C$: $\tan A+\tan B=\tan C$: $2\tan C=6$

Now, $\tan C=3$. So

$$\tan A=1,\ \tan B=2$$

40. Hint: $\sin C=3\sin B\Rightarrow C=90^\circ+B$

Now, $\sin(90^\circ+B)=3\sin B\Rightarrow\tan B=\frac{1}{3}$

41. Hint: $R=\frac{abc}{4\Delta}$

Now, $\Delta=\sqrt{36\times18\times12\times6}=6^3$. So

$$R=\frac{18\times24\times30}{4\times6^3}=15$$

42. See Fig. 4.49.

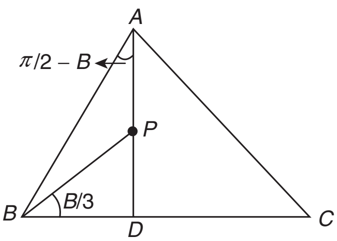


Figure 4.49

Hint: $\angle BPA=90^\circ+\frac{B}{3},\ \angle ABP=\frac{2B}{3}$

In ΔABP,

$$\frac{AP}{\sin\left(\frac{2B}{3}\right)}=\frac{AB}{\sin\left(90^\circ+\frac{B}{3}\right)}=\frac{BP}{\cos\frac{B}{3}}=AP=2c\sin\frac{B}{3}$$

43. Hint: $\cos A=\frac{b^2+c^2-a^2}{2bc}$; $a=1$.

Now,

$$r=\frac{\Delta}{s}=\frac{\frac{1}{2}bc\sin A}{\frac{a+b+c}{2}}=\frac{\sqrt{3}-1}{2}$$

44. $$\sin A+\sin B+\sin C=1+\sqrt{2}$$

and $$\cos A+\cos B+\cos B+\cos C=\sqrt{2}$$

Hint: Use odd one out rule

Take two angles $\frac{\pi}{4}$ and one angle $\frac{\pi}{2}$. This is satisfied.

$$B=\frac{\pi}{2},\ A=C=\frac{\pi}{4}$$

Therefore, it is right-angled isosceles.

45. $\tan\left(\frac{A-B}{2}\right)\cdot\cot\left(\frac{A+B}{2}\right)$

Hint: Use Napier's analogy,

$$\tan\left(\frac{A-B}{2}\right)=\left(\frac{a-b}{a+b}\right)\cdot\cot\frac{C}{2}$$

Now,

$$\tan\left(\frac{A-B}{2}\right)\cdot\cot\left(\frac{A+B}{2}\right)=\tan\left(\frac{A-B}{2}\right)\cdot\tan\frac{C}{2}=\frac{a-b}{a+b}$$

46. Hint: Convert into s: $\Delta=\frac{1}{2}bc\sin A$

Now,

$$\frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4b^2c^2}$$

$$= \frac{2s(2s-2a)(2s-2b)(2s-2c)}{4b^2c^2} = \frac{16\Delta^2}{4b^2c^2} = \sin^2 A$$

47. Hint: $r_1 = \dfrac{\Delta}{(s-a)}$

Now,

$$r_1 = 2r_2 = 3r_3 : \frac{1}{s-a} = \frac{2}{s-b} = \frac{3}{s-c}$$

$$\Rightarrow 3b = 3a - c \quad (1)$$

and $$a = 5b - 5c \quad (2)$$

From Eqs. (1) and (2), we get

$$b = \frac{4c}{3} \quad a = \frac{5b}{4}$$

Therefore,

$$a + b + c = 3b$$

48. Hint: Use cos A.

$$(b + a + c)(b + c - a) = 3bc$$

$$(b+c)^2 - a^2 = 3bc : \frac{b^2 + c^2 - a^2}{bc} = 1$$

$$\cos A = \frac{1}{2};\; A = \frac{\pi}{3}$$

49. Hint: $2(2R)^2 = a^2 + b^2 + c^2$

Now, use $\sin A = \dfrac{a}{2R}$.

$$2 = \left(\frac{a}{2R}\right)^2 + \left(\frac{b}{2R}\right)^2 + \left(\frac{c}{2R}\right)^2$$

$$\sin^2 A + \sin^2 B + \sin^2 C = 2$$

50. $PH = 2R' \cos P$, $QH = 2R' \cos Q$, $RH = 2R' \cos R$ (R': circumradius)

$$PH + QH + RH = 2R'(\cos P + \cos Q + \cos R) =$$

$$\frac{QR}{\sin P}\cos P + \frac{PR}{\sin Q}\cos Q + \frac{PQ}{\sin R}\cdot \cos R$$

51. Hint: $\tan\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$

Now,

$$\left(\frac{b+c-a}{2R}\right)\frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{\cos^2\left(\frac{A}{2}\right)} = \frac{(b+c-a)}{R}\tan\frac{A}{2}$$

$$\frac{2(s-a)}{R}\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{2\Delta}{s} = \frac{2rS}{SR} = \frac{2r}{R}$$

$$2r = R_{\text{maximum}} = 1$$

52. Hint: $\sin C = \dfrac{c}{a}\sin A$

Compare each option. We get option (B) correct.

53. Hint: $r_1 = \dfrac{\Delta}{(s-a)}$

Now,

$$(s-a)r_1 + (s-b)r_2 + (s-c)r_3 = 3\Delta = 3rs$$

54. Hint: $\angle B = 60° \; \angle A + \angle C = 120$

Hence, options (c) and (d) go automatically.

Now,

$$\frac{b}{c} = \frac{\sqrt{3}}{\sqrt{2}} \Rightarrow \frac{\sin B}{\sin A} = \frac{\sqrt{3}}{\sqrt{2}} \Rightarrow \sin C = \frac{1}{\sqrt{2}} \Rightarrow \angle C = 45°$$

Hence, $\angle A = 75°$.

55. Hint: Use cos A for both c_1 and c_2.

Now,

$$c_1c_2 = b^2 - a^2$$

$$\cos A = \frac{c_1 + c_2}{2b} \Rightarrow (c_1 + c_2) = 2b\cos A$$

Now,

$$(c_1 - c_2)^2 + (c_1 + c_2)^2 \cdot \tan^2 A$$

$$= (c_1 + c_2)^2 - 4c_1c_2 + (c_1 + c_2)^2 \tan^2 A$$

$$\Rightarrow (c_1 + c_2)^2 \sec^2 A - 4c_1c_2 = 4b^2 - 4b^2 + 4a^2 = 4a^2$$

56. Hint: $r_1r_2 + r_2r_3 + r_3r_1 = s^2$

$$4r_1r_2 + 4r_2r_3 + 4r_3r_1 = 4s^2 \Rightarrow 4d_1d_2 + 4d_2d_3 + 4d_3d_1 = 4s^2$$

57. See Fig. 4.50.

$$BD : DC = 1 : 3,$$

To calculate : $\dfrac{\sin\angle BAD}{\sin\angle CAD}$

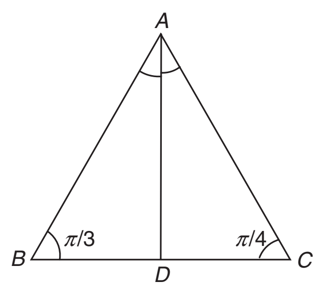


Figure 4.50

Hint: Apply sine rule

In ΔABD,

$$\frac{AD}{\sin\frac{\pi}{3}} = \frac{BD}{\sin\angle BAD} \quad (1)$$

In ΔCAD,

$$\frac{AD}{\sin\frac{\pi}{4}} = \frac{DC}{\sin\angle CAD} \quad (2)$$

From Eqs. (1) and (2), we get

$$\frac{\sqrt{3}}{2}\cdot\frac{BD}{\sin\angle BAD} = \frac{1}{\sqrt{2}}\cdot\frac{DC}{\sin\angle CAD}$$

$$\Rightarrow \frac{\sqrt{3}}{2}\cdot\left(\frac{BD}{DC}\right)\times\sqrt{2}=\frac{\sin\angle BAD}{\sin\angle CAD}$$

$$\Rightarrow \frac{\sqrt{3}}{2}\times\frac{1}{3}\times\sqrt{2}=\frac{\sin\angle BAD}{\sin\angle CAD}$$

$$\Rightarrow \frac{\sin\angle BAD}{\sin\angle CAD}=\frac{1}{\sqrt{6}}$$

58. See Fig. 4.51.

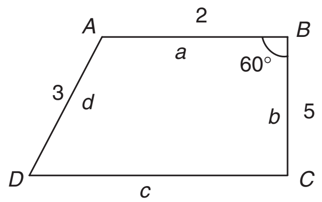


Figure 4.51

$$\cos B=\frac{a^2+b^2-c^2-d^2}{2(ab+cd)}$$

$$\Rightarrow \frac{1}{2}=\frac{4+25-c^2-9}{2(10+3c)}$$

$$\Rightarrow c^2+3c-10=0 \Rightarrow c=2$$

59. $3\sin x-4\sin^3 x=k \quad 0<k<1 \quad A>B$ (given)

Hint: Use $\sin\theta=\sin\alpha \Rightarrow \theta=n\pi+(-1)^n\alpha$

Now,

$$\sin 3x=k$$

$$\Rightarrow \sin 3A=\sin 3B$$

$$\Rightarrow 3A=n\pi+(-1)^n 3B$$

$$\Rightarrow A=\frac{n\pi}{3}+(-1)^n B$$

Therefore,

$$B+A=\frac{\pi}{3}\text{(only possibility)} \Rightarrow C=\frac{2\pi}{3}$$

60. $$\cos A=\frac{b^2+c^2-a^2}{2bc}$$

$$\Rightarrow 2k^2\cos A=b^2+\left(\frac{k^2}{b}\right)^2-a^2$$

$$\Rightarrow b^4-(a^2+2k^2\cos A)b^2+k^4=0$$

$$\Rightarrow b^2 \text{ is real}$$

Therefore, $D>0$, $(a^2+2k^2\cos A)^2-4k^4\geq 0$

$$\Rightarrow \left(a^2+2k^2\cdot 2\cos^2\frac{A}{2}\right)\left(a^2-2k^2\cdot 2\sin^2\frac{A}{2}\right)\geq 0$$

$$\Rightarrow a\geq 2k\sin A/2$$

Practice Exercise 2

1. See Fig. 4.52. Let triangle be right-angled at C. Then area $=\frac{1}{2}ab$, circumradius $R=\frac{c}{2}$ which are not necessarily integers. Again in square $ONCL$, $NC=OL=r$. We have

$$c=AB=AM+BM=AL+BN$$
$$=b-r+a-r\Rightarrow r=\frac{a+b-c}{2}$$

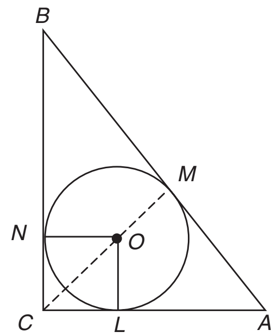


Figure 4.52

As $a^2+b^2=c^2$, we have following cases:

(i) If a and b are both odd or both even, then a^2+b^2 is even $\Rightarrow c^2$ is even
Therefore, c is even and so $(a+b)-c$ is even.

(ii) If one of a and b is odd and the other even, then a^2+b^2 is odd $\Rightarrow c^2$ is odd
Therefore, c is odd and so $(a+b)-c$ is even.
So, in every case if a, b, c are integers, we have $r=\frac{c+a-b}{2}=$ integer.

2. $$\text{Area of }\Delta ACB=\frac{1}{2}ab\sin 120°=\frac{\sqrt{3}}{4}ab \quad (1)$$

Also,

$$\text{area of }\Delta ACD=\frac{1}{2}bd\sin 60°=\frac{\sqrt{3}}{4}bd$$

$$\text{Area of }\Delta BCD=\frac{1}{2}ad\sin 60°=\frac{\sqrt{3}}{4}ad$$

Now

$$ar(\Delta ACD)+ar(\Delta BCD)=ar(\Delta ABC)$$

Therefore,

$$\frac{\sqrt{3}}{4}(a+b)d=\frac{\sqrt{3}}{4}ab\Rightarrow d=\frac{ab}{a+b}=\frac{h}{2}$$

3. Let the vertices of the triangle be $(\cos\theta_i, \sin\theta_i)$, $i=1, 2, 3$

$\Rightarrow$ Orthocentre is $((\cos\theta_1+\cos\theta_2+\cos\theta_3), (\sin\theta_1+\sin\theta_2+\sin\theta_3))$

$\Rightarrow$ Distance between the orthocentre and the circumcentre is

$$\sqrt{(\cos\theta_1+\cos\theta_2+\cos\theta_3)^2+(\sin\theta_1+\sin\theta_2+\sin\theta_3)^2}<3$$

4. Let h be the length of the third altitude falling on side c.
Let altitude of length 4 fall on side a and length 12 on side b.
Therefore,

$$\Delta=\frac{1}{2}\cdot 4\cdot a=\frac{1}{2}\cdot 12\cdot b=\frac{1}{2}\cdot h\cdot c$$

$$\Rightarrow a=\frac{2\Delta}{4},\ b=\frac{2\Delta}{12},\ c=\frac{2\Delta}{h}$$

Now,

$$c<a+b,\ c>|a-b|$$

$$\Rightarrow \frac{2\Delta}{h}<\frac{\Delta}{2}+\frac{\Delta}{6}\Rightarrow \frac{1}{h}<\frac{1}{4}+\frac{1}{12}$$

$$\Rightarrow \frac{2\Delta}{h}>\frac{\Delta}{2}-\frac{\Delta}{6}\Rightarrow \frac{1}{h}>\frac{1}{4}-\frac{1}{12}\Rightarrow 3<h<6$$

Therefore, greatest value possible is 5.

5. $$c \sin B \geq a \Rightarrow \sin C \sin B \geq \sin A$$

and $$a \sin C \geq b \Rightarrow \sin C \sin A \geq \sin B$$

$$\Rightarrow \sin C \geq 1 \Rightarrow \angle C \text{ is } \frac{\pi}{2}$$

6. Given $\alpha < \beta < \gamma < \delta$. Also $\sin\alpha = \sin\beta = \sin\gamma = \sin\delta = k$ and $\alpha, \beta, \gamma, \delta$ are smallest positive angles. Therefore

$$\beta = \pi - \alpha,\ \gamma = 2\pi + \alpha,\ \delta = 3\pi - \alpha$$

Now

$$\sin\beta = \sin\alpha \text{ and } \beta > \alpha$$
$$\sin\beta = \sin\gamma \text{ and } \gamma > \beta$$
$$\sin\gamma = \sin\delta \text{ and } \delta > \gamma$$

Putting these values in the given expansion, we get

$$2\left(\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}\right) = 2\sqrt{1 + \sin\alpha} = 2\sqrt{1+k}$$

7. $$R[\sin 2A + \sin 2B + \sin 2C] = 2r$$

$$\Rightarrow 4R \sin A \sin B \sin C = 8R \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\Rightarrow \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{1}{4}$$

8. See Fig. 4.53.

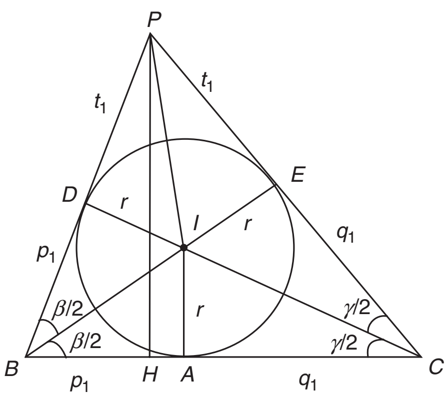


Figure 4.53

$$\angle PBA = \beta,\ \angle PCA = \gamma$$
$$AB = p_1,\ AC = q_1$$
$$p_1 q_1 = k^2$$

Let S touch BP and CP at E.
In ΔPEI

$$\angle EIP = \frac{1}{2}(\beta + \gamma)$$

$$t_1 = r\tan\frac{(\beta+\gamma)}{2} = \frac{(p_1+q_1)r^2}{p_1 q_1 - r^2}$$

Semi-perimeter of ΔBCP is

$$p_1 + q_1 + t_1 = p_1 + q_1 + \frac{(p_1+q_1)r^2}{p_1 q_1 - r^2} = \frac{p_1 q_1 (p_1 + q_1)}{p_1 q_1 - r^2}$$

$$\text{Area of } \Delta BCP = \frac{1}{2}(p_1 + q_1)\, PH = \frac{r p_1 q_1 (p_1 + q_1)}{p_1 q_1 - r^2}$$

$$PH = \frac{2k^2 r}{k^2 - r}$$

So, the locus of P is a line parallel to BC.

9. See Fig. 4.54.

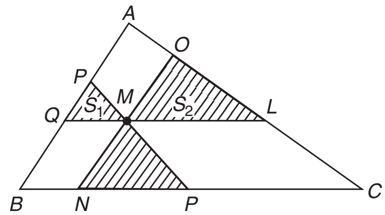


Figure 4.54

$$\frac{\text{Area of } ABC}{\text{Area of } PQM} = \frac{BC^2}{QM^2}$$

$$\frac{\text{Area of } ABC}{\text{Area of } MNP} = \frac{BC^2}{NP^2}$$

$$\frac{\text{Area of } ABC}{\text{Area of } MOL} = \frac{BC^2}{ML^2}$$

$$\sqrt{\text{Area of } \Delta ABC} = QM = \frac{BC\sqrt{S_1}}{\sqrt{S}}$$

$$PM = \frac{BC\sqrt{S_3}}{\sqrt{S}},\ ML = \frac{BC\sqrt{S_2}}{\sqrt{S}} \quad (QM = BN \text{ and } ML = PC)$$

$$BC = BC\left(\frac{\sqrt{S_1}}{\sqrt{S}} + \frac{\sqrt{S_2}}{\sqrt{S}} + \frac{\sqrt{S_3}}{\sqrt{S}}\right)$$

$$S = (\sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3})^2 \text{ where } S \text{ is area of } \Delta ABC$$

10. See Fig. 4.55. Point will lie inside the square $PQRS$

$$\text{Area of } PQRS = \frac{b}{\sqrt{2}} \cdot \frac{b}{\sqrt{2}} = \frac{b^2}{2} \text{ sq. units}$$

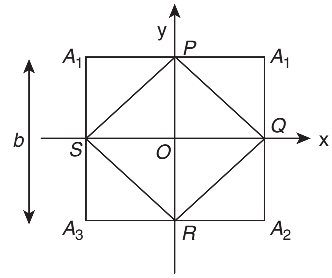


Figure 4.55

11. See Fig. 4.56.

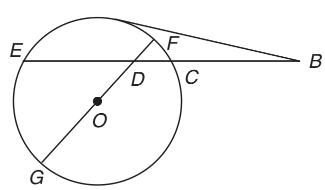


Figure 4.56

$$BC \times BE = AB^2 \Rightarrow 3 \times (6 + DE) = 36$$

Therefore, $DE = 6$.

Now $DE \times DC = DG \times DF$, so

$$6 \times 3 = (r+2)(r-2) \Rightarrow r^2 = 22$$

12. $$\sum \frac{\cos A}{\sin B \sin C} = \sum \frac{-\cos(B+C)}{\sin B \sin C}$$
$$= \sum (1 - \cot B \cot C) = 3 - \sum \cot A \cot B = 2$$

13. $$a + b > c$$

Adding c to both sides, we get

$$3b > 2c \Rightarrow \frac{b}{c} > \frac{2}{3}$$

Now,

$$b + c > a \Rightarrow b + c > 2b - c$$

and $$2c > b \Rightarrow \frac{b}{c} < 2$$

Therefore, $$\frac{b}{c} \in \left(\frac{2}{3}, 2\right)$$

14. Let centre of $ABCD$ be O. Then

$$OM = \frac{AA' + CC'}{2}, ON = \frac{BB' + DD'}{2}$$

$$\Rightarrow OM = 14 \text{ and } ON = 15$$

Also O, M, N are collinear. So $MN = 1$.

15. $$4 < k < 26$$

For triangle to be obtuse

Either $$11^2 + 15^2 < k^2$$

or $$11^2 + k^2 < 15^2$$

$$\Rightarrow k \in \{5, 6, 7, 8, 9, 10, 19, 20, 21, 22, 23, 24, 25\}$$

16. See Fig. 4.57. Since $r^2 + s^2 = e^2$, $e = 31$ or $e = 19$ is not possible. Therefore, e equal to 13, 20 or 25

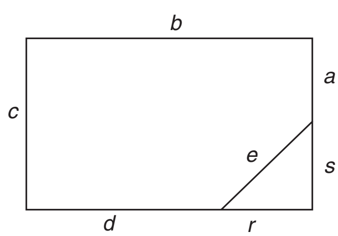


Figure 4.57

The possibilities for triplet $\{r, s, e\}$ are {5, 12, 13}, {12, 16, 20}, {15, 20, 25}, {7, 24, 25}.

Since 16, 15 and 24 do not appear among any of pair wise differences of 13, 19, 20, 25, 31, we have

$$a = 19, b = 25, c = 31, d = 20, e = 13.$$

Hence required area = 745 sq. units.

17. $$\frac{\sqrt{3}}{2\sin A \sin B \sin C} + \frac{1}{2}[\text{cosec}A \,\text{cosec}C + \text{cosec}A \,\text{cosec}B + \text{cosec}B \,\text{cosec}C]$$

Using AM $\geq$ GM and $\sin A \sin B \sin C \leq \frac{3\sqrt{3}}{8}$ we have

$$\geq \frac{\sqrt{3}}{2} \times \frac{8}{3\sqrt{3}} + \frac{3}{2}\left(\sqrt[3]{\text{cosec}^2 A \,\text{cosec}^2 B \,\text{cosec}^2 C}\right)$$

$$\geq \frac{4}{3} + \frac{3}{2}\frac{1}{(\sin A \sin B \sin C)^{2/3}}$$

$$\geq \frac{4}{3} + \frac{3}{2} \times \frac{4}{27^{1/3}}$$

$$= \frac{4}{3} + \frac{6}{3} = \frac{10}{3}$$

18. See Fig. 4.58. $\alpha + \gamma = \pi$ and $\beta + \gamma = \pi$,

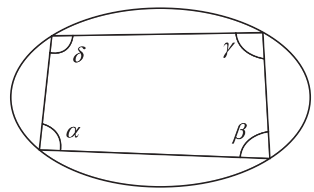


Figure 4.58

$$\Rightarrow \cos\alpha + \cos\beta + \cos\gamma + \cos\delta = 0$$
$$\Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta + 2\sum \cos\alpha\cos\beta = 0$$
$$\Rightarrow 2\sum \cos\alpha\cos\beta = -[2\cos^2\alpha + 2\cos^2\beta]$$
$$\Rightarrow 2 + \sum \cos\alpha\cos\beta = [\sin^2\alpha + \sin^2\beta]$$
$$= \sin^2\alpha + \sin^2\delta \text{ since, } (\delta = \pi - \beta)$$

19. $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$$\Rightarrow \tan C = \frac{\tan A + \tan B}{\tan A \tan B - 1}$$

$\Rightarrow \tan C$ must be rational

Since the slope of line AB is rational and $\tan A$ is also rational, therefore slope of line AC is also rational.

Similarly, slope of line BC is also rational.

$\Rightarrow C$ must be a rational point.

20. Given $\cos A$, $\cos B$, $\cos C$ are in AP. Then

$$2\cos B = \cos A + \cos C$$

$$\Rightarrow 2\left(1 - 2\sin^2\frac{B}{2}\right) = 2\cos\frac{A-C}{2}\sin\frac{B}{2}$$

$$\Rightarrow 1 - \sin^2\frac{B}{2} = \sin\frac{B}{2}\left[\sin\frac{B}{2} + \cos\left(\frac{A-C}{2}\right)\right]$$

$$\cos^2\frac{B}{2} = \sin\frac{B}{2}\left[\cos\left(\frac{A+C}{2}\right) + \cos\left(\frac{A-C}{2}\right)\right]$$

$$\Rightarrow \cot\frac{B}{2} = \frac{2\cos\frac{A}{2}\cos\frac{C}{2}}{\sin\left(\frac{A+C}{2}\right)}$$

$$\Rightarrow 2\tan\frac{B}{2} = \tan\frac{A}{2} + \tan\frac{C}{2}$$

$$\Rightarrow \cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2} \text{ are in HP}$$

$$\Rightarrow 2\left(s\tan\frac{B}{2}\right)=\left(s\tan\frac{A}{2}\right)+\left(s\tan\frac{C}{2}\right)$$

$$\Rightarrow 2r_2 = r_1 + r_2$$

$$\Rightarrow r_1, r_2, r_3 \text{ are in AP}$$

21. $YP \cdot YB = YZ \cdot YQ$ and $YA \cdot YP = YX \cdot YQ$

But $YA = YB$. Hence, $YX = YZ$

22. As a, b, c are in AP we have

$$a + c = 2b$$

Now,

$$a + b > c$$
$$b + c > a$$

and

$$a + b > c$$
$$\Rightarrow 3b > 2c$$

As $b + c > a$ we have

$$2c > b$$

Similarly for $\frac{b}{a}$.

Hence, (A) and (C) are the correct answers.

23. $c + a - b, b + c - a, a + b - c$ are all positive. Therefore

$$\frac{a}{c+a-b}+\frac{b}{a+b-c}+\frac{c}{b+c-a}$$

$$\geq 3\left[\frac{abc}{(c+a-b)(a+b-c)(b+c-a)}\right]^{1/3} \quad (1)$$

Also,

$$a^2 \geq a^2 - (b-c)^2 \Rightarrow a^2 \geq (a+b-c)(a-b+c)$$

Similarly,

$$b^2 \geq (b+c-a)(b-c+a)$$
$$c^2 \geq (c+a-b)(c-a+b)$$

Therefore,

$$a^2b^2c^2 \geq (a+b-c)^2(b+c-a)^2(c+a-b)^2$$

Thus,

$$abc \geq (a+b-c)(b+c-a)(c+a-b)$$

$$\Rightarrow \frac{abc}{(c+a-b)(a+b-c)(b+c-a)} \geq 1$$

So, from Eq. (1), we have

$$\frac{a}{c+a-b}+\frac{b}{a+b-c}+\frac{c}{b+c-a} \geq 3$$

24. $$l_1 = \frac{2\Delta}{a},\ l_2 = \frac{2\Delta}{b},\ l_3 = \frac{2\Delta}{c}$$

$\Rightarrow R = 2r$

$\Rightarrow$ Triangle is equilateral

$\Rightarrow l_1 = l_2 = l_3 = 3r$

25. We have

$$r(r_1r_2 + r_2r_3 + r_3r_1) = \frac{\Delta^3}{(s-a)(s-b)(s-c)}$$

and

$$r_1r_2r_3 = \frac{\Delta^3}{(s-a)(s-b)(s-c)}$$

Therefore, $x^2 - r(r_1r_2 + r_2r_3 + r_3r_1)x + r_1r_2r_3 - 1 = 0$ is satisfied by $x = 1$.

So, one root is $x = 1$ and other root is $r_1r_2r_3 - 1$.

26. $a = 2R \sin A$

$$\frac{da}{dA} = 2R\cos A,\ \frac{db}{dB} = 2R\cos B,\ \frac{dc}{dC} = 2R\cos C$$

Now,

$$\frac{da}{dA}+\frac{db}{dB}+\frac{dc}{dC} = 6$$

$$\Rightarrow 2R(\cos A + \cos B + \cos C) = 6$$

$$\Rightarrow 2R\, 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 6$$

$$\Rightarrow 2r = 6 \Rightarrow r = 3$$

27. $x_1 = 2R\cos A, x_2 = 2R\cos B, x_3 = 2R\cos C$

$\Rightarrow x_1 + x_2 + x_3 = 2R(\cos A + \cos B + \cos C) = 6$

Now,

$$\frac{x_1 + x_2 + x_3}{3} \geq (x_1x_2x_3)^{1/3}$$

$$\Rightarrow x_1x_2x_3 \leq 8$$

28. $\frac{dx_1}{dA} = -2R\sin A,\ \frac{dx_2}{dB} = -2R\sin B,\ \frac{dx_3}{dC} = -2R\sin C$

Also,

$$\frac{da}{dA} = 2R\cos A,\ \frac{db}{dB} = 2R\cos B,\ \frac{dc}{dC} = 2R\cos C$$

$$\frac{dx_1}{dA}+\frac{dx_2}{dB}+\frac{dx_3}{dC} = -(\tan A + \tan B + \tan C) \leq -3\sqrt{3}$$

29. Since $P(2, 4)$ is circumcentre and $O(1, 2)$ is orthocentre, $PE \perp BC$ and $OD \perp BC$.

Let R be the circumradius of ΔABC. Then

$$(OP)^2 = R^2(1 - 8\cos A\cos B\cos C)$$

Also,

$$PE = R\cos A = \frac{2}{\sqrt{5}}$$

$$OD = 2R\cos B\cos C = \frac{3}{\sqrt{5}}$$

Therefore,

$$5 = R^2 - 4\left(\frac{3}{\sqrt{5}}\right)^2$$

$$\Rightarrow R = \sqrt{\frac{61}{5}}$$

30. Now,

$$R\cos A = \frac{3}{\sqrt{5}}$$

$$\Rightarrow -R\cos B\cos C + R\sin B\sin C = \frac{3}{\sqrt{5}}$$

and

$$2R\cos B\cos C = \frac{3}{\sqrt{5}}$$

$$\Rightarrow R\sin B\sin C = \frac{9}{2\sqrt{5}}$$

Therefore, $\sin B\sin C = \frac{9}{2\sqrt{61}}$.

31. Distance of orthocentre from vertex $A = 2R\cos A = \dfrac{6}{\sqrt{5}}$.

32. See Fig. 4.59.

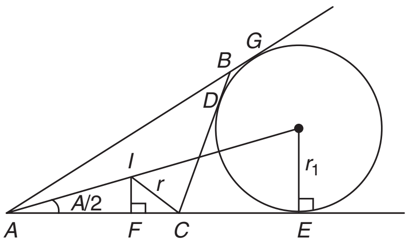


Figure 4.59

$$\frac{AI}{AI_1} = \frac{r}{r_1}$$

$$\Rightarrow II_1 = \left(\frac{r_1 - r}{r}\right) AI$$

$$= \frac{r\operatorname{cosec}\dfrac{A}{2}}{r}\left[4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right]$$

Therefore,

$$4R\left(\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}\right) \Rightarrow II_1 = 4R\sin\frac{A}{2}$$

Therefore, $II_1 + II_2 + II_3 = 4R\left(\sin\dfrac{A}{2} + \sin\dfrac{B}{2} + \sin\dfrac{C}{2}\right)$

33. $\alpha = \dfrac{BC}{2\sin\angle BIC} = \dfrac{a}{2\sin\left(\dfrac{\pi + A}{2}\right)} = \dfrac{a}{2\cos\dfrac{A}{2}}$

Therefore, $\beta = \dfrac{b}{2\cos\dfrac{B}{2}}$ and $\gamma = \dfrac{c}{2\cos\dfrac{C}{2}}$

$$\Rightarrow \alpha\beta\gamma = \frac{abc}{8\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}} = 2R^2 r$$

34. $\dfrac{II_1}{\alpha} + \dfrac{II_2}{\beta} + \dfrac{II_3}{\gamma} = \dfrac{4R\sin\dfrac{A}{2}}{2R\sin\dfrac{A}{2}} + \dfrac{4R\sin\dfrac{B}{2}}{2R\sin\dfrac{B}{2}} + \dfrac{4R\sin\dfrac{C}{2}}{2R\sin\dfrac{C}{2}} = 6$

35. Applying AM ≥ GM for $s - a, s - b, s - c, s - d$
We have

$$\frac{s - a + s - b + s - c + s - d}{4} \geq \sqrt{A}$$

$$\Rightarrow s \geq 2 \Rightarrow 2s \geq 4$$

$$\text{Area} = \frac{1}{2}d_1 d_2 \sin\theta \Rightarrow d_1 d_2 = \frac{2}{\sin\theta} \Rightarrow d_1 d_2 \geq 2$$

36. Also,

$$d_1 + d_2 \geq 2\sqrt{d_1 d_2} \geq 2\sqrt{2}$$

37. When the perimeter is minimum,

$$s - a = s - b = s - c = s - d \Rightarrow a = b = c = d$$

So $ABCD$ is a square.

Common Explanation Questions 38–40:

See Fig. 4.60.

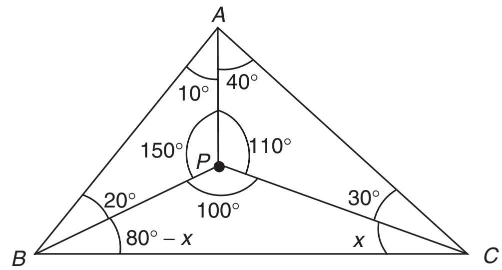


Figure 4.60

Now,

$$\frac{PA}{\sin 20^\circ} = \frac{PB}{\sin 10^\circ} \Rightarrow \frac{PA}{PB} = \frac{\sin 20^\circ}{\sin 10^\circ}$$

Similarly,

$$\frac{PB}{PC} = \frac{\sin x}{\sin(80^\circ - x)} \text{ and } \frac{PC}{PA} = \frac{\sin 40^\circ}{\sin 30^\circ}$$

Now,

$$\frac{PA}{PB}\cdot\frac{PB}{PC}\cdot\frac{PC}{PA} = \frac{2\sin x[\sin 50^\circ + \sin 30^\circ]}{\sin(80^\circ - x)}$$

$$\Rightarrow \sin(80^\circ - x) = 2\sin x\sin 50^\circ + \sin x$$

$$\Rightarrow x = 20^\circ$$

38. See Fig. 4.60. We have

$$\angle PBC = 80^\circ - 20^\circ = 60^\circ = \frac{\pi}{3}$$

39. $\angle BAC = \angle ACB = 50^\circ$

Therefore, ABC is an isosceles triangle.

40. $\angle ABC = 80^\circ$

Therefore, AC is the longest side, $AC > AB$.

41. (A) $\sin^2(A) + \sin^2(B) = \sin(A + B)$

$$\Rightarrow 1 - \frac{\cos 2A + \cos 2B}{2} = \sin C$$

$$\Rightarrow 1 - \cos(A + B)\cos(A - B) = \sin C$$

$$\Rightarrow \cos^2 C\cos^2(A - B) + 2\cos(C)\cos(A - B) = \cos^2 C$$

$\Rightarrow \cos C = 0 \Rightarrow$ The triangle is right angled.

(B) $\dfrac{bc}{2\cos A} = b^2 + c^2 - 2bc\cos A$

$$\Rightarrow \cos A = \frac{bc}{2a^2} \Rightarrow b^2c^2 = a^2(b^2 + c^2 - a^2)$$

$$\Rightarrow (a^2 - b^2)(a^2 - c^2) = 0 \Rightarrow a = b \text{ or } a = c.$$

Hence, the triangle is isosceles.

(C) $\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}=\sqrt{3}$

$$\Rightarrow\left(\tan\frac{A}{2}-\tan\frac{B}{2}\right)^2+\left(\tan\frac{B}{2}-\tan\frac{C}{2}\right)^2+\left(\tan\frac{C}{2}-\tan\frac{A}{2}\right)^2=0$$

The triangle is equilateral.

(D) If a, b, c are in AP. and h_a, h_b, h_c are in AP, where h_a, h_b, h_c are the altitudes, then $a=b=c$

The triangle is equilateral.

42. $$(G_3A)^2=(G_3M)\,(G_3C)$$

Therefore,

$$\left(\frac{c}{2}\right)^2=\frac{1}{3}(G_3C)^2$$

$$\Rightarrow\frac{c^2}{4}=\frac{1}{3}\left(\frac{2b^2+2a^2-c^2}{4}\right)$$

$$\Rightarrow a^2+b^2=2c^2 \quad (1)$$

So, $(C)\rightarrow(s)$.

Now in ΔAG_1C, we have

$$\sin\angle CAM=\frac{a\sin C}{2(AG_1)}$$

and in ΔBCG_2

$$\sin\angle CBM=\frac{b\sin C}{2(BG_2)}$$

Also,

$$AG_1=\sqrt{\frac{2b^2+2c^2-a^2}{4}}=\frac{\sqrt{3}}{2}b \quad \text{[using Eq. (1)]}$$

and $$BG_2=\sqrt{\frac{2a^2+2c^2-b^2}{4}}=\frac{\sqrt{3}}{2}a$$

So, $(A)\rightarrow(q)$.

Now,

$$\sin\angle CAM+\sin\angle CBM=\frac{2\Delta}{\sqrt{3}}\left(\frac{a^2+b^2}{a^2b^2}\right)$$

$$=\frac{1}{\sqrt{3}}\left(\frac{a^2+b^2}{ab}\right)\sin C \quad (2)$$

Also,

$$\cos C=\frac{a^2+b^2-c^2}{2ab}$$

$$\Rightarrow a^2+b^2=4ab\cos C$$

Now, from Eq. (2), we have

$$\sin\angle CAM+\sin\angle CBM=\frac{2}{\sqrt{3}}\sin 2C\le\frac{2}{\sqrt{3}}\left(\text{where } C=\frac{\pi}{4}\right)$$

From Eq. (1), we have

$$a^2+b^2=2c^2 \quad (3)$$

and from Eq. (2) and (3), we have

$$\sin\angle CAM+\sin\angle CBM=\frac{1}{\sqrt{3}ab}(a^2+b^2)\sin C$$
$$=\frac{1}{\sqrt{3}ab}(2c^2)\sin C \quad (4)$$

Also,

$$\frac{a^2+b^2-c^2}{2ab}=\cos C$$

Again using Eq. (1), we get

$$c^2=2ab\cos C \quad (5)$$

Putting this in Eq. (4), we get

$$\sin\angle CAM+\sin\angle CBM=\frac{1}{\sqrt{3}ab}(4ab\cos C)\sin C$$
$$=\frac{2}{\sqrt{3}}\sin 2C\le\frac{2}{\sqrt{3}}\left(\text{where}\angle C=\frac{\pi}{4}\right)$$

So, (B)$\rightarrow$ (p)

Again, $\sin\angle CAM+\sin\angle CBM$ is maximum when $C=\frac{\pi}{4}$

Also from Eq. (5), we have

$$c^2=2ab\cos C$$

which implies

$$\frac{c^2}{ab}=2\cos C=\sqrt{2}$$

So, (D)$\rightarrow$ (r).

43. Let centre of square (point of intersection of diagonals) be origin.

Vertices of square are $A(1, 1)$, $B(-1, 1)$, $C(-1, -1)$ and $D(1, -1)$. Radius of circumscribed or inscribed circles are $\sqrt{2}$ and 1, respectively.

Let any point P and Q on circumscribed and inscribed circles, respectively, be $(\sqrt{2}\cos\alpha, \sqrt{2}\sin\alpha)$ and $(\cos\beta, \sin\beta)$. Therefore,

$$\sum(PA)^2=\sum\{(\sqrt{2}\cos\alpha-1)^2+(\sqrt{2}\sin\alpha-1)^2\}=16$$
$$\sum(QA)^2=\sum\{(\cos\alpha-1)^2+(\sin\alpha-1)^2\}=12$$

44. Let A (0, 0), B (1, 0) and C (α, β) be the vertices of the triangle.

The product of the altitudes of the triangle is $\frac{|\beta|^3}{\alpha^2+\beta^2}$.

This is maximum for $\alpha=0$ and $\beta=\frac{1}{2}$.

The maximum values of 4 times the product of the altitudes $=2$.

5 Complex Number

5.1 Introduction

Whenever $\sqrt{x}$ is thought to give a real value, it has been, till now, insisted that $x \geq 0$. In other words, in the set of real numbers it is not possible to provide a value for the existence of $\sqrt{x}$ when $x < 0$. To make this possible, we extend the number system so as to include and cover yet another class of numbers called imaginary numbers.

Let us take the quadratic equation, $x^2 - 2x + 10 = 0$. The formal solution of this equation is $\frac{2 \pm \sqrt{4-40}}{2}$, that is, $1 \pm 3\sqrt{-1}$, which is not meaningful in the set of real numbers. So, a symbol $i = \sqrt{-1}$ is introduced.
The symbol i, is thought to possess the following properties:

1. It combines with itself and with real numbers satisfying the laws of algebra.
2. Whenever we come across -1 we may substitute i^2.

In the light of the foregoing, the roots of the equation discussed earlier may be taken as $1 + 3i$ and $1 - 3i$.

It is considered that 1 is the real part and 3 (or -3) is the imaginary part of the complex number $1 + 3i$ (or $1 - 3i$).

It has now to be mentioned that the "+" symbol appearing between 1 and $3i$ does not seem to be meaningful, though the following are true:

$$(x_1 + iy_1) \pm (x_2 + iy_2) = (x_1 \pm x_2) + i(y_1 \pm y_2) \qquad (5.1)$$

The real parts are added (or subtracted) separately and so in fact are the imaginary parts [Eq. (5.1)].

Also, $$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \qquad (5.2)$$

$$\frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + \frac{i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2} \qquad (5.3)$$

To make these operations really meaningful, a formal extension of the number system is presented in this lesson.

Illustration 5.1 If $x = -5 + 2\sqrt{-4}$, then find the value of $x^4 + 9x^3 + 35x^2 - x + 4$.

Solution:

$$x = -5 + 2 \cdot 2\sqrt{-1}$$

$$x = -5 + 4i \; (i = \sqrt{-1})$$

$$x + 5 = 4i$$

Squaring both sides, we get

$$x^2 + 10x + 25 = -16 \Rightarrow x^2 + 10x + 41 = 0$$

Now,

$$x^4 + 9x^3 + 35x^2 - x + 4 = (x^2 + 10x + 41)(x^2 - x + 4) - 160$$

We know,

$$x^2 + 10x + 41 = 0$$

$$\Rightarrow x^4 + 9x^3 + 35x^2 - x + 4 = 0 - 160 = -160$$

Hence, the value of given expression is -160.

5.2 Complex Numbers

A complex number, represented by an expression in the form $x + iy$ (where x, y are the real numbers), is considered to be an ordered pair (x, y) of two real numbers, combined to form a complex number, and an algebra is defined in the set of such numbers, represented by an ordered pair (x, y) to satisfy the following:

(addition) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

(subtraction) $(x_1, y_1) - (x_2, y_2) = (x_1 - x_2, y_1 - y_2)$

(multiplication) $(x_1, y_1) \times (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$

(division) $(x_1, y_1) \div (x_2, y_2) = \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}, \frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2} \right)$

For any real number α, $\alpha(x, y) = (\alpha x, \alpha y)$ and if $(x, y) = (x', y')$, then it must be $x' = x$, $y' = y$. In other words, the representation of a complex number in the form (x, y) has a uniqueness property; and for a complex number, it is not possible to have two different forms of the representation of the ordered pairs. In the light of the foregoing, it may be stated that the two representations (x, y) – in the ordered pair form – and $x + iy$ are indistinguishable.

Illustration 5.2 Find the sum and product of the two complex numbers $Z_1 = 2 + 3i$ and $Z_2 = -1 + 5i$.

Solution:

$$Z_1 + Z_2 = 2 + 3i + (-1 + 5i) = 2 - 1 + 8i = 1 + 8i$$

$$Z_1Z_2 = (2 + 3i)(-1 + 5i) = -2 + 15i^2 - 3i + 10i = -17 + 7i \quad (i^2 = -1)$$

Based on the above discussion, the following cases have been observed:

1. If $z = a + ib$, then the real part of $z = \text{Re}(z) = a$ and the imaginary part of $z = \text{Im}(z) = b$.
2. If $\text{Re}(z) = 0$, then the complex number is purely imaginary.
3. If $\text{Im}(z) = 0$, then the complex number is real.
4. The complex number $0 = 0 + 0i$ is both purely imaginary and real.

5. Two complex numbers are equal if and only if their real parts and imaginary parts are separately equal, that is, $a + ib = c + id \Leftrightarrow a = c$ and $b = d$.
6. There is no order relation between complex numbers, that is, $(a + ib) >$ or $< (c + id)$ is a meaningless expression.

Illustration 5.3 Express $\dfrac{1}{(1-\cos\theta + i\sin\theta)}$ in the form $a + ib$.

Solution:

$$\frac{1}{(1-\cos\theta+i\sin\theta)} = \frac{(1-\cos\theta)-i\sin\theta}{(1-\cos\theta+i\sin\theta)(1-\cos\theta-i\sin\theta)}$$

$$= \frac{\{(1-\cos\theta)-i\sin\theta\}}{\{(1-\cos\theta)^2+\sin^2\theta\}} = \frac{(1-\cos\theta)-i\sin\theta}{2-2\cos\theta}$$

$$= \frac{1-\cos\theta}{2(1-\cos\theta)} - \frac{i\cdot 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \frac{1}{2} - i.\cot\frac{\theta}{2}$$

5.3 Representation of a Complex Number

1. **Geometrical representation:** It is known, from the coordinate geometry, that the ordered pair (x, y) represents a point in the Cartesian plane.
   It is now seen that the ordered pair (x, y) considered as Z represents a complex number.
   It is therefore observed that to every complex number $Z \equiv (x, y)$, one can associate, a point $P \equiv (x, y)$ in the Cartesian plane. The point may be called to be a geometrical representation of Z. This association is a bijection – in the mapping language – whereby the correspondence between Z and P is ONE–ONE and ONTO. It is therefore possible to go over to a point from Z, or reversing the roles, come back to Z from the point.
2. **Argand diagram:** The graphical representation of a complex number $Z = (x, y)$ by a point $P(x, y)$ is called representation in the Argand diagram, which is also called Gaussian plane. In this representation, all complex numbers such as (2, 0), (3, 0), (–1, 0), $(\alpha, 0)$ with the imaginary part 0 will be represented by points on the x-axis. Since the real number α is represented as a complex number $(\alpha, 0)$, all real numbers will get marked on the x-axis. For this reason, the x-axis is called the real axis. Similarly, all purely imaginary numbers (with the real part 0) such as (0, 1), (0, 2), (0, –3), $(0, \beta)$ will be marked on the y-axis. Hence, the y-axis is also called the imaginary axis in this context. The Cartesian plane (two-dimensional plane) is also called the complex plane.
3. **Polar representation:** See Fig. 5.1. Let $P(x, y)$ be any point on the complex plane representing the complex number $z = (x, y)$, with $X'OX$ and $Y'OY$ as the axes of coordinates.
   Let $OP = r$ and $\angle XOP = \theta$ (measured anticlockwise).
   Then from ΔOMP, we find that

$$x = OM = r\cos\theta$$

and

$$y = MP = r\sin\theta$$

Thus,

$$z = (x, y) = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$$

Also,

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta \text{ by Euler's formula}$$

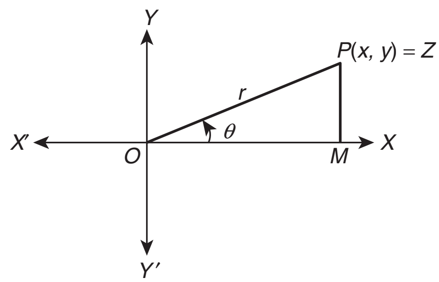


Figure 5.1

Thus, $z = r(\cos\theta + i\sin\theta)$ can be written as

$$z = re^{i\theta}$$

This form of representation of Z is called the *trigonometric form* or the *polar form* or the *modulus amplitude form*.
When z is written in the form $r(\cos\theta + i\sin\theta)$, r is called the modulus of z and is written as $|z|$, where

$$|z| = r = \sqrt{x^2 + y^2}$$

a non-negative number. $|z| = 0$ for the only number (0, 0).

Illustration 5.4 Represent the given complex numbers in the polar form:

(i) $(1+i\sqrt{3})^2/4i(1-i\sqrt{3})$ **(ii)** $\sin\alpha - i\cos\alpha$ (α acute)

(iii) $1+\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

Solution:

(i) $i(1-i\sqrt{3}) = i - i^2\sqrt{3} = \sqrt{3} + i$

Therefore,

$$\frac{(1+i\sqrt{3})^2}{4i(1-i\sqrt{3})} = \frac{(1+i\sqrt{3})^2}{4(\sqrt{3}+i)} = \frac{-2+2i\sqrt{3}}{4(\sqrt{3}+i)} = \frac{(-1+i\sqrt{3})(\sqrt{3}-i)}{2(\sqrt{3}+i)(\sqrt{3}-i)}$$

$$= \frac{-\sqrt{3}+\sqrt{3}+4i}{2(3+1)} = \frac{i}{2}$$

Now,

$$\frac{i}{2} \Rightarrow a + ib$$

$$\Rightarrow a = 0,\ b = \frac{1}{2}$$

$$a = r\cos\theta,\ b = r\sin\theta$$

$$\Rightarrow 0 = r\cos\theta,\ \frac{1}{2} = r\sin\theta$$

$$\Rightarrow \theta = \frac{\pi}{2}, r = \frac{1}{2}$$

So,

$$\frac{i}{2} = \frac{1}{2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

Hence,

$$\frac{(1+i\sqrt{3})^2}{4i(1-i\sqrt{3})} = \frac{1}{2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = \frac{1}{2}e^{i\pi/2}$$

(ii) Real part > 0 and imaginary part < 0.
So, argument of $\sin\alpha - i\cos\alpha$ is in the nature of a negative acute angle. Therefore,

$$\sin\alpha - i\cos\alpha = \cos\left(\alpha-\frac{\pi}{2}\right)+i\sin\left(\alpha-\frac{\pi}{2}\right)=e^{i\left(\alpha-\frac{\pi}{2}\right)}$$

(iii) $1+\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}=2\cos^2\frac{\pi}{6}+i\cdot 2\sin\frac{\pi}{6}\cos\frac{\pi}{6}$

$$=2\cos\frac{\pi}{6}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)=2\cos\frac{\pi}{6}\cdot e^{i\pi/6}$$

4. Vector representation of a complex number: In the Argand diagram, any complex number $Z = x + iy$ can be represented by a point P with coordinates (x, y). The vector $\overrightarrow{OP}$ can also be used to represent Z. The length of the vector $\overrightarrow{OP}$, that is, OP is the modulus of Z and the angle θ that OP makes with the positive x-axis is the amplitude of Z.

(a) *Representation of an algebraic operation on complex numbers*

(i) *Sum*: See Fig. 5.2(a). If two complex numbers Z_1 and Z_2 be represented by the points P and Q or by $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, then the sum $Z_1 + Z_2$ is represented by R or $\overrightarrow{OR}$, where $\overrightarrow{OR}=\overrightarrow{OP}+\overrightarrow{OQ}$ and OR is the diagonal of the parallelogram with OP and OQ as adjacent sides

(ii) *Difference*: See Fig. 5.2(b). $Z_1 - Z_2$ will be represented by $\overrightarrow{QP}$, where $\overrightarrow{QP}=\overrightarrow{OP}-\overrightarrow{OQ}$. $Z_2 - Z_1$ will be represented by $\overrightarrow{PQ}$.

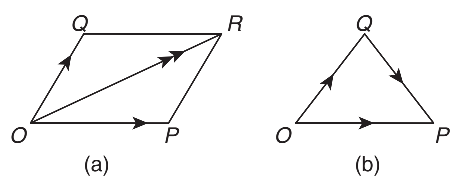


Figure 5.2

(iii) *Multiplication*: See Fig. 5.3. If $Z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $Z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then $Z_1 Z_2 = r_1 r_2\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}$.

If $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ represent Z_1 and Z_2, construct ΔOQR similar to ΔOEP where $OE = 1$.

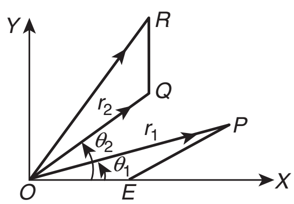


Figure 5.3

$$\angle XOR = \angle XOQ + \angle QOR = \angle XOQ + \angle EOP = \theta_2 + \theta_1$$

and

$$\frac{OR}{OQ}=\frac{OP}{OE}$$

Therefore,

$$OR = OP\cdot OQ = r_1 r_2 \qquad \{\text{as } OE = 1\}$$

Hence, $\overrightarrow{OR}$ represents the product $Z_1 Z_2$.

(iv) *Division*: See Fig. 5.4.

$$\frac{Z_1}{Z_2}=\left(\frac{r_1}{r_2}\right)\{\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)\}$$

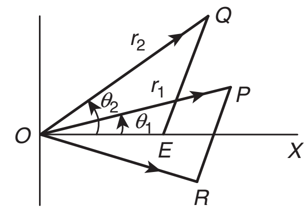


Figure 5.4

Construct ΔORP similar to ΔOEQ

Now

$$\frac{OR}{OE}=\frac{OP}{OQ}$$

$$\Rightarrow OR=\frac{r_1}{r_2}$$

and

$$\angle ROX = \angle ROP - \angle EOP = \angle EOQ - \angle EOP = \theta_2 - \theta_1$$

Therefore,

$$\angle XOR = \theta_1 - \theta_2$$

Hence, $\overrightarrow{OR}$ represents $\frac{Z_1}{Z_2}$.

Corollary 1: If Z_1, Z_2, Z_3 are the vertices of a triangle ABC described in the counter-clockwise direction, then

$$\frac{Z_3-Z_1}{Z_2-Z_1}=\frac{CA}{BA}(\cos\alpha+i\sin\alpha),$$

where

$$\alpha=\angle BAC$$

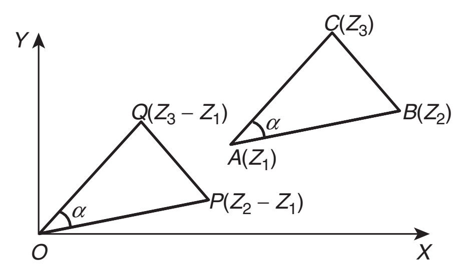


Figure 5.5

Let P and Q be the points representing $Z_2 - Z_1$ and $Z_3 - Z_1$, respectively. (See Fig. 5.5.)

Then, the triangles POQ and BAC are congruent.

Therefore,

$$\frac{CA}{BA}=\frac{OQ}{OP}$$

and

$$\angle QOP = \angle BAC = \alpha$$

Now $\frac{Z_3-Z_1}{Z_2-Z_1}$ has modulus $\frac{OQ}{OP}=\frac{CA}{BA}$ and argument $\angle POQ = \alpha$.

Hence,

$$\frac{Z_3 - Z_1}{Z_2 - Z_1} = \left(\frac{CA}{BA}\right)(\cos\alpha + i\sin\alpha)$$

In particular, if $\alpha = 90^\circ$ and $AB = AC$, then

$$\frac{Z_3 - Z_1}{Z_2 - Z_1} = i$$

$$\Rightarrow (Z_3 - Z_1) = i\,(Z_2 - Z_1)$$

Corollary 2: (See Fig. 5.6.) If Z_1, Z_2, Z_3 are represented by A, B, C, then

$$\arg\left(\frac{Z_3 - Z_1}{Z_2 - Z_1}\right) = \angle BAC$$

$$\arg\left(\frac{Z_2 - Z_3}{Z_1 - Z_3}\right) = \angle ACB$$

and

$$\arg\left(\frac{Z_1 - Z_2}{Z_3 - Z_2}\right) = \angle CBA$$

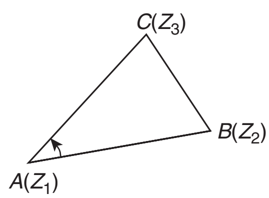


Figure 5.6

5.4 Conjugate of a Complex Number

The complex numbers $z = (a, b) = a + ib$ and $\bar{z} = (a, -b) = a - ib$, where a and b are the real numbers, $i = \sqrt{-1}$ and $b \neq 0$, are called to be complex conjugate of each other. (Here, the complex conjugate is obtained by just changing the sign of i).

Note that,

$$\text{sum} = (a + ib) + (a - ib) = 2a, \text{ which is real}$$

and

$$\text{product} = (a + ib)(a - ib) = a^2 - (ib)^2 = a^2 - i^2b^2$$
$$= a^2 - (-1)\,b^2 = a^2 + b^2, \text{ which is real}$$

Properties of conjugate

- $\overline{(\bar{z})} = z$
- $z = \bar{z} \Leftrightarrow z$ is real
- $z = -\bar{z} \Leftrightarrow z$ is purely imaginary
- $\text{Re}(z) = \text{Re}(\bar{z}) = \dfrac{z + \bar{z}}{2}$
- $\text{Im}(z) = \dfrac{z - \bar{z}}{2i}$
- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$
- $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}\ (z_2 \neq 0)$
- $z_1\overline{z_2} + \overline{z_1}z_2 = 2\text{Re}(\overline{z_1}z_2) = 2\text{Re}(z_1\overline{z_2})$
- $\overline{z^n} = (\bar{z})^n$
- If $z = f(z_1)$, then $\bar{z} = f(\overline{z_1})$

5.5 Modulus of a Complex Number

(See Fig. 5.7.) Modulus of a complex number $z = x + iy$ is a real number given by $|z| = \sqrt{x^2 + y^2}$. It is always non-negative and $|z| = 0$ only for $z = 0$, that is, origin of the Argand plane. Geometrically, it represents the distance of the point $z\,(x, y)$ from origin.

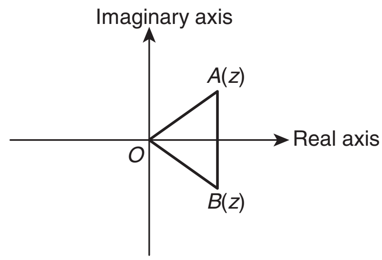


Figure 5.7

Properties of modulus

- $|z| \geq 0 \Rightarrow |z| = 0$ iff $z = 0$, and $|z| > 0$ iff $z \neq 0$.
- $-|z| \leq \text{Re}(z) \leq |z|$, and $-|z| \leq \text{Im}(z) \leq |z|$.
- $|z| = |\bar{z}| = |-z| = |-\bar{z}|$
- $z\bar{z} = |z|^2$
- $|z_1 z_2| = |z_1|\,|z_2|$
  In general, $|z_1 z_2 z_3 \ldots z_n| = |z_1|\,|z_2|\,|z_3| \ldots |z_n|$
- $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}\ (z_2 \neq 0)$
- $|z_1 \pm z_2| \leq |z_1| + |z_2|$
  In particular, if $|z_1 + z_2| = |z_1| + |z_2|$, then origin, z_1 and z_2 are collinear with origin at one of the ends.
- $|z_1 \pm z_2| \geq \big|\,|z_1| - |z_2|\,\big|$
  In particular, if $|z_1 - z_2| = \big|\,|z_1| - |z_2|\,\big|$, then origin, z_1 and z_2 are collinear with origin at one of the ends.
- $|z^n| = |z|^n$
- $\big|\,|z_1| - |z_2|\,\big| \leq |z_1| + |z_2|$
  Thus, $|z_1| + |z_2|$ is the greatest possible value of $|z_1 + z_2|$ and $\big|\,|z_1| - |z_2|\,\big|$ is the least possible value of $|z_1 + z_2|$.
- $|z_1 \pm z_2|^2 = (z_1 \pm z_2)(\overline{z_1} \pm \overline{z_2}) = |z_1|^2 + |z_2|^2 \pm (z_1\overline{z_2} + \overline{z_1}z_2)$ or
  $|z_1|^2 + |z_2|^2 \pm 2\text{Re}(z_1\overline{z_2})$
- $z_1\overline{z_2} + \overline{z_1}z_2 = 2|z_1|\,|z_2|\cos(\theta_1 - \theta_2)$ where $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$
- $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \dfrac{z_1}{z_2}$ is purely imaginary
- $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\{|z_1|^2 + |z_2|^2\}$
- $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$ where $a, b \in R$

- Unimodular, that is, unit modulus

If z is unimodular then $|z| = 1$. A unimodular complex number can always be expressed as $\cos\theta + i\sin\theta$, $\theta \in R$.

Note: $\frac{z}{|z|}$ is always a unimodular complex number if $z \neq 0$.

Some of the proofs are given as:

$|Z_1 Z_2| = |Z_1| \times |Z_2|$

Proof:

Let $Z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $Z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$
Then

$$Z_1Z_2 = r_1r_2\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\} = r(\cos\theta + i\sin\theta)$$

where

$$r = r_1 r_2 \text{ and } \theta = \theta_1 + \theta_2.$$

Therefore,

$$|Z_1 Z_2| = r = r_1 r_2 = |Z_1| \times |Z_2|$$

- $|Z_1 Z_2 \ldots Z_n| = |Z_1| \times |Z_2| \times |Z_3| \times \ldots \times |Z_n|$
  Proof follows by writing $Z_1 Z_2 \ldots Z_n$ as the product of $Z_1 Z_2 \ldots Z_{n-1}$ and Z_n and applying property (1) repeatedly.
- $|Z^n| = |Z|^n$
  Proof follows if we take $Z_1 = Z_2 = Z_3 = \ldots = Z_n$.

$\left|\frac{Z_1}{Z_2}\right| = \frac{|Z_1|}{|Z_2|}$

Proof:

Let $Z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $Z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$
Then

$$\frac{Z_1}{Z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)$$

$$\left(\text{since } \frac{1}{\cos\theta_2 + i\sin\theta_2} = \cos\theta_2 - i\sin\theta_2\right)$$

$$= \left(\frac{r_1}{r_2}\right)\{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)\}$$

$$= \left(\frac{r_1}{r_2}\right)\{\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\}$$

Hence,

$$\left|\frac{Z_1}{Z_2}\right| = \frac{r_1}{r_2} = \frac{|Z_1|}{|Z_2|}$$

First triangle inequality

$|Z_1| + |Z_2| \geq |Z_1 + Z_2|$

Proof:

$$|Z_1 + Z_2| = |r_1(\cos\theta_1 + i\sin\theta_1) + r_2(\cos\theta_2 + i\sin\theta_2)|$$

$$= |(r_1\cos\theta_1 + r_2\cos\theta_2) + i(r_1\sin\theta_1 + r_2\sin\theta_2)|$$

$$= \sqrt{(r_1\cos\theta_1 + r_2\cos\theta_2)^2 + (r_1\sin\theta_1 + r_2\sin\theta_2)^2}$$

$$= \sqrt{r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1-\theta_2)}$$

$$\leq \sqrt{r_1^2 + r_2^2 + 2r_1r_2}, \text{ since } \cos(\theta_1-\theta_2) \leq 1$$

Therefore,

$$|Z_1 + Z_2| \leq \sqrt{(r_1+r_2)^2}$$

$$\Rightarrow |Z_1 + Z_2| \leq r_1 + r_2$$

Thus,

$$|Z_1 + Z_2| \leq |Z_1| + |Z_2|$$

Note: Equality occurs only when $\theta_1 = \theta_2$, that is, when Z_1 and Z_2 have the same amplitude.

Second triangle inequality

$|Z_1 - Z_2| \geq |Z_1| - |Z_2|$

Proof:

$$Z_1 - Z_2 = r_1\cos\theta_1 - r_2\cos\theta_2 + i(r_1\sin\theta_1 - r_2\sin\theta_2)$$

Therefore,

$$|Z_1 - Z_2| = \sqrt{(r_1\cos\theta_1 - r_2\cos\theta_2)^2 + (r_1\sin\theta_1 - r_2\sin\theta_2)^2}$$

$$= \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1-\theta_2)}$$

$$\geq \sqrt{r_1^2 + r_2^2 - 2r_1r_2}, \text{ since } \cos(\theta_1-\theta_2) \leq 1$$

Therefore,

$$|Z_1 - Z_2| \geq \sqrt{(r_1-r_2)^2} = |r_1 - r_2|$$

$$|Z_1 - Z_2| \geq r_1 \sim r_2 = |Z_1| \sim |Z_2|$$

$|\bar{Z}| = |Z|$

Proof:

If $Z = x + iy$, then

$$|Z| = \sqrt{x^2 + y^2}$$

Now,

$$\bar{Z} = x - iy$$

$$\Rightarrow |\bar{Z}| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2 + y^2}$$

Therefore,

$$|\bar{Z}| = |Z|$$

Illustration 5.5 If $|z - 2 + i| \leq 2$, then find the greatest and the least value of $|z|$.

Solution: Given that

$$|z - 2 + i| \leq 2 \quad (1)$$

(Using $|z_1 - z_1| \geq ||z_1| - |z_2||$)

$$|z - 2 + i| \geq ||z| - |2 - i||$$

$$\Rightarrow |z - 2 + i| \geq ||z| - \sqrt{5}| \quad (2)$$

From Eqs. (1) and (2), we get

$$||z| - \sqrt{5}| \leq |z - 2 + i| \leq 2$$

Therefore,

$$||z| - \sqrt{5}| \leq 2$$

$$\Rightarrow -2 \leq |z| - \sqrt{5} \leq 2$$

$$\Rightarrow \sqrt{5} - 2 \leq |z| \leq \sqrt{5} + 2$$

Hence, the greatest value of $|z|$ is $\sqrt{5}+2$ and the least value of $|z|$ is $\sqrt{5}-2$.

Illustration 5.6 If $\left|Z+\frac{1}{Z}\right|=a$, where Z is a complex number and a is a positive real number, then find the greatest and least value of $|Z|$.

Solution: Let us first find greatest value of $|Z|$.

If $|Z|$ is greatest, then $\frac{1}{|Z|}$ is least. Hence,

$$|Z|>\frac{1}{|Z|}$$

Write $$a=\left|Z+\frac{1}{Z}\right|=\left|Z-\left(-\frac{1}{Z}\right)\right|\geq|Z|-\frac{1}{|Z|}$$

$$\Rightarrow |Z|^2-a|Z|-1\leq 0$$

Hence, $|Z|$ lies between the roots of the equation

$$|Z|^2-a|Z|-1=0.$$

Roots of the equation are $\frac{a\pm\sqrt{a^2+4}}{2}$.

Hence,

$$\frac{a-\sqrt{a^2+4}}{2}\leq|Z|\leq\frac{a+\sqrt{a^2+4}}{2} \quad (1)$$

It is known that $|Z|\geq 0$ while $\frac{a-\sqrt{a^2+4}}{2}$ is <0 and hence, Eq. (1) gets modified as

$$0\leq|Z|\leq\frac{a+\sqrt{a^2+4}}{2}$$

Thus, the greatest value of $|Z|$ is $\frac{a+\sqrt{a^2+4}}{2}$.

Now for the least value of $|Z|$.

In this case $\frac{1}{|Z|}$ is greatest and hence,

$$\frac{1}{|Z|}-|Z|>0$$

Write $$a=\left|Z+\frac{1}{Z}\right|=\left|\frac{1}{Z}-(-Z)\right|\geq\frac{1}{|Z|}-|Z|$$

$$\Rightarrow |Z|^2+a|Z|-1\geq 0$$

This is possible for all $|Z|$ lying outside the roots of $|Z|^2+a|Z|-1=0$

Roots of the equation are $\frac{-a\pm\sqrt{a^2+4}}{2}$, and of these $\frac{-a-\sqrt{a^2+4}}{2}$ is negative, hence $|Z|$ cannot be less than this negative value. Therefore,

$$|Z|\geq\frac{-a+\sqrt{a^2+4}}{2}$$

Thus, the least value of $|Z|$ is $\frac{-a+\sqrt{a^2+4}}{2}$.

Illustration 5.7 If Z_1 and Z_2 be two complex numbers such that $\left|\frac{Z_1-2Z_2}{2-Z_1\bar{Z}_2}\right|=1$ and $|Z_2|\neq 1$. What is the value of $|Z_1|$?

Solution:

$$|Z_1-2Z_2|=|2-Z_1\bar{Z}_2|$$

Therefore,

$$|Z_1-2Z_2|^2=|2-Z_1\bar{Z}_2|^2$$

$$\Rightarrow (Z_1-2Z_2)(\bar{Z}_1-2\bar{Z}_2)=(2-Z_1\bar{Z}_2)(2-\bar{Z}_1Z_2)$$

$$\Rightarrow Z_1\bar{Z}_1-2\bar{Z}_1Z_2-2Z_1\bar{Z}_2+4Z_2\bar{Z}_2=4-2Z_1\bar{Z}_2-2\bar{Z}_1Z_2+Z_1\bar{Z}_1Z_2\bar{Z}_2$$

$$\Rightarrow Z_1\bar{Z}_1+4Z_2\bar{Z}_2-4-Z_1\bar{Z}_1Z_2\bar{Z}_2=0$$

$$|Z_1|^2+4|Z_2|^2-|Z_1|^2|Z_2|^2-4=0$$

$$\Rightarrow (|Z_1|^2-4)(|Z_2|^2-1)=0$$

Since $$|Z_2|\neq 1 \Rightarrow |Z_1|^2=4$$

$$\Rightarrow |Z_1|=2$$

5.6 Argument of a Complex Number

If $z=x+iy=r(\cos\theta+i\sin\theta)$, where $r=\sqrt{x^2+y^2}$, then θ is called the argument of Z or the amplitude of Z. Since $x=r\cos\theta$ and $y=r\sin\theta$, θ is such that $\cos\theta=\frac{x}{\sqrt{x^2+y^2}}$ and $\sin\theta=\frac{y}{\sqrt{x^2+y^2}}$. Since there can be many values of θ satisfying these conditions, by convention, θ such that $-\pi<\theta\leq\pi$ is defined as the principal argument of Z and is denoted by arg Z. The argument of a complex number $a+ib$ is given by α, $\pi-\alpha$, $-\pi+\alpha$ or $-\alpha$ if $a+ib$ is in the first, second, third or fourth quadrant, respectively, where $\alpha=\tan^{-1}\left|\frac{b}{a}\right|$. For example,

- $Z=1+i=(1, 1)$ and is marked by point $P(1, 1)$ that lies in first quadrant. Therefore,
  $|Z|=\sqrt{2}$ and arg $Z=\pi/4$
- If $Z=1-i=(1,-1)$, then P lies in the fourth quadrant and $|Z|=\sqrt{2}$ and arg $Z=-\pi/4$.
- If $Z=-1+i=(-1, 1)$, then P lies in the second quadrant and arg $Z=\frac{3\pi}{4}$.
- If $Z=-1-i$, then P lies in the third quadrant and arg $Z=-\frac{3\pi}{4}$.
- Argument of all positive real numbers such as 1, 2, 3, $\frac{1}{2}$, … is 0 since they are marked on the positive x-axis. The argument of all negative real numbers such as $-1, -2, -3$, … is π since they are marked on negative x-axis. The argument of purely imaginary numbers such as $i, 2i, 3i$, … is $\frac{\pi}{2}$ since these are marked on the positive y-axis. The argument of purely imaginary numbers like $-i, -2i, -3i$, … is $-\frac{\pi}{2}$. Since these are marked on negative y-axis.

Illustration 5.8 Among the complex numbers z which satisfies $|z-25i| \le 15$, find the complex numbers z having

(i) least positive argument **(ii)** maximum positive argument
(iii) least modulus **(iv)** maximum modulus

Solution: The complex numbers z satisfying the condition $|z-25i| \le 15$ are represented by the points inside and on the circle of radius 15 and centre at the point $C(0, 25)$, Fig. 5.8.

The complex number having least positive argument and maximum positive arguments in this region are the points of contact of tangents drawn from origin to the circle.
Here,

$$\theta = \text{least positive argument}$$

and

$$\phi = \text{maximum positive argument}$$

Therefore, In ΔOCP,

$$OP = \sqrt{(OC)^2-(CP)^2} = \sqrt{(25)^2-(15)^2} = 20$$

and

$$\sin\theta = \frac{OP}{OC} = \frac{20}{25} = \frac{4}{5}$$

Therefore,

$$\tan\theta = \frac{4}{3}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{4}{3}\right)$$

Thus, complex number at P has modulus 20 and argument $\theta = \tan^{-1}\left(\frac{4}{3}\right)$.

Therefore,

$$Z_P = 20(\cos\theta + i\sin\theta) = 20\left(\frac{3}{5}+i\frac{4}{5}\right)$$

$$\Rightarrow Z_P = 12+16i$$

Similarly,

$$Z_Q = -12+16i$$

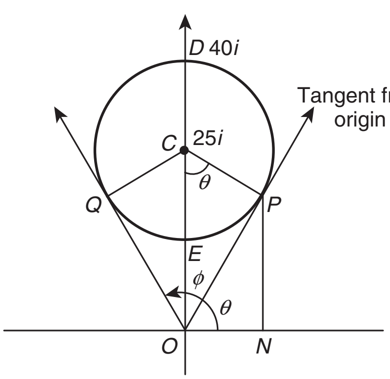


Figure 5.8

From the Fig. 5.8, E is the point with the least modulus and D is the point with the maximum modulus.
Hence,

$$Z_E = \overrightarrow{OE} = \overrightarrow{OC} - \overrightarrow{EC} = 25i - 15i = 10i$$

and

$$Z_D = \overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD} = 25i + 15i = 40i$$

Properties of arguments

- $\arg(z_1z_2) = \arg(z_1) + \arg(z_2) + 2k\pi$ ($k = 0$ or 1 or -1)
  In general $\arg(z_1z_2z_3 \ldots z_n) = \arg(z_1) + \arg(z_2) + \arg(z_3) + \cdots + \arg(z_n) + 2k\pi$ (where $k \in \boldsymbol{I}$)
- $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 + 2k\pi$ ($k = 0$ or 1 or -1)
- $\arg\left(\frac{z}{\bar{z}}\right) = 2\arg z + 2k\pi$ ($k = 0$ or 1 or -1)
- $\arg(z^n) = n\arg z + 2k\pi$ ($k = 0$ or 1 or -1)
- If $\arg\left(\frac{z_2}{z_1}\right) = \theta$, then $\arg\left(\frac{z_1}{z_2}\right) = 2k\pi - \theta$ where $k \in \boldsymbol{I}$.
- $\arg \bar{z} = -\arg z$
- If $\arg z = 0$, then z is real.

Note: Proper value of k must be chosen in above results so that arguments lies in $(-\pi, \pi]$.

All the above formulae are written on the basis of the principal argument.

Illustration 5.9 Let z, z_0 be two complex numbers. It is given that $|z| = 1$ and the numbers z, z_0, $z\bar{z}_0$, 1 and 0 are represented in an Argand diagram by the points P, P_0, Q, A and the origin O, respectively. Show that the triangles POP_0 and AOQ are congruent. Hence, or otherwise, prove that $|z - z_0| = |z\bar{z}_0 - 1|$.

Solution: See Fig. 5.9. Given $OA = 1$ and $|z| = 1$

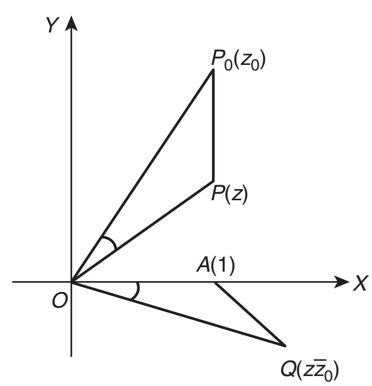


Figure 5.9

Therefore,

$$OP = |z-0| = |z| = 1$$

So,

$$OP = OA$$
$$OP_0 = |z_0 - 0| = |z_0|$$

and

$$OQ = |z\bar{z}_0 - 0| = |z\bar{z}_0| = |z|\,|\bar{z}_0| = 1\,|z_0| = |z_0|$$

Therefore,

$$OP_0 = OQ$$

and $\angle P_0OP = \arg\left(\frac{z_0 - 0}{z - 0}\right) = \arg\left(\frac{z_0}{z}\right)$

$$= \arg\left(\frac{\bar{z}z_0}{z\bar{z}}\right) = \arg\left(\frac{\bar{z}z_0}{|z|^2}\right) = \arg\left(\frac{\bar{z}z_0}{1}\right) = -\arg(\overline{\bar{z}z_0})$$

$$= -\arg(z\bar{z}_0) = \arg\left(\frac{1}{z\bar{z}_0}\right)$$

$$= \arg\left(\frac{1-0}{z\bar{z}_0 - 0}\right) = \angle AOQ$$

Thus, the triangles POP_0 and AOQ are congruent.
Also,

$$PP_0 = AQ$$
$$\Rightarrow |z - z_0| = |z\bar{z}_0 - 1|$$

Illustration 5.10 If $\arg(Z^{1/3}) = \frac{1}{2}\arg(z^2 + \bar{z}z^{1/3})$, then find the value of $|z|$.

Solution:

We have

$$\arg(z^{1/3}) = \frac{1}{2}\arg(z^2 + \bar{z}z^{1/3})$$

$$\Rightarrow 2\arg(z^{1/3}) = \arg(z^2 + \bar{z}z^{1/3})$$

$$\Rightarrow \arg(z^{2/3}) = \arg(z^2 + \bar{z}z^{1/3})$$

$$\Rightarrow \arg(z^2 + \bar{z}z^{1/3}) - \arg(z^{2/3}) = 0$$

$$\Rightarrow \arg\left(\frac{z^2 + \bar{z}z^{1/3}}{z^{2/3}}\right) = 0$$

$$\Rightarrow \arg\left(z^{4/3} + \frac{\bar{z}}{z^{1/3}}\right) = 0$$

$$\Rightarrow z^{4/3} + \frac{\bar{z}}{z^{1/3}} \text{ is real} \Rightarrow \operatorname{Im}\left(z^{4/3} + \frac{\bar{z}}{z^{1/3}}\right) = 0$$

$$\Rightarrow \frac{\left(z^{4/3} + \frac{\bar{z}}{z^{1/3}}\right) - \overline{\left(z^{4/3} + \frac{\bar{z}}{z^{1/3}}\right)}}{2i} = 0$$

$$\Rightarrow z^{4/3} + \frac{\bar{z}}{z^{1/3}} = (\bar{z})^{4/3} + \frac{(\bar{\bar{z}})}{(\bar{z})^{1/3}}$$

$$\Rightarrow z^{4/3} + \frac{(\bar{z})(\bar{z})^{1/3}}{|z|^{2/3}} = (\bar{z})^{4/3} + \frac{z(z)^{1/3}}{|z|^{2/3}}$$

Since

$$[z^{1/3}(\bar{z})^{1/3} = (z\bar{z})^{1/3} = |z|^{2/3}]$$

$$\Rightarrow z^{4/3} - (\bar{z})^{4/3} - \frac{1}{|z|^{2/3}}((z)^{4/3} - (\bar{z})^{4/3}) = 0$$

$$\Rightarrow \{z^{4/3} - (\bar{z})^{4/3}\}\left[1 - \frac{1}{|z|^{2/3}}\right] = 0$$

Therefore,

$$|z|^{2/3} = 1 \qquad (\text{since } z \neq \bar{z})$$
$$\Rightarrow |z| = 1$$

Illustration 5.11 If $|Z| \le 1$ and $|W| \le 1$, then show that $|Z - W|^2 \le (|Z| - |W|)^2 + (\arg Z - \arg W)^2$

Solution: Let

$$Z = |Z|(\cos\theta + i\sin\theta)$$

and $W = |W|(\cos\varphi + i\sin\varphi)$

$$|Z - W|^2 = (|Z|\cos\theta - |W|\cos\phi)^2 + (|Z|\sin\theta - |W|\sin\phi)^2$$

$$= |Z|^2(\cos^2\theta + \sin^2\theta) + |W|^2(\cos^2\phi + \sin^2\phi)$$
$$-2|Z||W|(\cos\theta\cos\phi + \sin\theta\sin\phi)$$

$$= |Z|^2 + |W|^2 - 2|Z||W|\cos(\theta - \phi)$$

$$= (|Z| - |W|)^2 + 2|Z||W|(1 - \cos(\theta - \phi))$$

$$= (|Z| - |W|)^2 + 4|Z||W|\sin^2\left(\frac{\theta - \phi}{2}\right)$$

$$\le (|Z| - |W|)^2 + (\theta - \phi)^2$$

As $|Z| \le 1, |W| \le 1$, and $\sin^2(\theta - \phi) \le \left(\frac{\theta - \phi}{2}\right)^2$.

Hence,

$$|Z - W|^2 \le (|Z| - |W|)^2 + (\arg Z - \arg W)^2$$

Your Turn 1

1. If $(a + 2b) - i(2a - b) = 2i - 6$, then find a and b. **Ans.** $a = -2, b = -2$
2. Find the value of $\sum_{k=1}^{4n+7} i^k$. **Ans.** -1
3. If $a = \frac{1+i}{\sqrt{2}}$, then prove that the value of a^{1929} is also equal to $\frac{1+i}{\sqrt{2}}$.
4. If $z_1 = 2 - 3i$ and $z_2 = 2 + 7i$, then find $|z_1 - z_2|$ and $\arg(z_1 - z_2)$. **Ans.** $|z_1 - z_2| = 10$ and $\arg(z_1 - z_2) = -\pi/2$
5. What is the polar form of $z = 1 - i\sqrt{3}$? **Ans.** $z = 2e^{i(-\pi/3)}$
6. If $a + ib = \frac{(2+3i)^2}{2+i}$, then find a and b. **Ans.** $a = \frac{2}{5}, b = \frac{29}{5}$
7. Find the value of $i^{13} + i^{14} + i^{15} + i^{16}$. **Ans.** 0
8. Find the least non-zero positive integer n such that $\left(\frac{1+i}{1-i}\right)^n = 1$. **Ans.** $n = 4$
9. If $X + iY = (x + iy)^{1/3}$, then prove that $4(X^2 - Y^2) = \frac{x}{X} + \frac{y}{Y}$.

10. If $|z_1| = |z_2| = |z_3| \ldots = |z_n| = 1$, then prove that

$$|z_1 + z_2 + z_3 + \cdots + z_n| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \cdots + \frac{1}{z_n}\right|$$

5.7 De Moivre's Theorem

For any rational number n, the value or one of the values of $(\cos\theta + \sin\theta)^n$ is $(\cos n\theta + \sin n\theta)$. The following may also be noted:

1. $(\cos\theta + i\sin\theta)^{-n} = (\cos n\theta - \sin n\theta) = (\cos\theta - i\sin\theta)^n$
2. $(\cos\theta + i\sin\theta)^n = (\cos n\theta + \sin n\theta) = (\cos\theta - i\sin\theta)^{-n}$
3. If $x + \frac{1}{x} = 2\cos\theta$, and if the equation is solved for x, then

$$x = \cos\theta + i\sin\theta = e^{i\theta} \Rightarrow \frac{1}{x} = \cos\theta - i\sin\theta = e^{-i\theta}$$

or

$$x = \cos\theta - i\sin\theta = e^{-i\theta}$$

$$\Rightarrow \frac{1}{x} = \cos\theta + i\sin\theta = e^{i\theta}$$

Illustration 5.12 If $2\cos\theta = x + \frac{1}{x}$ and $2\cos\phi = y + \frac{1}{y}$, prove the following:

(i) $x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\theta + n\varphi)$

(ii) $\frac{x^m}{y^n} + \frac{y^n}{x^m} = 2\cos(m\theta - n\varphi)$

Solution:

(i) Given $x + \frac{1}{x} = 2\cos\theta \Rightarrow x^2 - 2x\cos\theta + 1 = 0$. Solving this, $x = \cos\theta \pm i\sin\theta$.

In fact, if $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$. It may also be noted that $x + \frac{1}{x} = 2\cos\theta$ is symmetrical w.r.t. $\frac{1}{x}$ and hence if one root is the value for x, the other root is $\frac{1}{x}$ and vice versa.

Similarly, given that $2\cos\theta = y + \frac{1}{y}$, we have $y = \cos\varphi + i\sin\varphi$

Therefore,

$$x^m = (\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta;$$

and

$$y^n = (\cos\varphi + i\sin\varphi)^n = \cos n\varphi + i\sin n\varphi$$

$$x^m y^n = (\cos m\theta + i\sin m\theta)(\cos n\varphi + i\sin n\varphi)$$

$$= \cos(m\theta + n\varphi) + i\sin(m\theta + n\varphi)$$

and

$$\frac{1}{x^m y^n} = \cos(m\theta + n\varphi) - i\sin(m\theta + n\varphi)$$

By adding we get

$$x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\theta + n\varphi)$$

(ii) By similar reasoning

$$\frac{x^m}{y^n} + \frac{y^n}{x^m} = 2\cos(m\theta - n\varphi)$$

Illustration 5.13 If n be a positive integer, prove that

$$(1+i)^{2n} + (1-i)^{2n} = \begin{cases} 0 & \text{if } n \text{ be odd} \\ 2^{n+1} & \text{if } \frac{n}{2} \text{ be even} \\ -2^{n+1} & \text{if } \frac{n}{2} \text{ be odd} \end{cases}$$

Solution:

$$(1+i)^{2n} = 2^n\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{2n} = 2^n\left(\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2}\right)$$

$$(1-i)^{2n} = 2^n\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)^{2n} = 2^n\left(\cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right)$$

Therefore,

$$(1+i)^{2n} + (1-i)^{2n} = 2^n\left(\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2} + \cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right)$$

$$= 2^{n+1}\cos\left(\frac{n\pi}{2}\right)$$

If n be odd $= 2m + 1$, then

$$\text{RHS} = 2^{n+1}\cos(2m+1)\frac{\pi}{2} = 0$$

If n be even and $\frac{n}{2}$ also even so that $n = 4k$, then

$$\text{RHS} = 2^{n+1}\cos(2k\pi) = 2^{n+1}$$

If $\frac{n}{2}$ is odd, then

$$\text{RHS} = 2^{n+1}\cos\left(\frac{n\pi}{2}\right) = -2^{n+1}$$

If $z = r(\cos\theta + i\sin\theta)$, and n is a positive integer, then

$$z^{1/n} = r^{1/n}\left[\cos\frac{2k\pi + \theta}{n} + i\sin\frac{2k\pi + \theta}{n}\right], k = 0, 1, 2, \ldots, n-1$$

5.8 Roots of Unity

1. **Cube roots of unity**
   Consider the cubic (third degree) equation

$$x^3 = 1 = \cos 0 + i\sin 0 = \cos 2k\pi + i\sin 2k\pi$$

Therefore,

$$x = \sqrt[3]{1} = (\cos 2k\pi + i\sin 2k\pi)^{1/3}$$

$$= \cos\left(\frac{2k\pi}{3}\right) + i\sin\left(\frac{2k\pi}{3}\right)$$

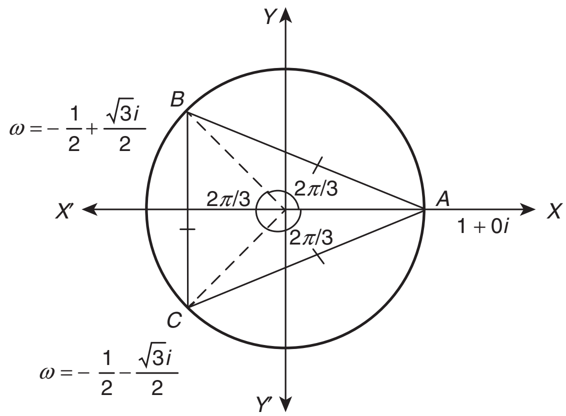


Figure 5.10

(See Fig. 5.10.) To get three roots of the cubic equation, we give $k = 0$, giving the real root, $\cos 0 + i \sin 0 = 1$

$k = 1$, giving one imaginary root, $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \omega$

$k = 2$, giving the other imaginary root, $\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \omega^2$

It is said that 1, ω, ω^2 are the three cubic roots of unity satisfying

(a) $1 + \omega + \omega^2 = 0$

(b) $\omega^3 = 1$

(c) 1, ω, ω^2 are represented respectively by points A, B, C lying on the unit circle $|Z| = 1$ and forming the corners of an equilateral triangle with each side of length $\sqrt{3}$.

Illustration 5.14 If α, β, γ are the roots of $x^3 - 3x^2 + 3x + 7 = 0$ (and ω is the cube roots of unity), then find the value of $\frac{\alpha-1}{\beta-1} + \frac{\beta-1}{\gamma-1} + \frac{\gamma-1}{\alpha-1}$.

Solution: We have

$$x^3 - 3x^2 + 3x + 7 = 0$$
$$\Rightarrow (x-1)^3 + 8 = 0$$
$$\Rightarrow (x-1)^3 = (-2)^3$$
$$\Rightarrow \left(\frac{x-1}{-2}\right)^3 = 1$$
$$\Rightarrow \frac{x-1}{-2} = (1)^{1/3}$$
$$= 1, \omega, \omega^2 \text{ (cube roots of unity)}$$

Therefore,

$$x = -1, 1 - 2\omega, 1 - 2\omega^2$$

Here,

$$\alpha = -1, \beta = 1 - 2\omega, \gamma = 1 - 2\omega^2$$

So,

$$\alpha - 1 = -2, \beta - 1 = -2\omega, \gamma - 1 = -2\omega^2$$

Then

$$\frac{\alpha-1}{\beta-1} + \frac{\beta-1}{\gamma-1} + \frac{\gamma-1}{\alpha-1} = \left(\frac{-2}{-2\omega}\right) + \left(\frac{-2\omega}{-2\omega^2}\right) + \left(\frac{-2\omega^2}{-2}\right)$$
$$= \frac{1}{\omega} + \frac{1}{\omega} + \omega^2$$
$$= \omega^2 + \omega^2 + \omega^2 = 3\omega^2$$

2. Some useful results

$(x^3 + y^3) = (x + y)(x + \omega y)(x + \omega^2 y)$

$(x^3 - y^3) = (x - y)(x - \omega y)(x - \omega^2 y)$

$(x^3 + y^3 + z^3 - 3xyz) = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$

Illustration 5.15 If $a = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \cdots\infty$,

$$b = x + \frac{x^4}{4!} + \frac{x^7}{7i} + \cdots\infty$$

and $$c = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \cdots\infty$$

Then, find the value of $a^3 + b^3 + c^3 - 3abc$.

Solution:

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$$
$$= e^x e^{\omega x} e^{\omega^2 x}$$
$$= e^{x(1+\omega+\omega^2)}$$
$$= e^0 = 1$$

3. n^{th} Roots of Unity

Generally, the n^{th} degree equation $x^n = 1$ has 'n' n^{th} roots of unity given by

$$\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}, \cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n}, \cdots, \cos\frac{2(n-1)\pi}{n} + i\sin\frac{2(n-1)\pi}{n}$$

that is,

1, α, α^2, …, α^{n-1} satisfying

(a) $1 + \alpha + \alpha^2 + \cdots + \alpha^{n-1} = 0$

(b) $\alpha^n = 1$

(c) 1, α, …, α^{n-1} represent n points in the Argand plane situated on the unit circle $|Z| = 1$ and forming the corners of a regular n sides polygon. (See Fig. 5.11.)

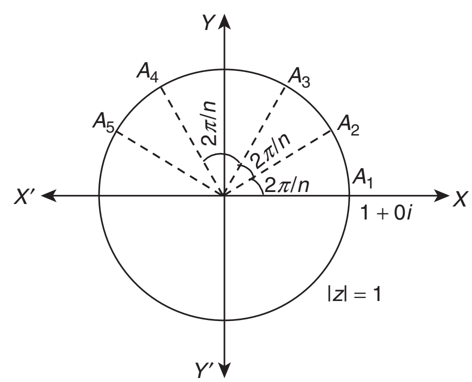


Figure 5.11

As the sum of n^{th} roots of unity = 0

$$\Rightarrow \sum_{k=0}^{n-1} \alpha^k = 0$$

$$\Rightarrow \sum_{k=0}^{n-1} \left(\cos\left(\frac{2k\pi}{n}\right) + i\left(\sin\frac{2k\pi}{n}\right)\right) = 0$$

$$\Rightarrow \sum_{k=0}^{n-1} \cos\left(\frac{2k\pi}{n}\right) = 0$$

and

$$\sum_{k=0}^{n-1} \sin\left(\frac{2k\pi}{n}\right) = 0$$

Generally, equation like $x^n = a + ib$ can be solved by using this method.

First write $a + ib = r\,[\cos\theta + i\sin\theta] = r\,[\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)]$ and hence the n n^{th} roots of $x^n = a + ib$ are $\sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right)$ where $k = 0, 1, 2, \ldots, (n-1)$.

Illustration 5.16 Solve $2\sqrt{2}\,x^5 = (\sqrt{3} - 1) + i(\sqrt{3} + 1)$.

Solution:

$$2\sqrt{2}\,x^5 = (\sqrt{3} - 1) + i\,(\sqrt{3} + 1)$$

$$x^5 = \left(\frac{\sqrt{3} - 1}{2\sqrt{2}}\right) + i\left(\frac{\sqrt{3} + 1}{2\sqrt{2}}\right)$$

$$x^5 = \cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}$$

$$\left(\text{Since, } \cos\frac{5\pi}{12} = \cos 75^\circ = \frac{\sqrt{3} - 1}{2\sqrt{2}};\ \sin\frac{5\pi}{12} = \sin 75^\circ = \frac{\sqrt{3} + 1}{2\sqrt{2}}\right)$$

$$\Rightarrow x^5 = \cos\left(2k\pi + \frac{5\pi}{12}\right) + i\sin\left(2k\pi + \frac{5\pi}{12}\right)$$

Therefore, the five roots of the given equation are

$$x = \cos\left(\frac{2k\pi + \frac{5\pi}{12}}{5}\right) + i\sin\left(\frac{2k\pi + \frac{5\pi}{12}}{5}\right) (k = 0, 1, 2, 3, 4)$$

Illustration 5.17 If $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$ are nth roots of unity, then prove that

(a) $(1 - \alpha)(1 - \alpha^2) \ldots (1 - \alpha^{n-1}) = n$

(b) $\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\sin\frac{3\pi}{n}\cdots\sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}, n \geq 2$

Solution: If $1, \alpha, \alpha^2 \ldots \alpha^{n-1}$ are roots of $x^n = 1$, then

$$x^n - 1 = (x - 1)(x - \alpha)(x - \alpha^2) \ldots (x - \alpha^{n-1})$$

$$(x - \alpha)(x - \alpha^2) \ldots (x - \alpha^{n-1}) = \frac{x^n - 1}{x - 1} = 1 + x + x^2 + \cdots + x^{n-1}$$

Put $x = 1$, then

$$(1 - \alpha)(1 - \alpha^2) \ldots (1 - \alpha^{n-1}) = n$$

Also,

$$\alpha^k = e^{\frac{i2k\pi}{n}}$$

$$\Rightarrow \left|1 - \alpha^k\right| = \left|2\sin\frac{k\pi}{n}\right|$$

Taking modulus of the first result, we get

$$|1 - \alpha|\,\left|1 - \alpha^2\right| \cdots \left|1 - \alpha^{n-1}\right| = |n|$$

$$\Rightarrow \left(2\sin\frac{\pi}{n}\right)\left(2\sin\frac{2\pi}{n}\right)\cdots\left(2\sin\frac{(n-1)\pi}{n}\right) = n$$

$$\Rightarrow \sin\frac{\pi}{n}\sin\frac{2\pi}{n}\cdots\sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}$$

The n^{th} Root of Unity

Let x be the n^{th} root of unity. Then

$$x^n = 1 = \cos 2k\pi + i\sin 2k\pi \text{ (where } k \text{ is an integer)}$$

$$\Rightarrow x = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} \qquad k = 0, 1, 2, \ldots, n-1$$

Let $\alpha = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. Then the n^{th} roots of unity are α^t $(t = 0, 1, 2, \ldots, n-1)$, that is, the n^{th} roots of unity are $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$.

Sum of the Roots

$$1 + \alpha + \alpha^2 + \cdots + \alpha^{n-1} = \frac{1 - \alpha^n}{1 - \alpha} = 0$$

$$\Rightarrow \sum_{k=0}^{n-1} \cos\frac{2k\pi}{n} = 0 \text{ and } \sum_{k=0}^{n-1} \sin\frac{2k\pi}{n} = 0$$

Thus, the sum of the roots of unity is zero.

Product of the Roots

$$\alpha.\alpha^2.\ldots.\alpha^{n-1} = (-1)^n(-1) = (-1)^{n+1}$$

5.9 Rotation Theorem

1. **Coni method:** This method gives the angle between two intersecting lines.

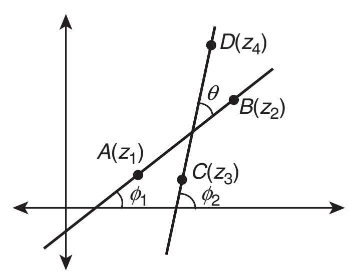


Figure 5.12

See Fig. 5.12. Let z_1, z_2, z_3 and z_4 be complex numbers representing points A, B, C and D, respectively. Then

$$\overrightarrow{AB} = z_2 - z_1$$

$$\overrightarrow{CD} = z_4 - z_3$$

Let $\arg \overrightarrow{AB} = \phi_1$ and $\arg \overrightarrow{CD} = \phi_2$. Then angle of intersection

$$\theta = \phi_2 - \phi_1 = \arg \overrightarrow{CD} - \arg \overrightarrow{AB}$$

$$= \arg(z_4 - z_3) - \arg(z_2 - z_1) = \arg\left(\frac{z_4 - z_3}{z_2 - z_1}\right)$$

(a) If $\theta = 0$ or $\pm\pi$, then $\left(\dfrac{z_4 - z_3}{z_2 - z_1}\right)$ is real. Points are collinear as the two lines coincide. It follows that if $\left(\dfrac{z_4 - z_3}{z_2 - z_1}\right)$ is real, points are collinear.

(b) If $\theta = \pm\dfrac{\pi}{2}$, then $\left(\dfrac{z_4 - z_3}{z_2 - z_1}\right)$ is purely imaginary. It follows that if $\left(\dfrac{z_4 - z_3}{z_2 - z_1}\right)$ is purely imaginary, then the line joining z_1, z_2 is perpendicular to the line joining z_3, z_4.

(c) (See Fig. 5.13.) Hence, the angle between the lines passing through z_2 and z_3 and intersecting at z_1 is given by

$$\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \theta$$

Also,

$$z = |z|\, e^{i\theta}$$

$$\Rightarrow \frac{z_3 - z_1}{z_2 - z_1} = \left|\frac{z_3 - z_1}{z_2 - z_1}\right| e^{i\theta}$$

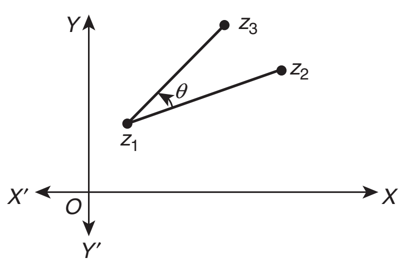


Figure 5.13

Illustration 5.18 *ABCD* is a rhombus. Its diagonals *AC* and *BD* intersect at *M* such that *BD* = 2*AC*. If the points *D* and *M* represent the complex number 1 + *i* and 2 – *i*, respectively, then find the complex number(s) representing *A*.

Solution: See Fig. 5.14. Let *A* be *z*. The position *MA* can be obtained by rotating *MD* anticlockwise through an angle $\dfrac{\pi}{2}$; simultaneously the length gets halved.

Therefore,

$$z - (2-i) = \frac{1}{2}\left[(1+i) - (2-i)\right]e^{i\pi/2}$$

$$= \frac{1}{2}(1+i-2+i)(i)$$

$$= \frac{1}{2}(-2-i) = -1 - \frac{1}{2}i$$

$$\Rightarrow z = -1 - \frac{1}{2}i + 2 - i = 1 - \frac{3i}{2}$$

Another position of *A* corresponds to *A* and *C* getting interchanged and in that the complex number of *A* is

$$1 + \frac{1}{2}i + 2 - i = 3 - \frac{1}{2}i$$

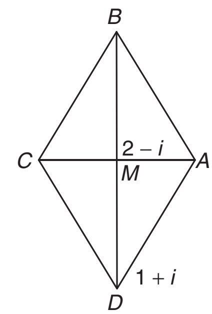


Figure 5.14

Therefore, the complex number of *A* is either $1 - \dfrac{3i}{2}$ or $3 - \dfrac{1}{2}i$.

Illustration 5.19 See Fig. 5.15. Show that the triangles whose vertices are Z_1, Z_2, Z_3 and a, b, c (Z_1, Z_2, Z_3 and a, b, c are complex) are similar if $\begin{vmatrix} Z_1 & a & 1 \\ Z_2 & b & 1 \\ Z_3 & c & 1 \end{vmatrix} = 0$.

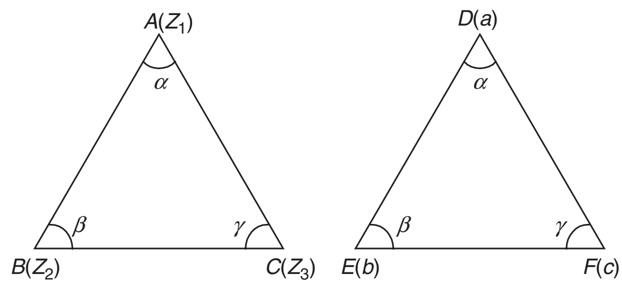


Figure 5.15

Solution: The two triangles are similar if

$$\frac{AB}{DE} = \frac{BC}{EF}$$

and

$$\angle ABC = \angle DEF = \beta \text{ (say)}$$

So,

$$\frac{Z_1 - Z_2}{Z_3 - Z_2} = \frac{AB}{BC}(\cos\beta + i\sin\beta)$$

Similarly,

$$\frac{a-b}{c-b} = \frac{DE}{EF}(\cos\beta + i\sin\beta)$$

Therefore,

$$\frac{Z_1 - Z_2}{Z_3 - Z_2} = \frac{a-b}{c-b}$$

$$\Rightarrow \begin{vmatrix} Z_1 - Z_2 & a-b \\ Z_3 - Z_2 & c-b \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} Z_1 - Z_2 & a-b & 0 \\ Z_2 & b & 1 \\ Z_3 - Z_2 & c-b & 0 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} Z_1 & a & 1 \\ Z_2 & b & 1 \\ Z_3 & c & 1 \end{vmatrix} = 0 \text{ adding } R_2 \text{ to } R_1 \text{ and } R_2 \text{ to } R_3$$

2. **Condition for four points to be concyclic:** See Fig. 5.16. Four points z_1, z_2, z_3 and z_4 in the Argand plane are concyclic if and only if

$$\arg\left(\frac{z_1 - z_3}{z_2 - z_3}\right) = \arg\left(\frac{z_1 - z_4}{z_2 - z_4}\right) = \theta \text{ (say)}$$

Applying the coni method, we get

$$\frac{z_1 - z_3}{z_2 - z_3} = \left|\frac{z_1 - z_3}{z_2 - z_3}\right| e^{i\theta}$$

$$\frac{z_1 - z_4}{z_2 - z_4} = \left|\frac{z_1 - z_4}{z_2 - z_4}\right| e^{i\theta}$$

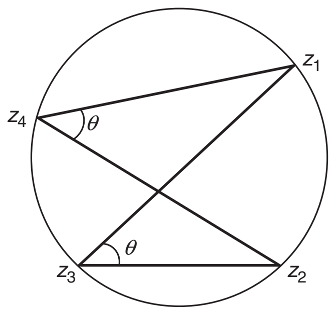


Figure 5.16

Solving the above two equations to eliminate θ, we get

$$\frac{z_1 - z_3}{z_2 - z_3} \cdot \frac{z_2 - z_4}{z_1 - z_4} = \left|\frac{z_1 - z_3}{z_2 - z_3}\, \frac{z_2 - z_4}{z_1 - z_4}\right|$$

This is possible only if the expression on the left-hand side is real (it may be positive or negative, depending upon whether the points are considered in a cyclic order or not).

3. **Complex number as a rotating arrow in the Argand plane:**
   (a) See Fig. 5.17. If a complex number z_1 is rotated in the anticlockwise sense by an angle θ and let z_2 be its new position, then

$$z_1 = re^{i\phi} \text{ and } z_2 = re^{i(\theta + \phi)} \text{ (as } |z_1| = |z_2| = r)$$

$$\Rightarrow z_2 = z_1 e^{i\theta}$$

Clearly, the multiplication of z with $e^{i\theta}$ rotates vector $\overrightarrow{OP}$ through an angle θ in the anticlockwise sense. Similarly, the multiplication of z with $e^{-i\theta}$ will the rotate vector $\overrightarrow{OP}$ in the clockwise sense.

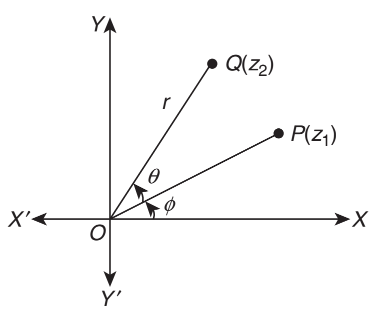


Figure 5.17

(b) See Fig. 5.18. Let z_1, z_2, z_3 be the affixes of three points A, B, C such that $AC = AB$ and $\angle CAB = \theta$. Then $\overrightarrow{AC} = z_3 - z_1$ will be obtained by rotating $\overrightarrow{AB} = z_2 - z_1$ through an angle θ in the anticlockwise sense and therefore,

$$(z_3 - z_1) = (z_2 - z_1)e^{i\theta}$$

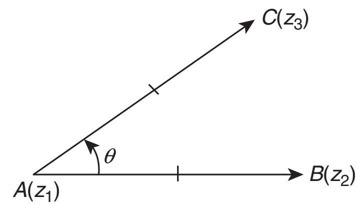


Figure 5.18

Illustration 5.20 Complex numbers Z_1, Z_2, Z_3 are the vertices A, B and C, respectively, of an isosceles right-angled triangle with $\angle C = 90°$. Show that $(Z_1 - Z_2)^2 = 2(Z_1 - Z_3)(Z_3 - Z_2)$ or equivalently $Z_1^2 + Z_2^2 + 2Z_3^2 = 2Z_1Z_3 + 2Z_2Z_3$.

Solution: See Fig. 5.19. It is seen that when CA is turned anticlockwise through an angle 90°, the position of CB is obtained. Lengthwise, $CA = CB$ because the triangle is isosceles.

Therefore,

$$Z_2 - Z_3 = (Z_1 - Z_3)\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

Squaring both sides, we get

$$(Z_2 - Z_3)^2 + (Z_1 - Z_3)^2 = 0$$

$$\Rightarrow Z_1^2 + Z_2^2 + 2Z_3^2 = 2Z_1Z_3 + 2Z_2Z_3$$

which is the second result.

To get the first from the second, we have

$$Z_1^2 + Z_2^2 = 2Z_1Z_3 + 2Z_2Z_3 - 2Z_3^2$$

$$Z_1^2 + Z_2^2 - 2Z_1Z_2 = 2Z_1Z_3 + 2Z_2Z_3 - 2Z_3^2 - 2Z_1Z_2$$

$$\Rightarrow (Z_1 - Z_2)^2 = 2(Z_1 - Z_3)(Z_3 - Z_2)$$

which is the desired form of the result.

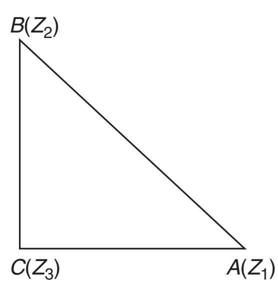


Figure 5.19

(c) See Fig. 5.20. In the above case if $AB \neq AC$, then we consider the rotation of unit vectors as

$$\frac{z_3 - z_1}{|z_3 - z_1|} = \frac{z_2 - z_1}{|z_2 - z_1|} e^{i\theta}$$

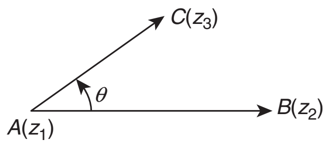


Figure 5.20

This concept has also been explained in terms of the coni method earlier.

Illustration 5.21 See Fig. 5.21. The points P, Q and R represent the complex numbers Z_1, Z_2 and Z_3, respectively, and the angles of the triangle PQR at Q and R are both $\frac{\pi}{2}-\frac{\alpha}{2}$. Prove that

$$(Z_3-Z_2)^2 = 4(Z_3-Z_1)(Z_1-Z_2)\sin^2\left(\frac{\alpha}{2}\right)$$

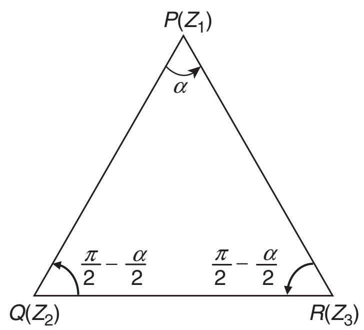


Figure 5.21

Solution: QP is obtained from QR by a rotation counter-clockwise through an angle $\frac{\pi}{2}-\frac{\alpha}{2}$, and the length PQ is different from the length of QR. Therefore,

$$Z_1-Z_2=\frac{PQ}{QR}(Z_3-Z_2)\left\{\cos\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)+i\sin\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)\right\}$$

Similarly,

$$Z_1-Z_3=\frac{PR}{QR}(Z_2-Z_3)\left\{\cos\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)-i\sin\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)\right\}$$

Multiplying the two equations

$$(Z_1-Z_2)(Z_1-Z_3)=\frac{PQ\cdot PR}{QR^2}(Z_3-Z_2)(Z_2-Z_3)\left\{\begin{array}{l}\cos^2\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)\\+\sin^2\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)\end{array}\right\}$$

Now,

$$\frac{QR}{\sin\alpha}=\frac{PQ}{\cos\frac{\alpha}{2}}=\frac{PR}{\cos\frac{\alpha}{2}} \quad \text{(By sine rule)}$$

Therefore,

$$\frac{PQ\cdot PR}{QR^2}=\frac{\cos^2\frac{\alpha}{2}}{4\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}}$$

So,

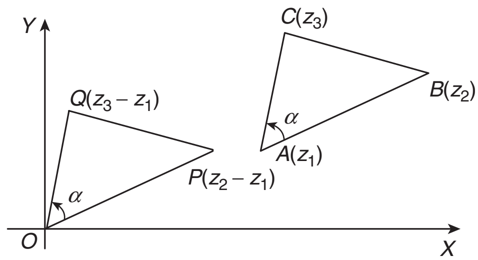
$$(Z_1-Z_2)(Z_1-Z_3)4\sin^2\frac{\alpha}{2}=(Z_3-Z_2)(Z_2-Z_3)$$

$$\Rightarrow (Z_3-Z_2)^2=4(Z_3-Z_1)(Z_1-Z_2)\sin^2\frac{\alpha}{2}$$

Concept of rotation: If z_1, z_2, z_3, are the three vertices of a triangle ABC described in the counter-clockwise sense, then

$$\frac{z_3-z_1}{z_2-z_1}=\frac{OQ}{OP}(\cos\alpha+i\sin\alpha)=\frac{CA}{BA}\cdot e^{i\alpha}=\frac{|z_3-z_1|}{|z_2-z_1|}\cdot e^{i\alpha}$$

Note that $\arg(z_3-z_1)-\arg(z_2-z_1)=\alpha$ is the angle through which OP must be rotated in the anticlockwise direction so that it becomes parallel to OQ.



5.10 Theory of Equations with Complex Coefficients

An n^{th} degree equation with complex coefficients $a_n, a_{n-1}, \ldots, a_0$ is given as

$$a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0=0$$

It has n roots say $\alpha_1, \alpha_2, \ldots \alpha_n$, and

$$\sum\alpha_1=-\frac{a_{n-1}}{a_n}$$

$$\sum\alpha_1\alpha_2=+\frac{a_{n-2}}{a_n}$$

$$\alpha_1\alpha_2\ldots\alpha_n=(-1)^n\frac{a_0}{a_n}$$

In the case of quadratic equations with complex coefficients having non-zero imaginary part, the discriminant has no role for the existence of roots.

Illustration 5.22 The roots z_1, z_2, z_3 of the equation $x^3+3ax^2+3bx+c=0$, in which a, b, c are complex numbers, corresponding to the points A, B, C on the Gaussian plane. Find the centroid of the triangle ABC and show that it will be equilateral if $a^2=b$.

Solution: Since z_1, z_2, z_3 are the roots of $x^3+3ax^2+3bx+c=0$. We have

$$z_1+z_2+z_3=-3a$$

$$\Rightarrow \frac{z_1+z_2+z_3}{3}=-a$$

and

$$z_1z_2+z_2z_3+z_3z_1=3b$$

Hence, the centroid of the triangle ABC is the point with affix $-a$. Now, the triangle will be equilateral if

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

$$\Rightarrow (z_1 + z_2 + z_3)^2 = 3(z_1z_2 + z_2z_3 + z_3z_1) \Rightarrow (-3a)^2 = 3(3b)$$

Therefore, the condition is $a^2 = b$.

Illustration 5.23 Find the value of $|Z|$ from the equation $2Z^3 - 3Z^2 - 18iZ + 27i = 0$.

Solution:

$$2Z^3 - 3Z^2 - 18iZ + 27i = 0$$

$$Z^2(2Z - 3) - 9i(2Z - 3) = 0$$

$$(2Z - 3)(Z^2 - 9i) = 0$$

Therefore,

$$2Z - 3 = 0 \Rightarrow |Z| = 3/2$$

or

$$Z^2 = 9i \Rightarrow |Z| = 3$$

5.11 Logarithms of a Complex Number

Let
$$\log_e(x + iy) = \alpha + i\beta \quad (5.1)$$
Suppose
$$x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta} \quad (5.2)$$
then
$$x = r\cos\theta, y = r\sin\theta$$
so that
$$r = \sqrt{(x^2 + y^2)}$$
and
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$
From Eq. (5.2) we get
$$\log_e(x + iy) = \log_e(re^{i\theta}) = \log_e r + \log_e e^{i\theta}$$
$$= \log_e r + i\theta$$
$$= \log_e\sqrt{(x^2 + y^2)} + i\tan^{-1}\left(\frac{y}{x}\right)$$
or
$$\log_e(z) = \log_e|z| + i \text{ amp } z$$

So, the general value of $\log(z) = \log_e|z| + 2n\pi i \ (-\pi < \text{amp}(z) < \pi)$.

Illustration 5.24 If $\sin(\log i^i) = a + ib$, then find a and b. Hence, find $\cos(\log i^i)$.

Solution:

$$a + ib = \sin(\log i^i) = \sin(i\log i)$$
$$= \sin[i(\log|i| + i \text{ amp } i)]$$
$$= \sin[i(\log 1 + i\pi/2)]$$
$$= \sin[i(0 + i\pi/2)]$$
$$= \sin(-\pi/2) = -1$$

Therefore,
$$a = -1, b = 0$$
So,
$$\sin(\log i^i) = -1$$
Now,
$$\cos(\log i^i) = \sqrt{\{1 - \sin^2(\log i^i)\}} = \sqrt{1 - 1} = 0$$

5.12 Section Formula

See Fig. 5.22. Let z_1 and z_2 represent any two complex number representing the points A and B, respectively, in the Argand plane. Let C be the point dividing the line AB internally in ratio $m{:}n$, that is, $\frac{AC}{BC} = m{:}n$ and let the complex number associated with point C be z.

Then, let us rotate the line BC about C so that it becomes parallel to CA. Then, the corresponding equation after rotation will be

$$\frac{z_1 - z}{z_2 - z} = \frac{|z_1 - z|}{|z_2 - z|}e^{i\pi} = \frac{m}{n}(-1)$$

$$\Rightarrow z = \frac{nz_1 + mz_2}{m + n}$$

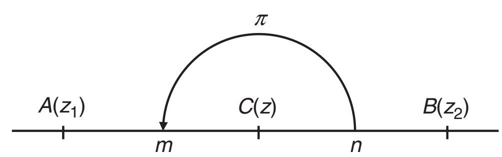


Figure 5.22

Thus,

1. If Z_1, Z_2 are divided at P in the ratio $m{:}n$ internally, then P has the complex number $\frac{mZ_2 + nZ_1}{m + n}$. Particularly, the mid-point of the join of Z_1 and Z_2 is $\frac{Z_1 + Z_2}{2}$.
2. If Z_1, Z_2, Z_3 be three points A, B, C forming a triangle ABC, then the centroid G of the triangle ABC has an associated complex number $\frac{Z_1 + Z_2 + Z_3}{3}$.

Illustration 5.25 If the vertices of a triangle ABC are represented by Z_1, Z_2 and Z_3, respectively, then prove that

(i) the centroid is $\frac{Z_1 + Z_2 + Z_3}{3}$

(ii) the orthocentre is $\frac{(a\sec A)Z_1 + (b\sec B)Z_2 + (c\sec C)Z_3}{a\sec A + b\sec B + c\sec C}$, or

$$\frac{(\tan A)z_1 + (\tan B)z_2 + (\tan C)z_3}{\tan A + \tan B + \tan C}$$

(iii) the circumcentre is $\frac{(\sin 2A)Z_1 + (\sin 2B)Z_2 + (\sin 2C)Z_3}{\sin 2A + \sin 2B + \sin 2C}$

Solution:

(i) See Fig. 5.23. The mid-point D of BC is $\frac{Z_2 + Z_3}{2}$ and the point G on AD, which divides AD in the ratio 2:1 is

$$\frac{2\left(\frac{Z_2 + Z_3}{2}\right) + Z_1}{2 + 1}$$

$$\Rightarrow \left(\frac{Z_2 + Z_3 + Z_1}{3}\right)$$

Symmetry in Z_1, Z_2, Z_3 of this result indicates that this point G also lies on the other two medians.

Therefore, the medians are concurrent at G, the centroid, the associated complex of which is $\dfrac{Z_1+Z_2+Z_3}{3}$.

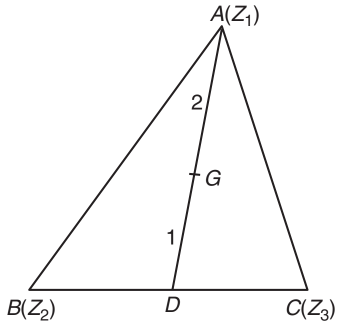


Figure 5.23

(ii) Orthocentre (see Fig. 5.24): Let the two altitudes AD and BE intersect at O.
Now,

$$\frac{BD}{DC}=\frac{c\cos B}{b\cos C}=\frac{c\sec C}{b\sec B}$$

The point D dividing BC in the ratio $\dfrac{BD}{DC}$ has a complex number

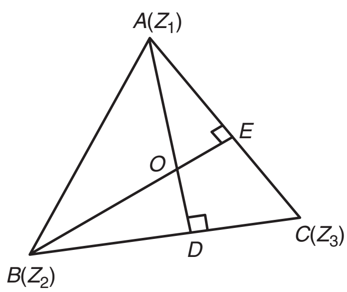


Figure 5.24

$$\frac{(c\sec C)Z_3+(b\sec B)Z_2}{b\sec B+c\sec C}$$

Again,

$$\frac{AO}{OD}=\frac{\text{Area of }\Delta ABO}{\text{Area of }\Delta OBD}\text{ (triangles of the same altitude)}$$

$$=\frac{\frac{1}{2}AB\cdot BO\sin\angle ABE}{\frac{1}{2}BO\cdot BD\sin\angle DBE}$$

$$=c\cos A/(c\cos B\cdot\cos C)$$

$$=\frac{a\cos A}{a\cos B\cos C}=\frac{b\cos C+c\cos B}{\cos B\cos C}\,\frac{1}{a\sec A}$$

$$=\frac{b\sec B+c\sec C}{a\sec A}$$

Therefore, the point O, dividing AD, in the ratio $\dfrac{AO}{OD}$ has a complex number

$$\frac{AO\text{ (complex number of }D)+OD\text{ (complex number of }A)}{AO+OD}$$

$$=\frac{(b\sec B+c\sec C)\left(\dfrac{b\sec B\cdot Z_2+c\sec C\cdot Z_3}{b\sec B+c\sec C}\right)+a\sec A\cdot Z_1}{b\sec B+c\sec C+a\sec A}$$

$$=\frac{(a\sec A)Z_1+(b\sec B)Z_2+(c\sec C)Z_3}{a\sec A+b\sec B+c\sec C}$$

The symmetry of this result in a, b, c and A, B, C indicates that O lies on the third altitude also. Hence O, the orthocentre, is

$$\frac{Z_1 a\sec A+Z_2 b\sec B+Z_3 c\sec C}{a\sec A+b\sec B+c\sec C}$$

To prove the other result substituting $a=2R\sin A$, $b=2R\sin B$ and $c=2R\sin C$ in the above result:

$$\frac{Z_1\tan A+Z_2\tan B+Z_3\tan C}{\tan A+\tan B+\tan C}$$

(iii) Circumcentre (see Fig. 5.25):

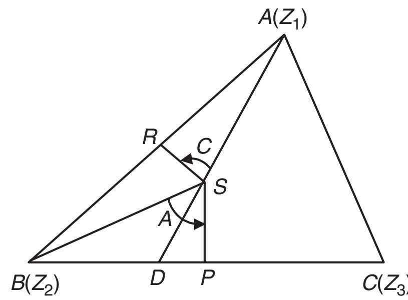


Figure 5.25

Let S be the point of intersection of perpendicular bisectors of BC and AB. S lies on the third perpendicular bisector also. Let AS produced meet BC at D. Now,

$$\frac{BD}{DC}=\frac{\text{area of }\Delta ABD}{\text{area of }\Delta ACD}\text{ (triangles of the same altitude)}$$

$$=\frac{AB\cdot AD\cdot\sin\angle BAD}{AC\cdot AD\cdot\sin\angle CAD}=\frac{c\sin(90^\circ-C)}{b\sin(90^\circ-B)}$$

$$=\frac{\sin 2C}{\sin 2B}\qquad(1)$$

Therefore, D is represented by the complex number,

$$\frac{(\sin 2C)Z_3+(\sin 2B)Z_2}{\sin 2B+\sin 2C}$$

$$\frac{AS}{SD}=\frac{\text{area of }\Delta ASB}{\text{area of }\Delta BSD}=\frac{AS\cdot BS\cdot\sin 2C}{BS\cdot BD\cdot\sin(90^\circ-A)}$$

$$=\frac{R\sin 2C}{BD\cos A}\qquad(2)$$

From Eq. (1),

$$\frac{BD}{\sin 2C}=\frac{DC}{\sin 2B}=\frac{BD+DC}{\sin 2B+\sin 2C}=\frac{a}{\sin 2B+\sin 2C}$$

Substituting Eq. (1) into Eq. (2), we get

$$\frac{AS}{SD}=\frac{R\sin 2C}{\dfrac{a\sin 2C}{\sin 2B+\sin 2C}\cdot\cos A}$$

$$= \frac{R \sin 2C}{\dfrac{2R \sin A \cos A \sin 2C}{\sin 2B + \sin 2C}} = \frac{\sin 2B + \sin 2C}{\sin 2A}$$

Therefore, S is represented by

$$\frac{(\sin 2A)Z_1 + (\sin 2B + \sin 2C)\left(\dfrac{\sin 2C \cdot Z_3 + \sin 2B \cdot Z_2}{\sin 2B + \sin 2C}\right)}{\sin 2A + \sin 2B + \sin 2C}$$

$$\Rightarrow \frac{Z_1 \sin 2A + Z_2 \sin 2B + Z_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}$$

5.13 Locus in an Argand Plane

It has been pointed that there is a bijective correspondence between a complex number $Z \equiv (x, y)$ and a point $P(x, y)$ in the complex plane or an Argand diagram.

Coordinate geometry theory gives us the concept of a locus as a curve, every point $P(x, y)$ on the curve satisfies a relation between x and y termed as the equation to a curve.

But $P(x, y)$ is also equivalent to $Z = (x, y)$, and hence this relation between x and y – representing the equation – can also be put in the form of a condition on Z.

To cite an example, $x^2 + y^2 = 1$, expressed in terms of Z, is $|Z| = 1$, and it is said that the condition $|Z| = 1$, being satisfied by all points Z at unit distance from (0, 0), represents a circle with a centre at (0, 0) and radius = 1. We therefore assert that any condition imposed on Z automatically places a restriction on the possible locations in the Argand diagram of the point P representing Z, and hence all such points lie on a curve. Such a curve traced in the Argand diagram by $P \equiv Z$, because of a condition imposed on Z, is termed as *locus in an Argand diagram*.

1. **Straight line:** See Fig. 5.26. The equation of straight line passing through points A and B represented by complex numbers z_1 and z_2 is

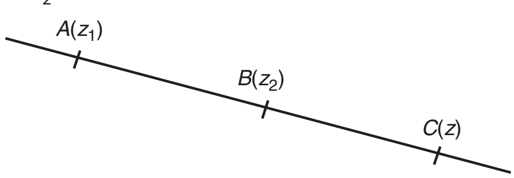


Figure 5.26

Let us take $C(z)$ as the general point on the line. Then

$$\arg\left(\frac{z - z_1}{z_2 - z_1}\right) = 0 \text{ or } \pi$$

$$\Rightarrow \frac{z - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1}$$

It can also be represented in the following form:

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

The general form of the straight line is

$$a\bar{z} + \bar{a}z + b = 0$$

where a is a complex number and b is a real number.

(a) ***Slope of a line*****:** Let the equation of line be $z\bar{a} + \bar{z}a + b = 0$

Replacing z by $x + iy$, we get

$$(a + \bar{a})x + i(\bar{a} - a)y + b = 0$$

Its real slope is

$$\frac{-(a + \bar{a})}{i(\bar{a} - a)} = \frac{-\text{Re}(a)}{\text{Im}(a)}$$

Its complex slope is

$$-\frac{a}{\bar{a}} = -\frac{\text{coeff of } \bar{z}}{\text{coeff of } z}$$

The equation of the line parallel to $\bar{a}z + a\bar{z} + b$ is $\bar{a}z + a\bar{z} + \lambda = 0$ (where λ is a real number) and that of the line perpendicular to it is $z\bar{a} - \bar{z}a + i\lambda = 0$.

(b) ***Ray***

- $\arg Z = \theta$ is a ray (or a straight line) from the origin and pointed in such a direction that any point Z situated on the line has an argument θ.
- $\arg(Z - \alpha) = \theta$ is a ray (or a straight line) from the point α and pointed in such a direction that the line joining α to Z is inclined at an angle θ to the positive direction of the real axis (x-axis).

(c) ***Perpendicular bisector***

- $|Z - \alpha| = |Z - \beta|$ represents the perpendicular bisector of the line joining the two points $\alpha \equiv (p, q)$ and $\beta \equiv (r, s)$.
- The perpendicular distance of a point z_0 from the line $\bar{a}z + a\bar{z} + b = 0$ is $\dfrac{|\bar{a}z_0 + a\bar{z}_0 + b|}{2|a|}$.

Equation of a Straight Line

- **Equation of a straight line with the help of rotation formula:** Let $A(z_1)$ and $B(z_2)$ be any two points lying on any line and we have to obtain the equation of this line. For this purpose, let us take any point $C(z)$ on this line. Since

$$\arg\left(\frac{z - z_1}{z_2 - z_1}\right) = 0 \text{ or } \pi \Rightarrow \frac{z - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1}$$

- **General equation of the line:**

$$z\bar{a} + \bar{z}a + b = 0,$$

where

$$\bar{a} = (z_1 - z_2),\ b = z_1\bar{z}_2 - z_2\bar{z}_1$$

This is the general equation of a line in the complex plane.

- **Slope of a given line:** If $z\bar{a} + \bar{z}a + b = 0$ is the given line, then its slope $= -\dfrac{\text{Re}(a)}{\text{Im}(a)}$.
- Equation of a line parallel to the line $z\bar{a} + \bar{z}a + b = 0$ is $z\bar{a} + \bar{z}a + \lambda = 0$ (where λ is a real number).
- Equation of a line perpendicular to the line $z\bar{a} + \bar{z}a + b = 0$ is $z\bar{a} - \bar{z}a + i\lambda = 0$ (where λ is a real number).

- **Equation of perpendicular bisector:** Consider a line segment joining $A(z_1)$ and $B(z_2)$. Let the line 'L' be its perpendicular bisector. If $P(z)$ be any point on the 'L', we have $PA = PB$.

$$\Rightarrow |z - z_1| = |z - z_2|$$

$$\Rightarrow z(\bar{z}_2 - \bar{z}_1) + \bar{z}(z_2 - z_1) + z_1\bar{z}_1 - z_2\bar{z}_2 = 0$$

- **Distance of a given point from a given line:** Let the given line be $z\bar{a} + \bar{z}a + b = 0$ and the given point be z_c, then the distance of z_c from this line is $\dfrac{|z_c\bar{a} + \bar{z}_c a + b|}{2|a|}$.

2. Circle

- $|Z| = r$ is a circle, centre (0, 0) and radius r.
- $|Z - \alpha| = r$ (α, complex) is a circle, centre at $\alpha \equiv (p, q)$ and radius $= r$ since $|Z - \alpha|$ represents the absolute distance of Z from α.
- $|Z - \alpha| = k\,|Z - \beta|$ (k real > 0, ≠ 1) is the circle and any point P on the circle, with reference to the points $A(\alpha)$ and $B(\beta)$, satisfies the condition $\dfrac{AP}{PB} = k\ (k \neq 1)$.

  Let us take, for exactness, $0 < k < 1$. Let L and M divide the join of $A(\alpha)$ and $B(\beta)$ internally at L and externally at M in the ratio k, so that $\dfrac{AL}{LB} = \dfrac{MA}{MB} = k < 1$.

Draw the circle with LM as the diameter. Any point P on this circle will satisfy the requirement $\dfrac{AP}{PB} = k$. The locus of the point $P(Z)$ satisfying the condition $\dfrac{|Z - \alpha|}{|Z - \beta|} = k \neq 1$ is the circle on LM as diameter and is called the **Apollonius circle** of A and B with respect to the ratio k. The circle meets AB at L and M and these two points, being points on the circle, trivially satisfy the condition to be satisfied by any point P to lie on the circle. In fact, the choice of L and M has been made to satisfy this requirement. It may be also pointed, as a property, that PL and PM bisect $\angle APB$ internally and externally (Fig. 5.27).

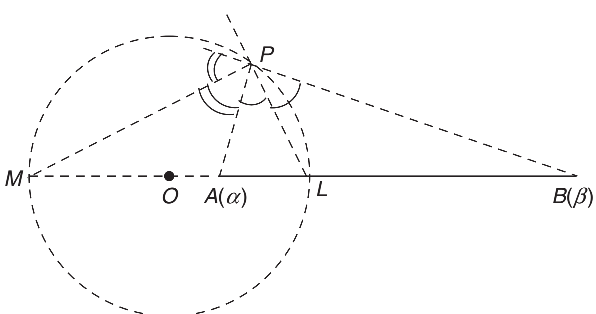


Figure 5.27

- $\arg\left(\dfrac{Z - Z_1}{Z - Z_2}\right) = 0$ is a straight line – that part of the segment of the line through Z_1 and Z_2 which is outside the segment joining Z_1 and Z_2.
- $\arg\left(\dfrac{Z - Z_1}{Z - Z_2}\right) = \pi$ represents the line segment joining Z_1 and Z_2.
- In fact the condition $\arg\left(\dfrac{Z_1 - Z_2}{Z_1 - Z_3}\right) = 0$ or π is the condition for Z_1, Z_2 and Z_3 to be collinear.
- $\arg\left(\dfrac{Z - Z_1}{Z - Z_2}\right) = \theta \neq 0 \neq \pi$. The equation $\arg\left(\dfrac{Z - Z_1}{Z - Z_2}\right) = \theta$ geometrically expresses the fact that the join of Z_1 and Z_2 subtends the angle θ at Z. Hence, the condition represents the segment of a circle described on the join if Z_1 and Z_2 as a chord and containing at any point $P(Z)$ on the segment the angle θ. If $0 < \theta < \pi/2$ the segment is a major segment. If $\pi/2 < \theta < \pi$, the segment is a minor segment. If $\theta = \pi/2$ the locus is the semi-circle on the join of Z_1 and Z_2 with the circle being appropriately chosen.

  It has already been pointed out that every point can be taken to be represented by a complex number Z. Thus, just as in coordinate geometry where we have for every point a pair of numbers (its coordinates), in complex number theory every point has an associate complex number, of which the point is a geometrical representation.

Equation of a Circle

Equation of a circle of radius r and having its centre at z_0 is

$$|z - z_0| = r$$

$$\Rightarrow |z - z_0|^2 = r^2 \Rightarrow (z - z_0)(\bar{z} - \bar{z}_0) = r^2 \Rightarrow z\bar{z} + a\bar{z} + \bar{a}z + b = 0,$$

where $-a = z_0$ and $b = z_0\bar{z}_0 - r^2$.

It represents the general equation of a circle in the complex plane.

- Equation of a circle described on a line segment AB, ($A(z_1)$, $B(z_2)$) as diameter is $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$.
- Let z_1 and z_2 be two given complex numbers and z be any complex number such that $\arg\left(\dfrac{z - z_1}{z - z_2}\right) = \alpha$, where $\alpha \in (0, \pi)$. Then z will lie on the arc of a circle.
- Let $ABCD$ be a cyclic quadrilateral such that $A(z_1)$, $B(z_2)$, $C(z_3)$ and $D(z_4)$ lie on a circle. Then $\dfrac{(z_4 - z_1)(z_2 - z_3)}{(z_2 - z_1)(z_4 - z_3)}$ is purely real.

Illustration 5.26 Show that the equation $Z\bar{Z} + a\bar{Z} + \bar{a}Z + b^2 = 0$ (b is real) is the complex from of the equation to a circle.

Solution:

$$Z\bar{Z} + a\bar{Z} + \bar{a}Z + b^2 = 0$$

Therefore,

$$Z\bar{Z} + a\bar{Z} + \bar{a}Z + a\bar{a} = a\bar{a} - b^2$$

$$\Rightarrow (Z + a)(\bar{Z} + \bar{a}) = |a|^2 - b^2$$

$$\Rightarrow |Z + a|^2 = r^2 \quad (1)$$

where

$$r^2 = |a|^2 - b^2$$

Equation (1) represents a circle with centre at $-a$ (complex) and radius $r = \sqrt{|a|^2 - b^2}$, and for the circle to be real we need the condition $|a|^2 > b^2$.

Illustration 5.27 Examine the locus that is represented by $|Z-a|^2+|Z-b|^2=k$ (where k is real).

Solution:

$$|Z-a|^2=(Z-a)(\bar{Z}-\bar{a})=Z\bar{Z}+a\bar{a}-(Z\bar{a}+\bar{Z}a)$$

$$=|Z|^2+|a|^2-2\text{Re}(Z\bar{a})$$

Similarly,

$$|Z-b|^2=|Z|^2+|b|^2-2\text{Re}(Z\bar{b})$$

The given equation becomes

$$2|Z|^2+|a|^2+|b|^2-2\text{Re}(Z(\bar{a}+\bar{b}))=k$$

Dividing by 2 and adding $\dfrac{|a+b|^2}{4}$ on both sides

$$|Z|^2-2\text{Re}\left[\frac{Z(\bar{a}+\bar{b})}{2}\right]+\frac{|a+b|^2}{4}=\frac{k}{2}+\frac{1}{4}|a+b|^2-\frac{|a|^2}{2}-\frac{|b|^2}{2}$$

$$\Rightarrow\left|Z-\frac{a+b}{2}\right|^2=\frac{1}{2}\left\{k-\frac{1}{2}[|a|^2+|b|^2-2\text{Re}\,ab]\right\}$$

$$\Rightarrow\left|Z-\frac{a+b}{2}\right|^2=\frac{1}{2}\left\{k-\frac{1}{2}|a-b|^2\right\}$$

This will represent a circle with centre at $\dfrac{a+b}{2}$ and radius $\dfrac{1}{2}\sqrt{2k-|a-b|^2}$.

Illustration 5.28 Let $Z_1=10+6i$ and $Z_2=4+6i$. If Z be any complex number such that $\arg\left(\dfrac{Z-Z_1}{Z-Z_2}\right)=\pi/4$. Prove that the locus of Z is $|Z-7-9i|=3\sqrt{2}$.

Solution: See Fig. 5.28. Since

$$\arg\left(\frac{Z-Z_1}{Z-Z_2}\right)=\arg(Z-Z_1)-\arg(Z-Z_2)=\frac{\pi}{4}$$

So, the join of $A(Z_1)$ and $B(Z_2)$ subtends an angle $=\dfrac{\pi}{4}$ at $P(Z)$.

Hence, the locus of Z is a segment of a circle drawn on AB to contain the angle $\dfrac{\pi}{4}$.

The point M – the mid-point of AB – is (7, 6) and the centre is $C(7, 9)$.

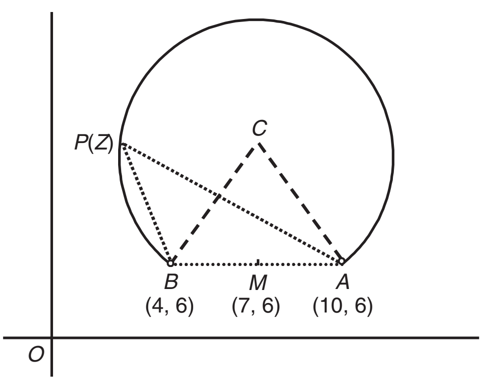


Figure 5.28

The locus of Z is the circle (segment) drawn to contain angle at $\pi/4$. The radius of the circle is $=\sqrt{9+9}=3\sqrt{2}$.

It is therefore true that Z lies on $|Z-(7+9i)|=3\sqrt{2}$.

But it is not true that every point Z on $|Z-(7+9i)|=3\sqrt{2}$ satisfies that condition $\arg\left(\dfrac{Z-Z_1}{Z-Z_2}\right)=\pi/4$.

Therefore, the locus of Z subject to the condition $\arg\left(\dfrac{Z-Z_1}{Z-Z_2}\right)=\pi/4$ can only be the major segment drawn on AB.

The part of the (minor) segment lying below AB may be found to satisfy the condition $\arg\left(\dfrac{Z-Z_1}{Z-Z_2}\right)=-\left(\pi-\dfrac{\pi}{4}\right)=\dfrac{-3\pi}{4}$.

3. **Conic section**
   (a) ***Parabola*:** Equation of parabola with focus at z_0 and directrix as $\bar{a}z+a\bar{z}+b=0$ is given by

$$|z-z_0|=\frac{|a\bar{z}+\bar{a}z+b|}{2|a|}$$

   (b) ***Ellipse*:** Equation of ellipse with foci at z_1 and z_2 and length of the major axes as $2a$ is

$$|z-z_1|+|z-z_2|=2a$$

   where

$$2a>|z_1-z_2|$$

   (c) ***Hyperbola*:** Equation of hyperbola with foci at z_1 and z_2 and length of the transverse axes as $2a$ is

$$||z-z_1|-|z-z_2||=2a$$

   where

$$2a<|z_1-z_2|$$

Illustration 5.29 If $||z+2|-|z-2||=a^2, z\in C$ represents a hyperbola for $a\in R$, then find the values of a.

Solution: Here, foci are at –2 and 2 at a distance of 4. Hence, the given equation represents a hyperbola if $a^2<4$, that is, $a\in(-2, 2)$.

Illustration 5.30 Locate the points representing the complex numbers Z in the Argand diagram for which

(a) $|i-1-2Z|>9$ **(b)** $4\le|2Z+i|\le 6$

(c) $|Z+i|=|Z-1|$ **(d)** $|Z-1|^2+|Z+1|^2=4$

Solution:

(a)
$$i-1-2Z=-2\left(Z+\frac{1}{2}-\frac{i}{2}\right)$$

$$\Rightarrow|i-1-2Z|=\left|-2\left[Z-\left(\frac{-1+i}{2}\right)\right]\right|$$

$$=2\left|Z-\left(\frac{-1+i}{2}\right)\right|$$

Therefore, the given condition becomes $\left|Z-\left(\dfrac{-1+i}{2}\right)\right|>\dfrac{9}{2}$.

This represents all points represented by Z and lying outside the circle with centre $\frac{-1+i}{2}$, that is, $\left(-\frac{1}{2},\frac{1}{2}\right)$ and radius 9/2.

(b)
$$2Z+i=2\left(Z+\frac{i}{2}\right)$$
$$\Rightarrow |2Z+i|=2\left|Z+\frac{1}{2}i\right|$$

Therefore,
$$4\le |2Z+i|\le 6$$
$$\Rightarrow 4\le 2\left|Z+\frac{1}{2}i\right|\le 6$$
$$\Rightarrow 2\le \left|Z+\frac{i}{2}\right|\le 3$$

This represents the locations of all points Z on or outside the circle with centre $-\frac{i}{2}$, that is, $\left(0,-\frac{1}{2}\right)$ and radius 2, and on or inside the circle with centre at $-\frac{1}{2}i\left(\text{i.e.}\left(0,-\frac{1}{2}\right)\right)$ and radius 3.

Thus, it denotes the circular strip lying between two concentric circles.

(c)
$$|Z+i|=|Z-1|$$

$|Z+i|=|Z-(-i)|$ denotes the distance of Z from $-i$, that is, $(0,-1)$, and $|Z-1|$ denotes the distance of Z from 1, that is, (1, 0).

Therefore, $|Z+i|=|Z-1|$ is satisfied for all Z equidistant from (0, –1) and (1, 0), and thus it is perpendicular bisector of the join of (0, –1) and (1, 0), whose Cartesian equation is $x+y=0$.

(d)
$$|Z-1|^2+|Z+1|^2=4$$
$$|Z-1|^2+|Z+1|^2=(Z-1)(\bar{Z}-1)+(Z+1)(\bar{Z}+1)\quad (\because |Z|^2=Z\bar{Z})$$
$$=Z\bar{Z}-(Z+\bar{Z})+1+Z\bar{Z}+(Z+\bar{Z})+1$$
$$=2Z\bar{Z}+2$$
$$\Rightarrow 2Z\bar{Z}+2=4$$
$$\Rightarrow Z\bar{Z}=1$$
$$\Rightarrow |Z|^2=1$$
$$\Rightarrow |Z|=1$$

Thus, the locus of Z subject to the given condition is the unit circle $|Z|=1$.

Your Turn 2

1. Solve $x^7+1=0$. **Ans.** $x=e^{\frac{i(2k+1)\pi}{7}}$ $k=0,1,2,\ldots,6$
2. Find all non-zero complex number satisfying $|z|+z^2=0$. **Ans.** $z=i$
3. If 1, ω, ω^2 are the cube roots of unity prove that $(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)\ldots$ up to $2n$ factors $=2^{2n}$.
4. If 1, α, α^2, …, α^{n-1} are the n^{th} roots of unity, then find the value of $(3-\alpha)(3-\alpha^2)(3-\alpha^3)\ldots(3-\alpha^{n-1})$. **Ans.** $\frac{3^n-1}{2}$
5. If $\frac{z_1-z_2}{z_3-z_4}$ is purely real for four complex numbers, then these complex numbers are collinear. (True/False) **Ans.** True
6. The quadratic equation $|z|^2+z|z|+z^2=0$ represents pair of rays. (True/False) **Ans.** True
7. $|z-i|+|z+i|=2$ is the equation of an ellipse. (True/False) **Ans.** False
8. $1<|z-2-3i|<4$ represents circular strip between two concentric circles with centre $(2+3i)$ and radii 1 and 4. (True/False) **Ans.** True

Some Important Results to Remember

The triangle whose vertices are the points represented by complex numbers z_1, z_2, z_3 is equilateral if $\frac{1}{z_2-z_3}+\frac{1}{z_3-z_1}+\frac{1}{z_1-z_2}=0$, that is, if $z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1$.

- $|z-z_1|+|z-z_2|=\lambda$ represents an ellipse if $|z_1-z_2|<\lambda$, having the points z_1 and z_2 as its foci. And if $|z_1-z_2|=\lambda$, then z lies on a line segment connecting z_1 and z_2.
- $|z-z_1|-|z-z_2|=\lambda$ represents a hyperbola if $|z_1-z_2|>\lambda$, having the points z_1 and z_2 as its foci. And if $|z_1-z_2|=\lambda$, then z lies on the line passing through z_1 and z_2 excluding the points between z_1 and z_2.

Additional Solved Examples

1. Find the value of $(x^2+5x)^2+x(x+5)$ for $x=\frac{-5+i\sqrt{3}}{2}$.

Solution:
$$x+5=\frac{-5+i\sqrt{3}}{2}+5=\frac{5+i\sqrt{3}}{2}$$

Therefore,
$$x(x+5)=\left(\frac{-5+i\sqrt{3}}{2}\right)\left(\frac{5+i\sqrt{3}}{2}\right)=\frac{(-5)5+3i^2}{4}$$
$$=\frac{-25-3}{4}=-7$$

Therefore, the required value is
$$(-7)^2-7=49-7=42$$

2. Find two complex numbers satisfying the given conditions.
   (i) the sum of their real parts is 3
   (ii) the product of their real parts is 2
   (iii) their product is $5-i$

Solution: Take the complex numbers as $a+ib$, $p+iq$. So, as per the given conditions
$$a+p=3;\ ap=2$$
$$\Rightarrow \left.\begin{matrix}a=2\\p=1\end{matrix}\right\}\text{ or }\left.\begin{matrix}a=1\\p=2\end{matrix}\right\}$$

Also,

$$(a+ib)\,(p+iq) = ap - bq + i(bp+aq) = 5 - i$$

So,

$$ap - bq = 5;\ aq + bp = -1$$

Taking

$$a = 2, p = 1;$$
$$bq = -3 \text{ and } b + 2q = -1$$

This gives

$$\left.\begin{matrix} b=-3 \\ q=1 \end{matrix}\right\} \text{ or } \left.\begin{matrix} 2+2i \\ 1-\frac{3}{2}i \end{matrix}\right\}$$

The numbers are

$$\left.\begin{matrix} 2-3i \\ 1+i \end{matrix}\right\} \text{ or } \left.\begin{matrix} 2+2i \\ 1-\frac{3}{2}i \end{matrix}\right\}$$

Thus, there are two pairs of a complex numbers satisfying the requirements. It may be verified that $a = 1$, $p = 2$, give the same set of numbers.

3. Prove that

(i) $|Z_1+Z_2|^2 + |Z_1-Z_2|^2 = 2(|Z_1|^2 + |Z_2|^2)$

(ii) Using above result, prove that $\left|\alpha-\sqrt{\alpha^2-\beta^2}\right| + \left|\alpha+\sqrt{\alpha^2-\beta^2}\right| = |\alpha+\beta| + |\alpha-\beta|$, where α, β are complex numbers.

Solution:

$$|Z_1+Z_2|^2 = (Z_1+Z_2)(\bar{Z}_1+\bar{Z}_2) = Z_1\bar{Z}_1 + Z_2\bar{Z}_2 + Z_1\bar{Z}_2 + Z_2\bar{Z}_1 \quad (1)$$

$$|Z_1-Z_2|^2 = (Z_1-Z_2)(\bar{Z}_1-\bar{Z}_2) = Z_1\bar{Z}_1 + Z_2\bar{Z}_2 - Z_1\bar{Z}_2 - Z_2\bar{Z}_1 \quad (2)$$

Adding Eqs. (1) and (2)

$$|Z_1+Z_2|^2 + |Z_1-Z_2|^2 = 2(Z_1\bar{Z}_1 + Z_2\bar{Z}_2) = 2(|Z_1|^2 + |Z_2|^2)$$

Now, for the second part,

$$\left|\alpha-\sqrt{\alpha^2-\beta^2}\right| + \left|\alpha+\sqrt{\alpha^2-\beta^2}\right|$$

$$= \frac{1}{2}\left\{\left|2\alpha-2\sqrt{\alpha^2-\beta^2}\right| + \left|2\alpha+2\sqrt{\alpha^2-\beta^2}\right|\right\}$$

$$= \frac{1}{2}\left\{\left|\alpha+\beta+\alpha-\beta-2\sqrt{\alpha^2-\beta^2}\right| + \left|\alpha+\beta+\alpha-\beta+2\sqrt{\alpha^2-\beta^2}\right|\right\}$$

$$= \frac{1}{2}\{|\sqrt{\alpha+\beta}-\sqrt{\alpha-\beta}|^2 + |\sqrt{\alpha+\beta}+\sqrt{\alpha-\beta}|^2\}$$

$$= \frac{1}{2}\{|Z_1-Z_2|^2 + |Z_1+Z_2|^2\} = \frac{1}{2}\{2(|Z_1|^2 + |Z_2|^2)\}$$

$$= \left|\sqrt{\alpha+\beta}\right|^2 + \left|\sqrt{\alpha-\beta}\right|^2 = |\alpha+\beta| + |\alpha-\beta|$$

4. If Z be a complex number with $|Z| = 1$, imaginary part of $Z \neq 0$, then show that Z can be represented as $\frac{c+i}{c-i}$ where c is real.

Solution: Since $|Z| = 1$, Z can be represented as $(\cos\theta + i\sin\theta)$.

Therefore,

$$Z = (\cos\theta + i\sin\theta)$$

$$= \left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)$$

$$= \left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$$

$$= \frac{\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}}{\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}}$$

$$= \frac{\cot\frac{\theta}{2} + i}{\cot\frac{\theta}{2} - i}\left(\text{dividing by } \sin\frac{\theta}{2}\right)$$

$$= \frac{c+i}{c-i} \text{ where } c = \cot\frac{\theta}{2} \text{ is real}$$

5. For every real $b \geq 0$, find all the complex numbers Z satisfying $2|Z| - 4bZ + 1 + ib = 0$.

Solution: Let $Z = x + iy$. The equation becomes

$$2\sqrt{x^2+y^2} - 4b(x+iy) + 1 + ib = 0$$

Real part: $\quad 2\sqrt{x^2+y^2} - 4bx + 1 = 0 \quad (1)$

Imaginary part: $\quad -4by + b = 0 \quad (2)$

From (2) either $b = 0$ and in that case from (1), $2\sqrt{x^2+y^2} + 1 = 0$ and this equation is not satisfied for any (x, y)

Therefore, $b = 0$, there is no solution for the equation. If $b \neq 0$ but > 0, then

$$-4y + 1 = 0$$
$$\Rightarrow y = \frac{1}{4}$$

From Eq. (2) substituting $y = \frac{1}{4}$ in Eq. (1)

$$2\sqrt{x^2+\frac{1}{16}} = 4bx - 1 \quad (3)$$

This requires that $4bx - 1 > 0$, that is, $x > \frac{1}{4b}$ and $b > 0$ and hence $x > 0$.

Squaring Eq. (3)

$$4\left(x^2+\frac{1}{16}\right) = 16b^2x^2 - 8bx + 1$$

$$x^2(16b^2-4) - 8bx + \frac{3}{4} = 0$$

So, roots are

$$x = \frac{8b \pm \sqrt{16b^2+12}}{2(16b^2-4)} = \frac{4b \pm \sqrt{4b^2+3}}{16b^2-4}$$

If $16b^2 - 4 < 0$, which in effect means that $b < \frac{1}{2}$ (note already $b > 0$), the values of x are such that

(1) for the + sign $x < 0$ while the requirement is $x > 0$

(2) for the – sign, even if $x > 0$, the condition $x > \frac{1}{4b}$ is not satisfied.

Therefore, for $0 < b < \frac{1}{2}$, there is no solution.

For $b > \frac{1}{2}$, the solution is

$$Z = \frac{4b + \sqrt{4b^2 + 3}}{16b^2 - 4} + \frac{i}{4}$$

6. For complex numbers z and w, prove that $|z|^2 w - |w|^2 z = z - w$ if and only if $z = w$ or $z\bar{w} = 1$.

Solution:

$$\frac{z}{w} = \frac{|z|^2 + 1}{|w|^2 + 1} = \text{purely real number}$$

$$\Rightarrow \frac{z}{w} \text{ is purely real, that is, } \frac{z}{w} = \overline{\left(\frac{z}{w}\right)} \Rightarrow \quad z\bar{w} = \bar{z}w \qquad (1)$$

$$|z|^2 w - |w|^2 z = z - w$$

$$z\bar{z}w - w\bar{w}z = z - w$$

from Eq. (1),

$$z\bar{w}(z - w) = z - w \qquad (2)$$

$$\Rightarrow (z\bar{w} - 1)(z - w) = 0 \Rightarrow z = w \text{ or } z\bar{w} = 1$$

Conversely, if $z = w$, then

$$\text{LHS} = \text{RHS} = 0$$

If $z\bar{w} = 1$, then from Eq. (2)

$$\text{LHS} = \text{RHS} = z - w$$

7. Show that the triangle whose vertices are the points represented by the complex numbers Z_1, Z_2 and Z_3 on the Argand diagram is equilateral if and only if $\frac{1}{Z_2 - Z_3} + \frac{1}{Z_3 - Z_1} + \frac{1}{Z_1 - Z_2} = 0$ (OR) equivalently $Z_1^2 + Z_2^2 + Z_3^2 = Z_1Z_2 + Z_2Z_3 + Z_3Z_1$.

Solution: See Fig. 5.29. *ABC* is the equilateral triangle formed of $A(Z_1)$, $B(Z_2)$ and $C(Z_3)$. So,

$$Z_3 - Z_1 = (Z_2 - Z_1)\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

(*AC* is obtained from *AB* by a rotation anticlockwise through an angle $\pi/3$)

Lengthwise, $AC = AB$

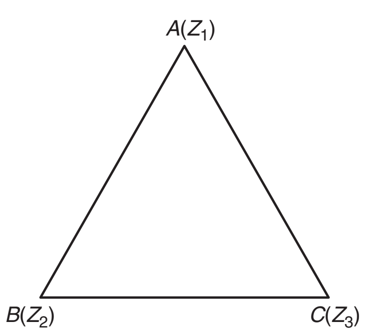


Figure 5.29

$$\frac{1}{Z_3 - Z_1} = \frac{1}{Z_1 - Z_2}\left\{-\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right\} \qquad (1)$$

Similarly,

$$\frac{1}{Z_2 - Z_3} = \frac{1}{Z_1 - Z_2}\left\{-\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right\} \qquad (2)$$

Adding Eqs. (1) and (2)

$$\frac{1}{Z_3 - Z_1} + \frac{1}{Z_2 - Z_3} = -\frac{1}{Z_1 - Z_2}$$

$$\Rightarrow \frac{1}{Z_3 - Z_1} + \frac{1}{Z_2 - Z_3} + \frac{1}{Z_1 - Z_2} = 0 \qquad (3)$$

This may be equivalently written in the form

$$\sum (Z_1 - Z_2)(Z_3 - Z_1) = 0$$

$$\Rightarrow \sum Z_1(Z_3 - Z_1) - \sum Z_2(Z_3 - Z_1) = 0$$

$$\Rightarrow Z_1^2 + Z_2^2 + Z_3^2 = Z_1Z_2 + Z_3Z_1 + Z_2Z_3 \qquad (4)$$

$$\left(\because \sum Z_2(Z_3 - Z_1) = 0\right)$$

The condition for Z_1, Z_2 and Z_3 to form an equilateral triangle is given in one of the two equivalent forms given by Eqs. (3) and (4).

Let us prove the converse also

Assume

$$\frac{1}{Z_2 - Z_3} + \frac{1}{Z_3 - Z_1} + \frac{1}{Z_1 - Z_2} = 0$$

If $p = Z_2 - Z_3, q = Z_3 - Z_1, r = Z_1 - Z_2$, then $p + q + r = 0$

Therefore,

$$p(q + r) + rq = 0 \Rightarrow p(-p) + qr = 0 \Rightarrow p^2 = qr$$

So,

$$p^2 = qr; \quad \bar{p}^2 = \bar{q}\,\bar{r}$$

Multiplying above two equations we get,

$$p^2\bar{p}^2 = q\bar{q}\,r\bar{r} \Rightarrow (p\bar{p})^2\,(p\bar{p}) = (q\bar{q})\,(r\bar{r})\,(p\bar{p})$$

Similarly, it is possible to prove $p\bar{p}\,q\bar{q}\,r\bar{r} = (q\bar{q})^3 = (r\bar{r})^3$. This gives

$$p\bar{p} = q\bar{q} = r\bar{r}$$

$$\Rightarrow |p^2| = |q|^2 = |r|^2$$

$$\Rightarrow |Z_2 - Z_3| = |Z_3 - Z_1| = |Z_1 - Z_2|$$

Therefore, the triangle is an equilateral triangle.

Let us also prove the converse from the other condition,

$$Z_1^2 + Z_2^2 + Z_3^2 - Z_1Z_2 - Z_2Z_3 - Z_3Z = 0 \qquad (5)$$

ω, ω^2 being the two imaginary cube roots of unity, Eq. (5) may be written as

$$(Z_1 + \omega Z_2 + \omega^2 Z_3)(Z_1 + \omega^2 Z_2 + \omega Z_3) = 0$$

Hence,

$$Z_1 - Z_2 = -Z_2 - \omega Z_2 - \omega^2 Z_3$$

$$= -Z_2(1 + \omega) - \omega^2 Z_3$$

$$= -Z_2(-\omega^2) - \omega^2 Z_3$$

$$Z_1 - Z_2 = \omega^2 (Z_2 - Z_3)$$

Therefore,

$$|Z_1 - Z_2| = |\omega^2||Z_2 - Z_3| \Rightarrow |Z_1 - Z_2| = |Z_2 - Z_3|$$

Similarly, it can be proved by combining the terms differently

$$|Z_1 - Z_3| = |Z_2 - Z_3|$$

Hence,

$$|Z_1 - Z_2| = |Z_2 - Z_3| = |Z_3 - Z_1|$$

Therefore, the triangle is an equilateral triangle.

8. Find all non-zero complex numbers satisfying $\bar{Z} = iZ^2$.

Solution: Let

$$Z = x + iy;\ \bar{Z} = x - iy;\ Z^2 = x^2 - y^2 + 2ixy$$

Therefore, the equation is

$$x - iy = i(x^2 - y^2 + 2ixy)$$

Equating real and imaginary parts, we get

$$x = -2xy \quad (1)$$

$$-y = x^2 - y^2 \quad (2)$$

Equation (1) gives either $x = 0$, in that case $y = 0$; $y = 1$, or

$$y = -\frac{1}{2}, \text{ in that case } \frac{1}{4} + \frac{1}{2} = x^2$$

Therefore,

$$x = \pm\frac{\sqrt{3}}{2}$$

Hence, the non-zero Z, satisfying the equation are

$$Z_1 = i;\ Z_2 = \frac{\sqrt{3}}{2} - \frac{1}{2}i;\ Z_3 = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$$

9. If A, B, C, D are four points in a plane forming a quadrilateral $ABCD$, then prove that $AC \cdot BD \le AB \cdot CD + AD \cdot BC$. When does the equality exist?

Solution: Let the four points A, B, C and D have associate complex numbers Z_1, Z_2, Z_3 and Z_4.

First factor Z_4 fixed, Z_1, Z_2, Z_3 cyclically changed. We have

$$(Z_1 - Z_4)(Z_2 - Z_3) + (Z_2 - Z_4)(Z_3 - Z_1) + (Z_3 - Z_4)(Z_1 - Z_2) = 0$$

$$\Rightarrow -(Z_3 - Z_1)(Z_2 - Z_4) = (Z_1 - Z_4)(Z_2 - Z_3) + (Z_3 - Z_4)(Z_1 - Z_2)$$

$$\Rightarrow |(Z_3 - Z_1)(Z_2 - Z_4)| = |(Z_1 - Z_4)(Z_2 - Z_3) + (Z_3 - Z_4)(Z_1 - Z_2)|$$

$$\Rightarrow |(Z_3 - Z_1)(Z_2 - Z_4)| \le |Z_1 - Z_4||Z_2 - Z_3| + |Z_3 - Z_4||Z_1 - Z_2|$$

Therefore,

$$AC \cdot BD \le AD \cdot BC + AB \cdot CD$$

When equality exists, we have

$$|(Z_1 - Z_4)(Z_2 - Z_3) + (Z_3 - Z_4)(Z_1 - Z_2)|$$

$$= |Z_1 - Z_4||Z_2 - Z_3| + |Z_3 - Z_4||Z_1 - Z_2|$$

$$\Rightarrow \arg\{(Z_1 - Z_4)(Z_2 - Z_3)\} = \arg\{(Z_3 - Z_4)(Z_1 - Z_2)\}$$

$$\Rightarrow \arg\{(Z_1 - Z_4)(Z_2 - Z_3)\} - \arg\{(Z_3 - Z_4)(Z_1 - Z_2)\} = 0$$

$$\Rightarrow \arg\left\{\frac{(Z_1 - Z_4)(Z_2 - Z_3)}{(Z_3 - Z_4)(Z_1 - Z_2)}\right\} = 0$$

This is possible, when A, B, C, D are concylic points, that is, when Z_1, Z_2, Z_3 and Z_4 represent four points which are concylic.

10. Solve the equation $Z + a|Z + 1| + i = 0$ (a is a real number ≥ 1).

Solution: Taking $Z = x + iy$, the equation reduces to

$$x + iy + a\sqrt{x^2 + 2x + 1 + y^2} + i = 0$$

$$\text{Imaginary} = 0 \Rightarrow y = -1$$

$$\text{Real part} = 0 \Rightarrow x + a\sqrt{x^2 + 2x + 1 + y^2} = 0$$

Eliminating y, the equation in x is

$$x^2 = a^2(x^2 + 2x + 2)$$

$$x^2(a^2 - 1) + 2a^2x + 2a^2 = 0$$

This gives real x only if

$$4a^4 - 8a^2(a^2 - 1) \ge 0$$

$$\Rightarrow -a^4 + 2a^2 \ge 0$$

$$\Rightarrow 0 \le a^2 \le 2$$

Also, $a \ge 1$(given), the value of a are $1 \le a \le \sqrt{2}$.

Therefore,

$$x = \frac{-a^2 \pm a\sqrt{2 - a^2}}{a^2 - 1},$$

$x < 0$ for the negative sign and for the positive sign also $x < 0$.

Hence, the solutions are

$$Z = \left\{\frac{-a^2 \pm a\sqrt{2 - a^2}}{a^2 - 1}\right\} - i$$

where

$$1 \le a \le \sqrt{2}$$

Previous Years' Solved JEE Main/AIEEE Questions

1. If ω is an imaginary cube root of unity then $(1 + \omega - \omega^2)^7$ equals

(A) 128ω **(B)** -128ω
(C) $128\omega^2$ **(D)** $-128\omega^2$

[AIEEE 2002]

Solution:

$$(1 + \omega - \omega^7)^4 = (-\omega^2 - \omega^2)^7 = (-2\omega^2)^7 = -128\omega^{14}$$

$$= -128(\omega^3)^4\omega^2 = -128\omega^2$$

Hence, the correct answer is option (D).

2. If $|z - 4| < |z - 2|$, then its solution is given by

(A) Re $z > 0$ **(B)** Re $z < 0$
(C) Re $z > 3$ **(D)** Re $z > 2$

[AIEEE 2002]

Solution: Let $z = x + iy$. Therefore,

$$|z - 4| < |z - 2|$$

$$\Rightarrow |x+iy-4| < |x+iy-2|$$
$$\Rightarrow |(x-4)+iy| < |(x-2)+iy|$$
$$\Rightarrow (x-4)^2+y^2 < (x-2)^2+y^2$$
$$\Rightarrow -4x < -12$$
$$\Rightarrow x > 3$$

Therefore, $|z-4| < |z-2|$ if $\text{Re}\, z > 3$.

Hence, the correct answer is option (C).

3. z and ω are two non-zero complex numbers such that $|z| = |\omega|$ and $\arg z + \arg \omega = \pi$, then z is equal to

(A) $\bar{\omega}$ **(B)** $-\bar{\omega}$
(C) ω **(D)** $-\omega$

[AIEEE 2002]

Solution: Let $|\omega| = r$ and $\arg\omega = \theta$. Therefore,

$$|z| = r \text{ and } \arg z = \pi - \theta$$
$$\Rightarrow |z| = r\cos(\pi-\theta) = r[\cos(\pi-\theta)+i\sin(\pi-\theta)]$$
$$= r(-\cos\theta + i\sin\theta) = -r(\cos\theta - i\sin\theta) = -\bar{\omega}$$

Therefore, $z = -\bar{\omega}$.

Hence, the correct answer is option (B).

4. If $\left(\frac{1+i}{1-i}\right)^x = 1$, then

(A) $x = 2n+1$, where n is any positive integer
(B) $x = 4n$, where n is any positive integer
(C) $x = 2n$, where n is any positive integer
(D) $x = 4n+1$, where n is any positive integer

[AIEEE 2003]

Solution:

$$\left(\frac{1+i}{1-i}\right)^x = \left(\frac{(1+i)(1+i)}{1+1}\right)^x = \left(\frac{1-1+2i}{2}\right)^x = i^x$$

So,

$$i^x = 1 \text{ or } i^x = i^{4n},\ n \in Z$$

Therefore, $x = 4n$, $n \in Z$.

Hence, the correct answer is option (B).

5. Let z_1 and z_2 be the roots of the equation $z^2 + az + b = 0$, z being complex number. Further, assume that the origin, z_1 and z_2 form an equilateral triangle, then

(A) $a^2 = 4b$ **(B)** $a^2 = b$
(C) $a^2 = 2b$ **(D)** $a^2 = 3b$

[AIEEE 2003]

Solution: We have

$$z_1 + z_2 = -\frac{a}{1} = -a \text{ and } z_1 z_2 = \frac{b}{1} = b$$

From the Fig. 5.30

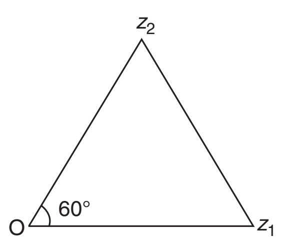


Figure 5.30

$$z_2 = z_1(\cos 60° + i\sin 60°)$$
$$\Rightarrow z_2 = z_1\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$
$$\Rightarrow 2z_2 = (1+\sqrt{3}i)z_1$$
$$\Rightarrow 2z_2 - z_1 = \sqrt{3}z_1 i$$
$$\Rightarrow 4z_2^2 + z_1^2 - 4z_1z_2 = -3z_1^2$$
$$\Rightarrow z_1^2 + z_2^2 = z_1 z_2$$
$$\Rightarrow (z_1+z_2)^2 = 3z_1z_2$$
$$\Rightarrow (-a)^2 = 3b \Rightarrow a^2 = 3b$$

Hence, the correct answer is option (D).

6. If z, ω are two non-zero complex numbers such that $|z\omega| = 1$ and $\arg(z) - \arg(\omega) = \pi/2$, then $\bar{z}\omega$ is equal to

(A) –1 **(B)** 1 **(C)** $-i$ **(D)** i

[AIEEE 2003]

Solution: Let $z = r_1 e^{i\theta_1}$, $\omega = r_2 e^{i\theta_2}$. Therefore,

$$z\omega = r_1 r_2 e^{i(\theta_1+\theta_2)}$$
$$\Rightarrow |z\omega| = 1 \Rightarrow r_1 r_2 = 1$$

So,

$$\arg z - \arg\omega = \frac{\pi}{2} \Rightarrow \theta_1 - \theta_2 = \frac{\pi}{2}$$

Therefore,

$$\bar{z}\omega = r_1 e^{-i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{-i(\theta_1-\theta_2)}$$
$$= 1\cdot e^{-i(\pi/2)} = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)$$
$$= 0 - i = -i$$

Hence, the correct answer is option (C).

7. Let z, ω be complex numbers such that $\bar{z} + i\bar{\omega} = 0$ and $\arg(z\omega) = \pi$. Then $\arg(z)$ is equal to

(A) $\frac{\pi}{4}$ **(B)** $\frac{\pi}{2}$ **(C)** $\frac{3\pi}{4}$ **(D)** $\frac{5\pi}{4}$

[AIEEE 2004]

Solution:

$$\bar{z} - i\bar{\omega} = 0 \Rightarrow \overline{\bar{z} + i\bar{\omega}} = 0 \Rightarrow z - i\bar{\bar{\omega}} = 0 \Rightarrow z + (-i)\omega = 0$$

Also,

$$\arg(z\omega) = \pi$$
$$\Rightarrow \arg[z - (-iz)] = \pi$$
$$\Rightarrow \arg(-i) + \arg z + \arg z = \pi$$
$$\Rightarrow -\frac{\pi}{2} + 2\arg z = \pi \Rightarrow \arg z = \frac{3\pi}{4}$$

Hence, the correct answer is option (C).

8. If $z = x - iy$ and $z^{1/3} = p + iq$, then $\dfrac{\frac{x}{p}+\frac{x}{q}}{p^2+q^2}$ is equal to

(A) 1 **(B)** –1 **(C)** 2 **(D)** –2

[AIEEE 2004]

Solution:

$$z^{1/3} = p + iq$$
$$\Rightarrow z = (p+iq)^3$$
$$\Rightarrow z - iy = p^3 + 3ip^2q - 3pq^2 - iq^3$$
$$\Rightarrow x = p^3 - 3pq^2 \text{ and } y = q^3 - 3p^2q$$

Therefore,

$$\frac{x}{p} + \frac{y}{q} = (p^2 - 3q^2) + (q^2 - 3p^2) = 2(p^2+q^2)$$
$$\Rightarrow \frac{\frac{x}{p}+\frac{y}{q}}{p^2+q^2} = -2$$

Hence, the correct answer is option (D).

9. If $|z^2 - 1| = |z|^2 + 1$, then z lies on

(A) the real axis **(B)** the imaginary axis
(C) a circle **(D)** an ellipse

[AIEEE 2004]

Solution: Let $z = x + iy$. Now,

$$|z^2 - 1| = |z|^2 + 1 \Rightarrow |(x+iy)^2 - 1| = |x+iy|^2 + 1$$
$$\Rightarrow |(x^2 - y^2 - 1) + 2ixy| = x^2 + y^2 + 1$$
$$\Rightarrow (x^2 - y^2 - 1)^2 + 4x^2y^2 = (x^2 + y^2 + 1)^2$$
$$\Rightarrow x^4 + y^4 + 1 - 2x^2y^2 + 2y^2 - 2x^2 + 4x^2y^2$$
$$= x^4 + y^4 + 1 + 4x^2y^2 + 2x^2 + 2y^2$$
$$\Rightarrow 4x^2 + 2x^2y^2 = 0 \Rightarrow 2x^2(2+y^2) = 0$$
$$\Rightarrow x^2 = 0 \Rightarrow x = 0$$

Therefore, z lies on the imaginary axis.
Hence, the correct answer is option (B).

10. If the cube roots of unity are 1, ω, ω^2, then the roots of the equation $(x-1)^3 + 8 = 0$ are

(A) $-1, 1+2\omega, 1+2\omega^2$ **(B)** $-1, 1-2\omega, 1-2\omega^2$
(C) $-1, -1, -1$ **(D)** $-1, -1+2\omega, -1, -2\omega^2$

[AIEEE 2005]

Solution: We have

$$(x-1)^3 + 8 = 0$$
$$\Rightarrow (x-1)^3 = -8 \Rightarrow \frac{(x-1)^3}{-8} = 1$$
$$\Rightarrow \left(\frac{x-1}{-2}\right)^3 = 1 \Rightarrow \frac{x-1}{-2} = 1^{1/3} = 1, \omega, \omega^2$$
$$\Rightarrow x - 1 = -2, -2\omega, -2\omega^2$$
$$\Rightarrow x = -1, 1-2\omega, 1-2\omega^2$$

Hence, the correct answer is option (B).

11. If z_1 and z_2 are two non-zero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then $\arg(z_1) - \arg(z_2)$ is equal to

(A) $-\frac{\pi}{2}$ **(B)** 0
(C) $-\pi$ **(D)** $\frac{\pi}{2}$

[AIEEE 2005]

Solution: Let $z_1 = a + ib$ and $z_2 = c + id$. Therefore,

$$z_1 + z_2 = a + c + i(b+d)$$

We are given $|z_1 + z_2| = |z_1| + |z_2|$. Then

$$\sqrt{(a+c)^2 + (b+d)^2} = \sqrt{a^2+b^2} + \sqrt{c^2+d^2}$$
$$\Rightarrow a^2 + c^2 + 2ac + b^2 + d^2 + 2bd = a^2 + b^2 + c^2 + d^2 + 2\sqrt{a^2 - b^2}\sqrt{c^2+d^2}$$
$$\Rightarrow ac + bd = \sqrt{(a^2+b^2)(c^2+d^2)}$$
$$\Rightarrow a^2c^2 + b^2d^2 + 2acbd = a^2c^2 + b^2d^2 + b^2c^2 + a^2d^2$$
$$\Rightarrow b^2c^2 + a^2d^2 - 2acbd = 0$$
$$\Rightarrow bc - ad = 0 \Rightarrow \frac{b}{a} = \frac{d}{c}$$
$$\Rightarrow \tan^{-1}\frac{b}{a} = \tan^{-1}\frac{d}{c}$$
$$\Rightarrow \arg z_1 = \arg z_2 \Rightarrow \arg z_1 - \arg z_2 = 0$$

Hence, the correct answer is option (B).

12. If $\omega = \dfrac{z}{z - \frac{1}{3}i}$ and $|\omega| = 1$, then z lies on

(A) a parabola **(B)** a straight line
(C) a circle **(D)** an ellipse

[AIEEE 2005]

Solution:

$$|\omega| = 1 \Rightarrow \left|\frac{z}{z - 1/3\, i}\right| = 1$$
$$\Rightarrow |z| = \left|z - \frac{1}{3}i\right|$$
$$\Rightarrow |z - (0+0i)| = \left|z - \left(0 + \frac{1}{3}i\right)\right|$$

So, z lies on the bisector of the line joining (0, 0) and $\left(0, \frac{1}{3}\right)$.
Therefore, z lies on a line.
Hence, the correct answer is option (B).

13. If $z^2 + z + 1 = 0$, where z is a complex number, then the value of $\left(z+\frac{1}{z}\right)^2 + \left(z^2+\frac{1}{z^2}\right)^2 + \left(z^3+\frac{1}{z^3}\right)^2 + \cdots + \left(z^6+\frac{1}{z^6}\right)^2$ is

(A) 18 **(B)** 54
(C) 6 **(D)** 12

[AIEEE 2006]

Solution:

$$z^2 + z + 1 = 0 \Rightarrow z = \frac{-1 \pm \sqrt{3}i}{2} = \omega, \omega^2$$

Let $z = \omega$. Therefore,

$$z + \frac{1}{z} = \omega + \frac{1}{\omega} = \omega + \omega^2 = -1$$
$$\Rightarrow z^2 + \frac{1}{z^2} = \omega^2 + \frac{1}{\omega^2} = \omega^2 + \omega = -1$$
$$\Rightarrow z^3 + \frac{1}{z^3} = \omega^3 + \frac{1}{\omega^3} = 1 + 1 = 2$$

$$\Rightarrow z^4+\frac{1}{z^4}=\omega^4+\frac{1}{\omega^4}=\omega+\omega^2=-1$$

$$\Rightarrow z^5+\frac{1}{z^5}=\omega^5+\frac{1}{\omega^5}=\omega+\omega^2=-1$$

$$\Rightarrow z^6+\frac{1}{z^6}=\omega^6+\frac{1}{\omega^6}=1+1=2$$

So, the required sum $=(-1)^2+(-1)^2+(2)^2+(-1)^2+(-1)^2+(2)^2=1+1+4+1+1+4=12$

Hence, the correct answer is option (D).

14. If $|z+4|\le 3$, then the maximum value of $|z+1|$ is

(A) 0 **(B)** 4
(C) 10 **(D)** 6

[AIEEE 2007]

Solution:

$$|z+1|=|z+4+(-3)|\le|z+4|+|-3|=|z+1|+3\le 3+3=6$$

Hence, the correct answer is option (D).

15. The conjugate of a complex number is $\frac{1}{i-1}$. Then that complex number is

(A) $\frac{-1}{i-1}$ **(B)** $\frac{1}{i+1}$

(C) $\frac{-1}{i+1}$ **(D)** $\frac{1}{i-1}$

[AIEEE 2008]

Solution: Let the required complex number be z. Therefore,

$$\bar{z}=\frac{1}{i-1}\Rightarrow \bar{\bar{z}}=\overline{\left(\frac{1}{i-1}\right)}$$

$$\Rightarrow z=\left(\frac{\bar{1}}{\overline{i-1}}\right)=\frac{1}{-i-1}=\frac{-1}{i+1}$$

Hence, the correct answer is option (C).

16. If $\left|z-\frac{4}{z}\right|=2$, then the maximum value of $|z|$ is equal to

(A) $\sqrt{3}+1$ **(B)** $\sqrt{5}+1$

(C) 2 **(D)** $2+\sqrt{2}$

[AIEEE 2009]

Solution:

$$|z|=\left|\left(z-\frac{4}{z}\right)+\frac{4}{z}\right|$$

$$\Rightarrow |z|\le\left|z-\frac{4}{z}\right|+\frac{4}{|z|}$$

$$\Rightarrow |z|\le 2+\frac{4}{|z|}$$

$$\Rightarrow |z|^2-2|z|-4\le 0$$

$$\Rightarrow (|z|-(\sqrt{5}+1))(|z|-(1-\sqrt{5}))\le 0$$

$$\Rightarrow 1-\sqrt{5}\le|z|\le\sqrt{5}+1$$

Hence, the correct answer is option (B).

17. The number of complex number z such that $|z-1|=|z+1|=|z-i|$ equals

(A) ∞ **(B)** 0
(C) 1 **(D)** 2

[AIEEE 2010]

Solution: Only one solution and that is circumcentre of triangle formed by (1, 0), (0, 1) and (–1, 0).

Hence, the correct answer is option (D).

18. If α and β are the roots of the equation $x^2-x+1=0$, then $\alpha^{2009}+\beta^{2009}=?$

(A) 2 **(B)** –2
(C) –1 **(D)** 1

[AIEEE 2010]

Solution:

$$\alpha=-\omega,\ \beta=-\omega^2$$

$$\Rightarrow \alpha^{2009}+\beta^{2009}=(-\omega)^{2009}+(-\omega^2)^{2009}=-\omega^2-\omega=1$$

Hence, the correct answer is option (D).

19. If $\omega(\ne 1)$ is a cube root of unity, and $(1+\omega)^7=A+B\omega$. Then (A, B) equals

(A) (1, 1) **(B)** (1, 0)
(C) (–1, 1) **(D)** (0, 1)

[AIEEE 2011]

Solution:

$$(1+\omega)^7=A+B\omega$$
$$\Rightarrow(-\omega^2)^7=A+B\omega$$
$$\Rightarrow -\omega^2=A+B\omega$$
$$\Rightarrow 1+\omega=A+B\omega$$
$$\Rightarrow A=1, B=1$$

Hence, the correct answer is option (A).

20. Let α, β be real and z be a complex number. If $z^2+\alpha z+\beta=0$ has two distinct roots on the line Rz = 1, then it is necessary that

(A) $\beta\in(-1,0)$ **(B)** $|\beta|=1$

(C) $\beta\in(1,\infty)$ **(D)** $\beta\in(0,1)$

[AIEEE 2011]

Solution: Since coefficients are real, so roots occur in conjugate pair, that is, $(1+ki)$ and $(1-ki)$, where $k\ne 0$ (since distinct roots).

Therefore,

$$S=\text{sum}=2=-\alpha$$
$$\alpha=-2$$
$$P=\text{product}=1+k^2=\beta$$

Therefore,

$$\beta\in(1,\infty)$$

Since,

$$(k\ne 0)$$

Hence, the correct answer is option (C).

21. If $z \neq 1$ and $\dfrac{z^2}{z-1}$ is real, then the point represented by the complex number z lies

(A) either on the real axis or on a circle passing through the origin.
(B) on a circle with centre at the origin.
(C) either on the real axis or on a circle not passing through the origin.
(D) on the imaginary axis.

[AIEEE 2012]

Solution: Let $z = x + iy$. Therefore,

$$z^2 = x^2 - y^2 + 2ixy$$

Now,

$$\frac{z^2}{z-1} \text{ is real} \Rightarrow \text{Im}\left(\frac{z^2}{z-1}\right) = 0$$

$$\Rightarrow \text{Im}\left(\frac{x^2 - y^2 + 2ixy}{(x-1)+iy}\right) = 0$$

$$\Rightarrow \text{Im}\left[(x^2 - y^2 + 2ixy)(x-1) - iy\right] = 0$$

$$\Rightarrow 2xy(x-1) - y(x^2 - y^2) = 0$$

$$\Rightarrow y(x^2 + y^2 - 2x) = 0$$

$$\Rightarrow y = 0;\ x^2 + y^2 - 2x = 0$$

Therefore, z lies either on real axis or on a circle through origin.
Hence, the correct answer is option (A).

22. If z is a complex number of unit modulus and argument θ, then $\arg\left(\dfrac{1+z}{1+\bar{z}}\right)$ equals

(A) $-\theta$ **(B)** $\dfrac{\pi}{2} - \theta$
(C) θ **(D)** $\pi - \theta$

[AIEEE 2013]

Solution: Given $|z| = 1$, $\arg z = \theta$

As we know, $\bar{z} = \dfrac{1}{z}$. Therefore,

$$\arg\left(\frac{1+z}{1+\bar{z}}\right) = \arg\left(\frac{1+z}{1+\frac{1}{z}}\right) = \arg(z) = \theta$$

Hence, the correct answer is option (C).

23. If z is a complex number such that $|z| \geq 2$, then the minimum value of $\left|z + \dfrac{1}{2}\right|$

(A) is strictly greater than $\dfrac{5}{2}$
(B) is strictly greater than $\dfrac{3}{2}$ but less than $\dfrac{5}{2}$
(C) is equal to $\dfrac{5}{2}$
(D) lies in the interval (1, 2)

[JEE MAIN 2014 (OFFLINE)]

Solution: See Fig. 5.31. Min$\left|z + \dfrac{1}{2}\right|$ is when z is at B.

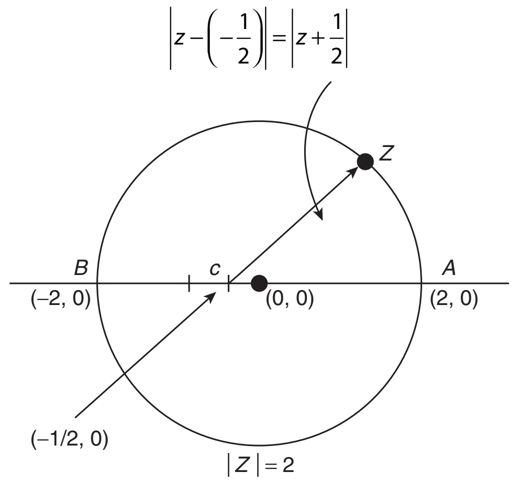


Figure 5.31

Therefore,

$$\text{Min}\left|z + \frac{1}{2}\right| = |BC| = 1 + \frac{1}{2} = \frac{3}{2}$$

So, Min$\left|z + \dfrac{1}{2}\right|$ lies in the interval (1, 2).

Hence, the correct answer is option (D).

24. If z_1, z_2 and z_3, z_4 are two pairs of complex conjugate numbers, then $\arg\left(\dfrac{z_1}{z_4}\right) + \arg\left(\dfrac{z_2}{z_3}\right)$ equals

(A) 0 **(B)** $\dfrac{\pi}{2}$ **(C)** $\dfrac{3\pi}{2}$ **(D)** π

[JEE MAIN 2014 (ONLINE SET 2)]

Solution: Let

$$z_1 = a + ib \Rightarrow z_2 = a - ib,\ z_3 = c + id \Rightarrow z_4 = c - id$$

$$\arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right) = \arg\left(\frac{z_1}{z_4} \times \frac{z_2}{z_3}\right)$$

$$= \arg\left(\frac{(a+ib)(a-ib)}{(c+id)(c+id)}\right) = \arg\left(\frac{a^2+b^2}{c^2+d^2}\right) = 0$$

$$\left(\text{since, } \frac{a^2+b^2}{c^2+d^2} \text{ is real.}\right)$$

Hence, the correct answer is option (A).

Previous Years' Solved JEE Advanced/ IIT-JEE Questions

1. A man walks a distance of 3 units from the origin towards the northeast (N45°E) direction. From there, he walks a distance of 4 units towards the northwest (N45°W) direction to reach a point P. Then, the position of P in the Argand plane is

(A) $3e^{i\pi/4} + 4i$ **(B)** $(3 - 4i)e^{i\pi/4}$
(C) $(4 + 3i)e^{i\pi/4}$ **(D)** $(3 + 4i)e^{i\pi/4}$

[IIT-JEE 2007]

Solution: See Fig. 5.32.

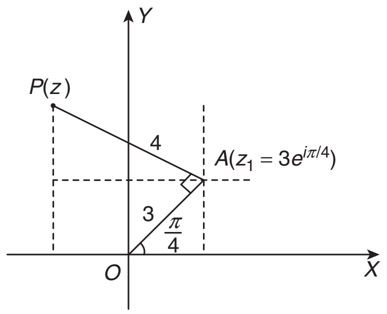


Figure 5.32

Since ΔOAP is a right-angle triangle, we get

$$\frac{z-z_1}{0-z_1}=\frac{|z-z_1|}{|0-z_1|}\times e^{-i(\pi/2)}$$

$$\frac{z-z_1}{(-z_1)}=\frac{4}{3}\times e^{-i(\pi/2)}$$

$$\Rightarrow z-z_1=-z_1\times\frac{4}{3}e^{-i(\pi/2)}$$

Therefore,

$$z=3e^{i(\pi/4)}-3e^{i(\pi/4)}\times\frac{4}{3}\times e^{-i(\pi/2)}$$

$$(\text{since } e^{-i(\pi/2)}=-i)$$

$$=3e^{i(\pi/4)}-4e^{-i(\pi/4)}$$

$$=(3-4e^{-i(\pi/2)})e^{i(\pi/4)}$$

$$=(3+4i)e^{i(\pi/4)}$$

$$(\text{since } e^{-i(\pi/2)}=-i)$$

Hence, the correct answer is option (D).

2. If $|z|=1$ and $z\neq\pm1$, then all the value of $\frac{z}{1-z^2}$ lie on

(A) a line not passing through the origin
(B) $|z|=\sqrt{2}$
(C) the x-axis
(D) the y-axis

[IIT-JEE 2007]

Solution: We have $|z|=1$. Let $z=\cos\theta+i\sin\theta$. Now,

$$\frac{z}{1-z^2}=\frac{\cos\theta+i\sin\theta}{1-(\cos\theta+i\sin\theta)^2}$$

$$=\frac{\cos\theta+i\sin\theta}{1-(\cos2\theta+i\sin2\theta)}$$

$$=\frac{\cos\theta+i\sin\theta}{(2\sin^2\theta)-i(2\sin\theta\cos\theta)}$$

$$=\frac{\cos\theta+i\sin\theta}{2\sin\theta(\sin\theta-i\cos\theta)}$$

$$=\frac{(\cos\theta+i\sin\theta)}{-2i\sin\theta(\cos\theta+i\sin\theta)}$$

$$=\frac{-1}{2i\sin\theta}=\frac{i}{2\sin\theta}$$

which is purely imaginary. Therefore, $\frac{z}{1-z^2}$ lies on y-axis, that is, $x=0$.

Hence, the correct answer is option (D).

3. A particle P starts from the point $z_0 = 1 + 2i$, where $i=\sqrt{-1}$. It moves first horizontally away from origin by 5 units and then vertically away from origin by 3 units to reach a point z_1. From z_1 the particle moves $\sqrt{2}$ units in the direction of the vector $\hat{i}+\hat{j}$ and then it moves through an angle $\frac{\pi}{2}$ in the anticlockwise direction on a circle with centre at origin, to reach a point z_2. The point z_2 is given by

(A) $6+7i$ **(B)** $-7+6i$
(C) $7+6i$ **(D)** $-6+7i$ **[IIT-JEE 2008]**

Solution: We have

$$z_0=(1+2i)$$

Since P moves first horizontally away from origin by 5 units and then vertically away from origin by 3 units, we have

$$z_1=(6+5i)$$

Since from z_1 the particle moves $\sqrt{2}$ units in the direction of $(i+j)$, then p reaches at $(7+6i)$.

Now, point $(7+6i)$ rotates $\frac{\pi}{2}$ in the anticlockwise direction on a circle with centre at origin.

Therefore,

$$z_2=(7+6i)e^{i\frac{\pi}{2}}$$
$$=-6+7i$$

Hence, the correct answer is option (D).

4. Let ω be the complex number $\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$. Then the number of distinct complex numbers z satisfying $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix}=0$ is equal to

[IIT-JEE 2010]

Solution:

$$\omega=e^{i2\pi/3}$$

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix}=0$$

$$R_1\to R_1+R_2+R_3$$

$$\Rightarrow\begin{vmatrix} z+1+\omega+\omega^2 & z+1+\omega+\omega^2 & z+1+\omega+\omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix}$$

$$=z\begin{vmatrix} 1 & 1 & 1 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix}$$

$$\Rightarrow z[(z+\omega^2)(z+\omega)-1-\omega(z+\omega-1)+\omega^2(1-z-\omega^2)]=0$$

$$\Rightarrow z^3 = 0$$

Therefore, $z = 0$ is the only solution.

Hence, the correct answer is (1).

5. If z is any complex number satisfying $|z-3-2i| \le 2$, then the minimum value of $|2z-6+5i|$ is ______.

[IIT-JEE 2011]

Solution: See Fig. 5.33.

$$\text{Length } AB = \frac{5}{2}$$

$$\Rightarrow \text{Minimum value} = 5$$

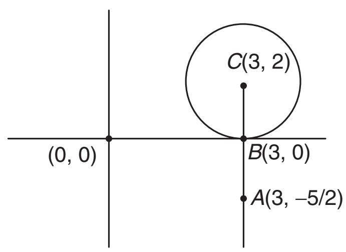


Figure 5.33

Hence, the correct answer is (5).

6. Let $\omega = e^{i\pi/3}$, and a, b, c, x, y, z be non-zero complex numbers such that

$$a+b+c = x$$
$$a+b\omega+c\omega^2 = y$$
$$a+b\omega^2+c\omega = z$$

Then the value of $\dfrac{|x|^2+|y|^2+|z|^2}{|a|^2+|b|^2+|c|^2}$ is ______.

[IIT-JEE 2011]

Solution: The expression may not attain integral value for all a, b, c. If we consider $a = b = c$, then

$$x = 3a$$
$$y = a(1+\omega+\omega^2) = a(1+i\sqrt{3})$$
$$z = a(1+\omega^2+\omega) = a(1+i\sqrt{3})$$

Therefore,

$$|x|^2+|y|^2+|z|^2 = 9|a|^2+4|a|^2+4|a|^2 = 17|a|^2$$

Therefore,

$$\frac{|x|^2+|y|^2+|z|^2}{|a|^2+|b|^2+|c|^2} = \frac{17}{3}$$

Note: However if $\omega = e^{i(2\pi/3)}$, then the value of the expression = 3.
Hence, the correct answer is (17/3).

7. Let z be a complex number such that the imaginary part of z is non-zero and $a = z^2+z+1$ is real. Then a cannot take the value

(A) -1 **(B)** $\frac{1}{3}$

(C) $\frac{1}{2}$ **(D)** $\frac{3}{4}$

[IIT-JEE 2012]

Solution: Given equation is $z^2+z+1-a=0$.

Clearly, this equation does not have real roots if

$$D < 0$$
$$\Rightarrow 1-4(1-a) < 0$$
$$\Rightarrow 4a < 3$$
$$\Rightarrow a < \frac{3}{4}$$

Hence, the correct answer is option (D).

8. Let $\omega = \dfrac{\sqrt{3}+i}{2}$ and $P = \{\omega^n : n = 1, 2, 3, \ldots\}$. Further $H_1 = \left\{z \in \mathrm{C} : \mathrm{Re}\, z > \dfrac{1}{2}\right\}$ and $H_2 = \left\{z \in \mathrm{C} : \mathrm{Re}\, z < \dfrac{-1}{2}\right\}$, where C is the set of all complex numbers. If $z_1 \in P \cap H_1$, $z_2 \in P \cap H_2$ and O represents the origin, then $\angle z_1Oz_2 = ?$

(A) $\frac{\pi}{2}$ **(B)** $\frac{\pi}{6}$

(C) $\frac{2\pi}{3}$ **(D)** $\frac{5\pi}{6}$

[JEE ADVANCED 2013]

Solution: See Fig. 5.34. We note that $|\omega| = 1$. We also note that α_i are possible values of z_1 and β_i are possible values of z_2, where i = 1, 2, 3. Therefore,

$$\omega = \frac{\sqrt{3}}{2}+\frac{i}{2},$$
$$\Rightarrow \omega = e^{i\frac{\pi}{6}},$$
$$\omega^2 = e^{i\frac{\pi}{3}},$$
$$\omega^3 = e^{i\frac{\pi}{2}},$$
$$\omega^4 = e^{i\frac{2\pi}{3}},$$
$$\omega^5 = e^{i\frac{5\pi}{6}}$$

Thus, $\angle z_1Oz_2$ can take the values $\frac{2\pi}{3}, \frac{5\pi}{6}$.

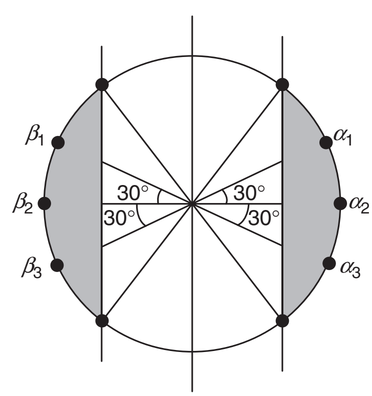


Figure 5.34

Hence, the correct answers are options (C) and (D).

9. $\min_{z \in S} |1-3i-z| =$

(A) $\frac{2-\sqrt{3}}{2}$ **(B)** $\frac{2+\sqrt{3}}{2}$

(C) $\frac{3-\sqrt{3}}{2}$ **(D)** $\frac{3+\sqrt{3}}{2}$

[JEE ADVANCED 2013]

Solution: We have $\min|1-3i-z|$. The minimum distance of z from $(1, -3)$ from $y+\sqrt{3}x=0$ is

$$\left|\frac{-3+\sqrt{3}}{2}\right| = \frac{3-\sqrt{3}}{2}$$

Hence, the correct answer is option (C).

10. Let $z_k = \cos\left(\frac{2k\pi}{10}\right) + i\sin\left(\frac{2k\pi}{10}\right); k = 1, 2, \ldots, 9.$

List I	List II
P. For each z_k there exists a z_j such that $z_k \cdot z_j = 1$	**1.** True
Q. There exists a $k \in \{1, 2, \ldots, 9\}$ such that $z_1 \cdot z = z_k$ has no solution z in the set of complex numbers.	**2.** False
R. $\frac{\|1-z_1\|\|1-z_2\|\ldots\|1-z_9\|}{10}$ equals	**3.** 1
S. $1-\sum_{k=1}^{9}\cos\left(\frac{2k\pi}{10}\right)$ equals	**4.** 2

	P	Q	R	S
(A)	1	2	4	3
(B)	2	1	3	4
(C)	1	2	3	4
(D)	2	1	4	3

[JEE ADVANCED 2014]

Solution:

For (P) in List I:

$$z_k = \cos\frac{2k\pi}{10} + i\sin\frac{2k\pi}{10}, \quad k = 1, 2, \cdots, 9$$

Let us take

$$z_k \times z_{10-k} = e^{\frac{i2k\pi}{10}} \times e^{\frac{i2(10-k)\pi}{10}} = e^{\frac{i2\pi}{10}(k+10-k)}$$

$$= e^{\frac{i2\pi}{10}\times 10} = \cos 2\pi + i\sin 2\pi = 1 + i(0) = 1$$

Therefore,

$$z_k \times z_{10-k} = 1$$
$$z_1 \times z_9 = 1$$
$$z_2 \times z_8 = 1$$
$$\vdots$$
$$z_9 \times z_1 = 1$$
$$\Rightarrow z_k \times z_j = 1 \qquad \text{(where } z_j = z_{10-k}\text{)}$$

Therefore,

$$(P) \to (1)$$

For (Q) in List I:

$$z_1 \cdot z = z_k \Rightarrow z = \frac{z_k}{z_1} = \frac{e^{i\frac{2k\pi}{10}}}{e^{i\frac{2\pi}{10}}} = e^{i\frac{2(k-1)\pi}{10}} = z_{k-1}$$

Therefore,

$$\text{for } k = 1,\ z = z_0 = \cos 0 + i\sin 0 = 1$$
$$\text{for } k = 2,\ z = z_1$$
$$\text{for } k = 9,\ z = z_8$$

So, solutions are there for $z_1 \cdot z = z_k$.

Therefore,

$$(Q) \to (2)$$

For (R) in List I: We know if $1, z_1, z_2, \ldots, z_n$ are n, n^{th} roots of unity, then they are the roots of $z^n - 1 = 0$.

Therefore,

$$(z-1)(z-z_1)(z-z_2)\cdots(z-z_{n-1}) = (z^n - 1)$$

$$= (z-1)(z^{n-1} + z^{n-2} + \cdots + z' + 1)$$

$$\Rightarrow (z-z_1)(z-z_2)\cdots(z-z_{n-1}) = z^{n-1} + 2^{n-2} + \cdots + z' + 1$$

Putting $z = 1$

$$(1-z_1)(1-z_2)\cdots(1-z_{n-1}) = 1+1+\cdots+1 = n$$

Therefore,

$$(1-z_1)(1-z_2)\cdots(1-z_9) = 10$$

So,

$$|1-z_1|\,|1-z_2|\cdots|1-z_9| = 10$$

$$\Rightarrow \frac{|1-z_1|\,|1-z_2|\cdots|1-z_9|}{10} = 1$$

Therefore,

$$(R) \to (3)$$

For (S) in List I: We know

$$1 + \propto + \propto^2 \cdots + \propto^{n-1} = 0$$

the sum of n, n^{th} roots of 1

$$z_0 + z_1 + z_2 + \cdots + z_9 = 0$$

$$\Rightarrow 1 - \{z_1 + z_2 + \cdots + z_9\} = 1 - (-z_0) = 1 - (-1) = 2$$

Therefore,

$$(S) \to (4)$$

Hence, the correct answer is option (C).

11. For any integer k, let $\alpha_k = \cos\left(\frac{k\pi}{7}\right) + i\sin\left(\frac{k\pi}{7}\right)$, where $i = \sqrt{-1}$. The value of the expression $\frac{\sum_{k=1}^{12}|\alpha_{k+1}-\alpha_k|}{\sum_{k=1}^{3}|\alpha_{4k-1}-\alpha_{4k-2}|}$ is ______.

[JEE ADVANCED 2015]

Solution:

$$\alpha_k = \cos\left(\frac{k\pi}{7}\right) + i\sin\frac{k\pi}{7} = e^{ik\pi/7}$$

Therefore,

$$\frac{\sum_{k=1}^{12}|\alpha_{k+1}-\alpha_k|}{\sum_{k=1}^{3}|\alpha_{4k-1}-\alpha_{4k-2}|} = \frac{\sum_{k=1}^{12}|e^{i(k+1)\pi/7}-e^{ik\pi/7}|}{\sum_{k=1}^{3}|e^{i(4k-1)\pi/7}-e^{i(4k-2)\pi/7}|}$$

$$= \frac{\sum_{k=1}^{12}|e^{ik\pi/7}||e^{i\pi/7}-1|}{\sum_{k=1}^{3}|e^{i(4k-2)\pi/7}||e^{i\pi/7}-1|}$$

$$= \frac{\sum_{k=1}^{12}(1)}{\sum_{k=1}^{3}|1|} = \frac{12}{3} = 4$$

Hence, the correct answer is (4).

Practice Exercise 1

1. For any integer n, the argument of $z = \frac{(\sqrt{3}+i)^{4n+1}}{(1-i\sqrt{3})^{4n}}$ is

(A) $\frac{\pi}{6}$ **(B)** $\frac{\pi}{3}$
(C) $\frac{\pi}{2}$ **(D)** $\frac{2\pi}{3}$

2. If α, β, γ are the cube roots of p, $(p < 0)$, then for any x, y, z $\frac{\alpha x+\beta y+\gamma z}{\beta x+\gamma y+\alpha z}$ is equal to

(A) $\alpha\omega+\beta\omega^2+\gamma$ **(B)** $\alpha\beta\gamma$
(C) ω, ω^2 **(D)** None of these

3. For any two complex numbers z_1, z_2 and real numbers a and b, $|az_1+bz_2|^2+|bz_1-az_2|^2$ is equal to

(A) $(a^2+b^2)(|z_1|^2+|z_2|^2)$ **(B)** $(a^2-b^2)(|z_1|^2+|z_2|^2)$
(C) $(a^2+b^2)(|z_1|+|z_2|)^2$ **(D)** None of these

4. If a and b are real numbers between 0 and 1 such that the points $z_1 = a+i$, $z_2 = 1+bi$, $z_3 = 0$ form an equilateral triangle, then a and b are equal to

(A) $a=b=2+\sqrt{3}$ **(B)** $a=b=2-\sqrt{3}$
(C) $a=b=-2+\sqrt{3}$ **(D)** $a=b=-2-\sqrt{3}$

5. If $z = re^{i\theta}$, then $|e^{iz}|$ is equal to

(A) $e^{-r\cos\theta}$ **(B)** $e^{r\cos\theta}$
(C) $e^{r\sin\theta}$ **(D)** $e^{-r\sin\theta}$

6. If $(1+i)(1+2i)(1+3i)\ldots(1+ni) = x+iy$, then $2.5.10\ldots(1+n^2)$ is equal to

(A) $1+n^2$ **(B)** x^2-y^2
(C) x^2+y^2 **(D)** None of these

7. If z_1 and z_2 are two complex numbers, then the equation of the perpendicular bisector of the segment joining z_1 and z_2 is

(A) $(\bar{z}_2-\bar{z}_1)z+(z_2-z_1)\bar{z} = |z_2|^2-|z_1|^2$
(B) $(\bar{z}_2-\bar{z}_1)z = (z_2-z_1)\bar{z}$
(C) $(\bar{z}_2-\bar{z}_1)z = |z_2|^2-|z_1|^2$
(D) None of these

8. If z lies on the circle $|z| = 1$, then $2/z$ lies on a

(A) Circle **(B)** Straight line
(C) Parabola **(D)** None of these

9. If p, q and r are positive integers and ω be an imaginary cube root of unity and $f(x) = x^{3p}+x^{3q+1}+x^{3r+2}$, then $f(\omega)$ is equal to

(A) 1 **(B)** 0 **(C)** -1 **(D)** None of these

10. If P and Q are represented by the complex numbers z_1 and z_2, such that $\left|\frac{1}{z_2}+\frac{1}{z_1}\right| = \left|\frac{1}{z_2}-\frac{1}{z_1}\right|$, then the circumcentre of ΔOPQ (where O is the origin) is

(A) $\frac{1}{2}(z_1-z_2)$ **(B)** $\frac{1}{3}(z_1+z_2)$
(C) $\frac{1}{2}(z_1+z_2)$ **(D)** $\frac{1}{3}(z_1-z_2)$

11. The complex numbers $\sin x + i\cos 2x$ and $\cos x - i\sin 2x$ are conjugate to each other for

(A) $x = n\pi$ **(B)** $x = 0$
(C) $x = (n+1/2)\pi$ **(D)** No value of x

12. For any complex number z, maximum value of $|z| - |z-1|$ is

(A) 0 **(B)** ½
(C) 1 **(D)** 3/2

13. If $(a_1+ib_1)(a_2+ib_2)\ldots(a_n+ib_n) = A+iB$, then $\sum_{j=1}^{n}\tan^{-1}\left(\frac{b_j}{a_j}\right)$ is equal to

(A) B/A **(B)** $\tan(B/A)$
(C) $\tan^{-1}\left(\frac{B}{A}\right)$ **(D)** $\tan^{-1}\left(\frac{A}{B}\right)$

14. The value of the expression $2\left(1+\frac{1}{\omega}\right)\left(1+\frac{1}{\omega^2}\right)+3\left(2+\frac{1}{\omega}\right)\left(2+\frac{1}{\omega^2}\right)+4\left(3+\frac{1}{\omega}\right)\left(3+\frac{1}{\omega^2}\right)+\cdots+(n+1)\left(n+\frac{1}{\omega}\right)\left(n+\frac{1}{\omega^2}\right)$, where ω is an imaginary cube root of unity, is

(A) $\frac{n(n^2+2)}{3}$ **(B)** $\frac{n(n^2-2)}{3}$
(C) $\frac{n^2(n+1)^2+4n}{4}$ **(D)** None of these

15. If $z = x+iy$, $z^{1/3} = a-ib$ and $\frac{x}{a}-\frac{y}{b} = \lambda(a^2-b^2)$, then λ is equal to

(A) 1 **(B)** 2
(C) 3 **(D)** 4

16. The roots of the cubic equation $(z+ab)^3 = a^3$, where $a \neq 0$ represents the vertices of a triangle. If α, β and γ are the roots, then find the value of $|\alpha - \beta|$.

(A) $\frac{1}{\sqrt{3}}|ab|$ **(B)** $\sqrt{3}|a|$

(C) $\sqrt{3}|b|$ **(D)** $\sqrt{3}|ab|$

17. If $x + \frac{1}{x} = 1$, then $x^{2000} + \frac{1}{x^{2000}}$ is equal to

(A) 1 **(B)** −1

(C) 0 **(D)** None of these

18. If $w = \alpha + i\beta$, where $\beta \neq 0$ and $z \neq 1$, satisfies the condition that $\left(\frac{w - \bar{w}z}{1-z}\right)$ is purely real, then the set of values of z is

(A) $\{z: |z| = 1\}$ **(B)** $\{z: z = \bar{z}\}$

(C) $\{z: z \neq 1\}$ **(D)** $\{z: |z| = 1, z \neq 1\}$

19. If $|z - i| < 1$, then the value of $|z + 12 - 6i|$ is less than

(A) 14 **(B)** 2

(C) 28 **(D)** None of these

20. If a, b and c are integers not all equal and w is a cube root of unity ($w \neq 1$), then the minimum value of $|a + bw + cw^2|$ is

(A) 0 **(B)** 1

(C) $\sqrt{3}/2$ **(D)** 1/2

21. If $\theta_i \in [0, \pi/6]$, $i = 1, 2, 3, 4, 5$ and $\sin\theta_1 z^4 + \sin\theta_2 z^3 + \sin\theta_3 z^2 + \sin\theta_4 z + \sin\theta_5 = 2$, then z satisfies

(A) $|z| > 3/4$ **(B)** $|z| < 1/2$

(C) $1/2 < |z| < 3/4$ **(D)** None of these

22. If $|z - 2| = \min\{|z - 1|, |z - 3|\}$, where z is a complex number, then

(A) $\text{Re}(z) = 3/2$ **(B)** $\text{Re}(z) = 5/2$

(C) $\text{Re}(z) \in \left\{\frac{3}{2}, \frac{5}{2}\right\}$ **(D)** None of these

23. If a complex number x satisfies $\log_{1/\sqrt{2}}\left(\frac{|z|^2 + 2|z| + 6}{2|z|^2 - 2|z| + 1}\right) < 0$, then locus/region of the point represented by z is

(A) $|z| = 5$ **(B)** $|z| < 5$

(C) $|z| > 1$ **(D)** $2 < |z| < 3$

24. The points representing complex number z for which $|z - 3| = |z - 5|$ lie on the locus given by

(A) Circle **(B)** Ellipse

(C) Straight line **(D)** None of these

25. If z is a complex number lying in the first quadrant such that $\text{Re}(z) + \text{Im}(z) = 3$, then the maximum value of $\{\text{Re}(z)\}^2\,\text{Im}(z)$ is

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4

26. The equation $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$, $b \in R$ represents a circle, if

(A) $|a|^2 = |b|^2$ **(B)** $|a|^2 \geq b$

(C) $|a|^2 < b$ **(D)** None of these

27. If z_1, z_2, z_3 are the vertices of an equilateral triangle inscribed in the circle $|z| = 1$, then the area of the triangle having $z_1, -z_2, z_3$, as its vertices is

(A) $\frac{\sqrt{3}}{2}$ **(B)** $\frac{\sqrt{3}}{4}$ **(C)** $\frac{3\sqrt{3}}{4}$ **(D)** None of these

28. Let z_1 and z_2 be two complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$. Then

(A) $\arg(z_1) \neq \arg(z_2)$ **(B)** $\arg(z_1) + \arg(z_2) = 0$

(C) $\arg\left(\frac{z_1}{z_2}\right) = 0$ **(D)** None of these

29. If $|z_1| + |z_2| + |z_3| = |z_1 + z_2 + z_3|$, if z is defined as

$z = \frac{z_1 z_2}{z_3^2} + \frac{z_2 z_3}{z_1^2} + \frac{z_1 z_3}{z_2^2}$, then

(A) z is a purely real number

(B) z is a purely imaginary number

(C) $\text{Re}(z) = \text{Im}(z)$

(D) None of these

30. The point of intersection of the curves $\arg(z + 3 - 4i) = \frac{2\pi}{3}$ and $\arg(3z + 2 - 3i) = \frac{\pi}{4}$ is

(A) $\frac{1}{4}(5 - 7i)$ **(B)** $\frac{1}{4}(7 - 5i)$

(C) $(1 - i)$ **(D)** None of these

31. If 'α' be the non-real n^{th} root of unity, then $1 + 3\alpha + 5\alpha^2 + \cdots + (2n-1)\alpha^{n-1}$ is equal to

(A) $\frac{2n}{1-\alpha}$ **(B)** $\frac{n}{1-\alpha}$

(C) $\frac{n}{2(1-\alpha)}$ **(D)** None of these

32. If z_1 and z_2 are two complex numbers satisfying the equation $\left|\frac{z_1 + iz_2}{z_1 - iz_2}\right| = 1$, then $\frac{z_1}{z_2}$ is

(A) Purely real **(B)** Unit modulus

(C) Purely imaginary **(D)** None of these

33. If $(1 + x + x^2)^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_{2n} x^{2n}$, then $a_0 + a_3 + a_6 + a_9 + \cdots$ is equal to

(A) 3^n **(B)** 3^{n-1}

(C) 3^{n+1} **(D)** None of these

34. If $5 < |z|^2 \leq 12$ and $z^2 + \bar{z}^2 - 2z\bar{z} + 8z + 8\bar{z} > 0$, then

(A) $1 < \text{Re}(z) \leq 2\sqrt{3}$ and $|\text{Im}(z)| < 2\sqrt{2}$

(B) $-2\sqrt{3} \leq \text{Re}(z) < -1$ and $|\text{Im}(z)| < 2\sqrt{2}$

(C) $1 < \text{Re}(z) \leq 2\sqrt{3}$ and $|\text{Im}(z)| < 2\sqrt{3}$

(D) $-2\sqrt{3} \leq \text{Re}(z) < -1$ and $|\text{Im}(z)| < 2\sqrt{3}$

35. The maximum area of the triangle formed by the complex coordinates z, z_1, z_2 which satisfy the relations $|z - z_1| = |z - z_2|$, $\left|z - \left(\frac{z_1 + z_2}{2}\right)\right| \leq r$, where $r > |z_1 - z_2|$ is

(A) $\frac{1}{2}|z_1 - z_2|^2$ **(B)** $\frac{1}{2}|z_1 - z_2|r$

(C) $\frac{1}{2}|z_1 - z_2|^2 r^2$ **(D)** $\frac{1}{2}|z_1 - z_2|r^2$

36. Locus of z if $\arg[z-(1+i)] = \begin{cases} \dfrac{3\pi}{4} & \text{when } |z| \le |z-2| \\ \dfrac{-\pi}{4} & \text{when } |z| > |z-2| \end{cases}$ is

(A) Straight lines passing through (2, 0)
(B) Lines passing through (2, 0), (1, 1)
(C) A line segment
(D) A set of two rays

37. If $z = \left(\dfrac{\sqrt{3}+i}{2}\right)^5 + \left(\dfrac{\sqrt{3}-i}{2}\right)^5$, then

(A) $\text{Re}(z) = 0$ **(B)** $\text{Im}(z) = 0$
(C) $\text{Re}(z), \text{Im}(z) > 0$ **(D)** $\text{Re}(z) > 0, \text{Im}(z) < 0$

38. See Fig. 5.35. The locus of Z which lies in the shaded region is best represented by

(A) $z:|z+1| > 2, |\arg(z+1)| < \pi/4$
(B) $z:|z-1| > 2, |\arg(z-1)| < \pi/4$
(C) $z:|z+1| < 2, |\arg(z+1)| < \pi/2$
(D) $z:|z-1| < 2, |\arg(z-1)| < \pi/2$

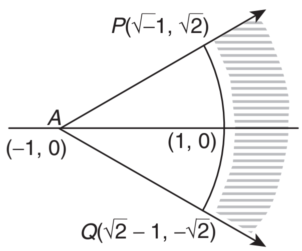


Figure 5.35

39. The number of complex numbers satisfying $|z+2| + |z-2| = 8$ and $|z-1| + |z+1| = 2$ is

(A) 4 **(B)** 2
(C) 0 **(D)** None of these

40. If z_1, z_2, z_3 are complex number such that $|z_1| = 2, |z_2| = 3, |z_3| = 4$, then maximum value of $|z_1 - z_2|^2 + |z_2 - z_3|^2 + |z_3 - z_1|^2$ is

(A) 58 **(B)** 29
(C) 87 **(D)** None of these

41. The complex number z and ω satisfying $z^3 + \bar{\omega}^7 = 0$ and $z^5 . \omega^{11} = 1$ are

(A) $\omega = \pm i, z = i$ **(B)** $\omega = \pm 1, z = 1$
(C) $\omega = \pm i, z = \pm i$ **(D)** None of these

42. The greatest positive argument of complex number satisfying $|z-4| = \text{Re}(z)$ is

(A) $\dfrac{\pi}{3}$ **(B)** $\dfrac{2\pi}{3}$
(C) $\dfrac{\pi}{2}$ **(D)** $\dfrac{\pi}{4}$

43. The complex number associated with the vertices A, B, C of the ΔABC are $e^{i\theta}$, ω, $\bar{\omega}$, respectively (where ω, $\bar{\omega}$ are the complex cube roots of unity and $\cos\theta > \text{Re}(\omega)$), then the complex number of the point where angle bisector of A meets the circumcircle of the triangle is

(A) $-e^{i\theta}$ **(B)** $-e^{-i\theta}$
(C) $\omega\bar{\omega}$ **(D)** $\omega + \bar{\omega}$

44. If $k + |k + z^2| = |z|^2$ $(k \in R^-)$, then argument of z is

(A) 0 **(B)** π
(C) $\pi/2$ **(D)** None of these

45. Let $\alpha_r = e^{i\frac{2\pi r}{n}}$ $(1 \le r \le n)$ be the complex number associated with the point A_r on Argand plane, and point B is (2, 0). Then the value of $BA_1 \cdot BA_2 \cdot BA_3 \ldots BA_n$ is equal to

(A) n **(B)** $2^n - 1$
(C) $2n$ **(D)** $2n - 1$

46. If x and y are complex numbers, then the system of equations $(1+i)x + (1-i)y = 1$, $2ix + 2y = 1 + i$ has

(A) Unique solution
(B) No solution
(C) Infinite number of solutions
(D) None of these

47. If $z = x + iy$ $(x, y \in R, x \ne -1/2)$, then the number of values of z satisfying $|z|^n = z^2|z|^{n-2} + z|z|^{n-2} + 1$ $(n \in N, n > 1)$ is

(A) 0 **(B)** 1
(C) 2 **(D)** 3

48. The value of $i\log(x-i) + i^2\pi + i^3\log(x+i) + i^4(2\tan^{-1}x)$ (where $x > 0$) is (where $i = \sqrt{-1}$)

(A) 0 **(B)** 1
(C) 2 **(D)** 3

Practice Exercise 2

Single/Multiple Correct Choice Type Questions

1. $\displaystyle\sum_{j=1}^{n-1} \frac{1}{1 - e^{\left(\frac{2i\pi j}{n}\right)}}$ is equal to

(A) $\dfrac{n+1}{2}$ **(B)** $\dfrac{n}{2}$
(C) $\dfrac{n-1}{2}$ **(D)** $\dfrac{n+1}{2} - 1$

2. If $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \cdots + {}^nC_nx^n$ where ${}^nC_0, {}^nC_1, {}^nC_2, \ldots$ are binomial coefficients, then $2(C_0 + C_3 + C_6 + \cdots) + (C_1 + C_4 + C_7 + \cdots)(1 + \omega) + (C_2 + C_5 + C_8 + \cdots)(1 + \omega^2)$ where ω is the cube root of unity and n is a multiple of 3 is equal to

(A) $2^n + 1$ **(B)** $2^{n-1} + 1$
(C) $2^{n+1} - 1$ **(D)** $2^n - 1$

3. The perpendicular distance of line $(1-i)z + (1+i)\bar{z} + 3 = 0$ from $(3 + 2i)$ will be

(A) 13 **(B)** $\dfrac{13}{2}$
(C) 26 **(D)** None of these

4. The complex number associated with the centre of the circle represented by $\arg\left(\dfrac{z-3i}{z-2i+4}\right) = \dfrac{\pi}{4}$ is

(A) $\dfrac{1}{2}(5i+5)$ **(B)** $\dfrac{1}{2}(5i-5)$
(C) $\dfrac{1}{2}(9i+5)$ **(D)** $\dfrac{1}{2}(9i-5)$

5. If arg(z) < 0, then arg $\left(\frac{z-\bar{z}}{2}\right)$ is equal to

(A) 0 **(B)** $\frac{\pi}{2}$

(C) $-\frac{\pi}{2}$ **(D)** π

6. If complex number z satisfies $|z - 6i| = \text{Im}(z)$, then range of ($\arg z - \arg\bar{z}$) will be

(A) $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ **(B)** $\left[\frac{2\pi}{3}, \frac{4\pi}{3}\right]$

(C) $\left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$ **(D)** $\left[\frac{3\pi}{4}, \frac{5\pi}{3}\right]$

7. If $z = x + iy$ is a complex number with $x, y \in Q$ and $|z| = 1$, then $|z^{2n} - 1|$ is a _____ for every $n \in N$.

(A) Real number **(B)** Rational number

(C) Irrational number **(D)** Cannot say anything

8. Let z_k; k = 1, 2, 3, 4 be four complex numbers such that $|z_k| = \sqrt{k+1}$ and $|30z_1 + 20z_2 + 15z_3 + 12z_4| = k|z_2z_3z_4 + z_3z_4z_1 + z_1z_2z_4 + z_1z_2z_3|$. Then k is equal to

(A) $|z_1z_2z_4|$ **(B)** $|z_2z_3z_4|$

(C) $|z_1z_3z_4|$ **(D)** $|z_1z_2z_3|$

9. For a complex number z, the minimum value of $|z| + |z - \cos\alpha - i\sin\alpha|$ is

(A) 0 **(B)** 1

(C) 2 **(D)** None of these

10. If α is a complex constant such that $\alpha z^2 + z + \bar{\alpha} = 0$ has a real root, then

(A) $\alpha + \bar{\alpha} = 0$ **(B)** $\alpha - \bar{\alpha} = 0$

(C) $\alpha + \bar{\alpha} = -1$ **(D)** None of these

11. Suppose $\alpha + i\beta$ is a solution of the polynomial equation $a_4z^4 + i\,a_3z^3 + a_2z^2 + i\,a_1z + a_0 = 0$ where $\alpha, \beta, a_i \in R\ \forall\ i \in \{0, 1, 2, 3, 4\}$. Which one of the following must also be a solution?

(A) $-\alpha - \beta i$ **(B)** $\alpha - \beta i$

(C) $-\alpha + \beta i$ **(D)** $\beta + \alpha i$

12. Let a, b, c be distinct complex numbers with $|a| = |b| = |c| = 1$ and z_1, z_2 be the roots of the equation $az^2 + bz + c = 0$ with $|z_1| = 1$. Also, let P and Q points represent the complex numbers z_1 and z_2 in the complex plane with $\angle POQ = \theta$ where O being the origin. Then

(A) $b^2 = ac$, $\theta = \frac{2\pi}{3}$ **(B)** $\theta = \frac{2\pi}{3}$, $PQ = \sqrt{3}$

(C) $PQ = 2\sqrt{3}$, $b^2 = ac$ **(D)** $\theta = \frac{\pi}{3}$, $b^2 = ac$

13. If $0 < c < b < a$ and the roots α, β of the equation $cx^2 + bx + a = 0$ are imaginary, then

(A) $\frac{|\alpha| + |\beta|}{2} = |\alpha|\,|\beta|$ **(B)** $\frac{1}{|\alpha|} = \frac{1}{|\beta|}$

(C) $\frac{1}{|\alpha|} + \frac{1}{|\beta|} < 2$ **(D)** None of these

14. Let z, z_0 be two complex numbers $\bar{z}_0$ being the conjugate of z_0. The numbers z, z_0, $z\bar{z}_0$, 1 and 0 are represented in an Argand diagram by P, P_0, Q, A and origin, respectively. If $|z| = 1$, then

(A) POP_0 and AOQ are congruent

(B) $|z - z_0| = |z\bar{z}_0 - 1|$

(C) $|z - z_0| = \frac{1}{2}|z\bar{z}_0 - 1|$

(D) None of these

15. The complex number z satisfying $|z + \bar{z}| + |z - \bar{z}| = 2$ and $|iz - 1| + |z - i| = 2$ is/are

(A) i **(B)** $-i$

(C) $\frac{1}{i}$ **(D)** $\frac{1}{i^3}$

16. A complex number z is rotated in anticlockwise direction by an angle α and we get z' and if the same complex number z is rotated by an angle α in clockwise direction and we get z'' then

(A) z', z, z'' are in GP **(B)** $z'^2 + z''^2 = 2z^2 \cos 2\alpha$

(C) $z' + z'' = 2z \cos \alpha$ **(D)** z', z, z'' are in HP

Comprehension Type Questions

Paragraph for Questions 17–19: Let $A(z_1)$, $B(z_2)$, $C(z_3)$ and $D(z_4)$ be the vertices of a trapezium in an Argand plane. Let $|z_1 - z_2| = 4$, $|z_3 - z_4| = 10$ and the diagonals AC and BD intersect at P.

It is given that $\arg\left(\frac{z_4 - z_2}{z_3 - z_1}\right) = \frac{\pi}{2}$ and $\arg\left(\frac{z_3 - z_2}{z_4 - z_1}\right) = \frac{\pi}{4}$.

17. Area of the trapezium $ABCD$ is equal to

(A) $\frac{130}{3}$ **(B)** $\frac{160}{3}$

(C) $\frac{190}{3}$ **(D)** None of these

18. Area of triangle PCB is equal to

(A) $\frac{100}{21}$ **(B)** $\frac{200}{21}$

(C) $\frac{100}{7}$ **(D)** $\frac{400}{21}$

19. $|CP - DP|$ is equal to

(A) $\frac{10}{\sqrt{21}}$ **(B)** $\frac{16}{\sqrt{21}}$

(C) $\frac{17}{\sqrt{21}}$ **(D)** $\frac{19}{\sqrt{21}}$

Paragraph for Questions 20–22: On the side AB and BC of a ΔABC, squares are drawn with centre D and E such that points C and D lies on the same side of line AB and points A and E lies in the opposite side of line BC. If A, B and C are represented by the complex numbers 1, ω and ω^2, respectively, then

20. Angle between AC and DE is equal to

(A) $\frac{\pi}{3}$ **(B)** $\frac{\pi}{6}$

(C) $\frac{\pi}{4}$ **(D)** $\frac{\pi}{2}$

21. The length of DE is

(A) $\sqrt{\frac{3}{2}}$ **(B)** $\sqrt{3}$

(C) $\sqrt{2}$ **(D)** $\sqrt{6}$

22. The length of AE is

(A) $\frac{3-\sqrt{3}}{2}$ **(B)** $\frac{3+\sqrt{3}}{2}$

(C) $3-\sqrt{3}$ **(D)** $3+\sqrt{3}$

Paragraph for Questions 23–25: Consider a triangle having vertices at the points $A(2e^{i\pi/4})$, $B(2e^{11i\pi/12})$ and $C(2e^{-5i\pi/12})$. Let the incircle of ΔABC touches the sides BC, CA and AB at D, E and F, respectively, which are represented by the complex number Z_d, Z_e, Z_f in order. If $P(z)$ be any point on the incircle, then

23. $AP^2 + BP^2 + CP^2$ is equal to

(A) 12 **(B)** 15

(C) 16 **(D)** $\frac{27}{2}$

24. $\text{Re}\left(\frac{1}{z_d}+\frac{1}{z_e}+\frac{1}{z_f}\right)$ is equal to

(A) $\sqrt{2}$ **(B)** $\frac{1}{\sqrt{2}}$

(C) $-\frac{1}{\sqrt{2}}$ **(D)** None of these

25. If the altitude through vertex A cuts the circumcircle of ΔABC at Q, then the complex number representing Q is

(A) $-\sqrt{2}(1+i)$ **(B)** $-(1+i)$

(C) $\frac{-(1+i)}{\sqrt{2}}$ **(D)** $-\frac{1}{2}(1+i)$

Paragraph for Questions 26–28: In an Argand plane z_1, z_2 and z_3 are respectively the vertices of an isosceles triangle ABC with $AC = BC$ and $\angle CAB = \theta$. If z_4 is the incentre of the triangle, then

26. The value of $\left(\frac{AB}{IA}\right)^2\left(\frac{AC}{AB}\right)$

(A) $\frac{(z_2-z_1)(z_1-z_3)}{(z_4-z_1)^2}$ **(B)** $\frac{(z_2-z_1)(z_3-z_1)}{(z_4-z_1)}$

(C) $\frac{(z_2-z_1)(z_3-z_1)}{(z_4-z_1)^2}$ **(D)** None of these

27. The value of $(z_4-z_1)^2\ (1+\cos\theta)\sec\theta$ is

(A) $(z_2-z_1)(z_3-z_1)$ **(B)** $\frac{(z_2-z_1)(z_3-z_1)}{(z_4-z_1)}$

(C) $\frac{(z_2-z_1)(z_3-z_1)}{(z_4-z_1)^2}$ **(D)** $(z_2-z_1)(z_3-z_1)^2$

28. The value of $(z_2-z_1)^2 \tan\theta\cdot\tan\frac{\theta}{2}$ is

(A) $(z_1+z_2-2z_3)(z_1+z_2-2z_4)$

(B) $(z_1+z_2-z_3)(z_1+z_2-z_4)$

(C) $-(z_1+z_2-2z_3)(z_1+z_2-2z_4)$

(D) None of these

Paragraph for Questions 29–31: Consider the quadratic equation $az^2+bz+c=0$ where a, b, c and z are complex numbers, then

29. The condition that the equation has both real roots is

(A) $\frac{a}{\bar{a}}=-\frac{b}{\bar{b}}=\frac{c}{\bar{c}}$ **(B)** $\frac{a}{\bar{a}}=\frac{b}{\bar{b}}=\frac{c}{\bar{c}}$

(C) $\frac{a}{\bar{a}}=\frac{b}{\bar{b}}=-\frac{c}{\bar{c}}$ **(D)** None of these

30. The condition that equation has both roots purely imaginary is

(A) $\frac{a}{\bar{a}}=-\frac{b}{\bar{b}}=-\frac{c}{\bar{c}}$ **(B)** $\frac{a}{\bar{a}}=-\frac{b}{\bar{b}}=\frac{c}{\bar{c}}$

(C) $\frac{a}{\bar{a}}=\frac{b}{\bar{b}}=-\frac{c}{\bar{c}}$ **(D)** None of these

31. The Condition that equation has one complex root m such that $|m|=1$ is

(A) $\frac{\bar{b}c-b\bar{a}}{a\bar{a}-c\bar{c}}=\frac{a\bar{a}+c\bar{c}}{\bar{c}b+a\bar{b}}$

(B) $\frac{\bar{b}c+b\bar{a}}{a\bar{a}+c\bar{c}}=\frac{a\bar{a}+c\bar{c}}{\bar{c}b+a\bar{b}}$

(C) $(\bar{b}c-b\bar{a})(\bar{c}b-a\bar{b})=(a\bar{a}-c\bar{c})^2$

(D) None of these

Matrix Match Type Questions

32. Match the following:

Column I	Column II
(A) $f(z)$ is a complex valued function $f(z) = (a+ib)z$ where $a, b \in R$ and $\lvert a+ib\rvert = \frac{1}{\sqrt{2}}$. It has the property that $f(z)$ is always equidistant from 0 and z, then $a-b=$	**(p)** 5
(B) Let $z_1 = 6+i$ and $z_2 = 4-3i$ and z be a complex number such that $\arg\left(\frac{z-z_1}{z_2-z}\right)=\frac{\pi}{2}$ and z satisfies $\lvert z-(5-i)\rvert = \sqrt{a}$. Then a is	**(q)** 0
(C) If A is the region of the complex plane $\{z: z/4$ and $4/\bar{z}$ have real andimaginary part in $(0, 1)\}$, then $[p]$ (where p is the area of the region A and [.] denotes the greatest integer function) is	**(r)** 6
(D) Let z be a root of $x^5-1=0$ with $z\neq 1$. The value of $z^{15}+z^{16}+\cdots+z^{50}$ is	**(s)** 1

33. Match the following:

List I	List II
(A) $\arg\left(\frac{z^2-1}{z^2+1}\right)=0;\ z\neq\pm i, \pm 1$	**(p)** portions of a line
(B) $\left\lvert\, \lvert z-\cos^{-1}\cos 12\rvert - \lvert z-\sin^{-1}\sin 12\rvert \,\right\rvert = 8(\pi-3)$	**(q)** point of intersection of hyperbolae
(C) $z^2+k_1 = i\lvert z_1\rvert^2+k_2;\ k_1\neq k_2 \in R-\{0\}$ and z_1 is fixed $\neq 0$	**(r)** pair of open rays
(D) $\left\lvert z-1-\sin^{-1}\frac{1}{\sqrt{3}}\right\rvert+\left\lvert z+\cos^{-1}\frac{1}{\sqrt{3}}-\frac{\pi}{2}\right\rvert=1$	**(s)** line segment

Answer Key

Practice Exercise 1

1. (A)	**2.** (C)	**3.** (A)	**4.** (B)	**5.** (D)	**6.** (C)	**7.** (A)	**8.** (A)
9. (B)	**10.** (A)	**11.** (D)	**12.** (C)	**13.** (C)	**14.** (C)	**15.** (D)	**16.** (D)
17. (B)	**18.** (B)	**19.** (A)	**20.** (B)	**21.** (A)	**22.** (C)	**23.** (B)	**24.** (C)
25. (D)	**26.** (B)	**27.** (B)	**28.** (C)	**29.** (A)	**30.** (A)	**31.** (D)	**32.** (A)
33. (B)	**34.** (A)	**35.** (B)	**36.** (D)	**37.** (B)	**38.** (A)	**39.** (C)	**40.** (C)
41. (C)	**42.** (D)	**43.** (D)	**44.** (C)	**45.** (B)	**46.** (C)	**47.** (B)	**48.** (A)

Practice Exercise 2

1. (C)	**2.** (D)	**3.** (B)	**4.** (D)	**5.** (C)	**6.** (A)	**7.** (B)	**8.** (A)
9. (B)	**10.** (C)	**11.** (C)	**12.** (A), (B)	**13.** (A), (B), (C)	**14.** (A), (B)	**15.** (A), (B), (C), (D)	**16.** (A), (B), (C)
17. (D)	**18.** (B)	**19.** (A)	**20.** (C)	**21.** (A)	**22.** (B)	**23.** (B)	**24.** (D)
25. (A)	**26.** (C)	**27.** (A)	**28.** (C)	**29.** (B)	**30.** (B)	**31.** (C)	**32.** (A) → (q),
(B) → (p), (C) → (p), (D) → (s)		**33.** (A) → (p), (B) → (p), (C) → (q), (D) → (s)					

Solutions

Practice Exercise 1

1. $$z=\left(\frac{\sqrt{3}+i}{1-i\sqrt{3}}\right)^{4n}(\sqrt{3}+i)=(i)^{4n}(\sqrt{3}+i)=\sqrt{3}+i$$

$$\Rightarrow \arg z=\frac{\pi}{6}$$

2. $\dfrac{\alpha x+\beta y+\gamma z}{\beta x+\gamma y+\alpha z}$ $\alpha=1, \beta=\omega, \gamma=\omega^2$ as α, β, γ are cube roots of unity

$$\frac{x+y\omega+z\omega^2}{\omega x+\omega^2 y+z}$$

$$\Rightarrow \frac{\omega}{\omega}\frac{(x+y\omega+z\omega^2)}{\omega x+\omega^2 y+z}=\frac{1}{\omega}\frac{\omega x+y\omega^2+z\omega^3}{\omega x+\omega^2 y+z}=\frac{1}{\omega}=\omega^2$$

Similarly answer can also be ω by varying the values of α, β, γ.

3. $|az_1+bz_2|^2+|bz_1-az_2|^2=(az_1+bz_2)(a\bar{z}_1+b\bar{z}_2)+(bz_1-az_2)$

$(b\bar{z}_1-a\bar{z}_2)$

$=a^2z_1\bar{z}_1+\cancel{abz_1\bar{z}_2}+\cancel{abz_2\bar{z}_1}+b^2z_2\bar{z}_2+b^2z_1\bar{z}_1-\cancel{baz_1\bar{z}_2}$

$-\cancel{abz_2\bar{z}_1}+a^2z_2\bar{z}_2$

$=a^2(|z_1|^2+|z_2|^2)+b^2(|z_1|^2+|z_2|^2)=\left(a^2+b^2\right)\left(|z_1|^2+|z_2|^2\right)$

4. See Fig. 5.36.

$$z_1=a+i, z_2=1+bi, z_3=0$$

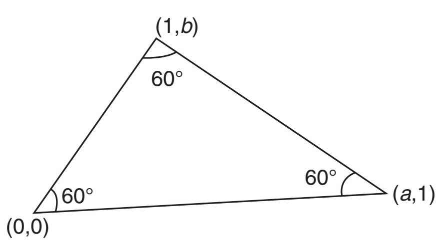


Figure 5.36

$$(a+i)e^{i\frac{\pi}{3}}=1+bi \Rightarrow (a+i)\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=1+bi$$

$$\Rightarrow (a+i)\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)=1+bi \Rightarrow \frac{a}{2}-\frac{\sqrt{3}}{2}+i\left(\frac{1}{2}+\frac{\sqrt{3}}{2}a\right)=1+bi$$

Comparing both sides we get,

$$\frac{a}{2}-\frac{\sqrt{3}}{2}=1 \Rightarrow a=\sqrt{3}+2 \text{, and}$$

$$\frac{1}{2}+\frac{\sqrt{3}}{2}a=b \Rightarrow b=\frac{1}{2}+\frac{\sqrt{3}}{2}(\sqrt{3}+2)$$

$$\Rightarrow b=\frac{1+3+2\sqrt{3}}{2}=2+\sqrt{3}$$

So, $a=b=2+\sqrt{3}$.

5. $$z=re^{i\theta}$$

$$\left|e^{iz}\right|=e^{i\{r\cos\theta+i\sin\theta\}}=e^{-r\sin\theta+ir\cos\theta}$$

$$e^{iz}=e^{-r\sin\theta}\cdot e^{ir\cos\theta}$$

So, $\left|e^{iz}\right|=e^{-r\sin\theta}$.

6. $$(1+i)(1+2i)(1+3i)\ldots(1+ni)=x+iy \quad (1)$$

Taking conjugate, we get

$$(1-i)(1-2i)(1-3i)\ldots(1-ni)=x-iy \quad (2)$$

Multiplying Eqs. (1) and (2), we get

$$(1^2+1^2)(1^2+2^2)\ldots.(1^2+n^2)=x^2+y^2$$

So, $2.5.10\ldots(1+n^2)=x^2+y^2$.

7. Eq. of perpendicular bisector of z_1 and z_2 (See Fig. 5.37)

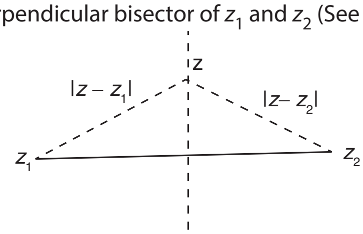


Figure 5.37

$$|z-z_2|=|z-z_1| \Rightarrow |z-z_2|^2=|z-z_1|^2$$
$$\Rightarrow (z-z_2)(\bar{z}-\bar{z}_2)=(\bar{z}-\bar{z}_1)(z-z_1)$$
$$\Rightarrow z\bar{z}-z_2\bar{z}-z\bar{z}_2+z_2\bar{z}_2=z\bar{z}-\bar{z}z_1-\bar{z}_1z+z_1\bar{z}_1$$
$$\Rightarrow z(\bar{z}_2-\bar{z}_1)+\bar{z}(z_2-z_1)=|z_2|^2-|z_1|^2$$

8. $$|z|=1$$
$$z=\cos\theta+i\sin\theta$$

Now,
$$\frac{2}{z}=x+iy=\frac{2}{\cos\theta+i\sin\theta}$$
$$\Rightarrow x+iy=2(\cos\theta-i\sin\theta)$$
$$\Rightarrow x=2\cos\theta,\ y=-\sin\theta$$
$$\Rightarrow x^2+y^2=4$$

So, $\frac{2}{z}$ lies on a circle of radius 2.

9. $$f(x)=x^{3p}+x^{3q+1}+z^{3r+2}$$

Put $x=\omega$, we get
$$f(\omega)=\omega^{3p}+\omega^{3q+1}+\omega^{3r+2}=(\omega^3)^p+(\omega^3)^q\cdot\omega+(\omega^3)^r\cdot\omega^2$$
$$=1+\omega+\omega^2=0$$

10. Since,
$$\left|\frac{1}{z_2}+\frac{1}{z_1}\right|=\left|\frac{1}{z_2}-\frac{1}{z_1}\right| \Rightarrow |z_1+z_2|=|z_1-z_2|$$

Squaring both sides, we get
$$|z_1|^2+|z_2|^2+2(z_1\bar{z}_2+\bar{z}_1z_2)=|z_1|^2+|z_2|^2-2(z_1\bar{z}_2+\bar{z}_1z_2)$$
$$\Rightarrow 4(z_1\bar{z}_2+\bar{z}_1z_2)=0 \Rightarrow \left(\frac{z_1}{z_2}\right)=-\left(\frac{\bar{z}_1}{\bar{z}_2}\right)$$

Therefore,

$\left(\frac{z_1}{z_2}\right)$ is purely imaginary $\Rightarrow \arg\left(\frac{z_1}{z_2}\right)=\frac{\pi}{2}=\arg\left(\frac{z_1-0}{z_2-0}\right)$,

that is, angle between z_2, O and z_1 is a right angle, taken in order.

As shown in the above arrangement. Now, the circumcentre of the above arrangement will lie on the line PQ as diameter and is represented by C which is the centre of PQ (Fig. 5.38), such that $z=\left(\frac{z_1+z_2}{2}\right)$, where z is the affix of circumcentre.

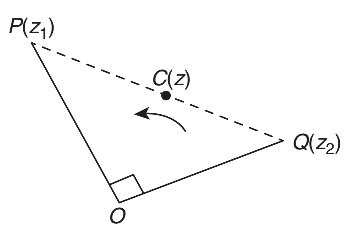


Figure 5.38

11. $$\sin x+i\cos 2x=\sin x-i\cos 2x=\cos x-i\sin 2x$$
$$\Rightarrow \sin x=\cos x \text{ and } \cos 2x=\sin 2x$$
$$\Rightarrow x=n\pi+\pi/4 \text{ and } 2x=m\pi+\pi/4$$

Thus, these equations cannot be true simultaneously.

12. $|z|-|z-1|\le|z-(z-1)|=1$

13. $$(a_1+ib_1)(a_2+ib_2)\dots(a_n+ib_n)=A+iB$$
$$\Rightarrow \arg(A+iB)=\arg(a_1+ib_1)+\arg(a_2+ib_2)\dots\arg(a_n+ib_n)$$
$$\Rightarrow \tan^{-1}(B/A)=\tan^{-1}(b_1/a_1)+\tan^{-1}(b_2/a_2)+\cdots+\tan^{-1}\frac{b_n}{a_n}$$
$$=\sum_{j=1}^{n}\tan^{-1}\left(\frac{b_j}{a_j}\right)$$

14. $t_n=(n+1)\left(n+\frac{1}{\omega}\right)\left(n+\frac{1}{\omega^2}\right)=n^3+n^2\left(\frac{1}{\omega^2}+\frac{1}{\omega}+1\right)$

$n\left(1+\frac{1}{\omega^2}+\frac{1}{\omega}\right)+1$

$=n^3+n^2(\omega+\omega^2+1)+n(\omega+\omega^2+1)+1=n^3+1$

Therefore,
$$S_n=\sum_{r=1}^{n}t_r=\sum_{r=1}^{n}(r^3+1)=\frac{n^2(n+1)^2}{4}+n$$

15. $z=x+iy \Rightarrow z^{1/3}=(x+iy)^{1/3}=(a-ib)$

$\Rightarrow x+iy=(a-ib)^3=(a^3-3ab^2)+i(b^3-3a^2b)$

$\Rightarrow x=a^3-3ab^2,\ y=b^3-3a^2b$

$$\Rightarrow \frac{x}{a}=a^2-3b^2,\ \frac{y}{b}=b^2-3a^2 \Rightarrow \frac{x}{a}-\frac{y}{b}=4(a^2-b^2) \Rightarrow \lambda=4$$

16. Since α, β, γ are roots of $(z+ab)^3=a^3$
$$\Rightarrow \alpha,\beta,\gamma \in z+ab=a, a\omega, a\omega^2$$

Therefore,
$$\alpha,\beta,\gamma=a-ab,\ a\omega-ab,\ a\omega^2-ab \text{ (say)}$$

Let $d=|\alpha-\beta|=|a-ab-(a\omega-ab)|$
$$\Rightarrow d=|\alpha-\beta|=|a-ab-a\omega+ab|=|a(1-\omega)|$$

Therefore,
$$d=|\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\sin\theta|=3\times1\times1=3$$

17. $$x^2-x+1=0$$
$$\Rightarrow x=\omega,\ \omega^2$$

Now,
$$\omega^{2000}+\frac{1}{\omega^{2000}} \Rightarrow \omega^2+\frac{1}{\omega^2}=\omega^2+\omega=-1$$

18. $$\frac{w-\bar{w}z}{1-z}=\frac{\bar{w}-w\bar{z}}{1-\bar{z}}$$
$$\Rightarrow (z\bar{z}-1)(\bar{w}-w)=0$$
$$\Rightarrow z\bar{z}=1 \Rightarrow |z|^2=1 \Rightarrow |z|=1$$

19. See Fig. 5.39.

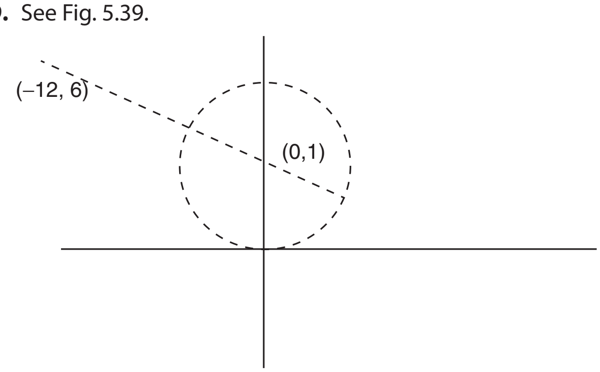


Figure 5.39

$$d-r<|z+12-6i|<d+r$$
$$\Rightarrow 13-1<|z+12-6i|<13+1$$
$$\Rightarrow 12<|z+12-6i|<14$$

20. $|a+bw+cw^2| = \sqrt{\left(a-\frac{b}{2}-\frac{c}{2}\right)^2+\frac{3}{4}(c-b)^2}$

$$= \sqrt{a^2+\frac{b^2}{4}+\frac{c^2}{4}-ab-\frac{bc}{2}-ac+\frac{3}{4}(c^2+b^2-2bc)}$$

$$= \sqrt{a^2+b^2+c^2-ab-bc-ca}$$

$$= \sqrt{\frac{1}{2}\left[(a-b)^2+(b-c)^2+(c-a)^2\right]}$$

This is minimum when $a = b$ and $(b - c)^2 = (c - a)^2 = 1$
Hence, the minimum value is 1.

21. Given that $\sin\theta_1 z^4 + \sin\theta_2 z^3 + \sin\theta_3 z^2 + \sin\theta_4 z + \sin\theta_5 = 2$
or, $2 = |\sin\theta_1 z^4 + \sin\theta_2 z^3 + \sin\theta_3 z^2 + \sin\theta_4 z + \sin\theta_5|$

$\le \frac{1}{2}\left[|z|^4+|z|^3+|z|^2+|z|+1\right]$

or,

$$3 \le |z|^4 + |z|^3 + |z|^2 + |z| \qquad (1)$$

Clearly, $|z| \ge 1$ satisfies Eq. (1). If $|z| < 1$, then
$3 < |z|^4 + |z|^3 + |z|^2 + |z| \le |z| + |z|^2 + |z|^3 + |z|^4 \ldots \infty = \frac{|z|}{1-|z|}$

$$\Rightarrow 3 - 3|z| < |z| \Rightarrow |z| > \frac{3}{4}$$

22. Obvious, after drawing the locus of z in the Argand plane.

23. Given that

$$\log_{1/\sqrt{2}}\left(\frac{|z|^2+2|z|+6}{2|z|^2-2|z|+1}\right) < 0$$

$$\Rightarrow \frac{|z|^2+2|z|+6}{2|z|^2-2|z|+1} > 1$$

$$\Rightarrow |z|^2 - 4|z| - 5 < 0$$

$$\Rightarrow (|z| - 5)(|z| + 1) < 0$$

Thus ,

$$|z| < 5$$

24. $|z - 3| = |z - 5| \Rightarrow (z - 3)(\bar{z} - 3) = (z - 5)(\bar{z} - 5)$
$\Rightarrow z + \bar{z} = 8 \Rightarrow x = 4 \Rightarrow$ locus of z is a straight line

25. Let $z = a + ib$ where $a > 0, b > 0$ (given)
Since,

$$\text{Re}(z) + \text{Im}(z) = 3 \qquad \text{(given)}$$

$$\Rightarrow a + b = 3 \qquad (1)$$

Now, let $E = \text{Re}(z)^2\,\text{Im}(z)$

$$\Rightarrow E = a^2 b = a^2(3 - a) \qquad (2)$$

Now, E is maximum or minimum if

$$\frac{dE}{da} = 0$$

$$\Rightarrow 6a - 3a^2 = 0 \Rightarrow 3a(2 - a) = 0$$

either

$$a = 0$$

or

$$a = 2$$

Again, $\frac{d^2E}{da^2} < 0$, when $a = 2$.

Hence, E will attain its maximum value if $a = 2$.
Therefore, the maximum value $= (2)^2(3 - 2) = 4$.

26. $z\bar{z} + a\bar{z} + \bar{a}z + b = 0 \qquad (1)$

Let $z = x + iy$ and $a = \alpha + i\beta$. So, Eq. (1) becomes

$$x^2 + y^2 + (\alpha + i\beta)(x - iy) + (\alpha - i\beta)(x + iy) + b = 0$$

$$\Rightarrow x^2 + y^2 + 2(\alpha x + \beta y) + b = 0$$

$$\Rightarrow R = \sqrt{\alpha^2 + \beta^2 - b}$$

$$\Rightarrow \alpha^2 + \beta^2 - b \ge 0$$

$$\Rightarrow \alpha^2 + \beta^2 \ge b$$

$$\Rightarrow |a|^2 \ge b$$

27. See Fig. 5.40. $DE = BD - BE = 2 - 3/2 = 1/2$

$$BE = AB \sin 60$$

$$= \sqrt{3} \times \frac{\sqrt{3}}{2}$$

$$= \frac{3}{2}$$

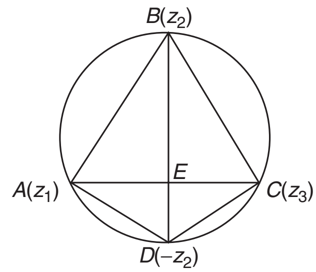


Figure 5.40

$$\text{Area of } \Delta ACD = \frac{1}{2} DE \times AC$$

$$= \frac{1}{2} \times \frac{1}{2}\sqrt{3} = \frac{\sqrt{3}}{4}$$

28. See Fig. 5.41

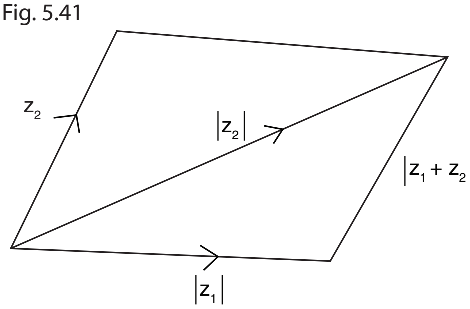


Figure 5.41

We have

$$|z_1| + |z_2| \ge |z_1 + z_2|$$

Now, $|z_1| + |z_2| = |z_1 + z_2|$ only if z_1 and z_2 are collinear. Therefore,

$$\arg z_1 - \arg z_2 = 0$$

$$\Rightarrow \arg\left(\frac{z_1}{z_2}\right) = 0$$

29. The equality $|z_1| + |z_2| + |z_3| = |z_1 + z_2 + z_3|$ is true if an only if z_1, z_2 and z_3 are of same signs, that is, either all positive or all negative, that is, they all must be comparable to additive identity. Thus, they all must be real quantities.

Hence, if $z = \frac{z_1 z_2}{z_3^2} + \frac{z_2 z_3}{z_1^2} + \frac{z_1 z_3}{z_2^2}$ then z must also be a real quantity.

Therefore, z is a purely real number.

30. $$\arg(z+3-4i)=\frac{2\pi}{3}$$

$$\Rightarrow \arg(z-(-3+4i))=\frac{2\pi}{3}$$

The above equation represents a locus of straight line passing through $-3+4i$ and inclined at an angle of $\frac{2\pi}{3}$ with the positive direction of the real axis in the anticlockwise direction.

Also,

$$\arg(3z+2-3i)=\frac{\pi}{4}\Rightarrow \arg(3)+\arg\left(z+\frac{2-3i}{3}\right)=\frac{\pi}{4}$$

Here z, represents the locus of a straight line through $\left(-\frac{2}{3}+i\right)$ and inclined at an angle of $\frac{\pi}{4}$ with the positive real axis in the anticlockwise direction.

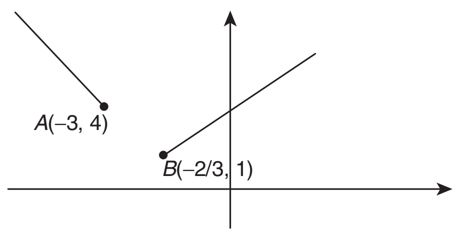


Figure 5.42

It can be seen from Fig. 5.42 that the given system does not possess a solution.

31. Let

$$S=1+3\alpha+5\alpha^2+\cdots+(2n-1)\alpha^{n-1} \quad (1)$$

$$\Rightarrow S\alpha=\alpha+3\alpha^2+\cdots+(2n-3)\alpha^{n-1}+(2n-1)\alpha^n \quad (2)$$

Subtracting Eq. (2) from Eq. (1), we get

$$S(1-\alpha)=1+2\alpha+2\alpha^2+\cdots+2.\,\alpha^{n-1}-(2n-1)\alpha^n$$

$$=1+2\alpha(1+\alpha+\cdots+\alpha^{n-2})-(2n-1)\alpha^n$$

$$=1+2\alpha\left(\frac{1-\alpha^{n-1}}{1-\alpha}\right)-(2n-1)\alpha^n$$

$$=1+\frac{2}{(1-\alpha)}(\alpha-\alpha^n)-(2n-1)\alpha^n$$

$$=1+\frac{2}{(1-\alpha)}(\alpha-1)-(2n-1)\alpha^n$$

$$=-(2n-1)-1=-2n\Rightarrow S=\frac{-2n}{(1-\alpha)}$$

32. $$(z_1+iz_2)(\bar{z}_1-i\bar{z}_2)=(z_1-iz_2)(\bar{z}_1+i\bar{z}_2)$$

$$\Rightarrow \bar{z}_1z_2=z_1\bar{z}_2\Rightarrow\frac{z_1}{z_2}=\frac{\bar{z}_1}{\bar{z}_2}\Rightarrow\frac{z_1}{z_2}\text{ is purely real.}$$

33. $$(1+x+x^2)^n=a_0+a_1x+a_2x^2+\cdots+a_{2n}x^{2n}$$

Putting $x=1,\ \omega,\ \omega^2$ in turn we get

$$3^n=a_0+a_1+a_2+a_3+\cdots+a_{2n} \quad (1)$$

$$0=a_0+a_1\omega+a_2\omega^2+\cdots+a_{2n}\omega^{2n} \quad (2)$$

and

$$0=a_0+a_1\omega^2+a_2\omega^4+\cdots+a_{2n}\omega^{2n} \quad (3)$$

Adding Eqs. (1), (2) and (3)

$$3^n=3(a_0+a_3+a_6+\cdots)\qquad(\text{as }1+\omega+\omega^2=0,\ \omega^{3n}=1)$$

$$a_0+a_3+a_6+\cdots=3^{n-1}$$

34. Let $z=x+iy$

Therefore, according to given inequations, we have

$$5<x^2+y^2\le 12$$

So, it represents the region bounded in between two concentric circles centred at origin of radii $\sqrt{5}$ and $2\sqrt{3}$ units.

and

$$(z-\bar{z})^2+8(z+\bar{z})>0$$

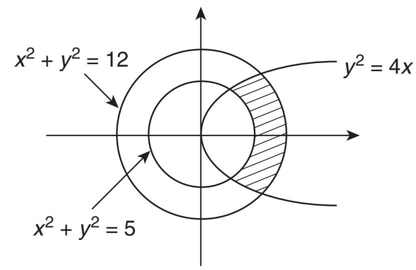


Figure 5.43

$$\Rightarrow\left(\frac{2y}{i}\right)^2+8(2x)>0$$

$$\Rightarrow -4y^2+16x>0\Rightarrow y^2<4x$$

represents the region inside the parabola $y^2=4x$.
The common region bounded is shown in Fig. 5.43.
The point of intersections are

$$x^2+y^2=5\text{ and }y^2=4x$$

$$\Rightarrow x^2+4x-5=0\Rightarrow x=1,-5$$

$$x^2+y^2=12\text{ and }y^2=4x\Rightarrow x^2+4x-12=0$$

$$\Rightarrow x=\frac{-4\pm\sqrt{16+48}}{2}\Rightarrow x=2,-6$$

35. See Fig. 5.44. By the given conditions, the area of the triangle ABC is

$$\frac{1}{2}|z_1-z_2|r$$

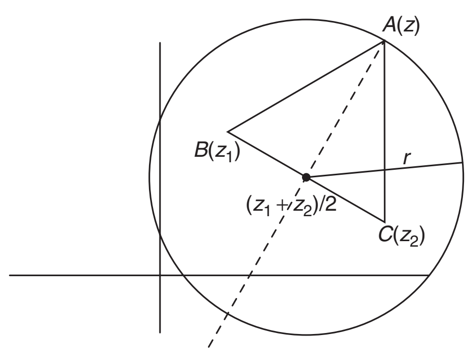


Figure 5.44

36. See Fig. 5.45. The given equation is written as

$$\arg[z-(1+i)]=\begin{cases}\frac{3\pi}{4} & \text{when } x\le 2\\ \frac{-\pi}{4} & \text{when } x>2\end{cases}$$

Therefore, the locus is a set of two rays.

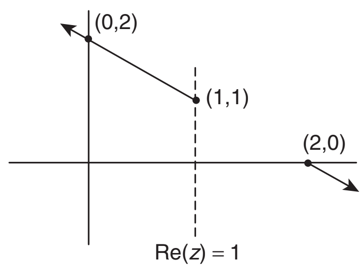


Figure 5.45

37. $z = \left(\dfrac{\sqrt{3}+i}{2}\right)^5 + \left(\dfrac{\sqrt{3}-i}{2}\right)^5 = (e^{i\pi/6})^5 + (e^{-i\pi/6})^5$

$= e^{i5\pi/6} + e^{-i5\pi/6} = 2\cos\dfrac{5\pi}{6} = -2\cos\dfrac{\pi}{6} = -\sqrt{3}$

Thus, Im $(z) = 0$.

38. The points (1, 0), $(\sqrt{2}-1, -\sqrt{2})$ and $(\sqrt{2}-1, \sqrt{2})$ are equidistant from the point (–1, 0). The shaded area belongs to the region outside the sector of circle $|z + 1| = 2$, lying between the line rays $\arg(z+1) = \dfrac{\pi}{4}$ and $\arg(z+1) = \dfrac{-\pi}{4}$.

39. First equation represents ellipse and the second one a line segment joining (–1, 0) and (1, 0) totally contained inside the ellipse.

40. $|z_1 - z_2|^2 + |z_2 - z_3|^2 + |z_3 - z_1|^2$

$= 2(|z_1|^2 + |z_2|^2 + |z_3|^2) - (z_1\bar{z}_2 + \bar{z}_1 z_2 + z_2\bar{z}_3 + \bar{z}_2 z_3 + z_3\bar{z}_1 + \bar{z}_3 z_1)$

$= 58 - (z_1\bar{z}_2 + \bar{z}_1 z_2 + z_2\bar{z}_3 + \bar{z}_2 z_3 + z_3\bar{z}_1 + \bar{z}_3 z_1)$ (1)

Now,

$$|z_1 + z_2 + z_3|^2 \geq 0$$

$\Rightarrow |z_1|^2 + |z_2|^2 + |z_3|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2 + z_2\bar{z}_3 + \bar{z}_2 z_3 + z_3\bar{z}_1 + \bar{z}_3 z_1 \geq 0$

$\Rightarrow -(z_1\bar{z}_2 + \bar{z}_1 z_2 + z_2\bar{z}_3 + \bar{z}_2 z_3 + z_3\bar{z}_1 + \bar{z}_3 z_1) \leq 29$ (2)

From Eqs. (1) and (2), maximum value = 58 + 29 = 87

41. $z^3 + \bar{\omega}^7 = 0 \Rightarrow z^3 = -\bar{\omega}^7 \Rightarrow z^{15} = -\bar{\omega}^{35}$

$z^5 \cdot \omega^{11} = 1 \Rightarrow z^5 = \dfrac{1}{\omega^{11}} \Rightarrow z^{15} = \dfrac{1}{\omega^{33}}$

Therefore,

$$-\bar{\omega}^{35} = \frac{1}{\omega^{33}} \Rightarrow \bar{\omega}^{35} \cdot \omega^{33} = -1$$

$\Rightarrow |\bar{\omega}^{35} \cdot \omega^{33}| = 1 \Rightarrow |\omega^{33}|\,|\bar{\omega}^{35}| = 1 \Rightarrow \omega^{68} = 1 \Rightarrow |\omega| = 1$

Again

$$\bar{\omega}^{35} \cdot \omega^{33} = -1$$

$\Rightarrow (\bar{\omega}^{33} \cdot \omega^{33}) \cdot \bar{\omega}^2 = 1 \Rightarrow (|\omega|^2)^{33} \cdot \bar{\omega}^2 = 1 \Rightarrow \omega = \pm i \Rightarrow z = \pm i$

42. See Fig. 5.46. The given relation represents the parabola with focus (4, 0) and the imaginary axis as the directrix. Pair of tangents from directrix is at right angle. By symmetry greatest positive argument of z is $\dfrac{\pi}{4}$.

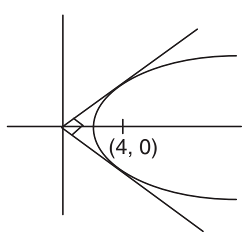


Figure 5.46

43. See Fig. 5.47. Clearly, $\angle DOB = \angle COD = A$

$\Rightarrow z = \omega e^{iA}$ and $\bar{\omega} = ze^{iA} \Rightarrow z^2 = \omega\bar{\omega} = 1$

$\Rightarrow z = -1$ (as A and D are on the opposite side of BC).

$= \omega + \omega^2$

$= \omega + \bar{\omega}$

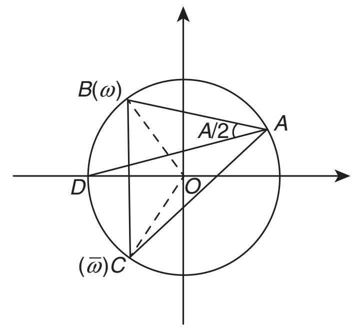


Figure 5.47

44. $|k + z^2| = |z|^2 - k = |z^2| + |k|$

$\arg(z^2) = \arg(k)$

$\Rightarrow 2\arg(z) = \pi \Rightarrow \arg(z) = \dfrac{\pi}{2}$

45. $x^n - 1 = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \ldots (x - \alpha_n)$

By Putting $x = 2$,

$2^n - 1 = (2 - \alpha_1)(2 - \alpha_2) \ldots (2 - \alpha_{n-1})$

$\Rightarrow |2 - \alpha_1|\,|2 - \alpha_2| \ldots |2 - \alpha_{n-1}| = |2^n - 1| = 2^n - 1$

46. Observing carefully the system of equations, we find

$$\frac{1+i}{2i} = \frac{1-i}{2} = \frac{1}{1+i}$$

Hence, the system of equations has infinite number of solutions.

47. The given equation is $|z|^n = (z^2 + z)\,|z|^{n-2} + 1$

So,

$z^2 + z = \text{real} = \bar{z}^2 + \bar{z}$

$\Rightarrow (z - \bar{z})(z + \bar{z} + 1) = 0$

$\Rightarrow z = \bar{z} = x$ as $z + \bar{z} + 1 \neq 0$ $(x \neq -1/2)$

Therefore,

$x^n = x^n + x\,|x|^{n-2} + 1$

$\Rightarrow x\,|x|^{n-2} = -1$

$\Rightarrow x = -1$

So, the number of solution is one.

48. Given expression $= i\left[\log\left(\dfrac{x-i}{x+i}\right)\right] - \pi + 2\tan^{-1} x = k$ (say)

$$\log\left(\frac{x+i}{x-i}\right) = (k + \pi - 2\tan^{-1} x)i$$

or $\dfrac{x+i}{x-i} = e^{i\theta}$ where $\theta = k + \pi - 2\tan^{-1} x$

$\Rightarrow (x + i) = (x\cos\theta + \sin\theta) + i\,(x\sin\theta - \cos\theta)$

$x = x\cos\theta + \sin\theta$

and

$1 = x\sin\theta - \cos\theta$

$\Rightarrow x = \cot(\theta/2) \Rightarrow \theta = 2\cot^{-1} x$

$\Rightarrow k + \pi - 2\tan^{-1} x = 2\cot^{-1} x$

$\Rightarrow k + \pi = 2[\tan^{-1} x + \cot^{-1} x] = 2(\pi/2)$

$\Rightarrow k + \pi = \pi \Rightarrow k = 0$

Practice Exercise 2

1. Let $\alpha^j = e^{i\left(\frac{2\pi j}{n}\right)}$

$$\sum_{j=1}^{n-1}\frac{1}{1-\alpha^j} = \frac{1}{1-\alpha}+\frac{1}{1-\alpha^2}+\cdots+\frac{1}{1-\alpha^{n-1}}$$

Now,

$$\frac{x^n-1}{x-1} = (x-\alpha)(x-\alpha^2)\cdots(x-\alpha^{n-1}) \text{ where}$$

$\alpha, \alpha^2, \ldots, \alpha^{n-1}$ are the n^{th} roots of unity

Taking log on both sides, we get

$$\log\left(\frac{x^n-1}{x-1}\right) = \log(x-\alpha) + \log(x-\alpha^2) + \cdots \log(x-\alpha^{n-1})$$

Differentiating both sides, we get

$$\frac{(n-1)x^n - nx^{n-1}+1}{x-1} = \frac{1}{x-\alpha}+\frac{1}{x-\alpha^2}+\cdots+\frac{1}{x-\alpha^{n-1}}$$

Taking $\lim\limits_{x\to 1}$ on both sides, we get

$$\frac{n-1}{2} = \frac{1}{1-\alpha}+\frac{1}{1-\alpha^2}+\cdots+\frac{1}{1-\alpha^{n-1}}$$

2. $(1+\omega)^n = C_0 + C_1\omega + C_2\omega^2 + \cdots + C_n\omega^n$

Now,

$$(1+1)^n = C_0 + C_1 + C_2 + \cdots + C_n$$

Adding both the above equations,

$(1+\omega)^n + (1+1)^n = 2C_0 + C_1(1+\omega) + C_2(1+\omega^2) + C_3(1+\omega^3)$
$+ C_4(1+\omega) + C_5(1+\omega^2) + C_6(1+\omega^3) + \cdots$
$+ C_n(1+\omega^n)$

$= 2(C_0 + C_3 + C_6 + \cdots) + (C_1 + C_4 + C_7 + \cdots)(1+\omega)$
$+ (C_2 + C_5 + C_6 + \cdots)(1+\omega^2) = -\omega^n + 2^n$

$= 2^n - 1$ (therefore, n in a multiple of 3, $\omega^n = 1$).

3. $(1-i)z + (1+i)\bar{z} + 3 = 0$, $(3+2i)$

Perpendicular distance between the given point to the given line is

$$\left|\frac{(1-i)(3+2i)+(1+i)(3-2i)+3}{\sqrt{(2+2)}}\right| = \left|\frac{3+2i-3i+2+3-2i+3i+2+3}{2}\right|$$

$$= \frac{13}{2}$$

4. The given equation implies that the points representing the complex numbers $3i$ and $-4+2i$ subtend an angle $\frac{\pi}{4}$ at the circumference of the circle. So, these points subtend an angle $\frac{\pi}{2}$ at the centre of the circle as points which subtends an angle θ on circumference subtends 2θ at center see Fig. 5.48. If z_0 is the corresponding complex number associated with the centre then

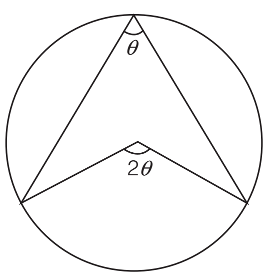


Figure 5.48

$$\frac{z_0 - 3i}{z_0 - (-4+2i)} = e^{i\pi/2} = i$$

$$\Rightarrow z_0 = \frac{1}{2}(9i-5)$$

5. $\left(\frac{z-\bar{z}}{2}\right)$ is purely imaginary. Also,

$$\arg(z) < 0$$

$$\Rightarrow \arg\left(\frac{z-\bar{z}}{2}\right) = -\frac{\pi}{2}$$

6. $|z-6i| = \text{Im}(z)$ is a parabola having focus $6i$ and directrix as real axis.

$$(\arg z)_{\min} = \frac{\pi}{4}, (\arg z)_{\max} = \frac{3\pi}{4}$$

$$\frac{\pi}{4} \le \arg z \le \frac{3\pi}{4}$$

$$\Rightarrow \arg z - \arg \bar{z} = 2\arg z \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$$

7. $$|z| = 1$$

$z = e^{i\theta} = x + iy$

$x = \cos\theta, y = \sin\theta$

$\cos\theta, \sin\theta \in Q$

$|z^{2n}-1|^2 = (z^{2n}-1)(\bar{z}^{2n}-1) = 2 - (z^{2n} + \bar{z}^{2n})$

$= 2 - (e^{2in\theta} + e^{-2in\theta})$

$= 2(1-\cos 2n\theta) = 4\sin^2 n\theta$

$|z^{2n}-1| = 2|\sin n\theta|$

$\sin n\theta = \text{Im}(e^{in\theta}) = \text{Im}(e^{i\theta})^n = \text{Im}(\cos\theta + i\sin\theta)^n$

$= {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + \cdots =$ rational number

8. $$\left|\frac{z_1}{2}+\frac{z_2}{3}+\frac{z_3}{4}+\frac{z_4}{5}\right| = \frac{k}{60}|z_1z_2z_3z_4|\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}+\frac{1}{z_4}\right|$$

Now,

$$z_1\bar{z}_1 = 2,\ z_2\bar{z}_2 = 3,\ z_3\bar{z}_3 = 4 \text{ and } z_4\bar{z}_4 = 5$$

$$(\text{since}, |z_k| = \sqrt{k+1})$$

$$\Rightarrow k = \frac{60}{|z_1z_2z_3z_4|} = \frac{60}{\sqrt{2}\sqrt{3}\sqrt{4}\sqrt{5}} = \sqrt{30} = |z_1z_2z_4|$$

9. We are finding out sum of distances of a complex number z from origin and $(\cos\alpha, \sin\alpha)$. This sum will be minimum if z lies on the line joining the two points and the minimum value of the sum will be the distance between the two points, i.e. 1. Hence, (B) is the correct answer.

10. Let $z = a$ be a real root. Then

$$\alpha a^2 + a + \bar{\alpha} = 0 \quad (1)$$

Let $\alpha = p + iq$. Then

$$(p+iq)a^2 + a + p - iq = 0$$

$$\Rightarrow pa^2 + a + p = 0 \text{ and } a^2q - q = 0$$

Therefore,

$$a = \pm 1 \text{ (since } q \ne 0)$$

From Eq. (1),

$$\alpha \pm 1 + \bar{\alpha} = 0, \text{ also } |a| = 1$$

11. $\alpha - i\beta$ will satisfy equation

$$a_4(-\bar{z})^4 + ia_3(-\bar{z})^3 + a_2(\bar{z})^2 + ia_1(-\bar{z}) + a_0 = 0$$

$\Rightarrow -\alpha + i\beta$ must be one of the roots

12. $$|z_1 z_2| = \left|\frac{c}{a}\right| = 1 \text{ and } |z_1 + z_2| = \left|-\frac{b}{a}\right| = 1$$

So,

$$(z_1 + z_2)(\bar{z}_1 + \bar{z}_2)\ |z_1 + z_2|^2 = 1$$
$$\Rightarrow 2 + \bar{z}_1 z_2 + z_1 \bar{z}_2 = 1$$
$$\Rightarrow 2 + \frac{z_2}{z_1} + \frac{z_1}{z_2} = 1$$
$$\Rightarrow \frac{(z_1 + z_2)^2}{z_1 z_2} = 1$$
$$\Rightarrow \frac{b^2}{a^2} = \frac{c}{a}$$
$$\Rightarrow b^2 = ac$$

Now,

$$z_2 = z_1 e^{i\theta}$$
$$\Rightarrow |z_1 + z_2| = |z_1|\ |1 + e^{i\theta}|$$
$$= 2\cos\frac{\theta}{2}\left|\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right|$$
$$|z_1 + z_2| = 2\cos\frac{\theta}{2} = 1 \Rightarrow \frac{\theta}{2} = \frac{\pi}{3} \Rightarrow \theta = \frac{2\pi}{3}$$

Now,

$$PQ = |z_2 - z_1| = |z_1|\ |e^{i\theta} - 1| = \left|2\sin\frac{\theta}{2}\right|$$

We know that $\theta = \frac{2\pi}{3}$. Therefore,

$$PQ = |z_2 - z_1| = \sqrt{3}$$

13. Since roots are imaginary.
So, discriminant < 0
Therefore,

$$\alpha = \frac{-b + i\sqrt{4ac - b^2}}{2c}$$
$$\beta = \frac{-b - i\sqrt{4ac - b^2}}{2c}$$
$$|\alpha| = |\beta| = \sqrt{\frac{b^2}{4c^2} + \frac{4ac - b^2}{4c^2}} = \sqrt{\frac{a}{c}} > 1$$

So,

$$\frac{1}{|\alpha|} + \frac{1}{|\beta|} < 1,\ \frac{1}{|\alpha|} = \frac{1}{|\beta|}$$

14. Given $OA = 1$, $OP = |z| = 1$. Therefore,

$$OA = OP$$

Now,

$$OP_0 = |z_0|$$
$$OQ = |z_0|$$
$$OP_0 = OQ$$
$$\angle POP_0 = \arg\frac{z_0}{z}$$
$$\angle AOQ = \arg\frac{1}{zz_0} = \arg\frac{z_0}{z}$$

15. See Fig. 5.49.

$$|x| + |y| = 1 \quad (1)$$
$$|z + i| + |z - i| = 2 \quad (2)$$

Eq. (1) represents square and Eq. (2) represents line segment.

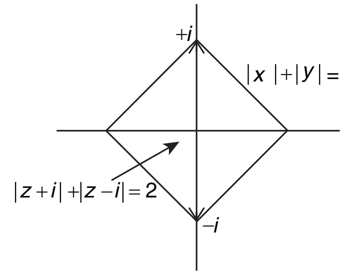


Figure 5.49

Therefore, solutions are $z = \pm i, \frac{1}{i}, \frac{1}{i^3}$.

16.
$$z' = ze^{i\alpha} \quad (1)$$
$$z'' = ze^{-i\alpha} \quad (2)$$

Therefore,

$$z'z'' = z^2$$
$$\Rightarrow z', z, z'' \text{ are in GP}$$

Now,

$$\left(\frac{z'}{z}\right)^2 + \left(\frac{z''}{z}\right)^2 = 2\cos 2\alpha$$
$$\Rightarrow z'^2 + z''^2 = 2z^2\cos 2\alpha$$

Adding Eqs. (1) and (2), we get

$$z' + z'' = 2z\cos\alpha$$

17. See Fig. 5.50. Triangles ABP and CDP are similar.
Let $AP = 2x$ and $BP = 2y$.
Then, $CP = 5x$, $DP = 5y$.
Therefore,

$$\text{Ar (Trapezium } ABCD) = \frac{49}{2}xy$$

Also,

$$\tan\alpha = \frac{2x}{5y},\ \tan\beta = \frac{2y}{5x}$$

and $$\alpha + \beta = 45^\circ$$

$$\Rightarrow \frac{10(x^2 + y^2)}{21xy} = 1 \Rightarrow xy = \frac{10}{21}(x^2 + y^2)$$

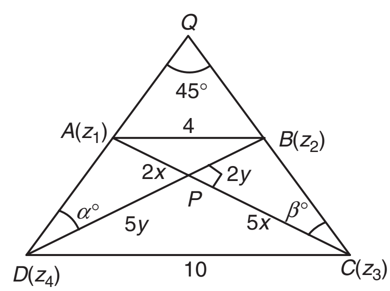


Figure 5.50

Also,

$$AB^2 = AP^2 + BP^2 \Rightarrow x^2 + y^2 = 4$$

Therefore,

$$xy = \frac{40}{21}$$

Hence,

$$\text{Area of } ABCD = \frac{49}{2}\cdot\frac{40}{21} = \frac{140}{3}$$

18. Area $(\Delta PCB) = \dfrac{1}{2}\cdot 2y\cdot 5x = 5xy = \dfrac{200}{21}$

19. $|CP - DP| = 5\,|x-y| = 5\sqrt{x^2+y^2-2xy} = 5\sqrt{4-\dfrac{80}{21}} = \dfrac{5\cdot 2}{\sqrt{21}} =$

$$= \frac{10}{\sqrt{21}}$$

20. See Fig. 5.51. By External division formula, we get

$$Z_D = \frac{Z_B - Z_A i}{1-i}$$

$$Z_E = \frac{Z_B - Z_C i}{1-i}$$

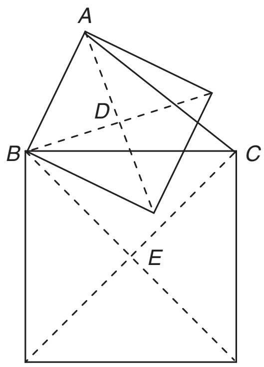


Figure 5.51

Angle between AC and DE is

$$\arg\left(\frac{Z_C - Z_A}{Z_E - Z_D}\right) = \arg\left(\frac{(Z_C - Z_A)}{(Z_A - Z_C)}\frac{1-i}{i}\right) = \frac{\pi}{4}$$

21. $DE = \left|\dfrac{(Z_A - Z_C)i}{1-i}\right| = \dfrac{|1-\omega^2|}{\sqrt{2}} = \dfrac{\sqrt{3}}{2}$

22.
$$Z_E = \frac{Z_B - Z_C i}{1-i} = \frac{\omega - \omega^2 i}{1-i}$$

$$\frac{(\omega+\omega^2) + i(\omega-\omega^2)}{2} = \frac{-1-\sqrt{3}}{2}$$

Therefore,

$$AE = \left|1 + \frac{1+\sqrt{3}}{2}\right| = \frac{3+\sqrt{3}}{2}$$

23.
$$AP^2 + BP^2 + CP^2 = |z-z_1|^2 + |z-z_2|^2 + |z-z_3|^2$$

$$= 3|z|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2 - z\left(\bar{z}_1 + \bar{z}_2 + \bar{z}_3\right) - \bar{z}\,(z_1+z_2+z_3)$$

Since ΔABC is equilateral, we have

$$|z_1| = |z_2| = |z_3| = 2$$

Therefore,

$$\frac{z_1+z_2+z_3}{3} = \frac{\bar{z}_1+\bar{z}_2+\bar{z}_3}{3} = 0$$

Also, $|z| = 1$ (since circumradius is 2).

Therefore,

$$AP^2 + BP^2 + CP^2 = 3\times 1 + 12 = 15$$

24.
$$\frac{1}{z_d} + \frac{1}{z_f} = \frac{2}{z_2}$$

$$\frac{1}{z_d} + \frac{1}{z_e} = \frac{2}{z_3}$$

$$\frac{1}{z_e} + \frac{1}{z_f} = \frac{2}{z_1}$$

$$\Rightarrow \frac{1}{z_d} + \frac{1}{z_e} + \frac{1}{z_f} = \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} = -\frac{i}{\sqrt{2}}$$

Therefore, $\text{Re}\left(\dfrac{1}{z_d} + \dfrac{1}{z_e} + \dfrac{1}{z_f}\right) = 0.$

25. $Z_Q = \dfrac{-z_2 z_3}{z_1} = -\sqrt{2}\,(1+i).$

26. See Fig. 5.52.

$$\angle IAB = \frac{\theta}{2},\ \angle IAC = \frac{\theta}{2}$$

$$\frac{z_2 - z_1}{|z_2 - z_1|} = \frac{z_4 - z_1}{|z_4 - z_1|} e^{-\frac{i\theta}{2}}$$

$$\frac{z_3 - z_1}{|z_3 - z_1|} = \frac{z_4 - z_1}{|z_2 - z_1|} e^{-\frac{i\theta}{2}}$$

$$\frac{(z_2 - z_1)(z_3 - z_1)}{|z_2 - z_1||z_3 - z_1|} = \frac{(z_4 - z_1)^2}{|z_4 - z_1|^2} e^{\theta}$$

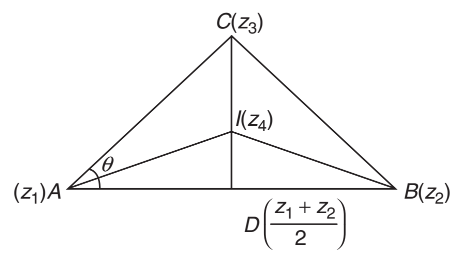


Figure 5.52

Therefore,

$$\frac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_1)^2} = \frac{AB\cdot AC}{(IA)^2} = \left(\frac{AB}{IA}\right)^2\left(\frac{AC}{AB}\right)$$

27. $\dfrac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_1)^2} = 2\left(\dfrac{AD}{IA}\right)^2\left(\dfrac{AC}{AD}\right)$ (since, $AB = 2AD$)

$$\Rightarrow (z_4 - z_1)^2\,(1+\cos\theta)\sec\theta = (z_2 - z_1)(z_3 - z_1)$$

28. $\dfrac{-(z_1 + z_2 - 2z_3)(z_1 + z_2 - 2z_4)}{(z_2 - z_1)^2} = \dfrac{CD}{AD}\cdot\dfrac{ID}{AD}$

$$\Rightarrow \frac{-(z_1 + z_2 - 2z_3)(z_1 + z_2 - 2z_4)}{(z_2 - z_1)^2} = \tan\theta\cdot\tan\frac{\theta}{2}$$

$$\Rightarrow (z_2 - z_1)^2\tan\theta\cdot\tan\frac{\theta}{2} = -(z_1 + z_2 - 2z_3)(z_1 + z_2 - 2z_4)$$

29. Let real roots be z_1 and z_2. Then

$$\bar{z}_1 = z_1,\ \bar{z}_2 = z_2$$

Now, equation $az^2 + bz + c = 0$ has z_1 and z_2 as roots. (1)

$$\bar{a}\bar{z}^2 + \bar{b}\bar{z} + \bar{c} = 0$$

So, the equation $\bar{a}z^2 + \bar{b}z + \bar{c} = 0$ also has z_1 and z_2 as roots. (2)

From Eqs. (1) and (2), we get

$$\frac{a}{\bar{a}} = \frac{b}{\bar{b}} = \frac{c}{\bar{c}}$$

30. Let the imaginary roots be z_1 and z_2. Then,

$$\bar{z}_1 = -z_1,\ \bar{z}_2 = -z_2$$

Now, equation $az^2 + bz + c = 0$ has z_1 and z_2 as roots (1)

Taking conjugate, we get

$$\bar{a}\bar{z}^2 + \bar{b}\bar{z} + \bar{c} = 0$$

That is, $\bar{a}z^2 - \bar{b}z + \bar{c} = 0$ has z_1 and z_2 as roots (2)

From Eqs. (1) and (2),

$$\frac{a}{\bar{a}} = -\frac{b}{\bar{b}} = \frac{c}{\bar{c}}$$

31. Given

$$|m| = 1 \Rightarrow m\bar{m} = 1$$

Now, m is a root of $az^2 + bz + c = 0$ (1)

Taking conjugate, we get

$$\bar{a}\bar{z}^2 + \bar{b}\bar{z} + \bar{c} = 0$$

$$\frac{\bar{a}}{z^2} + \frac{\bar{b}}{z} + \bar{c} = 0$$

$$\bar{a} + \bar{b}z + \bar{c}z^2 = 0$$

That is, m is a root of $\bar{c}z^2 + \bar{b}z + \bar{a} = 0$ (2)

That is, Eqs. (1) and (2) have m as common root.

32.

(A) $$|a + ib|\,|z| = |z|\,|(a-1) + ib|$$

$$\Rightarrow \frac{1}{\sqrt{2}} = \sqrt{(a-1)^2 + b^2} \text{ and } a^2 + b^2 = \frac{1}{2}$$

$$\Rightarrow 1 - 2a = 0 \Rightarrow a = \frac{1}{2}$$

And $$b^2 = \frac{1}{4} \Rightarrow b = \frac{1}{2}$$

$$\Rightarrow a - b = 0$$

(B) See Fig. 5.53.

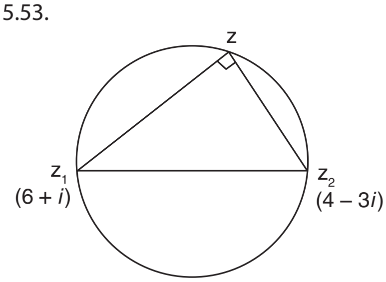


Figure 5.53

z_1 and z_2 are end points of a diameter and $z_0 = 5 - i$ is center of the circle.

So, $|z - (5 - i)|$ is distance of a point on circle to the center, whose radius is

$$r = \frac{|z_1 - z_2|}{2} = \frac{1}{2}\sqrt{2^2 + 4^2}$$

$$= \frac{2}{2}\sqrt{5} = \sqrt{5} = \sqrt{a} \Rightarrow a = 5$$

(C) $$\text{Re}\left(\frac{z}{4}\right) \in (0,\ 1),\ \text{Im}\left(\frac{z}{4}\right) \in [0,\ 1)$$

It means that if $z = a + ib$, then $a, b \in (0, 4)$

Now,

$$\frac{4}{a - ib} = \frac{4a}{a^2 + b^2} + \frac{4bi}{a^2 + b^2}$$

$$\Rightarrow 0 < a, b < \frac{a^2 + b^2}{4}$$

$$\Rightarrow (a-2)^2 + b^2 > 4 \text{ and } a^2 + (b-2)^2 > 4$$

So, we want area inside the square and outside the two circles. Therefore,

$$\text{Area} = 16 - 4\pi + (2\pi - 4) = 12 - 2\pi$$

(D) $$z^{15}(1 + z + z^2 + \cdots + z^{36}) = \frac{1(z^{36} - 1)}{z - 1} = \frac{z - 1}{z - 1} = 1$$

33.

(A) $$\arg\left(\frac{z^2 - 1}{z^2 + 1}\right) = 0;\ z \neq \pm i$$

$$\frac{z^2 - 1}{z^2 + 1} = \frac{\bar{z}^2 - 1}{\bar{z}^2 + 1} \Rightarrow z - \bar{z} = 0,\ z + \bar{z} = 0$$

$$y = 0,\ x = 0$$

Locus of z is portion of pair of lines $xy = 0$ $\left[\text{since}\left(\frac{z^2 - 1}{z^2 + 1}\right) > 0\right]$.

(B) $$||z - \cos^{-1}\cos 12| - |z - \sin^{-1}\sin 12|| = 8(\pi - 3)$$

Since

$$|\cos^{-1}\cos 12 - \sin^{-1}\sin 12| = 8(\pi - 3)$$

Therefore, locus of z is portion of a line joining z_1 and z_2 except the segment between z_1 and z_2.

(C) $$z^2 - i|z_1|^2 = k_2 - k_1$$

$$x^2 - y^2 + 2ixy - i\lambda_1 = \lambda_2$$

Therefore,

$$x^2 - y^2 = \lambda_2 \text{ and } xy = \frac{\lambda_1}{2}$$

Hence, locus of z is point of intersection of hyperbolae.

(D) $$\left|z - 1 - \sin^{-1}\frac{1}{\sqrt{3}}\right| + \left|z + \cos^{-1}\frac{1}{\sqrt{3}} - \frac{\pi}{2}\right| = 1$$

Since

$$1 + \sin^{-1}\frac{1}{\sqrt{3}} + \cos^{-1}\frac{1}{\sqrt{3}} - \frac{\pi}{2} = 1$$

Therefore, $|z - z_1| + |z - z_2| = |z_1 + z_2|$

Hence, locus of z is line segment joining z_1 and z_2.

Solved JEE 2017 Questions

JEE Main 2017

1. Let ω be a complex number such that $2\omega + 1 = z$, where $z = \sqrt{-3}$. If $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2-1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k$, then k is equal to:

(A) z **(B)** -1

(C) 1 **(D)** $-z$

(OFFLINE)

Solution: Applying column-reducing operation, $C_1 = C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} 3 & 1 & 1 \\ 0 & -(1+\omega^2) & \omega^2 \\ 0 & \omega^2 & \omega \end{vmatrix} = 3k$$

$$\begin{vmatrix} 3 & 1 & 1 \\ 0 & \omega & \omega^2 \\ 0 & \omega^2 & \omega \end{vmatrix} = 3k$$

Since $1 + \omega + \omega^2 = 0$, open by C_1:

$$3(\omega^2 - \omega^4) = 3k$$
$$3(-1 - \omega - \omega) = 3k$$
$$-3(1 + 2\omega) = 3k$$
$$-3z = 3k \Rightarrow k = -z$$

Hence, the correct answer is option (D).

2. Let $z \in \mathbb{C}$, the set of complex numbers, then the equation $2|z + 3i| - |z - i| = 0$ represents

(A) a circle with radius $\frac{8}{3}$.

(B) a circle with diameter $\frac{10}{3}$.

(C) an ellipse with length of major axis $\frac{16}{3}$.

(D) an ellipse with length of minor axis $\frac{16}{9}$.

(ONLINE)

Solution: Substituting $z = x + iy$, then the given equation of circle becomes

$$2|z+3i| - |z-i| = 0$$
$$\Rightarrow 2|x+iy+3i| - |x+iy-i| = 0$$
$$\Rightarrow 2|x+(y+3)i| - |x+(y-1)i| = 0$$
$$\Rightarrow 2|x+(y+3)i| = |x+(y-1)i|$$

Now,

$$|x+iy| = \sqrt{x^2+y^2}$$

$$\Rightarrow 2\sqrt{x^2+(y+3)^2} = \sqrt{x^2+(y-1)^2}$$

$$\Rightarrow 4(x^2+(y+3)^2) = x^2+(y-1)^2$$

$$\Rightarrow 4x^2+4(y+3)^2 = x^2+(y-1)^2$$

$$\Rightarrow 4x^2+4(y+3)^2 = x^2+(y-1)^2$$
$$\Rightarrow 4x^2+4y^2+36+24y = x^2+y^2+1-2y$$
$$\Rightarrow 4x^2+4y^2+36+24y-x^2-y^2-1+2y = 0$$
$$\Rightarrow 3x^2+3y^2+26y+35 = 0$$

Dividing this equation by 3, we get

$$x^2+y^2+\frac{26}{3}y+\frac{35}{3} = 0$$

Therefore, the radius of the circle is

$$r = \sqrt{\left(\frac{26}{3}\right)^2 - \frac{35}{3} + 0} = \sqrt{\frac{169}{9} - \frac{35}{3}} = \sqrt{\frac{169-105}{9}} = \sqrt{\frac{64}{9}} = \frac{8}{3}$$

Thus, the equation represents a circle with radius $\frac{8}{3}$.

Hence, the correct answer is option (A).

3. The equation $\text{Im}\left(\frac{iz-2}{z-i}\right) + 1 = 0$, $z \in \mathbb{C}$, $z \neq i$ represents a part of a circle having radius equal to

(A) 1 **(B)** $\frac{3}{4}$

(C) $\frac{1}{2}$ **(D)** 2

(ONLINE)

Solution: It is given that

$$\text{Im}\left(\frac{iz-2}{z-i}\right) + 1 = 0$$

Writing $z = x + iy$, we have

$$\text{Im}\left(\frac{ix-y-2}{x+i(y-1)}\right) + 1 = 0$$

$$\Rightarrow \text{Im}\left(\frac{ix-y-2}{x+i(y-1)} \times \frac{x-i(y-1)}{x-i(y-1)}\right) + 1 = 0$$

$$\Rightarrow \text{Im}\left(\frac{ix^2+i(y-1)(y+2)-x(y+2)+x(y-1)}{x^2+(y-1)^2}\right) + 1 = 0$$

$$\Rightarrow \frac{x^2+y^2+y-2}{x^2+(y-1)^2} = -1$$

$$\Rightarrow x^2+y^2+y-2 = -x^2-y^2+2y-1$$

$$\Rightarrow 2x^2+2y^2-y-1 = 0$$

$$\Rightarrow x^2+y^2-\frac{1}{2}y-\frac{1}{2} = 0$$

Comparing this with the general equation of circle, we get the radius of the circle (R) as

$$R = \sqrt{g^2+f^2-c}$$
$$= \sqrt{\frac{1}{16}+\frac{1}{2}} = \sqrt{\frac{9}{16}} = \frac{3}{4}$$

Hence, the correct answer is option (B).

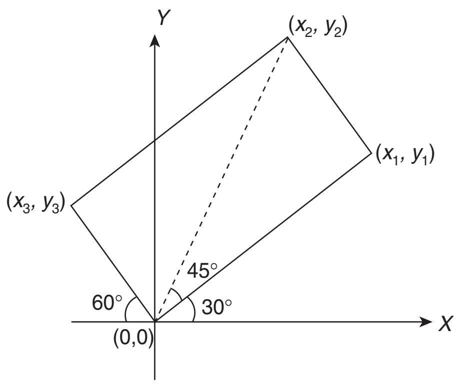
4. A square, of each side 2, lies above the x-axis and has one vertex at the origin. If one of the sides passing through the origin makes an angle 30° with the positive direction of the x-axis, then the sum of the x-coordinates of the vertices of the square is

(A) $\sqrt{3}-2$ **(B)** $2\sqrt{3}-1$

(C) $\sqrt{3}-1$ **(D)** $2\sqrt{3}-2$

(ONLINE)

Solution: The given geometrical situation is depicted in the following figure:



From the figure, we have

$$x_1 = 2\cos 30° = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3},$$

and $$y_1 = 2\sin 30° = 2\left(\frac{1}{2}\right) = 1$$

Therefore,

$$x_2 = \text{Re}\left[(x_1 + iy_1)\left(\frac{2\sqrt{2}}{2}(\cos 45° + 0° \sin 45°)\right)\right]$$

$$= \text{Re}[(\sqrt{3}+1)(1+i)]$$

$$= \text{Re}[\sqrt{3}+i+\sqrt{3}i-1]$$

Now,

$$x_2 = \sqrt{3}-1$$

and $$y_2 = \text{Im}[\sqrt{3}+i+\sqrt{3}i-1] \Rightarrow y_2 = 1+\sqrt{3}$$

Also,

$$x_3 = -2\cos 60° = -2\left(\frac{1}{2}\right) = -1$$

Therefore, the sum of the x-coordinates of the vertices of the square is

$$x_1 + x_2 + x_3 + x_4 = \sqrt{3}+\sqrt{3}-1-1+0$$

$$= 2\sqrt{3}-2$$

Hence, the correct answer is option (D).

JEE Advanced 2017

1. Let a, b, x and y be real numbers such that $a - b = 1$ and $y \neq 0$. If the complex number $z = x + iy$ satisfies $\text{Im}\left(\frac{az+b}{z+1}\right) = y$, then which of the following is(are) possible value(s) of x?

(A) $-1+\sqrt{1-y^2}$ **(B)** $-1-\sqrt{1-y^2}$

(C) $1+\sqrt{1+y^2}$ **(D)** $1-\sqrt{1+y^2}$

Solution: It is given that $z = x + iy$ satisfies $\text{Im}\left(\frac{az+b}{z+1}\right) = y$. Therefore,

$$\text{Im}\left(\frac{a(x+iy)+b}{(x+iy)+1}\right) = y$$

$$\Rightarrow \text{Im}\left(\frac{ax+iay+b}{x+iy+1}\right) = y$$

Rationalising the above equation: Multiplying and dividing LHS by $(x + 1 - iy)$, we get

$$\text{Im}\left(\frac{(ax+iay+b)}{(x+iy+1)} \times \frac{(x+1-iy)}{(x+1-iy)}\right) = y$$

Using $a^2 - b^2 = (a + b)(a - b)$, we get

$$\text{Im}\left(\frac{(ax+iay+b)(x+1-iy)}{(x+1)^2-(iy)^2}\right) = y$$

$$\text{Im}\left(\frac{ax^2+ax-iayx+iaxy-i^2ay^2+bx+b-iby}{(x+1)^2+y^2}\right) = y \quad (\text{as } i^2 = -1)$$

$$\text{Im}\left(\frac{(ax^2+ax+ay^2+bx+b)+i(axy-ayx+ay-by)}{(x+1)^2+y^2}\right) = y$$

Rearranging LHS, we get

$$\text{Im}\left(\frac{[(ax^2+bx)+(ax+b)+ay^2]+i(ay-by)}{(x+1)^2+y^2}\right) = y$$

$$\Rightarrow \frac{ay-by}{(x+1)^2+y^2} = y \text{ (as Im of the value in bracket is coefficient of } i)$$

$$\Rightarrow \cancel{y}(a-b) = \cancel{y}((x+1)^2+y^2) \Rightarrow (a-b) = (x+1)^2+y^2$$

It is given that $a - b = 1$ and $y \neq 0$. Therefore,

$$1 = (x + 1)^2 + y^2$$

$$\Rightarrow 1 - y^2 = (x + 1)^2$$

$$\Rightarrow (x+1) = \pm\sqrt{1-y^2} \qquad (\text{as } x^2 = b \Rightarrow x = \pm\sqrt{b})$$

$$\Rightarrow 1 = -1 \pm \sqrt{1-y^2}$$

or $$x = -1+\sqrt{1-y^2} \text{ and } x = -1-\sqrt{1-y^2}$$

Hence, the correct answers are options (A) and (B).

6 Quadratic Equations

6.1 Polynomial

Algebraic expression formed by terms of the form cx^n, n being a non-negative integer, is called a *polynomial*.

Or

An equation of the form

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0 = 0 \qquad (1)$$

(where $a_0, a_1, \ldots, a_n$ are real coefficients, $a_n \neq 0$ and x is a real variable or a complex variable) is called a *polynomial equation* of degree n (the highest power of variable x in a polynomial is called the degree of polynomial). This equation is called a *linear equation* if $n = 1$, *quadratic equation* if $n = 2$, *cubic equation* if $n = 3$, *biquadratic equation* if $n = 4$ and so on.

Example: $f(x) = 5x^5 + 2x^4 - 8x^3 - 2x^2 + 4x + 5$.

6.1.1 Real Polynomial

If a_i ($i = 1, 2, 3, \ldots, n$) are real numbers and x is a real variable, then Eq. (1) is known as a *real polynomial with real coefficients*.

Example: $f(x) = 2x^2 + 4x - 5$ is a real polynomial.

6.1.2 Complex Polynomial

If a_i ($i = 1, 2, 3, \ldots, n$) are complex numbers and x is a complex variable, then Eq. (1) is known as a *complex polynomial with complex coefficients*.

Example: $2x^2 - (3 + 4i)x + (6i - 4)$ is a complex polynomial.

6.2 Definition of a Quadratic Equation

An equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$ and a, b, c are real numbers, is called a *quadratic equation*. The numbers a, b and c are called the *coefficients of the quadratic equation*.

6.3 Root of a Quadratic Equation

A root of the quadratic equation is a number α (real or complex) such that $a\alpha^2 + b\alpha + c = 0$. The roots of the quadratic equation are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Let α and β be the two roots of the given quadratic equation. Then

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

6.4 Discriminant of a Quadratic Equation

The quantity D ($D = b^2 - 4ac$) is known as the *discriminant of a quadratic equation*.

6.5 Nature of Roots

For a quadratic equation,

$$ax^2 + bx + c = 0$$

the discriminant $D = b^2 - 4ac$.

The nature of roots are given as follows:

1. If $a, b, c \in R$ and $a \neq 0$, then
   - **(a)** The quadratic equation has complex roots with non-zero imaginary parts if and only if $D < 0$, that is, $b^2 - 4ac < 0$. If $p + iq$ (p and q being real) is a root of the quadratic equation where $i = \sqrt{-1}$, then $p - iq$ is also a root of the quadratic equation.
   - **(b)** The quadratic equation has real and distinct roots if and only if $D > 0$, that is, $b^2 - 4ac > 0$.

     Roots are, namely, $\alpha = \dfrac{-b + \sqrt{D}}{2a}$ and $\beta = \dfrac{-b - \sqrt{D}}{2a}$, and then $ax^2 + bx + c = a(x - \alpha)(x - \beta)$.
   - **(c)** The quadratic equation has real and equal roots, only if $D = 0$, that is, $b^2 - 4ac = 0$.

     Roots are, namely, $\alpha = \beta = -\dfrac{b}{2a}$ and then $ax^2 + bx + c = a(x - \alpha)^2$.
   - **(d)** The quadratic equation has real roots if $D \geq 0$.

2. If $a, b, c \in Q$ and $a \neq 0$, then
   - **(a)** Roots are unequal and rational if $D > 0$ and D is a perfect square.
   - **(b)** Roots are irrational and unequal if $D > 0$ and D is not a perfect square.

3. **Conjugate roots:** The irrational and complex roots of a quadratic equation always occur in pairs. Therefore
   - **(a)** If one root is $\alpha + i\beta$, then the other root will be $\alpha - i\beta$.
   - **(b)** If one root is $\alpha + \sqrt{\beta}$, then the other root will be $\alpha - \sqrt{\beta}$.

4. Let D_1 and D_2 be the discriminants of two quadratic equations. Now,
   - **(a)** If $D_1 + D_2 \geq 0$, then at least one of D_1 and $D_2 \geq 0$.
   - **(b)** If $D_1 + D_2 < 0$, then at least one of D_1 and $D_2 < 0$.

5. In a particular condition:

(a) If $b = 0$, then the roots are equal in magnitude and opposite in sign.

(b) If $c = 0$, then one root is zero and the other one is $-\frac{b}{a}$.

(c) If $b = c = 0$, then both the roots are zero.

(d) If $a = c$, then the roots are reciprocal to each other.

(e) If $a > 0$, $c < 0$ or $a < 0$, $c > 0$, then the roots are of opposite signs.

(f) If $a > 0$, $b > 0$ and $c > 0$ or $a < 0$, $b < 0$ and $c < 0$, then both the roots are negative, provided $D \ge 0$.

(g) If $a > 0$, $b < 0$ and $c > 0$ or $a < 0$, $b > 0$ and $c < 0$, then both the roots are positive, provided $D \ge 0$.

(h) If sign of a = sign of $b \ne$ sign of c, then the root greater in magnitude is negative.

(i) If sign of b = sign of $c \ne$ sign of a, then the root greater in magnitude is positive.

(j) If $a + b + c = 0$, then one root is 1 and second root is $\frac{c}{a}$.

(k) If $a = b = c = 0$, then the equation will become an identity and will be satisfied by every value of x.

(l) If $a = 1$ and $b, c \in I$ and the roots of equation $ax^2 + bx + c = 0$ are rational numbers, then these roots must be integers.

(m) If a, b and c are odd integers, then the roots of quadratic equation cannot be rational.

Illustration 6.1 If $c \ne 0$ and the equation $\frac{p}{2x} = \frac{a}{x+c} + \frac{b}{x-c}$ has equal roots, then find the value of p.

Solution: From the given equation, we can derive that

$$\frac{p}{2x} = \frac{(a+b)x + c(b-a)}{x^2 - c^2}$$

$$\Rightarrow (2a + 2b - p)x^2 - 2c(a-b)x + pc^2 = 0$$

For equal roots,

$$c^2(a-b)^2 - pc^2(2a + 2b - p) = 0$$

$$\Rightarrow (a-b)^2 - 2p(a+b) + p^2 = 0$$

$$\Rightarrow [p - (a+b)]^2 = (a+b)^2 - (a-b)^2 = 4ab$$

$$\Rightarrow p - (a+b) = \pm 2\sqrt{ab} \Rightarrow p = (\sqrt{a} \pm \sqrt{b})^2$$

Illustration 6.2 Let $a > 0$, $b > 0$ and $c > 0$. Then prove that both the roots of the equation $ax^2 + bx + c = 0$ have negative real parts.

Solution: We have

$$D = b^2 - 4ac$$

If $D \ge 0$, then the roots of the equation are given by

$$x = \frac{-b \pm \sqrt{D}}{2a}$$

As $D = b^2 - 4ac < b^2$ ($\because a > 0, c > 0$), it follows that the roots of the quadratic equation are negative.
If $D < 0$, the roots of the equation are given by

$$x = \frac{-b \pm i\sqrt{-D}}{2a}$$

which have negative real parts.

Illustration 6.3 Prove that both the roots of equation $(x-a)(x-b) + (x-b)(x-c) + (x-c)(x-a) = 0$ are always real.

Solution:
The given equation $(x-a)(x-b) + (x-b)(x-c) + (x-c)(x-a) = 0$ can be rewritten as $3x^2 - 2(a+b+c)x + (ab+bc+ca) = 0$. Now,

$$D = 4[(a+b+c)^2 - 3(ab+bc+ca)]$$

$$= 4[a^2 + b^2 + c^2 - ab - bc - ac]$$

$$= 2[(a-b)^2 + (b-c)^2 + (c-a)^2] \ge 0$$

Hence, both the roots are always real.

Illustration 6.4 If the roots of equation $(b-c)x^2 + (c-a)x + (a-b) = 0$ are equal, then prove that a, b and c are in HP.

Solution:
Using property 5(j) (given in Section 6.5).

$$b - c + c - a + a - b = 0$$

Hence, one root is 1. Also as roots are equal, other root will also be equal to 1.

Also,

$$\alpha \cdot \beta = \frac{a-b}{b-c} \Rightarrow 1 \cdot 1 = \frac{a-b}{b-c} \Rightarrow a - b = b - c \Rightarrow 2b = a + c$$

Hence a, b and c are in HP.

Illustration 6.5 If the roots of equation $\frac{1}{x+p} + \frac{1}{x+q} = \frac{1}{r}$ are equal in magnitude but opposite in sign, then find the value of $(p+q)$?

Solution: The given equation can be written as

$$x^2 + (p + q - 2r)x + [pq - (p+q)r] = 0$$

The roots are equal and opposite in sign. Therefore, sum of the roots = 0. Hence,

$$-(p + q - 2r) = 0 \Rightarrow p + q = 2r$$

Illustration 6.6 If α and β are roots of the equation $x^2 + 2ax + b = 0$, form a quadratic equation with rational coefficients one of whose roots is

$$\alpha + \beta + \sqrt{\alpha^2 + \beta^2}$$

Solution: Given

$$\alpha + \beta = -2a \text{ and } \alpha\beta = b.$$

Clearly roots of the required equation will be $\alpha + \beta + \sqrt{\alpha^2 + \beta^2}$ and $\alpha + \beta - \sqrt{\alpha^2 + \beta^2}$.
Hence, the sum of the roots of the required equation is $2(\alpha + \beta) = -4a$, and the product of the roots of the required equation is $(\alpha + \beta)^2 - (\alpha^2 + \beta^2) = 2\alpha\beta = 2b$.
Hence, the required equation is

$$x^2 + 4ax + 2b = 0$$

Illustration 6.7 Find the total number of values of a so that $x^2 - x - a = 0$ has integral roots, where $a \in N$, $6 \le a \le 100$.

Key Points: Since all the coefficients of the given equation are integers, hence for roots to be integer discriminant must be a perfect square.

Solution: From the given equation, we have

$$D = 1 + 4a$$

which is an odd integer.
Hence, it will be in the form of $D = (2\lambda + 1)^2$. This means

$$1 + 4a = 1 + 4\lambda^2 + 4\lambda \Rightarrow a = \lambda(\lambda + 1)$$

Hence, a should be in the form of product of two consecutive integers. Since $a \in [6, 100]$, therefore

$$a = 6, 12, 20, 30, 42, 56, 72, 90$$

Thus, a can attain 8 different values.

6.6 Identity

If the quadratic equation is satisfied by more than two distinct numbers (real or complex), then it becomes an identity, that is, $a = b = c = 0$.

For example, $\frac{(x-a)(x-b)}{(c-a)(c-b)} + \frac{(x-c)(x-b)}{(a-c)(a-b)} + \frac{(x-c)(x-a)}{(b-a)(b-c)} = 1$

is satisfied by three values of x which are a, b and c. Hence, this is an identity in x.

Illustration 6.8 Let $p(x) = \frac{(x-a)(x-b)}{(c-a)(c-b)}c^2 + \frac{(x-c)(x-b)}{(a-c)(a-b)}a^2 + \frac{(x-c)(x-a)}{(b-a)(b-c)}b^2$, $a \neq b \neq c$. Prove that $p(x)$ has the property that $p(y) = y^2$ for all $y \in R$.

Solution: Note that

$$P(a) = a^2, P(b) = b^2 \text{ and } P(c) = c^2$$

Consider the polynomial $Q(x) = P(x) - x^2$. $Q(x)$ has degree at most 2.
Also $Q(a) = Q(b) = Q(c) = 0 \Rightarrow Q(x)$ has 3 distinct roots.
It follows that $Q(x)$ is identically zero, that is, $Q(y) = 0 \;\; \forall\, y \in R$.

$$P(y) - y^2 = 0 \;\forall\, y \in R \Rightarrow P(y) = y^2 \;\; \forall\, y \in R$$

6.7 Formation of a Quadratic Equation

If α and β are the roots of a quadratic equation, then the equation will be

$$(x - \alpha)(x - \beta) = 0$$

or $\quad x^2 - x(\alpha + \beta) + \alpha\beta = 0$

or $\quad x^2 - Sx + P = 0$

where S is the sum of the roots and P is the product of the roots.

Illustration 6.9 Find the quadratic equation whose one root is $\frac{1}{2+\sqrt{5}}$.

Solution: Given root

$$\frac{1}{2+\sqrt{5}} = \frac{2-\sqrt{5}}{-1} = -2+\sqrt{5}$$

Therefore, other root is $-2-\sqrt{5}$.

Again, sum of roots $= -4$ and product of roots $= -1$.
Hence, the required equation is $x^2 + 4x - 1 = 0$.

Illustration 6.10 If $\alpha \neq \beta$, but $\alpha^2 = 5\alpha - 3, \beta^2 = 5\beta - 3$, then find the equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.

Solution:

$$S = \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta} = \frac{5\alpha-3+5\beta-3}{\alpha\beta} \quad \left[\begin{array}{l}\text{since, } \alpha^2 = 5\alpha - 3 \\ \qquad\quad \beta^2 = 5\beta - 3\end{array}\right]$$

$$= \frac{5(\alpha+\beta)-6}{\alpha\beta}, \; p = \frac{\alpha}{\beta}\cdot\frac{\beta}{\alpha} = 1 \Rightarrow p = 1$$

α and β are roots of $x^2 - 5x + 3 = 0$. Therefore, $\alpha + \beta = 5$, $\alpha\beta = 3$. So

$$S = \frac{5(5)-6}{3} = \frac{19}{3}$$

Therefore,

$$x^2 - \frac{19}{3}x + 1 = 0 \Rightarrow 3x^2 - 19x + 3 = 0$$

Illustration 6.11 Let α and β be the roots of the equation $x^2 + x + 1 = 0$. Then find the equation whose roots are α^{19}, β^7.

Solution: Roots of $x^2 + x + 1 = 0$ are

$$x = \frac{-1 \pm \sqrt{1-4}}{2} \text{ and } \frac{-1 \pm \sqrt{3}i}{2} = \omega, \omega^2$$

Take $\alpha = \omega, \beta = \omega^2$. Therefore,

$$\alpha^{19} = \omega^{19} = \omega, \beta^7 = (\omega^2)^7 = \omega^{14} = \omega^2$$

Hence, required equation is $x^2 + x + 1 = 0$.

Illustration 6.12 If α and β are the roots of $3x^2 + 2x + 1 = 0$, show that the equation whose roots are $\frac{1-\alpha}{1+\alpha}$ and $\frac{1-\beta}{1+\beta}$ is $x^2 - 2x + 3 = 0$.

Solution: We have

$$\alpha + \beta = -\frac{2}{3} \text{ and } \alpha\beta = \frac{1}{3}$$

Suppose

$$\gamma = \frac{1-\alpha}{1+\alpha} \text{ and } \delta = \frac{1-\beta}{1+\beta}$$

Then

$$S = \gamma + \delta = \frac{1-\alpha}{1+\alpha} + \frac{1-\beta}{1+\beta} = \frac{2-2\alpha\beta}{1+(\alpha+\beta)+\alpha\beta} = \frac{2-\frac{2}{3}}{1-\frac{2}{3}+\frac{1}{3}} = 2$$

$$P = \gamma\delta = \frac{(1-\alpha)(1-\beta)}{(1+\alpha)(1+\beta)} = \frac{1-(\alpha+\beta)+\alpha\beta}{1+(\alpha+\beta)+\alpha\beta} = \frac{1+\frac{2}{3}+\frac{1}{3}}{1-\frac{2}{3}+\frac{1}{3}} = 3$$

Hence, the required equation is

$$x^2 - Sx + P = 0 \Rightarrow x^2 - 2x + 3 = 0$$

Your Turn 1

1. If 3 is a root of equation $x^2 + kx - 24 = 0$, then it is also a root of

   (A) $x^2 + 5x + k = 0$ **(B)** $x^2 - 5x + k = 0$
   (C) $x^2 - kx + 6 = 0$ **(D)** $x^2 + kx + 24 = 0$ **Ans.** (C)

2. For what values of k will the equation $x^2 - 2(1+3k)x + 7(3+2k) = 0$ have equal roots?

   (A) 1, –10/9 **(B)** 2, –10/9
   (C) 3, –10/9 **(D)** 4, –10/9 **Ans.** (B)

3. Show that expression $x^2 + 2(a+b+c)x + 3(bc+ca+ab)$ will be a perfect square if $a = b = c$.
4. If a, b, c and d are four non-zero real numbers such that $(d+a-b)^2 + (d+b-c)^2 = 0$ and roots of the equation $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$ are real and equal, then show that $a = b = c$.
5. If $1 - i$ is a root of the equation $x^2 + ax + b = 0$, then the values of a and b are

   (A) 2, 1 **(B)** – 2, 2
   (C) 2, 2 **(D)** 2, – 2 **Ans.** (B)

6.8 Condition for Common Root(s)

1. **One root common root:** Suppose the quadratic equations $ax^2 + bx + c = 0$ and $a'x^2 + b'x + c' = 0$ (where $a, a' \neq 0$ and $ab' - a'b \neq 0$) have a common root. Let this common root be α. Then

   $$a\alpha^2 + b\alpha + c = 0 \quad \text{and} \quad a'\alpha^2 + b'\alpha + c' = 0$$

   Solving the above equations, we get

   $$\frac{\alpha^2}{bc' - b'c} = \frac{\alpha}{a'c - ac'} = \frac{1}{ab' - a'b}$$

   $$\Rightarrow \alpha^2 = \frac{bc' - b'c}{ab' - a'b} \quad \text{and} \quad \alpha = \frac{a'c - ac'}{ab' - a'b}$$

   Eliminating α, we get

   $$\frac{(a'c - ac')^2}{(ab' - a'b)^2} = \frac{bc' - b'c}{ab' - a'b}$$

   $$\Rightarrow (a'c - ac')^2 = (bc' - b'c)(ab' - a'b)$$

   This is the required condition for the equations to have a common root.
2. **Both roots common roots:** In this case,

   $$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

   Then both the quadratic equations will have the same roots. Given that the two quadratic equations have a common root. This root can be obtained by subtracting the equations after making the coefficients of x^2 same.

Illustration 6.13 If the equations $x^2 + 2x\sin y + 1 = 0$, where $y \in (0, \pi/2)$ and $ax^2 + x + 1 = 0$ have a common root, then find the values of a and y.

Solution: Since discriminant of $x^2 + 2x\sin y + 1 = 0$ is $4\sin^2 y - 4 < 0$, hence, roots of this equation are imaginary. Now this equation and $ax^2 + x + 1 = 0$ have a common root. Hence, roots of $ax^2 + x + 1 = 0$ are also imaginary. This implies that the given two equations have both the roots common. Hence, both the equations are identical and so,

$$\frac{a}{1} = \frac{1}{2\sin y} = \frac{1}{1} \Rightarrow a = 1, y = \frac{\pi}{6}$$

Illustration 6.14 If equations $ax^3 + 2bx^2 + 3cx + 4d = 0$ and $ax^2 + bx + c = 0$ have a non-zero common root, then prove that $(c^2 - 2bd)(b^2 - 2ac) \geq 0$.

Solution: Let α be the non-zero common root. So,

$$a\alpha^3 + 2b\alpha^2 + 3c\alpha + 4d = 0 \quad (1)$$
$$a\alpha^2 + b\alpha + c = 0 \quad (2)$$

Equation (1) – α Eq. (2) gives

$$b\alpha^2 + 2c\alpha + 4d = 0 \quad (3)$$

By Eqs. (2) and (3), we have

$$\frac{\alpha^2}{2c^2 - 4bd} = \frac{\alpha}{4ad - bc} = \frac{1}{b^2 - 2ac} \Rightarrow \frac{(4ad - bc)^2}{(b^2 - 2ac)^2} = \frac{2c^2 - 4bd}{b^2 - 2ac}$$

$$\Rightarrow (4ad - bc)^2 = 2(b^2 - 2ac)(c^2 - 2bd) \Rightarrow (b^2 - 2ac)(c^2 - 2bd) \geq 0$$

Illustration 6.15 If equations $ax^2 + bx + c = 0$ and $x^2 + 2x + 3 = 0$ have a common root, show that $a{:}b{:}c = 1{:}2{:}3$.

Solution: Since roots of equation $x^2 + 2x + 3 = 0$ are imaginary and equations $x^2 + 2x + 3 = 0$ and $ax^2 + bx + c = 0$ have a common root, both roots will be common. Hence, both equations are identical. Therefore,

$$\Rightarrow \frac{a}{1} = \frac{b}{2} = \frac{c}{3} \Rightarrow a : b : c = 1 : 2 : 3$$

Illustration 6.16 If a, b and c are in GP, then the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root. Show that $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in AP.

Solution: Given a, b, and c are in GP. Now $b^2 = ac$. So $ax^2 + 2bx + c = 0$ can be written as

$$ax^2 + 2\sqrt{ac}\,x + c = 0$$

$$\Rightarrow (\sqrt{a}x + \sqrt{c})^2 = 0 \Rightarrow x = -\sqrt{\frac{c}{a}}$$

This must be the common root by hypothesis. So it must satisfy the equation

$$dx^2 + 2ex + f = 0 \Rightarrow d\left(\frac{c}{a}\right) - 2e\sqrt{\frac{c}{a}} + f = 0$$

$$\Rightarrow \frac{d}{a} + \frac{f}{c} = \frac{2e}{b}$$

Hence, $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in AP.

Illustration 6.17 If α, β are the roots of $x^2 + px + q = 0$ and γ, δ are the roots of $x^2 + rx + s = 0$, evaluate $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$ in terms of p, q, r and s. Deduce the condition that the equation may have a common root.

Solution: We have

$$\alpha+\beta=-p;\ \alpha\beta=q;\ \gamma+\delta=-r;\ \gamma\delta=s$$

Now,

$$(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)$$
$$=[\alpha^2-\alpha(\gamma+\delta)+\gamma\delta]\,[\beta^2-\beta(\gamma+\delta)+\gamma\delta]$$
$$=(\alpha^2+\alpha r+s)(\beta^2+\beta r+s)$$
$$=(\alpha\beta)^2+r(\alpha^2\beta+\beta^2\alpha)+s(\alpha^2+\beta^2)+\alpha\beta r^2+rs(\alpha+\beta)+s^2$$
$$=q^2-prq+s(p^2-2q)+qr^2-prs+s^2$$
$$=(s-q)^2+q(r-p)^2-p(s-q)(r-p)=(s-q)^2+(r-p)(qr-ps)$$

If the two equations have a common root, then either $\alpha=\gamma$ or $\alpha=\delta$ or $\beta=\gamma$ or $\beta=\delta$.
Hence, one of the factors of $(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)$ must be zero. Therefore, $(s-q)^2+(r-p)(qr-ps)=0$ is the condition for the equations to have a common root.

6.9 Quadratic Expression

The expression ax^2+bx+c, where $a\neq 0$ and a, b, c are real numbers, is known as *real quadratic expression*.

6.9.1 Graph of a Quadratic Expression

Let $f(x)=ax^2+bx+c$ ($a\neq 0$, a, b, $c\in R$). It can be written as

$$y=f(x)=a\left[\left(x+\frac{b}{2a}\right)^2-\frac{D}{4a^2}\right]$$
$$\Rightarrow\left(y+\frac{D}{4a}\right)=a\left(x+\frac{b}{2a}\right)^2$$

where D is the discriminant.

This equation is in the form of

$$(x-\alpha)^2=4k(y-\beta)$$

which represents a parabola with vertex at (α,β). That is, $\left(\frac{-b}{2a},\frac{-D}{4a}\right)$ in this case.

1. If $a>0$, the parabola is concave upwards.
2. If $a<0$, the parabola is concave downwards. Depending upon the value of discriminant $D=b^2-4ac$, we have the following three cases:

Case 1.
(a) When $a>0$ and $D<0$:
In this case, the parabola is concave upwards and has no roots (Fig. 6.1).
$f(x)>0\ \forall\ x\in R$

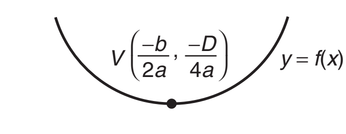


Figure 6.1

(b) When $a<0$ and $D<0$:
The parabola is concave downwards and has no roots. It always lies below x-axis (Fig. 6.2).
$f(x)<0\ \forall\ x\in R$

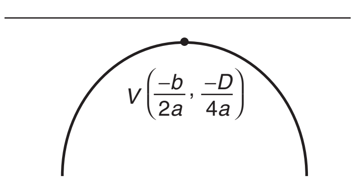


Figure 6.2

Case 2.
(a) When $a>0$ and $D=0$:
The parabola is concave upwards and touches the x-axis only at $x=\frac{-b}{2a}$ (Fig. 6.3).
$f(x)>0\ \forall\ x\in R-\left\{\frac{-b}{2a}\right\}$
and $f\left(\frac{-b}{2a}\right)=0$.

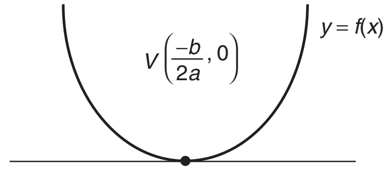


Figure 6.3

(b) When $a<0$ and $D=0$:
The parabola is concave downwards and touches x-axis at $x=\frac{-b}{2a}$ (Fig. 6.4).
$f(x)<0\ \forall\ x\in R-\left\{\frac{-b}{2a}\right\}$

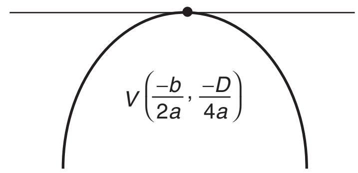


Figure 6.4

Case 3.
(a) When $a>0$ and $D>0$:
The parabola is concave upwards and cuts x-axis at α and β (roots of $ax^2+bx+c=0$). (Fig. 6.5).
$f(\alpha)=f(\beta)=0$ and $f(x)>0$ $\forall\ x\in(-\infty,\alpha)\cup(\beta,\infty)$
$f(x)<0\ \forall\ x\in(\alpha,\beta)$

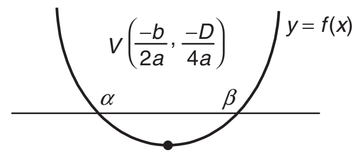


Figure 6.5

(b) When $a<0$ and $D>0$:
The parabola is concave downwards and cuts x-axis at α, β (roots of $ax^2+bx+c=0$) (Fig. 6.6).
$f(\alpha)=f(\beta)=0$ and $f(x)<0$ $\forall\ x\in(-\infty,\alpha)\cup(\beta,\infty)$
$f(x)>0\ \forall\ x\in(\alpha,\beta)$

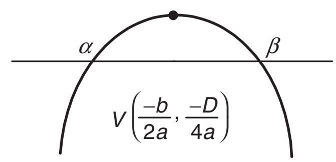


Figure 6.6

Illustration 6.18 If a, b, $c\in R$, $a\neq 0$ and the quadratic equation $ax^2+bx+c=0$ has no real root, then show that $(a+b+c)\,c>0$.

Solution: Let $f(x)=ax^2+bx+c$.
Given $ax^2+bx+c=0=f(x)=0$ has no real root, therefore, $f(x)$ will have same sign for all real values of x. So

$$\Rightarrow f(0) \text{ and } f(1) \text{ will have same sign}$$
$$\Rightarrow f(1)\cdot f(0)>0$$
$$\Rightarrow (a+b+c)\,c>0$$

Illustration 6.19 Suppose $f(x)$ is a quadratic expression which is positive for all real x and $g(x)=f(x)+f'(x)+f''(x)$ where $f'(x)$ and $f''(x)$ are the derivatives of $f(x)$ of first and second order. Show that $g(x)$ is positive for all real x.

Solution: Let $f(x)=ax^2+bx+c$.
If $f(x)$ is to be positive for all values of x, two conditions are necessary:
(i) a must be positive and
(ii) discriminant, b^2-4ac must be negative.

Now,

$$g(x) = (ax^2 + bx + c) + (2ax + b) + 2a$$

Discriminant of $g(x) = (b + 2a)^2 - 4a(2a + b + c)$ is

$$(b^2 - 4ac) - 4a^2 = \text{negative} \qquad (\text{since } b^2 - 4ac < 0)$$

Since, coefficient of x^2 in $g(x)$ is positive and discriminant of $g(x) = 0$ is negative, hence, $g(x)$ is positive for all values of x.

Illustration 6.20 If $ax^2 + bx + 8 = 0$ does not have 2 distinct real roots, then find the minimum value of $2a + b$.

Solution: Let $f(x) = ax^2 + bx + 8$.

Since $f(x) = 0$ does not have 2 distinct real roots, hence we have either $f(x) \geq 0\ \forall\ x \in R$ or $f(x) \leq 0\ \forall\ x \in R$.
Since $f(0) = 8$, therefore

$$\Rightarrow f(x) \geq 0\ \forall\ x \in R$$

In particular,

$$f(2) \geq 0$$
$$\Rightarrow 4a + 2b + 8 \geq 0$$
$$\Rightarrow 2a + b \geq -4$$

Hence, the minimum value is –4.

Illustration 6.21 Find the quadratic equation $f(x)$ such that $f(x) \leq 0\ \forall\ x \in [2, 3]$ and coefficient of x^2 is one of the roots of $x^2 - x - 2 = 0$.

Solution: Roots of $x^2 - x - 2 = 0$ are $x = 2, -1$ and $f(x) \leq 0\ \forall\ x \in [2, 3]$.
Hence, $a > 0 \Rightarrow f(x) = 2(x - 2)(x - 3) = 2(x^2 - 5x + 6)$.

Illustration 6.22 If $f(x) \leq 0\ \forall\ |x| \geq 2$ and $f(x)$ is a quadratic equation such that $f(1) = 6$, then find $f(x)$.

Solution: Given $f(x) \leq 0\ \forall\ |x| \geq 2$

$$\Rightarrow f(x) \leq 0\ \forall\ x \geq 2 \text{ or } x \leq -2$$

Let $f(x) = a(x - 2)(x + 2)$. So

$$f(1) = 6 \Rightarrow a = -2$$

Therefore,

$$f(x) = -2(x^2 - 4)$$

6.10 Range of a Quadratic or Rational Expression

6.10.1 Quadratic Expression

If $a > 0$, the minima of $f(x)$ occurs at $x = \dfrac{-b}{2a}$ and range is $\left[-\dfrac{D}{4a}, \infty\right)$, or if $a < 0$, then the maxima of $f(x)$ occurs at $x = \dfrac{-b}{2a}$ and range is $\left(-\infty, -\dfrac{D}{4a}\right]$.

Illustration 6.23 Find the following:

(a) Maximum value of $2 - 3x - 4x^2$.

(b) Minimum value of $x^2 - 8x + 17$.

Solution:

(a) $2 - 3x - 4x^2 = 2 - 4\left(x^2 + \dfrac{3}{4}x\right)$

$$= 2 - 4\left(x^2 + \frac{3}{4}x + \frac{9}{64}\right) + \frac{9}{16}$$

$$= \frac{41}{16} - 4\left(x + \frac{3}{8}\right)^2$$

Hence, the maximum value of $2 - 3x - 4x^2$ is $\dfrac{41}{16}$ and it takes this value when $x = -\dfrac{3}{8}$.

(b) Since $a = 1 > 0$, its minimum value is

$$\frac{4ac - b^2}{4a} = \frac{4(1)(17) - 64}{4} = \frac{4}{4} = 1$$

Illustration 6.24 Let α and β be the roots of the equation $ax^2 + 2bx + c = 0$ and $\alpha + k$, $\beta + k$ be the roots of $Ax^2 + 2Bx + C = 0$. Then prove that $A^2(b^2 - ac) = a^2(B^2 - AC)$.

Solution: Let

$$f(x) = x^2 + \frac{2bx}{a} + \frac{c}{a} \text{ and } g(x) = x^2 + \frac{2Bx}{A} + \frac{C}{A}$$

As roots of $f(x) = 0$ are α and β and roots of $g(x) = 0$ are $\alpha + k$ and $\beta + k$, graph of $g(x)$ will be obtained by translating the graph of $f(x)$ by k units on the x-axis (Fig. 6.7).

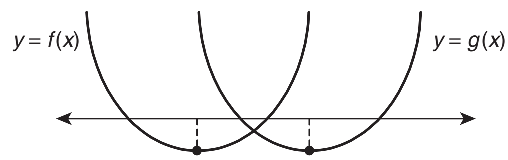


Figure 6.7

Hence,

Minimum value of $f(x)$ = Minimum value of $g(x)$

$$\Rightarrow f\left(\frac{-b}{a}\right) = g\left(\frac{-B}{A}\right)$$

$$\Rightarrow -\left(\frac{b^2 - ac}{a^2}\right) = -\left(\frac{B^2 - AC}{A^2}\right)$$

$$\Rightarrow A^2(b^2 - ac) = (B^2 - AC)a^2$$

Illustration 6.25 If min $[x^2 + (a - b)x + (1 - a - b)] >$ max $[-x^2 + (a + b)x - (1 + a + b)]$, prove that $a^2 + b^2 < 4$.

Solution: Given

$$\min [x^2 + (a - b)x + (1 - a - b)] > \max [-x^2 + (a + b)x - (1 + a + b)]$$

$$\Rightarrow \min\left[\left(x + \frac{a - b}{2}\right)^2 + (1 - a - b) - \frac{(a - b)^2}{4}\right]$$

$$> \max\left[\left(\frac{a + b}{2}\right)^2 - (1 + a + b) - \left(x - \frac{a + b}{2}\right)^2\right]$$

$$\Rightarrow 1-a-b-\frac{(a-b)^2}{4} > \left(\frac{a+b}{2}\right)^2 - (1+a+b)$$
$$\Rightarrow a^2+b^2<4$$

6.10.2 Rational Expression

To find the values attained by rational expression of the form $\frac{a_1x^2+b_1x+c_1}{a_2x^2+b_2x+c_2}$ for real values of x, use the following steps:

1. Equate the given rational expression to y.
2. Obtain a quadratic equation in x by simplifying the expression in step 1.
3. Obtain the discriminant of the quadratic equation in step 2.
4. Put discriminant ≥ 0 and solve the inequation for y. The values of y so obtained determine the set of values attained by the given rational expression.

Illustration 6.26 Find the range of $\frac{x^2-x+1}{x^2+x+1}$ for all real x.

Solution: Let $y=\frac{x^2-x+1}{x^2+x+1}$. So

$$x^2(y-1)+(y+1)x+(y-1)=0$$

Since x is real, $b^2-4ac \geq 0$. This means

$$(y+1)^2-4(y-1)(y-1)\geq 0 \Rightarrow 3y^2-10y+3\leq 0$$
$$\Rightarrow (3y-1)(y-3)\leq 0 \Rightarrow \left(y-\frac{1}{3}\right)(y-3)\leq 0 \Rightarrow \frac{1}{3}\leq y\leq 3$$

Thus, the greatest and least values of expression are 3 and 1/3, respectively.

6.11 Location of Roots

Let $f(x)=ax^2+bx+c$, where $a, b, c \in R$ be a quadratic expression and k, k_1, k_2 be real numbers such that $k_1<k_2$. Let α and β be the roots of the equation $f(x)=0$, that is, $ax^2+bx+c=0$. Then

$$\alpha=\frac{-b+\sqrt{D}}{2a},\ \beta=\frac{-b-\sqrt{D}}{2a}$$

where D is the discriminant of the equation.

Let α and β be the roots of the equation $ax^2+bx+c=0$. Then

$$\alpha+\beta=-\frac{b}{a} \text{ and } \alpha\beta=\frac{c}{a}$$

6.11.1 Location of Real Roots on the Number Line

1. **Both the roots are positive, that is, they lie in (0, ∞):** In this case, the sum of the roots as well as the product of the roots must be positive (Fig. 6.8). So

$$\alpha+\beta=-\frac{b}{a}>0 \text{ and } \alpha\beta=\frac{c}{a}>0 \text{ with } b^2-4ac\geq 0$$

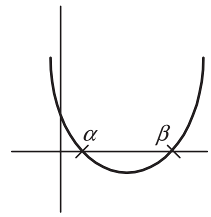


Figure 6.8

2. **Both the roots are negative, that is, they lie in (–∞, 0):** In this case, the sum of the roots must be negative and the product of the roots must be positive (Fig. 6.9). That is,

$$\alpha+\beta=-\frac{b}{a}<0 \text{ and } \alpha\beta=\frac{c}{a}>0 \text{ with } b^2-4ac\geq 0$$

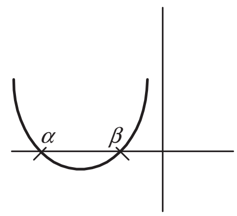


Figure 6.9

3. **One root is positive and the other is negative, that is, origin is lying between the roots:** Clearly, $af(0)<0$ is the necessary and sufficient condition (Fig. 6.10).

$$\alpha\beta=\frac{c}{a}<0 \text{ and } D>0$$

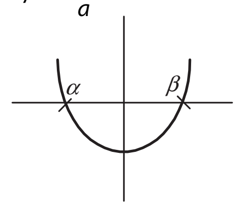


Figure 6.10

4. **Both the roots are greater than a real number k:** See Fig. 6.11.

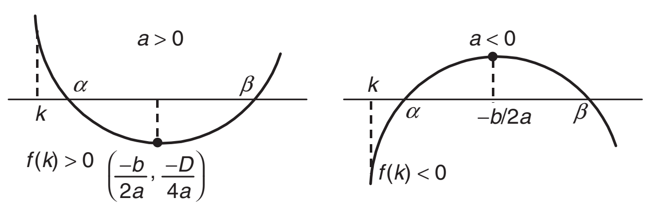


Figure 6.11

$$D\geq 0 \quad (6.1)$$
$$a\,f(k)>0 \quad (6.2)$$
$$-\frac{b}{2a}>k \quad (6.3)$$

These are the necessary and sufficient conditions.

Or

If both the roots are greater than k, then $\alpha-k>0$ and $\beta-k>0$. Therefore, $\alpha-k+\beta-k>0$ and $(\alpha-k)(\beta-k)>0$ with $D\geq 0$.

5. Both the roots are less than a real number *k*: See Fig. 6.12.

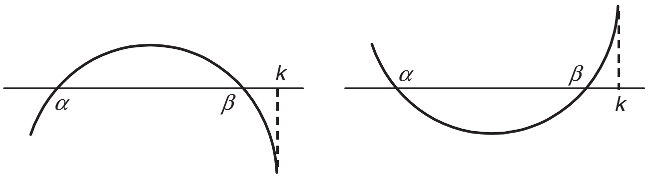


Figure 6.12

$$D \geq 0 \quad (6.4)$$

$$a\,f(k) > 0 \quad (6.5)$$

$$-\frac{b}{2a} < k \quad (6.6)$$

These are the necessary and sufficient conditions.

Or

If both the roots are less than k, then $\alpha - k < 0$ and $\beta - k < 0$.

Therefore, $\alpha + \beta - 2k < 0$ and $(\alpha - k)(\beta - k) > 0$ with $D \geq 0$.

6. A real number *k* is lying between the roots, that is, one root is less than *k* and the other is greater than *k*: See Fig. 6.13.

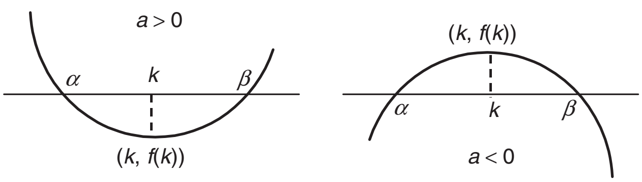


Figure 6.13

$$D > 0 \quad (6.7)$$

$$a\,f(k) < 0 \quad (6.8)$$

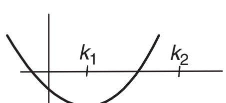
These are the necessary and sufficient conditions.

Or

If $\alpha < k$ and $\beta > k$, then $\alpha - k < 0$ and $\beta - k > 0$.

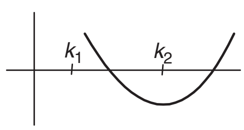
Therefore, $(\alpha - k)(\beta - k) < 0$ with $D > 0$.

7. Exactly one root is lying between k_1 and k_2: See Figs. 6.14 and 6.15.



$a\,f(k_1) < 0$ and $a\,f(k_2) > 0$

Figure 6.14



$a\,f(k_1) > 0$ and $a\,f(k_2) < 0$

Figure 6.15

Hence, the required condition is $f(k_1) \cdot f(k_2) < 0$.

Or

If $\alpha < k_1$ and $\beta < k_2$, then $\alpha - k_1 < 0$ and $\beta - k_2 < 0$.

If $\alpha > k_1$ and $\beta > k_2$, then $\alpha - k_1 > 0$ and $\beta - k_2 > 0$.

Therefore, $\alpha + \beta - (k_1 + k_2) < 0$, [or $\alpha + \beta - (k_1 + k_2) > 0$] and $(\alpha - k_1)(\beta - k_2) > 0$ with $D > 0$.

8. Both the roots lie between k_1 and k_2 ($k_1 < k_2$): See Figs. 6.16 and 6.17.

$$D \geq 0 \quad (6.9)$$

$$af(k_1) > 0 \text{ and } af(k_2) > 0 \quad (6.10)$$

$$k_1 < \frac{-b}{2a} < k_2 \quad (6.11)$$

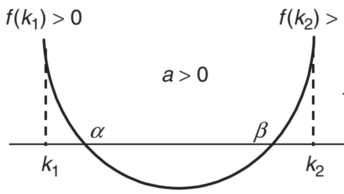


Figure 6.16

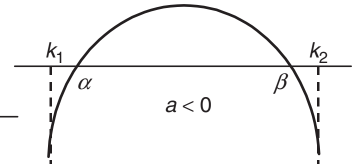


Figure 6.17

Or

If both the roots lie between k_1 and k_2, $k_1 < k_2$ then $\alpha - k_1 > 0$, $\beta - k_1 > 0$, $\alpha - k_2 < 0$, $\beta - k_2 < 0$.

Therefore, $\alpha - k_1 + \beta - k_1 > 0$, $\alpha - k_2 + \beta - k_2 < 0$ and $(\alpha - k_1)(\beta - k_1) > 0$ and $(\alpha - k_2)(\beta - k_2) > 0$ with $D \geq 0$.

9. k_1 and k_2 ($k_1 < k_2$) lie between the roots: See Figs. 6.18 and 6.19.

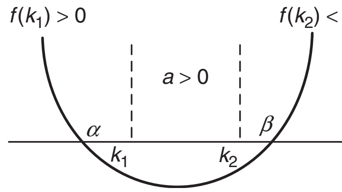


Figure 6.18

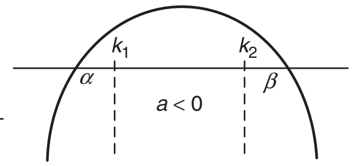


Figure 6.19

$$D > 0 \quad (6.12)$$

$$af(k_1) < 0 \text{ and } af(k_2) < 0 \quad (6.13)$$

or

If $\alpha < k_1$ and $\beta > k_2$, $k_1 < k_2$, then

$$\alpha - k_1 < 0,\ \beta - k_1 > 0,\ \alpha - k_2 < 0,\ \beta - k_2 > 0$$

Therefore,

$$(\alpha - k_1)(\beta - k_1) < 0 \text{ and } (\alpha - k_2)(\beta - k_2) < 0$$

Illustration 6.27 For the quadratic equation $x^2 - (m-3)x + m = 0$, find the value of m for which

(a) one root is smaller than 2 and the other is greater than 2.
(b) both roots are greater than 2.
(c) both roots lie in (1, 2).
(d) exactly one root lie in (1, 2).

Solution: Let $f(x) = x^2 - (m-3)x + m$ and $D = (m-1)(m-9)$.

(a) (a) $D > 0$ and (b) $f(2) < 0$
That is, $m < 1$ or $m > 9$ and $m > 10$
Therefore, $m \in (10, \infty)$.
See Fig. 6.20.

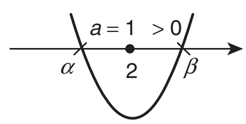


Figure 6.20

(b) The required necessary and sufficient conditions are

$$D \geq 0 \Rightarrow m \leq 1 \text{ or } m \geq 9 \quad (1)$$

$$f(2) > 0 \Rightarrow m < 10 \quad (2)$$

$$-\frac{b}{2a} > 2 \Rightarrow m > 7 \quad (3)$$

From Eqs. (1) – (3), we get $m \in [9, 10)$. See Fig. 6.21.

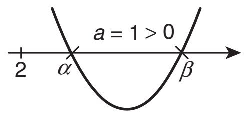


Figure 6.21

(c) $D \geq 0 \Rightarrow m \leq 1$ or $m \geq 9$ (1)

$af(1) > 0 \Rightarrow 4 > 0 \ \forall \ m \in R$ (2)

$af(2) > 0 \Rightarrow m < 10$ (3)

$1 < \dfrac{\alpha + \beta}{2} < 2 \Rightarrow m > 5$ and $m < 7$ (4)

Taking intersection of these four conditions , we get $m \in \phi$. See Fig. 6.22.

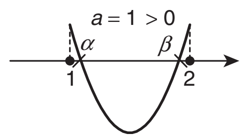


Figure 6.22

(d) $D > 0 \quad \Rightarrow m < 1$ or $m > 9$ (1)

$f(1)\cdot f(2) < 0 \Rightarrow m > 10$ (2)

Taking intersection of these two conditions, we get $m \in (10, \infty)$. See Fig. 6.23.

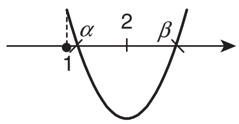


Figure 6.23

Illustration 6.28 Find all values of p so that 6 lies between roots of the equation $x^2 + 2(p - 3)x + 9 = 0$.

Solution: Let $f(x) = x^2 + 2(p - 3)x + 9$.

As 6 lies between the roots of $f(x) = 0$, we can take $D > 0$ and $af(6) < 0$. See Fig. 6.24.

(a) Consider $D > 0$:

$[-2(p - 3)]^2 - 4.1.9 > 0$

$\Rightarrow (p - 3)^2 - 9 > 0$

$\Rightarrow p(p - 6) > 0$

$\Rightarrow p \in (-\infty, 0) \cup (6, \infty)$ (1)

Figure 6.24

(b) Consider $af(6) < 0$:

$1\cdot[36 + 12(p - 3) + 9] < 0$

$\Rightarrow 12p + 9 < 0 \Rightarrow p + \dfrac{3}{4} < 0 \Rightarrow p \in \left(-\infty, -\dfrac{3}{4}\right)$

Hence, the values of p satisfying conditions (a) and (b) at the same time are $p \in \left(-\infty, -\dfrac{3}{4}\right)$.

Illustration 6.29 Find the values of m for which both roots of equation $x^2 - mx + 1 = 0$ are less than unity.

Solution: Let $f(x) = x^2 - mx + 1$.

As both roots of $f(x) = 0$ are less than 1, we can take $D \geq 0$, $af(1) > 0$ and $-\dfrac{b}{2a} < 1$.

(a) Consider $D \geq 0 : (-m)^2 - 4\cdot1\cdot1 \geq 0$

$\Rightarrow (m + 2)(m - 2) \geq 0$

$\Rightarrow m \in (-\infty, -2] \cup [2, \infty)$ (1)

(b) Consider $af(1) > 0 : 1\cdot(1 - m + 1) > 0$

$\Rightarrow m - 2 < 0 \Rightarrow m < 2$

$\Rightarrow m \in (-\infty, 2)$ (2)

(c) Consider $-\dfrac{b}{2a} < 1 \Rightarrow \dfrac{m}{2} < 1$

$\Rightarrow m < 2$

$\Rightarrow m \in (-\infty, 2)$ (3)

Hence, the values of m satisfying Eqs. (1), (2) and (3) at the same time are $m \in (-\infty, -2)$.

Illustration 6.30 For what values of $m \in R$, both roots of the equation $x^2 - 6mx + 9m^2 - 2m + 2 = 0$ exceed 3?

Solution: Let $f(x) = x^2 - 6mx + 9m^2 - 2m + 2$.

As both roots of $f(x) = 0$ are greater than 3, we can take $D \geq 0$, $af(3) > 0$ and $-\dfrac{b}{2a} > 3$.

(a) Consider $D \geq 0$:

$(-6m)^2 - 4\cdot1\cdot(9m^2 - 2m + 2) \geq 0$

$\Rightarrow 8m - 8 \geq 0 \Rightarrow m \geq 1$

So $m \in [1, \infty)$ (1)

(b) Consider $af(3) > 0$:

$\Rightarrow (9 - 18m + 9m^2 - 2m + 2) > 0$

$\Rightarrow 9m^2 - 20m + 11 > 0$

$\Rightarrow (9m - 11)(m - 1) > 0$

$\Rightarrow \left(m - \dfrac{11}{9}\right)(m - 1) > 0$

$\Rightarrow m \in (-\infty, 1) \cup \left(\dfrac{11}{9}, \infty\right)$ (2)

(c) Consider $-\dfrac{b}{2a} > 3$. Now

$\dfrac{6m}{2} > 3 \Rightarrow m > 1 \Rightarrow m \in (1, \infty)$ (3)

Hence, the values of m satisfying Eqs. (1)–(3) at the same time are $m \in \left(\dfrac{11}{9}, \infty\right)$.

Illustration 6.31 Find the value of a for which the equation $4x^2 - 2x + a = 0$ has two different roots lying in the interval $(-1, 1)$.

Solution: First of all, the roots must be real and different. Therefore,

$$4 - 16a > 0 \Rightarrow a < \frac{1}{4}$$

The roots are

$$\frac{2 \pm \sqrt{4 - 16a}}{8}$$

or $$\frac{1}{4} - \frac{\sqrt{1-4a}}{4} \text{ and } \frac{1}{4} + \frac{\sqrt{1-4a}}{4}$$

Now,

$$\frac{1}{4} - \frac{\sqrt{1 - 4a}}{4} > -1 \text{ and } \frac{1}{4} + \frac{\sqrt{1 - 4a}}{4} < 1$$

Hence,

$$1-\sqrt{1-4a} > -4 \Rightarrow 5 > \sqrt{1-4a}$$

and $$1+\sqrt{1-4a} < 4 \Rightarrow \sqrt{1-4a} < 3$$

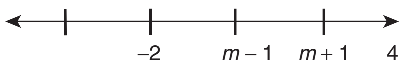
The second condition implies the first. Therefore,

$$1-4a < 9 \text{ or } 4a > -8 \text{ or } a > -2$$

Hence, $-2 < a < 1/4$.

Illustration 6.32 In what interval must m lie so that the roots of the equation $x^2 - 2mx + m^2 - 1 = 0$ lie between -2 and 4?

Solution: The given equation can be written as $(x-m)^2 = 1$ so that $x = m \pm 1$. The roots are $m-1$ and $m+1$.



Therefore, $m-1 > -2$ or $m > -1$ and $m+1 < 4$ or $m < 3$

Hence, m should lie between -1 and 3 or $-1 < m < 3$.

Illustration 6.33 Let a, b and c be real. If $ax^2 + bx + c = 0$ has two real roots α and β where $\alpha < -1$ and $\beta > 1$, then show that $1 + \frac{c}{a} + \left|\frac{b}{a}\right| < 0$.

Solution: Since the roots are real and different,

$$b^2 - 4ac > 0$$

$$\alpha < -1,\ \beta > -1 \Rightarrow \alpha + 1 < 0 \text{ and } \beta + 1 > 0 \quad (1)$$

$$\alpha < 1,\ \beta > 1 \Rightarrow \alpha - 1 < 0 \text{ and } \beta - 1 > 0 \quad (2)$$

From Eq. (1),

$$(\alpha+1)(\beta+1) < 0 \Rightarrow \alpha\beta + (\alpha+\beta) + 1 < 0$$

$$\Rightarrow \frac{c}{a} - \frac{b}{a} + 1 < 0 \quad (3)$$

From Eq. (2),

$$(\alpha-1)(\beta-1) < 0 \Rightarrow \alpha\beta - (\alpha+\beta) + 1 < 0$$

$$\Rightarrow \frac{c}{a} + \frac{b}{a} + 1 < 0 \quad (4)$$

Combining Eqs. (3) and (4), we get

$$\frac{c}{a} + \left|\frac{b}{a}\right| + 1 < 0$$

Illustration 6.34 Find a for which exactly one root of the equation $2^a x^2 - 4^a \cdot x + 2^a - 1 = 0$ lies between 1 and 2.

Solution:

$$F(1) \cdot F(2) < 0$$

$$\Rightarrow (2^a - 4^a + 2^a - 1)(4 \cdot 2^a - 2 \cdot 4^a + 2^a - 1) < 0$$

$$\Rightarrow \frac{1}{2} < 2^a < 1 \Rightarrow \log_2(1/2) < a < \log_2 1$$

$$\Rightarrow -1 < a < 0 \Rightarrow a \in (-1, 0)$$

6.12 Relation Between the Roots and Coefficients of Polynomial of Degree *n*

Consider the equation

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0 = 0 \quad (6.14)$$

where $a_0, a_1, \ldots, a_n$ are real coefficients and $a_n \neq 0$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of Eq. (6.14). Then

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$
$$\equiv a_n(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$$

Comparing the coefficients of like powers of x, we get

$$\alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n = -\frac{a_{n-1}}{a_n}$$

This can be written as

Sum of roots taken 1 root at a time $= (-1)^1 \dfrac{\text{Coefficient of } x^{n-1}}{\text{Coefficient of } x^n}$

$$\Rightarrow \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \cdots + \alpha_2\alpha_3 + \cdots + \alpha_{n-1}\alpha_n = \frac{a_{n-2}}{a_n}$$

Sum of roots taken 2 roots at a time $= (-1)^2 \dfrac{\text{Coefficientof } x^{n-2}}{\text{Coefficient of } x^n}$

$$\Rightarrow \alpha_1\alpha_2\cdots\alpha_r + \cdots + \alpha_{n-r+1}\alpha_{n-r+2}\cdots\ \alpha_n = (-1)^r \frac{a_{n-r}}{a_n}$$

……..…………………………

Sum of roots taken r roots at a time $= (-1)^r \dfrac{\text{Coefficient of } x^{n-r}}{\text{Coefficient of } x^n}$

$$\Rightarrow \alpha_1\alpha_2 \ldots \alpha_n = (-1)^n \frac{a_0}{a_n}$$

Sum of roots taken all roots at a time (product of roots) $= (-1)^n \dfrac{\text{Coefficient of } x^{n-n}}{\text{Coefficient of } x^n}$

For example, if α and β are the roots of the equation $ax^2 + bx + c = 0$, then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

If α, β, γ and δ are the roots of $ax^4 + bx^3 + cx^2 + dx + e = 0$, then

$$\alpha + \beta + \gamma + \delta = -b/a$$
$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = c/a$$
$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -d/a$$

and

$$\alpha\beta\gamma\delta = e/a$$

Key Points

1. A polynomial equation of degree n has n roots (real or imaginary).
2. If all the coefficients are real, then the imaginary roots occur in pairs, that is, the number of complex roots is always even.
3. If the degree of a polynomial equation is odd, then the number of real roots will also be odd. It follows that at least one of the roots will be real.
4. **Factor theorem:** If α is a root of the equation $f(x) = 0$, then $f(x)$ is exactly divisible by $(x - \alpha)$ and conversely, if $f(x)$ is exactly divisible by $(x - \alpha)$, then α is a root of the equation $f(x) = 0$.
5. Let $f(x) = 0$ be a polynomial equation and p and q be two real numbers. Then $f(x) = 0$ will have at least one real root or an odd number of roots between p and q if $f(p)$ and $f(q)$ are of opposite signs. But if $f(p)$ and $f(q)$ are of same signs, then $f(x) = 0$ has either no real roots or an even number of roots between p and q.
6. If α is a repeated root repeating r times of a polynomial equation $f(x) = 0$ of degree n, that is, $f(x) = (x-\alpha)^r g(x)$, where $g(x)$ is a polynomial of degree $n-r$ and $g(\alpha) \neq 0$, then $f(\alpha) = f'(\alpha) = f''(\alpha) = \ldots = f^{(r-1)}(\alpha) = 0$ and $f^r(\alpha) \neq 0$.

6.13 Descartes' Rule of Sign

1. Let $f(x) = 0$ be a polynomial equation. Then the number of positive roots of a polynomial equation $f(x) = 0$ (arrange in decreasing order of the degree) cannot exceed the number of changes of signs in $f(x) = 0$ as we move from left to right.
   For example, consider the equation
   $$2x^2 - 3x - x + 1 = 0$$
   The number of changes of signs from left to right is 2 (+ to − then − to +). So number of positive roots cannot exceed 2.
2. The number of negative roots of a polynomial equation $f(x) = 0$ cannot exceed the number of changes of signs in $f(-x)$.
   For example, consider the equation
   $$5x^4 + 3x^3 - 2x^2 + 5x - 8 = 0$$
   Let $f(x) = 5x^4 + 3x^3 - 2x^2 + 5x - 8$. Then
   $$f(-x) = 5x^4 - 3x^3 - 2x^2 - 5x - 8$$
   The number of changes of signs from left to right is 1 (+ to −). So, the number of negative roots cannot exceed 1.
3. If equation $f(x) = 0$ has at most r positive roots and at most t negative roots, then equation $f(x) = 0$ will have at most $(r + t)$ real roots. That is, it will have at least $n - (r + t)$ imaginary roots where n is the degree of polynomial.
   For example, consider the equation
   $$5x^6 - 8x^3 + 3x^5 + 5x^2 + 8 = 0$$
   The given equation can be written as
   $$5x^6 + 3x^5 - 8x^3 + 5x^2 + 8 = 0$$
   Let $f(x) = 5x^6 + 3x^5 - 8x^3 + 5x^2 + 8$.
   Here $f(x)$ has two changes in signs, so $f(x)$ has at most two positive real roots. Now
   $$f(-x) = 5x^6 - 3x^5 + 8x^3 + 5x^2 + 8$$
   Here $f(-x)$ has at most two changes in signs. So $f(x)$ has at most two negative real roots.
   And $x = 0$ cannot be a root of $f(x) = 0$.
   Also; $f(x) = 0$ has at most four real roots. Therefore, at least two imaginary roots.

6.14 Rolle's Theorem

This theorem is applicable to polynomials. It says that if $f(x)$ is a polynomial in the interval $[a, b]$ and $f(a) = f(b)$, then there is at least one point between a and b where $f'(x) = 0$.

Illustration 6.35 If $a, b, c \in R$ and $a + b + c = 0$, then prove that the quadratic equation $3ax^2 + 2bx + c = 0$ has at least one root in [0, 1].

Solution: Let $f(x) = ax^3 + bx^2 + cx$.

As $f(x)$ is a polynomial, it is continuous and derivable on R. Also $f(0) = 0$ and $f(1) = a + b + c = 0$.
Hence, by Rolle's theorem, there exists at least one root $\alpha \in (0, 1)$ such that $f'(\alpha) = 0$. So

$$3a\alpha^2 + 2b\alpha + c = 0$$

Thus, $3ax^2 + 2bx + c = 0$ has at least one root in [0, 1].

Illustration 6.36 If $a < b < c < d$, then show that $(x - a)(x - c) + 3(x - b)(x - d) = 0$ has real and distinct roots.

Solution: Let $f(x) = (x - a)(x - c) + 3(x - b)(x - d)$. Then

$$f(a) = 0 + 3(a - b)(a - d) > 0 \qquad \{\because a - b < 0, a - d < 0\}$$

and $$f(b) = (b - a)(b - c) < 0 \qquad \{\because b - a > 0, b - c < 0\}$$

Thus, one root will lie between a and b. Now,

$$f(c) = 0 + 3(c - b)(c - d) < 0 \qquad \{\because c - b > 0, c - d < 0\}$$

and $$f(d) = (d - a)(d - c) + 0 > 0 \qquad \{\because d - a > 0, d - c > 0\}$$

Thus, one root will lie between c and d. Hence, roots of equation are real and distinct.

Illustration 6.37 If α and β are the roots of the equation $6x^2 - 6x + 1 = 0$, then prove that

$$\frac{1}{2}(a + b\alpha + c\alpha^2 + d\alpha^3) + \frac{1}{2}(a + b\beta + c\beta^2 + d\beta^3) = \frac{a}{1} + \frac{b}{2} + \frac{c}{3} + \frac{d}{4}$$

Solution: For the given equation, we have

$$\alpha + \beta = 1 \text{ and } \alpha\beta = \frac{1}{6}$$

Now,

$$\frac{1}{2}(a + b\alpha + c\alpha^2 + d\alpha^3) + \frac{1}{2}(a + b\beta + c\beta^2 + d\beta^3)$$
$$= \frac{1}{2}[(a + a) + b(\alpha + \beta) + c(\alpha^2 + \beta^2) + d(\alpha^3 + \beta^3)]$$
$$= \frac{1}{2}\{2a + b + c[(\alpha + \beta)^2 - 2\alpha\beta] + d[(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)]\}$$
$$= \frac{1}{2}\left[2a + b + c\left(1 - \frac{1}{3}\right) + d\left(1 - \frac{1}{2}\right)\right]$$
$$= a + \frac{b}{2} + \frac{c}{3} + \frac{d}{4}$$

Illustration 6.38 If α and β are the roots of the equation $(a - 2)x^2 - (5 - a)x - 5 = 0$. Find a if $|\alpha - \beta| = 2\sqrt{6}$.

Solution: We have

$$\alpha + \beta = \frac{5 - a}{a - 2} \text{ and } \alpha\beta = \frac{-5}{a - 2}$$

But $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$. So

$$24 = \left(\frac{5 - a}{a - 2}\right)^2 + \frac{20}{a - 2}$$
$$\Rightarrow 23a^2 - 106a + 111 = 0$$
$$\Rightarrow (a - 3)(23a - 37) = 0$$
$$\Rightarrow a = 3 \text{ or } 37/23$$

Illustration 6.39 Find the sum of the squares and cubes of the roots of the equation $x^3 - px^2 + qx - r = 0$.

Solution:

Part 1: If α, β and γ are the roots of the equation, we have

$$\alpha + \beta + \gamma = p, \alpha\beta + \beta\gamma + \gamma\alpha = q \text{ and } \alpha\beta\gamma = r$$

Hence,

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) = p^2 - 2q$$

Part 2: Since α is a root of the equation and α satisfies the equation, hence,

$$\alpha^3 - p\alpha^2 + q\alpha - r = 0$$
$$\Rightarrow \alpha^3 = p\alpha^2 - q\alpha + r$$

Similar equation can be written for β and γ. Adding these, we get

$$\alpha^3+\beta^3+\gamma^3 = p(\alpha^2+\beta^2+\gamma^2) - q(\alpha+\beta+\gamma)+3r$$
$$= p(p^2-2q) - qp + 3r = p^3 - 3pq + 3r$$

Illustration 6.40 If α, β and γ are the roots of $7x^3 - x - 2 = 0$, then find the value of $\sum\left(\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}\right)$.

Solution:

$$\sum\alpha = 0,\ \sum\alpha\beta = -\frac{1}{7} \text{ and } \alpha\beta\gamma = \frac{2}{7}$$

Now, $(\alpha+\beta+\gamma)\left(\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma}\right) = 3 + \sum\left(\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}\right)$ on expansion.

Therefore,

$$3+\sum\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right) = 0 \quad (\text{since } \alpha+\beta+\gamma = 0)$$

$$\Rightarrow \sum\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right) = -3$$

Illustration 6.41 If $b^2 < 2ac$, then prove that $ax^3+bx^2+cx+d=0$ has exactly one real root.

Solution: Let all the roots of $f(x) = ax^3 + bx^2 + cx + d$ and $f(x) = 0$ be real. So

$f'(x) = 3ax^2 + 2bx + c = 0$ has two real roots.

But its discriminant is $(2b)^2 - 4\cdot 3\cdot ac = (b^2 - 2ac) - ac < 0$ (as $b^2 < 2ac$) which is a contradiction. So $f(x) = 0$ will not have all the roots real.

6.15 Transformation of Roots

If α and β are roots of the equation $ax^2+bx+c=0$, then the equation whose roots are

1. $-\alpha, -\beta$ is $ax^2 - bx + c = 0$ (Replace x by $-x$)
2. $1/\alpha, 1/\beta$ is $cx^2+bx+a=0$ (Replace x by $1/x$)
3. $\alpha^n, \beta^n; n\in N$ is $a(x^{1/n})^2 + b(x^{1/n}) + c = 0$ (Replace x by $x^{1/n}$)
4. $k\alpha, k\beta$ is $ax^2+kbx+k^2c=0$ (Replace x by x/k)
5. $k+\alpha,\ k+\beta$ is $a(x-k)^2+b(x-k)+c=0$ [Replace x by $(x-k)$]
6. $\dfrac{\alpha}{k}, \dfrac{\beta}{k}$ is $k^2ax^2+kbx+c=0$ (Replace x by kx)
7. $\alpha^{1/n}, \beta^{1/n}; n\in N$ is $a(x^n)^2+b(x^n)+c=0$ (Replace x by x^n).

6.16 Roots of Symmetric Equation

The symmetric expressions of the roots α and β of an equation are those expressions which do not change by interchanging α and β. To find the value of such an expression, we generally express that in terms of $\alpha+\beta$ and $\alpha\beta$.

Some examples of symmetric expressions are as follows:

1. $\alpha^2+\beta^2$
2. $\alpha^2+\alpha\beta+\beta^2$
3. $\dfrac{1}{\alpha}+\dfrac{1}{\beta}$
4. $\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}$
5. $\alpha^2\beta+\beta^2\alpha$
6. $\left(\dfrac{\alpha}{\beta}\right)^2+\left(\dfrac{\beta}{\alpha}\right)^2$
7. $\alpha^3+\beta^3$
8. $\alpha^4+\beta^4$

Illustration 6.42 If the roots of the equation $(x-a)(x-b)-k=0$ are c and d, prove that the roots of $(x-c)(x-d)+k=0$ are a and b.

Solution: Given c and d are the roots of $x^2 - (a+b)x + ab - k = 0$. Hence

$$c+d = a+b \text{ and } cd = ab-k$$
$$\Rightarrow a+b = c+d \text{ and } ab = cd+k \quad (1)$$

But a and b are the roots of $(x-a)(x-b)=0$. So

That is,

$$x^2-(a+b)x+ab=0$$

Using Eq. (1) we get

$$x^2-(c+d)x+cd+k=0$$
$$\Rightarrow (x-c)(x-d)+k=0$$

Illustration 6.43 If the difference between the corresponding roots of $x^2+ax+b=0$ and $x^2+bx+a=0$ is same and $a\neq b$, then find the value of $a+b+4$.

Solution:

If α and β are roots of the first equation and γ, δ are roots of the second equation, then

$$\alpha+\beta=-a,\ \alpha\beta=b \Rightarrow \alpha-\beta=\sqrt{a^2-4b}$$
$$\gamma+\delta=-b,\ \gamma\delta=a$$
$$\Rightarrow \gamma-\delta=\sqrt{b^2-4a}$$

According to question,

$$\alpha-\beta=\gamma-\delta$$
$$\Rightarrow \sqrt{a^2-4b}=\sqrt{b^2-4a}$$
$$\Rightarrow a+b+4=0$$

Illustration 6.44 If the sum of the roots of the quadratic equation $ax^2+bx+c=0$ is equal to the sum of the squares of their reciprocals, then prove that a/c, b/a, c/b are in HP.

Solution: As given, if α and β be the roots of the quadratic equation, then

$$\alpha+\beta=\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha^2\beta^2}\Rightarrow -\frac{b}{a}=\frac{b^2/a^2-2c/a}{c^2/a^2}=\frac{b^2-2ac}{c^2}$$

$$\Rightarrow \frac{2a}{c}=\frac{b^2}{c^2}+\frac{b}{a}=\frac{ab^2+bc^2}{ac^2}\Rightarrow 2a^2c=ab^2+bc^2\Rightarrow \frac{2a}{b}=\frac{b}{c}+\frac{c}{a}$$

$$\frac{c}{a},\frac{a}{b},\frac{b}{c} \text{ are in AP} \Rightarrow \frac{a}{c},\frac{b}{a},\frac{c}{b} \text{ are in HP.}$$

Illustration 6.45 Let α and β be the roots of $x^2-x+p=0$ and γ and δ be the roots of $x^2-4x+q=0$. If α, β, γ and δ are in GP, then find the integral value of p and q.

Solution: We have

$$\alpha+\beta=1,\ \alpha\beta=p,\ \gamma+\delta=4,\ \gamma\delta=q$$

Since $\alpha, \beta, \gamma, \delta$ are in GP.

$$r=\beta/\alpha=\gamma/\beta=\delta/\gamma$$

$$\alpha+\alpha r=1 \Rightarrow \alpha(1+r)=1,\ \alpha(r^2+r^3)=4 \Rightarrow \alpha\cdot r^2(1+r)=4$$

[Using $\alpha(1+r)=1$]

If $r=2$, $\alpha+2\alpha=1 \Rightarrow \alpha=1/3$ and if $r=-2$, $\alpha-2\alpha=1 \Rightarrow \alpha=-1$

But $p=\alpha\beta\in I$. So $r=-2, \alpha=-1$.

Therefore, $p=-2,\ q=\alpha^2r^5=1(-2)^5=-32$.

Your Turn 2

1. If the roots of the equation $x^2-5x+16=0$ are α, β and the roots of equation $x^2+px+q=0$ are $\alpha^2+\beta^2$, $\alpha\beta/2$, then
   (A) $p=1, q=-56$ (B) $p=-1, q=-56$
   (C) $p=1, q=56$ (D) $p=-1, q=56$ **Ans.** (B)
2. If α and β are roots of the equation $x^2-ax+b=0$ and $V_n=\alpha^n+\beta^n$,
   (A) $V_{n+1}=aV_n-bV_{n-1}$ (B) $V_{n+1}=bV_n-aV_{n-1}$
   (C) $V_{n+1}=aV_n+bV_{n-1}$ (D) $V_{n+1}=bV_n+aV_{n-1}$
   Ans. (A)
3. Let α be a root of the equation $ax^2+bx+c=0$ and β be a root of the equation $-ax^2+bx+c$. Show that there exists a root of the equation $\frac{a}{2}x^2+bx+c=0$ that lies between α and β or β and α as the case may be ($\alpha, \beta \neq 0$).
4. If $f(x)$ is quadratic expression which is positive for all real values of x and $g(x)=f(x)+f'(x)+f''(x)$, then for any real value of x
   (A) $g(x)<0$ (B) $g(x)>0$
   (C) $g(x)=0$ (D) $g(x)\geq 0$ **Ans.** (B)
5. If α, β $(\alpha<\beta)$ are roots of the equation $x^2+bx+c=0$ where $(c<0<b)$ then
   (A) $0<\alpha<\beta$ (B) $\alpha<0<\beta<|\alpha|$
   (C) $\alpha<\beta<0$ (D) $\alpha<0<|\alpha|<\beta$ **Ans.** (B)
6. If one of the roots of the equation $x^2+ax+b=0$ and $x^2+bx+a=0$ is coincident, then find the numerical value of $(a+b)$. **Ans.** -1

6.17 Wavy Curve Method (Sign Scheme)

This method is based on the fact that the sign of a polynomial does not change on any value between two of its consecutive roots.

Type 1: If $f(x)=0$ is a polynomial equation with roots $a_1, a_2, \ldots, a_n$ and $g(x)=0$ is a polynomial equation with roots $b_1, b_2, \ldots, b_n$ then

$$\frac{f(x)}{g(x)}=\frac{(x-a_1)(x-a_2)\cdots(x-a_n)}{(x-b_1)(x-b_2)\cdots(x-b_n)} \text{ (where } a_i\neq b_i, i=1,2,\ldots,n)$$

Then solution of $\frac{f(x)}{g(x)}\geq 0$ or $\frac{f(x)}{g(x)}\leq 0$ is given by wavy curve method as follows:

Arrange roots α_i's and β_i's in increasing order and plot them as $A_1, A_2, \ldots, A_m$ on the number line. Take one point in an interval and check the sign of $\frac{f(x)}{g(x)}$. If it is positive then the curve of $\frac{f(x)}{g(x)}$ is above x-axis in this interval else below x-axis. Plot a rough graph of the curve (known as wavy curve, which is not the actual graph of curve; Fig. 6.25).

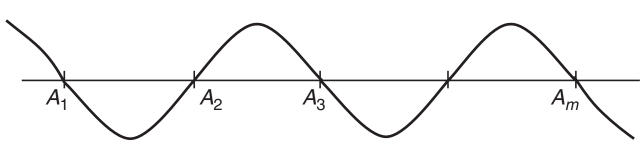


Figure 6.25

If the inequality also has equality sign then the roots of $f(x)$ are included in intervals, else all the roots are excluded.

Or

Let

$$f(x)=(x-a_1)^{k_1}(x-a_2)^{k_2}(x-a_3)^{k_3}\cdots(x-a_{n-1})^{k_{n-1}}(x-a_n)^{k_n} \quad (6.15)$$

where $k_1, k_2, k_3, \ldots, k_n \in N$ and $a_1, a_2, a_3, \ldots, a_n$ are fixed natural numbers satisfying the condition $a_1<a_2<a_3<\ldots<a_{n-1}<a_n$.

First we mark the numbers $a_1, a_2, a_3, \ldots, a_n$ on the real axis and the plus sign in the interval of the right of the largest of these numbers, that is, on the right of a_n.

1. If k_n is even, then we put plus sign on the left of a_n and

2. If k_n is odd, then we put minus sign on the left of a_n.

In the next interval, we put a sign according to the following rule:

When passing through the point a_{n-1} the polynomial $f(x)$ changes sign if k_{n-1} is an odd number and the polynomial $f(x)$ has same sign if k_{n-1} is an even number. Then, we consider the next interval and put a sign in it using the same rule. Thus, we consider all the intervals. The solution of $f(x)>0$ is the union of all intervals in which we have put the plus sign and solution of $f(x)<0$ is the union of all intervals in which we have put minus sign.

Type 2: If $f(x)=0$ and $g(x)=0$ have a common root say $x=\alpha$ then this root will not contribute to the intervals, but it may sometime be included in the solution as a single point.

Type 3: If any of the root of $f(x)=0$ is repeated it does not contribute to the intervals but it is included in the solution, if inequality is $\geq$ or $\leq$.

Illustration 6.46 Find the interval satisfied by $\frac{x^2-8x+12}{x-6}>0$.

Solution:

$$\frac{x^2-8x+12}{x-6}=\frac{(x-6)(x-2)}{(x-6)}=x-2 \text{ for } x\neq 6$$

Now,

$$\frac{x^2-8x+12}{x-6}>0 \Rightarrow (x-2)>0 \Rightarrow x>2$$

But $x\neq 6$. So $x\in(2,6)\cup(6,\infty)$.

Illustration 6.47 Find the interval satisfied by $\frac{x^2}{1-x}\geq 0$.

Solution: As $x^2\geq 0\ \forall\ x\in R$

$$\frac{1}{1-x}\geq 0 \Rightarrow 1-x>0 \Rightarrow 1>x$$

This means $x\in(-\infty, 1)$ which includes the point $x=0$.

Illustration 6.48 Find the solution of equations:

(a) $\frac{2x}{2x^2+5x+2}>\frac{1}{x+1}$

(b) $x^2-3x+2>0$ and $x^2-3x-4\leq 0$.

Solution:

(a) Given $\frac{2x}{2x^2+5x+2}-\frac{1}{x+1}>0 \Rightarrow \frac{2x^2+2x-2x^2-5x-2}{(2x+1)(x+2)(x+1)}>0$

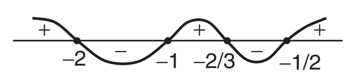


Figure 6.26

$$\Rightarrow \frac{-3x-2}{(2x+1)(x+2)(x+1)} > 0 \Rightarrow \frac{-3(x+2/3)}{(x+1)(x+2)(2x+1)} > 0$$

$$\Rightarrow \frac{(x+2/3)}{(x+1)(x+2)(2x+1)} < 0$$

Equating each factor equal to 0, we get $x = -2, -1, -2/3, -1/2$. (See Fig. 6.26.)

Therefore, $x \in (-2,-1) \cup (-2/3, -1/2) \Rightarrow -2/3 < x < -1/2$ or $-2 < x < -1$.

(b) $x^2 - 3x + 2 > 0$ or
$(x-1)(x-2) > 0$

Figure 6.27

Therefore,
$x \in (-\infty, 1) \cup (2, \infty)$ (1)
(See Fig. 6.27.)

Again $x^2 - 3x - 4 \le 0$ or
$(x-4)(x+1) \le 0$
Hence, $x \in [-1, 4]$ (2)
(See Fig. 6.28.)

Figure 6.28

From Eqs. (1) and (2), $x \in [-1, 1) \cup (2 4] \Rightarrow -1 \le x < 1$ or $2 < x \le 4$.

Illustration 6.49 Find the number of integral solution of $\frac{x+1}{x^2+2} > \frac{1}{4}$.

Solution: $\frac{x+1}{x^2+2} - \frac{1}{4} > 0 \Rightarrow \frac{x^2-4x-2}{x^2+2} < 0$

$\Rightarrow [x-(2+\sqrt{6})][x-(2-\sqrt{6})] < 0$

$\Rightarrow 2-\sqrt{6} < x < 2+\sqrt{6}$

(See Fig. 6.29.)

Figure 6.29

Approximately, $-0.4 < x < 4.4$
Therefore, integral values of x are 0, 1, 2, 3, 4.
Hence, number of integral solutions is 5.

Illustration 6.50 If $2a + 3b + 6c = 0$, then prove that at least one root of the equation $ax^2 + bx + c = 0$ lies in the interval (0, 1).

Solution: Let $f'(x) = ax^2 + bx + c$.
Therefore,

$$f(x) = \int f'(x)\,dx = \frac{ax^3}{3} + \frac{bx^2}{2} + cx$$

Clearly $f(0) = 0$, $f(1) = \frac{a}{3} + \frac{b}{2} + c = \frac{2a+3b+6c}{6} = \frac{0}{6} = 0$

Since $f(0) = f(1) = 0$, hence, there exists at least one point c in between 0 and 1, such that $f'(c) = 0$, by Rolle's theorem.

Trick: Put the value of $a = -3, b = 2, c = 0$ in given equation:

$$-3x^2 + 2x = 0 \Rightarrow 3x^2 - 2x = 0 \Rightarrow x(3x-2) = 0$$

$\Rightarrow x = 0, x = 2/3$, which lie in the interval (0, 1).

Illustration 6.51 Find all integral values of x for which $(5x - 1) < (x + 1)^2 < 7x - 3$.

Solution: There are two inequalities and both must hold simultaneously. The first inequality gives

$(x + 1)^2 > 5x - 1$
$\Rightarrow x^2 + 2x + 1 > 5x - 1$
$\Rightarrow x^2 - 3x + 2 > 0$
$\Rightarrow (x - 1)(x - 2) > 0$
$\Rightarrow x \in (-\infty, 1) \cup (2, \infty)$ (1)
(See Fig. 6.30.)

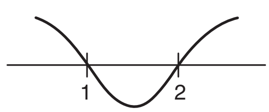


Figure 6.30

The second inequality gives
$(x + 1)^2 < 7x - 3$
$\Rightarrow x^2 - 5x + 4 < 0$
$\Rightarrow (x - 1)(x - 4) < 0$
$\Rightarrow x \in (1, 4)$ (2)
(See Fig. 6.31.)

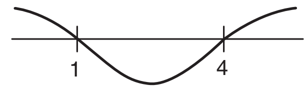


Figure 6.31

Since both inequalities are to hold good, combining results of Eq. (1) and (2), we get

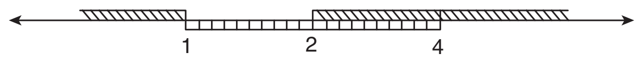


Figure 6.32

$$x \in (2, 4) \text{ or } 2 < x < 4$$

This can be inferred from Fig. 6.32. The shaded part above the line corresponds to Eq. (1) and the shaded part below the line corresponds to Eq. (2). The double shaded part in Fig 6.32 represents the solution.

Since the INTEGRAL (whole number) value of x is required, we have to find the integer in the interval (2, 4) and that is 3.

Illustration 6.52 Find all values of m for which $mx^2 + (m - 3)x + 1 < 0$ for at least one positive real x.

Solution: Let $f(x) = mx^2 + (m - 3)x + 1$.

Case (i): $f(x) < 0$ trivially if $m < 0$, as parabola will be concave downwards.

Case (ii): If $m > 0$, then the given condition is satisfied if $f(x)$ has distinct roots and at least one of them is positive real root (Fig. 6.33).

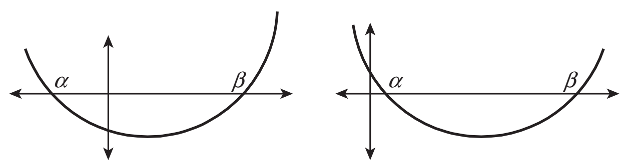


Figure 6.33

$$D > 0 \Rightarrow (m-3)^2 - 4m > 0 \Rightarrow m < 1 \text{ or } m > 9$$

At least one root is positive, that is, R – intervals when both are non-positive. So

$$m < 3 \text{ (as sum} \le 0 \text{ and product} \ge 0)$$

Their intersection gives $m < 1$. Hence, from the above two cases $m \in (-\infty, 1)$.

Illustration 6.53 Find the values of x for which the inequality $\frac{8x^2+16x-51}{(2x-3)(x+4)} > 3$ holds.

Solution: We cannot write $8x^2 + 16x - 51 > 3(2x - 3)(x + 4)$ as this would be correct only if $(2x - 3)(x + 4)$ is positive.

Therefore, transfer 3 to L.H.S. and rewrite as

$$\frac{8x^2+16x-51}{2x^2+5x-12} - 3 > 0$$

$$\Rightarrow \frac{8x^2+16x-51-3(2x^2+5x-12)}{2x^2+5x-12} > 0$$

$$\Rightarrow \frac{2x^2+x-15}{2x^2+5x-12} > 0 \text{ or } \frac{(2x-5)(x+3)}{(2x-3)(x+4)} > 0$$

Multiplying both sides by the square of the denominator, we get

$$(2x-5)(x+3)(2x-3)(x+4) > 0$$

$$\Rightarrow 2\left(x-\frac{5}{2}\right)(x+3)\,2\left(x-\frac{3}{2}\right)(x+4)>0$$

Therefore, the inequality has the solution

$$(-\infty,-4)\cup\left(-3,\ \frac{3}{2}\right)\cup\left(\frac{5}{2},\ \infty\right)$$

Illustration 6.54 Find the greatest negative integer satisfying $x^2-4x-77<0$ and $x^2>4$.

Solution:

$$x^2-4x-77<0 \Rightarrow x^2-4x+4<81$$
$$\Rightarrow (x-2)^2<9^2$$
$$\Rightarrow -9<x-2<+9$$
$$\Rightarrow -7<x<11 \quad (1)$$

The second inequality

$$x^2-4>0 \Rightarrow x>2 \text{ or } x<-2 \quad (2)$$

The solution set satisfying both inequalities is the common part displayed in the diagram, namely, $(-7,-2)\cup(2,11)$.
Therefore, the greatest negative integer in this set is -3.

6.18 Equation and Inequation Containing the Absolute Value

1. **Equations containing absolute values:** By definition,

$$|x|=\begin{cases}x, & \text{if } x\ge 0\\ -x, & \text{if } x<0\end{cases}$$

Important forms containing absolute value:

Type I: The equation of the form $|f(x)+g(x)|=|f(x)|+|g(x)|$ is equivalent of the system $f(x)\cdot g(x)\ge 0$.

Type II: The equation of the form

$$|f_1(x)|+|f_2(x)|+|f_3(x)|+\cdots+|f_n(x)|=g(x) \quad (6.16)$$

where $f_1(x), f_2(x), f_3(x), \ldots, f_n(x), g(x)$ are functions of x and $g(x)$ may be a constant.

Equations of this form can be solved by the method of interval. We first find all critical points of $f_1(x), f_2(x), \ldots, f_n(x)$. If coefficient of x is positive, then graph starts with positive sign and if it is negative, then graph starts with negative sign. Then using the definition of the absolute value, we pass from Eq. (6.16) to a collection of system which does not contain the absolute value symbols.

2. **Equations containing absolute value:** By definition,

$|x|<a \Rightarrow -a<x<a\ (a>0)$, $|x|\le a \Rightarrow -a\le x\le a$

$|x|>a \Rightarrow x<-a$ or $x>a$

and $|x|\ge a \Rightarrow x\le -a$ or $x\ge a$

Illustration 6.55 Find the solution of the equation $|x-2|^2+|x-2|-6=0$.

Solution: We have

$$|x-2|^2+|x-2|-6=0.$$

Let $|x-2|=X$. Now,

$$X^2+X-6=0$$

$$\Rightarrow X=\frac{-1\pm\sqrt{1+24}}{2}=2,-3 \Rightarrow X=2 \text{ and } X=-3$$

Therefore, $|x-2|=2$ and $|x-2|=-3$. Now, $|x-2|=-3$ is not possible. So

$$x-2=2 \text{ or } x-2=-2$$

Therefore, $x=4$ or $x=0$.

Illustration 6.56 Find the set of all real numbers x for which $x^2-|x+2|+x>0$.

Solution:

Case I: If $x+2\ge 0$, that is, $x\ge -2$, we get

$$x^2-x-2+x>0 \Rightarrow x^2-2>0 \Rightarrow (x-\sqrt{2})(x+\sqrt{2})>0$$

$$\Rightarrow x\in(-\infty,-\sqrt{2})\cup(\sqrt{2},\infty)$$

(See Fig. 6.34.) But $x\ge -2$. Hence,

$$x\in[-2,-\sqrt{2})\cup(\sqrt{2}\ \infty) \quad (1)$$

Figure 6.34

Case II: $x+2<0$, that is, $x<-2$, then

$$x^2+x+2+x>0 \Rightarrow x^2+2x+2>0$$

$$\Rightarrow (x+1)^2+1>0$$

which is true for all x. Therefore,

$$x\in(-\infty,-2) \quad (2)$$

From Eqs. (1) and (2), we get

$$x\in(-\infty,-\sqrt{2})\cup(\sqrt{2},\infty)$$

6.19 Equation Reducible to Quadratic Equation

Some equations of higher degree or equations with power function or logarithmic function can be reduced to quadratic equation. A few of them are given in the following illustrations.

Illustration 6.57 Solve the equation $2x^4+x^3-11x^2+x+2=0$.

Solution: The given equation is

$$2x^4+x^3-11x^2+x+2=0 \quad (1)$$

Since $x=0$ is not a solution of the given equation, dividing by x^2 in both sides of Eq. (1), we get

$$2\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)-11=0 \quad (2)$$

Put $x+\dfrac{1}{x}=y$ in Eq. (2). Then Eq. (2) reduces to the form

$$2(y^2-2)+y-11=0$$
$$\Rightarrow 2y^2+y-15=0$$
$$\Rightarrow y_1=-3 \text{ and } y_2=\frac{5}{2}$$

Consequently, the original equation is equivalent to the collection of equations

$$\left[\begin{array}{l}x+\dfrac{1}{x}=-3\\[2ex] x+\dfrac{1}{x}=\dfrac{5}{2}\end{array}\right.$$

We find that

$$x_1=\frac{-3-\sqrt{5}}{2},\ x_2=\frac{-3+\sqrt{5}}{2},\ x_3=\frac{1}{2},\ x_4=2$$

Illustration 6.58 Solve the equation $(x + 2)(x + 3)(x + 8)(x+12) = 4x^2$.

Solution: Given equation is

$$(x+2)(x+3)(x+8)(x+12) = 4x^2 \quad (1)$$

Since, $(-2)(-12) = (-3)(-8)$ we can write Eq. (1) as

$$(x+2)(x+12)(x+3)(x+8) = 4x^2$$
$$\Rightarrow (x^2+14x+24)(x^2+11x+24) = 4x^2 \quad (2)$$

Now $x = 0$ is not a root of Eq. (1). So dividing by x^2 in both sides of Eq. (2) we get

$$\Rightarrow \left(x+\frac{24}{x}+14\right)\left(x+\frac{24}{x}+11\right) = 4 \quad (3)$$

Put $x+\frac{24}{x} = y$ in Eq. (3). Then Eq. (3) can be reduced to the form

$$(y+14)(y+11) = 4$$
$$\Rightarrow y_1 = -7 \text{ and } y_2 = -10$$

Thus, the original equation is equivalent to the collection of equations

$$\begin{cases} x+\frac{24}{x} = -7, \\ x+\frac{24}{x} = -10, \end{cases} \text{ that is, } \begin{cases} x^2+7x+24 = 0 \\ x^2+10x+24 = 0 \end{cases}$$

Solving these collections, we get

$$x_1 = \frac{-7+i\sqrt{47}}{2},\ x_2 = \frac{-7-i\sqrt{47}}{2}, x_3 = -6,\ x_4 = -4$$

Illustration 6.59 Solve the equation $(6-x)^4 + (8-x)^4 = 16$.

Solution: The given equation is

$$(6-x)^4 + (8-x)^4 = 16 \quad (1)$$

After a change of variable

$$y = \frac{(6-x)+(8-x)}{2}$$

Therefore,

$$y = 7 - x \text{ or } x = 7 - y$$

Now put $x = 7 - y$ in Eq. (1). We get

$$(y-1)^4 + (y+1)^4 = 16$$
$$\Rightarrow y^4 + 6y^2 - 7 = 0$$
$$\Rightarrow (y^2+7)(y^2-1) = 0$$

This implies

$$y^2 + 7 = 0 \text{ or } y^2 - 1 = 0$$

Now, $y^2 + 7 \neq 0$ (y gives imaginary values)

Therefore, $y^2 - 1 = 0 \Rightarrow y_1 = -1$ and $y_2 = 1$

Thus, $x_1 = 8$ and $x_2 = 6$ are the roots of Eq. (1) given in the question.

Illustration 6.60 Find all values of a for which the equation $4^x - a2^x - a + 3 = 0$ has at least one solution.

Solution: Putting $2^x = t > 0$, the original equation reduces to the form $t^2 - at - a + 3 = 0$ such that the quadratic equation should have at least one positive root ($t > 0$). The discriminant is

$$D = (-a)^2 - 4\cdot1(-a+3) \geq 0$$
$$\Rightarrow a^2 + 4a - 12 \geq 0$$
$$\Rightarrow (a+6)(a-2) \geq 0$$

Therefore, $a \in (-\infty, -6] \cup [2, \infty)$

If roots of Eq. (1) are t_1, t_2, then

$$\begin{cases} t_1 + t_2 = a \\ t_1 t_2 = 3 - a \end{cases}$$

For $a \in (-\infty, -6]$, $t_1 + t_2 < 0$ and $t_1 t_2 > 0$.

Therefore, both are negative and consequently, the original equation has no solutions.

For $a \in [2, \infty)$, $t_1 + t_2 > 0$ Consequently at least one of the roots. t_1 or t_2 is greater than zero.

Thus, for $a \in [2, \infty)$, the given equation has at least one solution.

Additional Solved Examples

1. If the minimum value of $f(x) = (1 + b^2)x^2 + 2bx + 1$ is $m(b)$, then the maximum value of $m(b)$ is

(A) 0 **(B)** –1
(C) 2 **(D)** 1

Solution:

$$f(x) = (1+b^2)\left(x^2 + \frac{2b}{1+b^2}x + \frac{b^2}{(1+b^2)^2}\right) + 1 - \frac{b^2}{1+b^2}$$

$$= (1+b^2)\left(x + \frac{b}{1+b^2}\right)^2 + \frac{1}{1+b^2}$$

for $f(x) = (1+b^2)x^2 + 2bx + 1$, $D < 0$ and $(1+b^2) > 0$

Therefore, $f(x)$ has minimum value at $x = -\frac{b}{1+b^2}$

Minimum value of $f(x)$, is given by

$$m(b) = \frac{b^2}{1+b^2} - 2\frac{b^2}{1+b^2} + 1 = 1 - \frac{b^2}{1+b^2} = \frac{1}{1+b^2}$$

Clearly, $0 < m(b) \leq 1$. Since $b^2 \geq 0$ maximum value of $m(b)$ is 1.

Hence, the correct answer is option (D).

2. If the larger root of equation $x^2 + (2-a^2)x + (1-a^2) = 0$ is less than both the roots of the equation $x^2 - (a^2+4a+1)x + a^2 + 4a = 0$, then the range of a is

(A) $(-\sqrt{2}, \sqrt{2})$ **(B)** $\left(-\frac{1}{4}, \sqrt{2}\right)$
(C) $\left(-\sqrt{2}, \frac{1}{4}\right)$ **(D)** None of these

Solution: The roots of first equation are –1 and $a^2 - 1$. Now the roots of second equation are 1, $a^2 + 4a$.

According to given condition $a^2 - 1 < 1$ and $a^2 - 1 < a^2 + 4a$. So

$$a \in (-\sqrt{2}, \sqrt{2}) \text{ and } a > -\frac{1}{4}$$

$$\Rightarrow a \in \left(-\frac{1}{4}, \sqrt{2}\right)$$

Hence, the correct answer is option (B).

3. Let $a, b, c, c_1, a_1, e \in R$ and satisfy the relations $a(b+c)^2 + a_1bc + e = 0$ and $a(b+c_1)^2 + a_1bc_1 + e = 0$. Then, the following is true?

(A) $acc_1 = ab^2 - e$ **(B)** $acc_1 = ab^2 + e$
(C) $a(c+c_1)+2ab+a_1b=0$ **(D)** $a(c+c_1)-2ab-a_1b=0$

Solution: We have,

$$a(b+c)^2 + a_1bc + e = 0 \quad (1)$$

and
$$a(b+c_1)^2 + a_1bc_1 + e = 0 \quad (2)$$

Equations (1) and (2) indicate that c and c_1 are the roots of

$$a(b+x)^2 + a_1xb + e = 0$$
$$\Rightarrow ax^2 + bx(a_1 + 2ba) + ab^2 + e = 0$$

So

$$\Rightarrow c + c_1 = \frac{-b(a_1+2ba)}{a} \text{ and } cc_1 = \frac{ab^2+e}{a}$$

From Eq. (2), we have

$$\Rightarrow acc_1 = ab^2 + e$$

Hence, the correct answer is option (B).

4. Let a, b, c be real numbers with $a \neq 0$ and let α and β be the roots of the equation $ax^2+bx+c=0$. Then the roots of $a^3x^2+abcx+c^3=0$ in terms of α, β are given by

(A) α, β **(B)** $\frac{c\alpha}{a}, \frac{c\beta}{a}$
(C) $a\alpha, c\beta$ **(D)** $c\alpha, \alpha\beta$

Solution: Given equation $a^3x^2+abcx+c+c^3=0$ can be written as

$$a\left(\frac{ax}{c}\right)^2 + b\left(\frac{ax}{c}\right) + c = 0$$

Clearly, roots of this equation are $\frac{ax}{c} = \alpha, \beta$. So

$$x = \frac{c\alpha}{a}, \frac{c\beta}{a}$$

Hence, the correct answer is option (B).

5. Let $\sqrt{x+3-4\sqrt{x-1}}+\sqrt{x+8-6\sqrt{x-1}}=1$. Solve for x.

(A) $\{5,10\}$ **(B)** $[1,\infty)$
(C) $[5,10]$ **(D)** None of these

Solution: Put $x - 1 = t^2$ in the given equation. We get

$$\sqrt{t^2+1+3-4t}+\sqrt{t^2+1+8-6t}=1$$
$$\Rightarrow |t-2|+|t-3|=1$$
$$\Rightarrow t \in [2, 3]$$
$$\Rightarrow x \in [5, 10]$$

Hence, the correct answer is option (C).

6. If $\frac{x^2-bx}{ax-c}=\frac{k-1}{k+1}$ has roots which are numerically equal but of opposite signs, then k is equal to

(A) $\frac{a-b}{a+b}$ **(B)** $\frac{a+b}{a-b}$
(C) c **(D)** $\frac{1}{c}$

Solution: The equation $\frac{x^2-bx}{ax-c}=\frac{k-1}{k+1}$ can be written as

$$(k+1)x^2 - [a(k-1)+b(k+1)]\,x + c(k-1) = 0$$

The roots are equal in magnitude but opposite in sign means sum of the roots = 0. Therefore,

$$a(k-1)+b(k+1)=0 \Rightarrow k = \frac{a-b}{a+b}$$

Hence, the correct answer is option (A).

7. For what integral value of a are the roots of the equation $ax^2 + (2a-1)x + (a-2) = 0$ rational?

Solution: The roots are rational if the discriminant is a perfect square. That is, $(2a-1)^2 - 4(a)(a-2)$ must be a perfect square. That is, $4a+1$ must be a perfect square.

Since a is an integer, $4a+1$ must be an odd number and a perfect square. The square of an odd number is an odd number. Therefore, $4a+1=(2n+1)^2$ for some n. This gives

$$a = \frac{4n^2+4n}{4} = n(n+1)$$

where n is a natural number.

8. If α and β are the roots of the equation $a(x^2+m^2)+amx+bm^2x^2=0$, then show that $a(\alpha^2+\beta^2)+a\alpha\beta+b\alpha^2\beta^2=0$.

Solution: The given equation may be written as

$$(a+bm^2)x^2 + amx + am^2 = 0$$

We have

$$\alpha+\beta = \frac{-am}{a+bm^2} \text{ and } \alpha\beta = \frac{am^2}{a+bm^2}$$

Now

$$\frac{\alpha+\beta}{\alpha\beta} = -\frac{1}{m} \text{ or } \frac{1}{\alpha}+\frac{1}{\beta} = -\frac{1}{m} \quad (1)$$

Also,

$$\frac{1}{\alpha\beta} = \frac{a+bm^2}{am^2} = \frac{1}{m^2}+\frac{b}{a} \quad (2)$$

Eliminating m between Eqs. (1) and (2), we get

$$\frac{1}{\alpha\beta} = \left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^2 + \frac{b}{a}$$
$$\Rightarrow \frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\alpha\beta}+\frac{b}{a}=0$$

Multiplying by $a\alpha^2\beta^2$, we get

$$a(\alpha^2+\beta^2)+a\alpha\beta+b\alpha^2\beta^2=0$$

9. If x is real, show that the expression $\frac{x^2+2x-11}{x-3}$ can take all values which do not lie in the open interval (4, 12).

Solution: Let

$$y = \frac{x^2+2x-11}{x-3}.$$

Writing this as a quadratic equation in x, we have

$$x^2 + x(2-y) + (3y-11) = 0 \quad (1)$$

The values of x and y are related by this equation and for each value of y, there is a value of x which is a root of this quadratic equation. In order to that this x (or root) is real, the discriminant ≥ 0.

$$(2-y)^2 - 4(3y-11) \geq 0$$

$$\Rightarrow y^2 - 16y + 48 \ge 0$$
$$\Rightarrow (y-4)(y-12) \ge 0$$
$$\Rightarrow y \le 4 \text{ or } y \ge 12$$

Hence, y (or the given expression) does not take any value between 4 and 12.

10. If x is real, find the values of k for which $\left|\dfrac{x^2+kx+1}{x^2+x+1}\right| < 2$.

Solution:

$$|x| < a \Rightarrow -a < x < +a$$

Hence, the given inequality implies

$$-2 < \frac{x^2+kx+1}{x^2+x+1} < 2 \quad (1)$$

Now $x^2 + x + 1 = \left(x + \dfrac{1}{2}\right)^2 + \dfrac{3}{4}$ is positive for all values of x.

Multiplying Eq. (1) by x^2+x+1, we get

$$-2(x^2+x+1) < x^2+kx+1 < 2(x^2+x+1)$$

This yields two inequalities

$$3x^2+(2+k)x+3 > 0 \text{ and } x^2+(2-k)x+1 > 0$$

For these quadratic expressions to be positive for all values of x, their discriminants must be negative. So

$$(2+k)^2 - 36 < 0 \quad (2)$$

and

$$(2-k)^2 - 4 < 0 \quad (3)$$

This gives $(k+8)(k-4) < 0$ and $k(k-4) < 0$. Hence, $-8 < k < 4$ and $0 < k < 4$. For both, these conditions to be satisfied, $0 < k < 4$.

11. For what real p do the roots of $x^2 - 2x - p^2 + 1 = 0$ lie between the roots of $x^2 - 2(p+1)x + p(p-1) = 0$?

Solution: The roots of the first equation are $\dfrac{2 \pm \sqrt{4p^2}}{2} = \dfrac{2 \pm 2p}{2}$.

The two roots may be taken as $1-p$ and $1+p$. Roots of the second equation are $p+1 \pm \sqrt{1+3p}$. The condition required when $p > 0$ is

$$1+p-\sqrt{1+3p} < 1-p < 1+p < 1+p+\sqrt{1+3p}$$

This reduces to $1+p-\sqrt{1+3p} < 1-p$. That is,

$$2p < \sqrt{1+3p}$$

Squaring both the sides and solving, we get

$$4p^2 - 3p - 1 < 0 \Rightarrow (p-1)(4p+1) < 0.$$

That is, $0 < p < 1$. When $p < 0$, it becomes

$$1-p < 1+p+\sqrt{1+3p} \Rightarrow -2p < \sqrt{1+3p}$$
$$\Rightarrow 4p^2 - 3p - 1 < 0 \text{ and } -\frac{1}{4} < p < 0$$

When $p = 0$, the first equation has two (equal) roots each equals to 1 while the roots of the second equation are 0, 2. The required condition is trivially satisfied here since $0 < 1 < 2$.

The values of p are $-\dfrac{1}{4} < p < 1$. So $p \in \left(-\dfrac{1}{4}, 1\right)$.

Alternative method

Let $f(x) = x^2 - 2(p+1)x + p(p-1) = 0$. Here

$$D > 0;\ f(1-p) < 0$$
$$f(1+p) < 0$$

Taking intersection of the above three cases we get $p \in \left(-\dfrac{1}{4}, 1\right)$

Previous Years' Solved JEE Main/AIEEE Questions

1. If the difference between the roots of the equation $x^2 + ax + 1 = 0$ is less than $\sqrt{5}$, then the set of possible values of a is

(A) $(-3, 3)$ **(B)** $(-3, \infty)$
(C) $(3, \infty)$ **(D)** $(-\infty, -3)$

[AIEEE 2007]

Solution: Let α and β be the roots of the equation $x^2 + ax + 1 = 0$. Then

$$\alpha + \beta = -a \text{ and } \alpha\beta = 1$$

Now

$$\Rightarrow |\alpha - \beta| = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$$
$$\Rightarrow |\alpha - \beta| = \sqrt{a^2 - 4}$$
$$\Rightarrow \sqrt{a^2-4} < \sqrt{5} \Rightarrow a^2 - 4 < 5$$
$$\Rightarrow a^2 - 9 < 0 \Rightarrow a \in (-3, 3)$$

Hence, the correct answer is option (A).

2. The quadratic equations $x^2 - 6x + a = 0$ and $x^2 - cx + 6 = 0$ have one root in common. The other roots of the first and second equations are integers in the ratio 4:3. Then the common root is

(A) 1 **(B)** 4
(C) 3 **(D)** 2

[AIEEE 2008]

Solution: Let α and 4β be roots of $x^2 - 6x + a = 0$ and α, 3β be the roots of $x^2 - cx + 6 = 0$. Then

$$\alpha + 4\beta = 6 \text{ and } 4\alpha\beta = a;\ \alpha + 3\beta = c \text{ and } 3\alpha\beta = 6$$

From $3\alpha\beta = 6$, we get $\alpha\beta = 2$. Using this in $4\alpha\beta = a$ we get $a = 8$. The first equation is $x^2 - 6x + 8 = 0 \Rightarrow x = 2, 4$. If $\alpha = 2$ and $4\beta = 4$, then $3\beta = 3$. If $\alpha = 4$ and $4\beta = 2$, then $3\beta = 3/2$ (non-integer). Therefore, common root is $x = 2$.

Hence, the correct answer is option (D).

3. If the roots of the equation $bx^2 + cx + a = 0$ are imaginary, then for all real values of x, the expression $3b^2x^2 + 6bcx + 2c^2$ is

(A) Greater than $4ab$ **(B)** Less than $4ab$
(C) Greater than $-4ab$ **(D)** Less than $-4ab$

[AIEEE 2009]

Solution: Given $bx^2 + cx + a = 0$

Roots are imaginary means, $c^2 - 4ab < 0 \Rightarrow c^2 < 4ab \Rightarrow -c^2 > -4ab$

Now,

$$f(x) = 3b^2x^2 + 6bcx + 2c^2 \Rightarrow D = (6bc)^2 - 4(3b^2)(2c^2)$$
$$\Rightarrow D = 36b^2c^2 - 24b^2c^2 = 12b^2c^2$$

When $\alpha x^2+\beta x+\gamma=0, \alpha>0$, minimum occurs at $-\dfrac{\beta}{2\alpha}$ and the minimum value is $\dfrac{4\alpha\gamma-\beta^2}{4\alpha}$.

Therefore, as $3b^2>0$, the given expression has minimum value

$$\frac{4(3b^2)(2c^2)-36b^2c^2}{4(3b^2)}=-\frac{12b^2c^2}{12b^2}=-c^2>-4ab$$

Hence, the correct answer is option (C).

4. If $\vec{u}, \vec{v}, \vec{w}$ are non-coplanar vectors and p, q are real numbers, then the equality $[3\vec{u} \quad p\vec{v} \quad p\vec{w}]-[p\vec{v} \quad \vec{w} \quad q\vec{u}]-[2\vec{w} \quad q\vec{v} \quad q\vec{u}]=0$ holds for

(A) exactly one value of (p, q)
(B) exactly two values of (p, q)
(C) more than two but not all values of (p, q)
(D) all values of (p, q)

[AIEEE 2009]

Solution:

$$3p^2\vec{u}\times\vec{v}\cdot\vec{w}-pq\vec{v}\times\vec{w}\cdot\vec{u}-2q^2\vec{w}\times\vec{v}\cdot\vec{u}=0$$

$$\Rightarrow 3p^2\vec{u}\times\vec{v}\cdot\vec{w}-\underbrace{pq\vec{u}\times\vec{v}\cdot\vec{w}}_{\because\text{ Cyclic permutation}}-\underbrace{2q^2\vec{u}\times\vec{v}\cdot\vec{w}}_{\because\text{ No Cyclic permutation}}=0$$

$$\Rightarrow (3p^2-pq+2q^2)[\vec{u} \quad \vec{v} \quad \vec{w}]=0$$

However, $[\vec{u} \quad \vec{v} \quad \vec{w}]\neq 0$. Therefore,

$$3p^2-pq+2q^2=0 \Rightarrow 2p^2+p^2-pq+\left(\frac{q}{2}\right)^2+\frac{7q^2}{4}=0$$

$$\Rightarrow 2p^2+\left(p-\frac{q}{2}\right)^2+\frac{7}{4}q^2=0 \Rightarrow p=0, q=0, p=\frac{q}{2}$$

This possible only when $p = 0$, $q = 0$, So the equality holds for exactly one value of (p, q).

Hence, the correct answer is option (A).

5. The equation $e^{\sin x}-e^{-\sin x}-4=0$ has

(A) infinite number of real roots
(B) no real roots
(C) exactly one real root
(D) exactly four real roots

[AIEEE 2012]

Solution: The given equation can be written as

$$e^{\sin x}-e^{-\sin x}=4$$

Let $e^{\sin x}=t$. Then

$$t-\frac{1}{t}=4 \Rightarrow t^2-4t-1=0 \Rightarrow t=\frac{4\pm\sqrt{16+4}}{2}=\frac{4\pm 2\sqrt{5}}{2}=2\pm\sqrt{5}$$

Therefore,

$$e^{\sin x}=2\pm\sqrt{5}$$

Also $-1\le \sin x\le 1$ and so $\dfrac{1}{e}\le e^{\sin x}\le e$. From this, we see that $e^{\sin x}=2+\sqrt{5}$ and $e^{\sin x}=2-\sqrt{5}$ are both not possible.

Hence, the correct answer is option (B).

6. If the equations $x^2+2x+3=0$ and $ax^2+bx+c=0$, $a, b, c \in R$, have a common root, then $a:b:c$ is

(A) 3:2:1 **(B)** 1:3:2
(C) 3:1:2 **(D)** 1:2:3

[JEE MAIN 2013]

Solution: Discriminant = 4 − 12 < 0 and $1, 2, 3 \in R$. Therefore, the equation has complex conjugate roots, which means both roots are common. Hence, all coefficients must be proportional,

$$\frac{a}{1}=\frac{b}{2}=\frac{c}{3}$$

Therefore, $a:b:c = 1:2:3$.

Hence, the correct answer is option (D).

7. Let α and β be the roots of the equation $px^2+qx+r=0, p\neq 0$. If p, q, r are in AP and $\dfrac{1}{\alpha}+\dfrac{1}{\beta}=4$, then the value of $|\alpha-\beta|$ is

(A) $\dfrac{\sqrt{34}}{9}$ **(B)** $\dfrac{2\sqrt{13}}{9}$

(C) $\dfrac{\sqrt{61}}{9}$ **(D)** $\dfrac{2\sqrt{17}}{9}$

[JEE MAIN 2014 (OFFLINE)]

Solution: Given that

$$\frac{1}{\alpha}+\frac{1}{\beta}=4 \quad (1)$$

Since p, q, r are in AP, we have

$$2q=p+r \quad (2)$$

Since α and β are the roots of equation $px^2+qx+r=0$, we have

$$\alpha+\beta=-\frac{q}{p} \quad (3)$$

$$\alpha\beta=\frac{r}{p} \quad (4)$$

From Eq. (1), we have $\dfrac{\alpha+\beta}{\alpha\beta}=4$

From Eqs. (2) and (3), we have

$$-2(\alpha+\beta)=\frac{p+r}{p}=1+\frac{r}{p}=1+\alpha\beta$$

Dividing both sides by $\alpha\beta$ we get

$$-2\frac{(\alpha+\beta)}{\alpha\beta}=\frac{1}{\alpha\beta}+1 \Rightarrow -2\times 4=\frac{1}{\alpha\beta}+1 \Rightarrow \frac{1}{\alpha\beta}=-9$$

Therefore, equation having roots $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$ is $x^2-4x-9=0$.

Hence, equation having roots α and β is $\dfrac{1}{x^2}-\dfrac{4}{x}-9=0 \Rightarrow 9x^2+4x-1=0$. Therefore,

$$|\alpha-\beta|=\sqrt{|\alpha+\beta|^2-4\alpha\beta}=\sqrt{\frac{16}{81}-4\left(\frac{-1}{9}\right)}$$

$$=\sqrt{\frac{16+36}{81}}=\frac{\sqrt{52}}{9}=\frac{2\sqrt{13}}{9}$$

Hence, the correct answer is option (B).

8. If equations $ax^2+bx+c=0$, $(a, b, c \in R, a \neq 0)$ and $2x^2+3x+4=0$ have a common root then a:b:c equals

(A) 1:2:3 **(B)** 2:3:4
(C) 4:3:2 **(D)** 3:2:1

[JEE MAIN 2014 (ONLINE SET-1)]

Solution: Since $2x^2+3x+4>0$, therefore, $2>0$ and discriminant $= 9 - 32 = -23 < 0$. Since coefficients 2, 3, 4 are real, we will have imaginary conjugate pair of roots. Since, both roots are common, therefore, a:b:c = 2:3:4

Hence, the correct answer is option (B).

9. If $\frac{1}{\sqrt{\alpha}}$ and $\frac{1}{\sqrt{\beta}}$ are the roots of the equation $ax^2+bx+1=0$ $(a \neq 0, a, b \in R)$, then the equation $x(x+b^3)+(a^3-3abx)=0$ has roots

(A) $\alpha^{3/2}$ and $\beta^{3/2}$ **(B)** $\alpha\beta^{1/2}$ and $\alpha^{1/2}\beta$

(C) $\sqrt{\alpha\beta}$ and $\alpha\beta$ **(D)** $\alpha^{-3/2}$ and $\beta^{-3/2}$

[JEE MAIN 2014 (ONLINE SET-1)]

Solution: For equation $ax^2+bx+1=0$, we have

$$\frac{1}{\sqrt{\alpha}}+\frac{1}{\sqrt{\beta}}=\frac{-b}{a} \quad (1)$$

$$\frac{1}{\sqrt{\alpha}}\cdot\frac{1}{\sqrt{\beta}}=\frac{1}{a} \quad (2)$$

Let α_1, β_1 be the roots of the equation, $x^2+(b^3-3ab)x+a^3=0$.
Then $\alpha_1+\beta_1=3ab-b^3$ and $\alpha_1\beta_1=a^3$.
Using Eqs. (1) and (2), we get

$$\alpha_1+\beta_1=3\sqrt{\alpha\beta}(-\sqrt{\alpha}-\sqrt{\beta})-(-\sqrt{\alpha}-\sqrt{\beta})^3$$

$$=-3\alpha\beta^{1/2}-3\alpha^{1/2}\beta+\alpha^{3/2}+\beta^{3/2}+3\sqrt{\alpha}\sqrt{\beta}(\sqrt{\alpha}+\sqrt{\beta})$$

$$=-\cancel{3\alpha\beta^{1/2}}-\cancel{3\alpha^{1/2}\beta}+\alpha^{3/2}+\beta^{3/2}+\cancel{3\alpha\beta^{1/2}}+\cancel{3\alpha\beta^{1/2}}$$

$$\Rightarrow \alpha_1+\beta_1=\alpha^{3/2}+\beta^{3/2} \quad (3)$$

Also $$\alpha_1\beta_1=(\alpha\beta)^{3/2}=\alpha^{3/2}\beta^{3/2} \quad (4)$$

From Eqs. (3) and (4), the required roots are $\alpha^{3/2}, \beta^{3/2}$.

Hence, the correct answer is option (A).

10. If $f(x)=\left(\frac{3}{5}\right)^x+\left(\frac{4}{5}\right)^x-1$, where $x \in R$, then the equation $f(x)=0$ has

(A) no solution **(B)** one solution
(C) two solutions **(D)** more than two solutions

[JEE MAIN 2014 (ONLINE SET-1)]

Solution: For $f(x)=0$, we have

$$f(x)=\left(\frac{3}{5}\right)^x+\left(\frac{4}{5}\right)^x-1=0 \Rightarrow 3^x+4^x=5^x$$

which is true when $x=2$. Therefore, there is only one solution.

Hence, the correct answer is option (B).

11. If α and β are roots of the equation $x^2-4\sqrt{2}kx+2e^{4\ln k}-1=0$ for some k and $\alpha^2+\beta^2=66$, then $\alpha^3+\beta^3$ is equal to

(A) $248\sqrt{2}$ **(B)** $280\sqrt{2}$
(C) $-32\sqrt{2}$ **(D)** $-280\sqrt{2}$

[JEE MAIN 2014 (ONLINE SET-2)]

Solution: Given

$$x^2-4\sqrt{2}kx+2k^4-1=0$$

Therefore, $\alpha+\beta=4\sqrt{2}k$. Squaring both sides we get

$$(\alpha+\beta)^2=\alpha^2+\beta^2+2\alpha\beta$$

Also

$$\alpha\beta=2k^4-1 \Rightarrow (4\sqrt{2}k)^2=66+2(2k^4-1) \Rightarrow 32k^2=66+4k^4-2$$

$$\Rightarrow 4k^4-32k^2+64=0 \Rightarrow (k^2)^2-8(k^2)+16=0 \Rightarrow k=\pm 2$$

Taking $k=2$, therefore $k>0$. Thus,

$$\alpha+\beta=8\sqrt{2} \text{ and } \alpha\beta=31$$

Now,

$$\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=(8\sqrt{2})^3-3\times 31(8\sqrt{2})$$

$$=(8\sqrt{2})(128-93)=(8\sqrt{2})(35)=280\sqrt{2}$$

Hence, the correct answer is option (B).

12. The sum of the roots of the equation $x^2+|2x-3|-4=0$ is

(A) 2 **(B)** -2
(C) $\sqrt{2}$ **(D)** $-\sqrt{2}$

[JEE MAIN 2014 (ONLINE SET-3)]

Solution:

Case I: When $x>\frac{3}{2}$, we have $x^2+2x-7=0$. The roots are

$$\Rightarrow x=\frac{-2\pm\sqrt{4+28}}{2}=\frac{-2\pm 4\sqrt{2}}{2}=-1\pm 2\sqrt{2}$$

Thus, root is $2\sqrt{2}-1$. Since $2\sqrt{2}-1>\frac{3}{2}$.

Case II: When $x<\frac{3}{2}=+1.5$, equation is $x^2-2x+3-4=0 \Rightarrow x^2-2x-1=0$. The roots are

$$\Rightarrow x=\frac{2\pm\sqrt{4+4}}{2}=\frac{2\pm 2\sqrt{2}}{2}=1\pm\sqrt{2}$$

Thus, $1-\sqrt{2}$ is a root because it is less than $\frac{3}{2}$. Now, when $x=\frac{3}{2}$, then $x^2-4=0$. That is, $x=\pm 2$. Thus, no root.

Therefore, sum of roots $=2\sqrt{2}-\cancel{1}+\cancel{1}-\sqrt{2}=\sqrt{2}$.

Hence, the correct answer is option (C).

13. The equation $\sqrt{3x^2+x+5}=x-3$, where x is real, has

(A) no solution **(B)** exactly one solution
(C) exactly two solutions **(D)** exactly four solutions

[JEE MAIN 2014 (ONLINE SET-4)]

Solution: Given

$$\sqrt{3x^2+x+5}=x-3$$

Squaring both sides we get

$$3x^2+x+5=x^2+9-6x \Rightarrow 2x^2+7x-4=0$$

The roots are

$$x=\frac{-7\pm\sqrt{49+32}}{4}=\frac{-7\pm\sqrt{81}}{4}=\frac{-7\pm 9}{4}=\frac{1}{2},-4$$

Both roots do not satisfy original equation, therefore no solution.

Hence, the correct answer is option (A).

14. If non-zero real numbers b and c are such that min $f(x) >$ max $g(x)$, where $f(x)=x^2+2bx+2c^2$ and $g(x)=-x^2-2cx+b^2$ ($x\in$ R), then $\left|\frac{c}{b}\right|$ lies in the interval

(A) $\left(0,\frac{1}{2}\right)$ **(B)** $\left[\frac{1}{2},\frac{1}{\sqrt{2}}\right)$

(C) $\left[\frac{1}{\sqrt{2}},\sqrt{2}\right]$ **(D)** $(\sqrt{2},\infty)$

[JEE MAIN 2014 (ONLINE SET-4)]

Solution: $f(x)=x^2+2bx+2c^2=x^2+2bx+b^2+2c^2-b^2=(x+b)^2+2c^2-b^2$

Therefore, min $f(x)=2c^2-b^2$ at $x=-b$.

$g(x)=-x^2-2cx+b^2=-\{x^2+2cx-b^2\}=-\{x^2+2cx+c^2-b^2-c^2\}$
$=-(x+c)^2+b^2+c^2$

Therefore, max $g(x)=b^2+c^2$ when $x=-c$.

Now according to question,

$$2c^2-b^2>b^2+c^2 \Rightarrow c^2>2b^2$$

$$\Rightarrow \left(\frac{c}{b}\right)^2>2 \Rightarrow \left|\frac{c}{b}\right|>\sqrt{2}$$

Hence, the correct answer is option (D).

15. The sum of coefficients of integral powers of x in the binomial expansion of $(1-2\sqrt{x})^{50}$ is

(A) $\frac{1}{2}(3^{50})$ **(B)** $\frac{1}{2}(3^{50}-1)$

(C) $\frac{1}{2}(2^{50}+1)$ **(D)** $\frac{1}{2}(3^{50}+1)$

[JEE MAIN 2015 (OFFLINE)]

Solution:

$$(1-2\sqrt{x})^{50}=\sum_{r=0}^{50}{}^{50}C_r(1)^{50-r}\left(-2\sqrt{x}\right)^r$$

$$=\sum_{r=0}^{50}{}^{50}C_r(-2)^r(x)^{r/2}$$

Therefore, for integer powers of x, $r\in\{0, 2, 4, 6, \ldots, 50\}$.

So the required sum of coefficients is

$${}^{50}C_0+{}^{50}C_2(2)^2+{}^{50}C_4(2)^4+\cdots+{}^{50}C_{50}(2)^{50} \quad (1)$$

Since,

$$(1+2)^{50}+(1-2)^{50}=2\left[{}^{50}C_0+{}^{50}C_2(2)^2+{}^{50}C_4(2)^4+\cdots+{}^{50}C_{50}(2)^{50}\right] \quad (2)$$

In view of Eqs. (1) and (2),

$$\text{Required sum}=\frac{1}{2}[(3)^{50}+1]$$

Hence, the correct answer is option (D).

16. Let α and β be the roots of equation $x^2-6x-2=0$. If $a_n=\alpha^n-\beta^n$, for $n\geq 1$, then the value of $\frac{a_{10}-2a_8}{2a_9}$ is equal to

(A) −6 **(B)** 3

(C) −3 **(D)** 6

[JEE MAIN 2015 (OFFLINE)]

Solution: α and β are roots of $x^2-6x-2=0$. Then $\alpha+\beta=6$, $\alpha\beta=-2$.

Now, $\alpha^n-\beta^n=a_n$

$$\Rightarrow \frac{a_{10}-2a_8}{2a_9}=\frac{(\alpha^{10}-\beta^{10})-2(\alpha^8-\beta^8)}{2(\alpha^9-\beta^9)}=\frac{\alpha^8(\alpha^2-2)-\beta^8(\beta^2-2)}{2(\alpha^9-\beta^9)}$$

$$=\frac{\alpha^8(6\alpha)-\beta^8(6\beta)}{2(\alpha^9-\beta^9)} \text{ [since } \alpha^2-6\alpha-2=0 \text{ and } \beta^2-6\beta-2=0]$$

$$=\frac{6(\alpha^9-\beta^9)}{2(\alpha^9-\beta^9)}=3$$

Hence, the correct answer is option (B).

17. If $2+3i$ is one of the roots of the equation $2x^3-9x^2+kx-13=0$, $k\in R$, then the real root of this equation

(A) does not exist

(B) exists and is equal to $\frac{1}{2}$

(C) exists and is equal to $-\frac{1}{2}$

(D) exists and is equal to 1

[JEE MAIN 2015 (ONLINE SET-1)]

Solution: Since imaginary roots always exist in pairs. If $2+3i$ is one root then $2-3i$ will be the second root. Let $\alpha=2+3i$ and $\beta=2-3i$ be roots of given cubic equation. Let r be its real root. Then $[x-(2+3i)]\,[x-(2-3i)]$ is a factor of

$$2x^3-9x^2+kx-13;\ k\in R$$

Now,

$$\alpha+\beta+\gamma=\text{sum of roots}=(-1)^1\left(\frac{-9}{2}\right)=\frac{9}{2}$$

$$\Rightarrow 4+\gamma=\frac{9}{2}\Rightarrow\gamma=\frac{9}{2}-4=\frac{1}{2}$$

Hence, the correct answer is option (B).

18. If the two roots of the equation $(a-1)(x^4+x^2+1)+(a+1)(x^2+x+1)^2=0$ are real and distinct, then the set of all values of a is

(A) $\left(-\frac{1}{2},0\right)$ **(B)** $(-\infty,-2)\cup(2,\infty)$

(C) $\left(-\frac{1}{2},0\right)\cup\left(0,\frac{1}{2}\right)$ **(D)** $\left(0,\frac{1}{2}\right)$

[JEE MAIN 2015 (ONLINE SET-2)]

Solution: Let α,β be real and distinct roots of equation

$$(a-1)(x^4+x^2+1)+(a+1)(x^2+x+1)^2=0$$

$\Rightarrow (a-1)(x^2+1+x)(x^2+1-x)+(a+1)(x^2+x+1)^2=0$

$\Rightarrow (x^2+x+1)[(a-1)(x^2-x+1)+(a+1)(x^2+x+1)]=0$

$\Rightarrow (x^2+x+1)[2a(x^2+1)+2x)=0$ $2a(x^2+1)+2x]=0$. This 2 real and distinct roots as $x^2+x+1=0$ has both imaginary roots.

$\Rightarrow (a(x^2) + x + a) = 0$. This has 2 real and distinct roots, if $a \neq 0$ and discriminant. $> 0 \Rightarrow 1 - 4a^2 > 0 \Rightarrow 4a^2 < 1 \Rightarrow a \in \left(\frac{-1}{2}, \frac{1}{2}\right) \sim \{0\}$, that is, $a \in \left(\frac{-1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right)$.

Hence, the correct answer is option (C).

19. The sum of all real values of x satisfying the equation $(x^2 - 5x + 5)^{x^2+4x-60} = 1$ is

(A) 5 **(B)** 3
(C) –4 **(D)** 6

[JEE MAIN 2016 (OFFLINE)]

Solution: We have

$$(x^2 - 5x + 5)^{x^2+4x-60} = 1$$

Case 1: $x^2 + 4x - 60 = 0$

$$(x + 10)(x - 6) = 0 \Rightarrow x = -10, 6$$

Case 2: $x^2 - 5x + 5 = 1$ or $x^2 - 5x + 5 = -1$

$\Rightarrow x^2 - 5x + 4 = 0$ $\Rightarrow x^2 - 5x + 6 = 0$

$\Rightarrow x = 1, 4$ $\Rightarrow x = 2$ and 3

$x^2 + 4x - 60$ must be even number for $x = 2$ and 3. Substituting the two values in $x^2 + 4x - 60$, we get

$$4 + 8 - 60 = -48$$
$$9 + 12 - 60 = -39$$

Value of x can not be 3 because $x^2 + 4x - 60$, is negative at $x = 3$. Thus, the sum of all value of x

$$= -10 + 6 + 1 + 4 + 2 = 3$$

Hence, the correct answer is option (B).

20. If the equations $x^2 + bx - 1 = 0$ and $x^2 + x + b = 0$ have a common root different from -1, then $|b|$ is equal to

(A) 2 **(B)** 3
(C) $\sqrt{3}$ **(D)** $\sqrt{2}$

[JEE MAIN 2016 (ONLINE SET-1)]

Solution: We have

$$x^2 + bx - 1 = 0 \quad (1)$$

and $x^2 + x + b = 0 \Rightarrow b = -x - x^2$

Substituting the value of b in Eq. (1), we get

$$x^2 - x(x + x^2) - 1 = 0$$
$$\Rightarrow x^3 + 1 = 0$$
$$\Rightarrow (x + 1)(x^2 - x + 1) = 0$$

Therefore, $x = -1$. Also,

$$x^2 - x + 1 = 0$$
$$\Rightarrow x = -\omega, -\omega^2$$

where ω is the cube root of unity. Now,

$$b = -(x + x^2)$$
$$\Rightarrow b = -(-\omega + \omega^2) = \omega - \omega^2 = i\sqrt{3}$$
$$\Rightarrow |b| = \sqrt{3}$$

Hence, the correct answer is option (C).

21. Let x, y, z be positive real numbers such that $x + y + z = 12$ and $x^3y^4z^5 = (0.1)(600)^3$. Then, $x^3 + y^3 + z^3$ is equal to

(A) 342 **(B)** 216
(C) 258 **(D)** 270

[JEE MAIN 2016 (ONLINE SET-1)]

Solution: We have

$$x + y + z = 12$$

By using weight arithmetic inequality

$$\frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n} \geq \left(x_1^{m_1} \cdot x_2^{m_2} \cdots x_n^{m_n}\right)^{\frac{1}{m_1 + \cdots + m_n}}$$

$$1 \geq \left(\frac{x^3y^4z^5}{21600000}\right)^{1/12} \Rightarrow 1 \geq 1$$

That is,

$$\left(\frac{x}{3}\right) = \left(\frac{y}{4}\right) = \left(\frac{z}{5}\right) = k$$
$$\Rightarrow x = 3k; y = 4k; z = 5k$$

Therefore,

$$3k + 4k + 5k = 12 \Rightarrow k = 1$$

So $x = 3; y = 4; z = 5$

Hence,

$$x^3 + y^3 + z^3 = 27 + 64 + 125 = 216$$

Hence, the correct answer is option (B).

22. If x is a solution of the equation $\sqrt{2x+1} - \sqrt{2x-1} = 1$, $\left(x \geq \frac{1}{2}\right)$, then $\sqrt{4x^2 - 1}$ is equal to

(A) $\frac{3}{4}$ **(B)** $\frac{1}{2}$
(C) $2\sqrt{2}$ **(D)** 2

[JEE MAIN 2016 (ONLINE SET-2)]

Solution: We have

$$\sqrt{2x+1} - \sqrt{2x-1} = 1 \quad \left(x \geq \frac{1}{2}\right)$$

(By Rationalization)

$$\frac{(\sqrt{2x+1} - \sqrt{2x-1})(\sqrt{2x+1} + \sqrt{2x-1})}{(\sqrt{2x+1} + \sqrt{2x-1})} = 1$$

$$\Rightarrow \frac{2x + 1 - (2x - 1)}{\sqrt{2x+1} + \sqrt{2x-1}} = 1$$

$$\Rightarrow \sqrt{2x+1} + \sqrt{2x-1} = 2$$

$$\Rightarrow \sqrt{2x+1} - \sqrt{2x-1} = 1$$

$$\Rightarrow \sqrt{2x+1} = \frac{3}{2}$$

$$\Rightarrow \sqrt{2x-1} = 2 - \frac{3}{2} = \frac{1}{2}$$

$$\Rightarrow \sqrt{4x^2 - 1} = \frac{3}{2} \times \frac{1}{2} = \frac{3}{4}$$

Hence, the correct answer is option (A).

Previous Years' Solved JEE Advanced/ IIT-JEE Questions

1. Let α and β be the roots of the equation $x^2 - px + r = 0$ and $\frac{\alpha}{2}$, 2β be the roots of the equation $x^2 - qx + r = 0$. The value of r is

(A) $\frac{2}{9}(p-q)(2q-p)$ **(B)** $\frac{2}{9}(q-p)(2p-q)$

(C) $\frac{2}{9}(q-2p)(2p-p)$ **(D)** $\frac{2}{9}(2p-q)(2q-p)$

[IIT 2007]

Solution: Since α and β are the roots of equation $x^2 - px + r = 0$, we get

$$\alpha + \beta = p \quad (1)$$

$$\alpha\beta = r \quad (2)$$

Also, since $\alpha/2$ and 2β are the roots of equation $x^2 - qx + r = 0$, we get

$$\frac{\alpha}{2} + 2\beta = q \Rightarrow \alpha + 4\beta = 2q$$

Solving Eqs. (1) and (2), we get

$$\alpha = \frac{2(2p-q)}{3} \text{ and } \beta = \frac{2q-p}{3}$$

Substituting α and β in Eq. (2), we get

$$r = \frac{2}{9}(2p-q)(2q-p)$$

Hence, the correct answer is option (D).

2. Let a, b, c, p and q be real numbers. Suppose α, β are the roots of the equation $x^2 + 2px + q = 0$ and $\alpha, \frac{1}{\beta}$ are roots of the equation $ax^2 + 2bx + c = 0$, where $\beta^2 \notin \{-1, 0, 1\}$.

Statement-1: $(p^2 - q)(b^2 - ac) \geq 0$

Statement-2: $p \neq pa$ or $c \neq qa$

(A) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1.

(B) Statement-1 is True, Statement-2 is True; Statement-2 is **NOT** a correct explanation for Statement-1.

(C) Statement-1 is True, Statement-2 is False.

(D) Statement-1 is False, Statement-2 is True.

[IIT-JEE 2008]

Solution: Given that, α, β are the roots of equation

$$x^2 + 2px + q = 0$$

Therefore

$$\alpha + \beta = -2p, \ \alpha\beta = q \quad (1)$$

Since $\alpha, \frac{1}{\beta}$ are the roots of equation.

$$ax^2 + 2bx + c = 0$$

We have

$$\alpha + \frac{1}{\beta} = \frac{-2b}{a}, \ \frac{\alpha}{\beta} = \frac{c}{a} \quad (2)$$

Using Eq. (1), if $\beta = 1$, then $\alpha = q$.
Using Eq. (2), if $\beta = 1$, then $\alpha = c/a$.

So $\alpha = q = \frac{c}{a} \Rightarrow c = qa$ (not possible)

Also

$$\alpha + 1 = -2p = \frac{-2b}{a} \Rightarrow b = pa \text{ (not possible)}$$

Therefore, Statement-2 is correct.
Now, if the roots are imaginary, we have

$$\beta = \bar{\alpha}, \frac{1}{\beta} = \bar{\alpha}$$

$$\Rightarrow \beta = \frac{1}{\beta} \text{ (not possible)}$$

Therefore, roots are real in both equations. So

$$(4p^2 - 4q)(4b^2 - 4ac) \geq 0$$

$$\Rightarrow (p^2 - q)(b^2 - ac) \geq 0$$

Hence, the correct answer is option (B).

3. Let $z = x + iy$ be a complex number where x and y are integers. Then the area of the rectangle whose vertices are the roots of the equation $\bar{z}z^3 + z\bar{z}^3 = 350$ is

(A) 48 **(B)** 32

(C) 40 **(D)** 80

[IIT-JEE 2009]

Solution: Given

$$z\bar{z}(\bar{z}^2 + z^2) = 350$$

Put $z = x + iy$. Then

$$(x^2 + y^2)(x^2 - y^2) = 175$$

$$\Rightarrow (x^2 + y^2)(x^2 - y^2) = 5.5.7$$

$$\Rightarrow x^2 + y^2 = 25$$

$$\Rightarrow x^2 - y^2 = 7$$

$$\Rightarrow x = \pm 4, y = \pm 3$$

$$\Rightarrow x, y \in I$$

Therefore, area $= 8 \times 6 = 48$ sq. unit.

Hence, the correct answer is option (A).

4. The smallest value of k for which both the roots of the equation $x^2 - 8kx + 16(k^2 - k + 1) = 0$ are real, distinct and have values at least 4 is ______.

[IIT-JEE 2009]

Solution: We have,

$$x^2 - 8kx + 16(k^2 - k + 1) = 0$$

Now,

$$D > 0 \Rightarrow k > 1 \quad (1)$$

$$\frac{-b}{2a} > 4 \Rightarrow \frac{8k}{2} > 4 \Rightarrow k > 1 \quad (2)$$

$$f(4) \geq 0 \Rightarrow 16 - 32k + 16\ (k^2 - k + 1) \geq 0$$

$$k^2 - 3k + 2 \geq 0$$
$$(k \leq 1) \cup (k \geq 2) \quad (3)$$

Using Eqs. (1)–(3), we get

$$k_{\min} = 2$$

Hence, the correct answer is (2).

5. Let p and q be real numbers such that $p \neq 0$, $p^3 \neq q$ and $p^3 \neq -q$. If α and β are non-zero complex numbers satisfying $\alpha + \beta = -p$ and $\alpha^3 + \beta^3 = q$, then a quadratic equation having $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ as its roots is

(A) $(p^3+q)x^2 - (p^3+2q)x + (p^3+q) = 0$

(B) $(p^3+q)x^2 - (p^3-2q)x + (p^3+q) = 0$

(C) $(p^3-q)x^2 - (5p^3-2q)x + (p^3-q) = 0$

(D) $(p^3-q)x^2 - (5p^3+2q)x + (p^3-q) = 0$

[IIT-JEE 2010]

Solution: We have

$$\alpha^3 + \beta^3 = q$$
$$\Rightarrow (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta) = q$$
$$\Rightarrow -p^3 + 3p\alpha\beta = q \Rightarrow \alpha\beta = \frac{q+p^3}{3p}$$

Now,

$$x^2 - \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)x + \frac{\alpha}{\beta}.\frac{\beta}{\alpha} = 0$$
$$\Rightarrow x^2 - \frac{(\alpha^2+\beta^2)}{\alpha\beta}x + 1 = 0$$
$$\Rightarrow x^2 - \left[\frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta}\right]x + 1 = 0$$
$$\Rightarrow x^2 - \frac{p^2 - 2\left(\frac{p^3+q}{3p}\right)}{\frac{p^3+q}{3p}}x + 1 = 0$$
$$\Rightarrow (p^3+q)x^2 - (3p^3 - 2p^3 - 2q)x + (p^3+q) = 0$$
$$\Rightarrow (p^3+q)x^2 - (p^3-2q)x + (p^3+q) = 0$$

Hence, the correct answer is option (B).

6. Let α and β be the roots of $x^2 - 6x - 2 = 0$ with $\alpha > \beta$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, then the value of $\frac{a_{10} - 2a_8}{2a_9}$ is

(A) 1 **(B)** 2
(C) 3 **(D)** 4

[IIT-JEE 2011]

Solution: Given

$$a_n = \alpha^n - \beta^n$$
$$\alpha^2 - 6\alpha - 2 = 0$$

Multiplying the above equation with α^8 on both sides we get

$$\alpha^{10} - 6\alpha^9 - 2\alpha^8 = 0 \quad (1)$$

Similarly,

$$\beta^{10} - 6\beta^9 - 2\beta^8 = 0 \quad (2)$$

Subtracting Eq. (2) from Eq. (1) we get

$$\alpha^{10} - \beta^{10} - 6(\alpha^9 - \beta^9) = 2(\alpha^8 - \beta^8)$$
$$\Rightarrow a_{10} - 6a_9 = 2a_8 \Rightarrow \frac{a_{10} - 2a_8}{2a_9} = 3$$

Hence, the correct answer is option (C).

7. A value of b for which the equations

$$x^2 + bx - 1 = 0$$
$$x^2 + x + b = 0$$

have one root in common is

(A) $-\sqrt{2}$ **(B)** $-i\sqrt{3}$
(C) $i\sqrt{5}$ **(D)** $\sqrt{2}$

[IIT-JEE 2011]

Solution: Given

$$x^2 + bx - 1 = 0$$
$$x^2 + x + b = 0 \quad (1)$$

Common root is

$$(b-1)x - 1 - b = 0 \Rightarrow x = \frac{b+1}{b-1}$$

This value of x satisfies Eq. (1). So

$$\frac{(b+1)^2}{(b-1)^2} + \frac{b+1}{b-1} + b = 0 \quad \Rightarrow b = \sqrt{3}i, -\sqrt{3}i, 0$$

Hence, the correct answer is option (B).

8. The number of distinct real roots of $x^4 - 4x^3 + 12x^2 + x - 1 = 0$ is ______.

[IIT-JEE 2011]

Solution: Let $f(x) = x^4 - 4x^3 + 12x^2 + x - 1 = 0$

$$f'(x) = 4x^3 - 12x^2 + 24x + 1 = 4(x^3 - 3x^2 + 6x) + 1$$
$$f''(x) = 12x^2 - 24x + 24 = 12(x^2 - 2x + 2)$$

Therefore, $f''(x)$ has 0 real roots.
Hence, $f(x)$ has maximum 2 distinct real roots as $f(0) = -1$.
Hence, the correct answer is (2).

9. Let $\alpha(a)$ and $\beta(a)$ be the roots of the equation $(\sqrt[3]{1+a} - 1)x^2 + (\sqrt{1+a} - 1)x + (\sqrt[6]{1+a} - 1) = 0$ where $a > -1$. Then $\lim_{a \to 0^+} \alpha(a)$ and $\lim_{a \to 0^+} \beta(a)$ are

(A) $-\frac{5}{2}$ and 1 **(B)** $-\frac{1}{2}$ and -1

(C) $-\frac{7}{2}$ and 2 **(D)** $-\frac{9}{2}$ and 3

[IIT-JEE 2012]

Solution: Let $1 + a = y$.

Substituting in the given equation we get

$$(y^{1/3} - 1)x^2 + (y^{1/2} - 1)x + y^{1/6} - 1 = 0$$

$$\Rightarrow \left(\frac{y^{1/3}-1}{y-1}\right)x^2+\left(\frac{y^{1/2}-1}{y-1}\right)x+\frac{y^{1/6}-1}{y-1}=0$$

Now taking $\lim_{y\to 1}$ on both the sides we get

$$\frac{1}{3}x^2+\frac{1}{2}x+\frac{1}{6}=0$$

$$\Rightarrow 2x^2+3x+1=0$$

$$\Rightarrow x=-1,-\frac{1}{2}$$

Hence, the correct answer is option (B).

10. The number of points in $(-\infty, \infty)$ for which $x^2 - x\sin x - \cos x = 0$ is

(A) 6 **(B)** 4
(C) 2 **(D)** 0

[JEE ADVANCED 2013]

Solution: Let us consider that

$$f(x)=x^2-x\sin x-\cos x$$

Therefore,

$$f'(x)=2x\cdot x\cos x-\sin x+\sin x=x(2-\cos x)$$

$f(x)$ is increasing when $x>0$; $f(x)$ is decreasing when $x<0$.

Therefore,

$$f(0)=-1$$
$$f(\infty)=\infty$$
$$f(-\infty)=\infty$$

Therefore, as shown in Fig. 6.35, the number of points in $(-\infty,\infty)$, for which $x^2-x\sin x-\cos x=0$, is two.

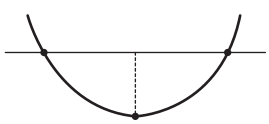


Figure 6.35

Hence, the correct answer is option (C).

11. Let $a\in R$ and let $f: R\to R$ be given by

$$f(x)=x^5-5x+a$$

Then

(A) $f(x)$ has three real roots if $a>4$
(B) $f(x)$ has only one real root if $a>4$
(C) $f(x)$ has three real roots if $a<-4$
(D) $f(x)$ has three real roots if $-4<a<4$

[JEE ADVANCED 2014]

Solution: Given

$f(x) = x^5 - 5x + a$. There will be different polynomials, depending on the parameter a. Now for roots of each of these in general are given by $f(x) = 0$. So

$$a=5x-x^5=x(5-x^4)$$

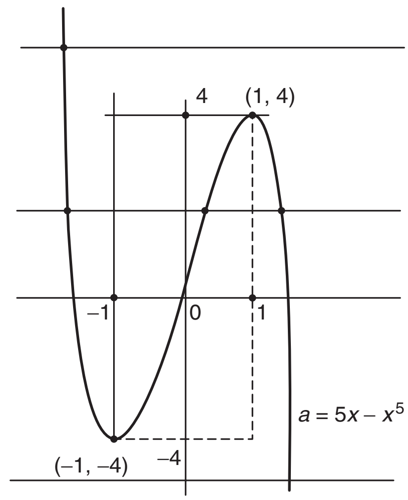


Figure 6.36

Therefore, parameter a is a function of x. That is

$$a(x)=x(5-x^4)$$

Differentiating w.r.t. x we get

$$a'(x)=5-5x^4$$

Hence, extrema occur at $a'(x)=0$, that is, when $x^4=1$ or $x=1$ and $x=-1$ (only real roots considered).

Again differentiating w.r.t. x we get

$$a''(x)=-20x^3$$
$$a''(1)<0 \text{ Max}$$
$$a''(-1)>0 \text{ Min}$$

Hence, max value $= a(1) = 4$ and min value $= a(-1) = -4$.

Therefore, when $-4<a<4$, there are three points, that is, x values where $f(x) = 0$, that is, 3 roots of $f(x)$ for any value of a lying in $(-4, 4)$. (1)

When $|a|>4$, there is only one x for which $f(x) = 0$. (See Fig. 6.36.) (2)

Hence, from conditions (1) and (2), we can conclude that (B) and (D) are correct options.

Hence, the correct answer is options (B) and (D).

12. The quadratic equation $p(x) = 0$ with real coefficients has purely imaginary roots. Then the equation $p[p(x)] = 0$ has

(A) only purely imaginary roots
(B) all real roots
(C) two real and two purely imaginary roots
(D) neither real nor purely imaginary roots

[JEE ADVANCED 2014]

Solution: Given equation is $p(x) = 0$. Let it be written as $x^2 + c = 0$ where $c > 0$ (since purely imaginary roots). Therefore,

$$p[p(x)]=[p(x)]^2+c=0$$
$$\Rightarrow (x^2+c)^2+c=0 \text{ or } x^4+c^2+2cx^2+c=0$$
$$\Rightarrow (x^2)^2+2cx^2+c^2+c=0$$

Therefore,

$$x^2=\frac{-2c\pm\sqrt{4c^2-4(c^2+c)}}{2}$$

$$=\frac{-2c\pm 2\sqrt{\cancel{c^2}-\cancel{c^2}-c}}{2}=-c\pm ic,\quad \text{where } c>0$$

$$\Rightarrow x = \pm\sqrt{-c \pm ic}$$

Hence, $p[p(x)] = 0$ has neither real nor purely imaginary roots.

Note: Let $c = 1$. Let us find square root of $-1 + 2'$, where $-1 + 2' = -1 + 2i$.

Let $x^2 = -1 + 2' = \sqrt{2}\left[\cos\left(\pi - \frac{\pi}{4}\right) + 2'\sin\left(\pi - \frac{\pi}{4}\right)\right]$

Hence,

$$x = 2^{1/4}\left[\cos\left(\frac{3\pi}{4} + 2k\pi\right) + 2'\sin\left(\frac{3\pi}{4} + 2k\pi\right)\right]^{1/2}$$

Putting $k = 0, 1$, we get

$$x = 2^{1/4}\left[\cos\frac{3\pi}{4} + 2'\sin\frac{3\pi}{4}\right],\quad 2^{1/2}\left[\cos\left(\frac{11\pi}{4}\right) + 2'\sin\left(\frac{11\pi}{4}\right)\right]$$

Hence, the correct answer is option (D).

13. Let S be the set of all non-zero real numbers α such that the quadratic equation $\alpha x^2 - x + \alpha = 0$ has two distinct real roots x_1 and x_2 satisfying the inequality $|x_1 - x_2| < 1$. Which of the following intervals is (are) a subset(s) of S?

(A) $\left(-\frac{1}{2}, -\frac{1}{\sqrt{5}}\right)$ **(B)** $\left(-\frac{1}{\sqrt{5}}, 0\right)$

(C) $\left(0, \frac{1}{\sqrt{5}}\right)$ **(D)** $\left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)$

[JEE ADVANCED 2015]

Solution: $\alpha x^2 - x + \alpha = 0$ has two distinct real roots x_1 and x_2; $|x_1 - x_2| < 1$.

$$D > 0 \Rightarrow 1 - 4\alpha^2 > 0 \Rightarrow \alpha^2 < \frac{1}{4}$$

$$\Rightarrow \alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right) \quad (1)$$

Also,

$$|x_1 - x_2|^2 = (x_1 + x_2)^2 - 4x_1x_2 < 1$$

$$\Rightarrow 1 > \left(\frac{1}{\alpha}\right)^2 - 4(1)$$

$$\Rightarrow \frac{1}{\alpha^2} < 5 \Rightarrow \alpha^2 > \frac{1}{5}$$

$$\Rightarrow \alpha \in \left(-\infty, \frac{-1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, \infty\right) \quad (2)$$

Therefore, intersection of Eqs. (1) and (2) gives

$$\alpha \in \left(-\frac{1}{2}, \frac{-1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)$$

Hence, the correct answer is options (A) and (D).

14. The least value of $\alpha \in R$ for which $4\alpha x^2 + \frac{1}{x} \geq 1$, for all $x > 0$, is

(A) $\frac{1}{64}$ **(B)** $\frac{1}{32}$

(C) $\frac{1}{27}$ **(D)** $\frac{1}{25}$

[JEE ADVANCED 2016]

Solution: It is given that

$$4\alpha x^2 + \frac{1}{x} \geq 1 \qquad \forall x > 0$$

$$\Rightarrow 4\alpha x^2 \geq 1 - \frac{1}{x}$$

$$\Rightarrow 4\alpha \geq \left(\frac{1}{x^2} - \frac{1}{x^3}\right) \qquad \forall\, x > 0$$

Now, let us consider that

$$f(x) = \frac{1}{x^2} - \frac{1}{x^3}$$

$$f'(x) = \frac{-2}{x^3} + \frac{3}{x^4} = 0$$

Differentiating w.r.t x we get

When $x = 3/2$, we have

$$(4\alpha) \geq \left(\frac{1}{x^2} - \frac{1}{x^3}\right)$$

$$\Rightarrow 4\alpha \geq \left(\frac{4}{9} - \frac{8}{27}\right) \Rightarrow \alpha \geq \frac{1}{27}$$

$$\Rightarrow \alpha \geq \frac{1}{27}$$

Hence the least value of α is

$$\alpha_{\text{least}} = \frac{1}{27}$$

Hence, the correct answer is option (C).

15. Let $-\frac{\pi}{6} < \theta < -\frac{\pi}{12}$. Suppose α_1 and β_1 are the roots of the equation $x^2 - 2x\sec\theta + 1 = 0$ and α_2 and β_2 are the roots of the equation $x^2 + 2x\tan\theta - 1 = 0$. If $\alpha_1 > \beta_1$ and $\alpha_2 > \beta_2$, then $\alpha_1 + \beta_2$ equals

(A) $2(\sec\theta - \tan\theta)$ **(B)** $2\sec\theta$

(C) $-2\tan\theta$ **(D)** 0

[JEE ADVANCED 2016]

Solution: We have $\theta \in \left(\frac{-\pi}{6}, \frac{-\pi}{12}\right)$.

- It is given that α_1 and β_1 are the roots of the equation

  $$x^2 - 2x\sec\theta + 1 = 0$$

  So, $$\alpha_1, \beta_1 = \frac{2\sec\theta \pm \sqrt{4\sec^2\theta - 4}}{2}$$

  $\Rightarrow \alpha_1, \beta_1 = \sec\theta \pm \tan\theta$ (since $\sec\theta > 0$ and $\tan\theta < 0$)

  Since it is given that $\alpha_1 > \beta_1$, we get

  $$\alpha_1 = \sec\theta - \tan\theta$$

- It is also given that α_2 and β_2 are the roots of the equation

  $$x^2 + 2x\tan - 1 = 0$$

  So, $$\alpha_2, \beta_2 = \frac{-2\tan\theta \pm \sqrt{4\tan^2\theta + 4}}{2}$$

  $$\Rightarrow \alpha_2, \beta_2 = -\tan\theta \pm \sec\theta$$

  Since it is given that $\alpha_2 > \beta_2$, we get

$$\alpha_2 = -\tan\theta + \sec\theta$$

and $$\beta_2 = -\tan\theta - \sec\theta$$

Therefore,

$$\alpha_1 + \beta_2 = \sec\theta - \tan\theta - \tan\theta - \sec\theta = -2\tan\theta$$

Hence, the correct answer is option (C).

Practice Exercise 1

1. If $a, b \in R, a \neq 0$ and the quadratic equation $ax^2 - bx + 1 = 0$ has imaginary roots, then $a + b + 1$ is

(A) positive **(B)** negative
(C) zero **(D)** depends on the sign of b

2. The number of real roots of the quadratic equation $\sum_{k=1}^{n}(x-k)^2 = 0$ $(n > 1)$ is

(A) 1 **(B)** 2
(C) n **(D)** 0

3. The set of values of a for which the equation $x^3 - 3x + a = 0$ has three distinct real roots is

(A) $(-\infty, \infty)$ **(B)** $(-2, 2)$
(C) $(-1, 1)$ **(D)** None of these

4. If roots of the equation $2x^2 - 4x + 2\sin\theta - 1 = 0$ are of opposite sign, then θ belongs to

(A) $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ **(B)** $\left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, \pi\right)$
(C) $\left(\frac{13\pi}{6}, \frac{17\pi}{6}\right)$ **(D)** None of these

5. Roots of the quadratic equation $(x^2 - 4x + 3) + \lambda(x^2 - 6x + 8) = 0$, $\lambda \in R$ will be

(A) always real
(B) real only when λ is positive
(C) real only when λ is negative
(D) always imaginary

6. The least integral value of k such that $(k-2)x^2 + 8x + k + 4$ is positive for all real values of x is

(A) 1 **(B)** 2
(C) 3 **(D)** 5

7. If all the real solutions of the equation $4^x - (a-3)2^x + (a-4) = 0$ are non-positive, then

(A) $4 < a \leq 5$ **(B)** $0 < a < 4$
(C) $a > 4$ **(D)** $a < 3$

8. If $x^2 + 2ax + b \geq c, \forall x \in R$, then

(A) $b - c \geq a^2$ **(B)** $c - a \geq b^2$
(C) $a - b \geq c^2$ **(D)** None of these

9. If the equation $5x^2 - 10x + \log_{1/5} a = 0$ has real roots, then the minimum value of a is

(A) $\frac{1}{5^5}$ **(B)** $\frac{1}{10^{10}}$
(C) $\frac{1}{5^{10}}$ **(D)** None of these

10. The roots of the quadratic equation $x^2 + 2(a+1)x + a^2 - 6a + 8 = 0$ will be of opposite sign if 'a' belongs to

(A) $(1, 4)$ **(B)** $(2, 4)$
(C) $(-2, -4)$ **(D)** $(2, -4)$

11. The minimum value of a for which $a^2x^2 + (2a-1)x + 1$ is non-negative for any real x is

(A) $\frac{1}{2}$ **(B)** $\frac{1}{4}$
(C) $-\frac{1}{2}$ **(D)** $-\frac{1}{4}$

12. The value of a for which the equation $(a^2 + 4a + 3)x^2 + (a^2 - a - 2)x + (a+1)a = 0$ has more than two roots is

(A) 1 **(B)** 2
(C) -2 **(D)** -1

13. Let S be the set of values of a for which 2 lies between the roots of quadratic equation $x^2 + (a+2)x - (a+3) = 0$. Then S is given by

(A) $(-\infty, -5)$ **(B)** $(5, \infty)$
(C) $(-\infty, -5]$ **(D)** $[5, \infty)$

14. If the equation $ax^2 + bx + 6 = 0$ does not have two distinct real roots, then the least value of $3a + b$ is

(A) 3 **(B)** -3
(C) 2 **(D)** -2

15. If $\frac{(x^2-1)(x+2)(x+1)^2}{(x-2)} < 0$, then x lies in the interval

(A) $(-2, -1) \cup (1, 2)$ **(B)** $(-\infty, -2) \cup (2, \infty)$
(C) $(-2, -1) \cup (2, \infty)$ **(D)** $(-2, -1) \cup (1, \infty)$

16. Let $p(x) = 0$ be a polynomial equation of least possible degree, with rational coefficients having $\sqrt[3]{7} + \sqrt[3]{49}$ as one of its roots. Then the product of all the roots of $p(x) = 0$ is

(A) 7 **(B)** 49
(C) 56 **(D)** 63

17. If $c > 0$ and $4a + c < 2b$, then $ax^2 - bx + c = 0$ has a root in the interval

(A) $(0, 2)$ **(B)** $(2, 4)$
(C) $(0, 1)$ **(D)** $(-2, 0)$

18. If $x^2 - 4x + \log_{1/2} a = 0$ does not have two distinct real roots, then maximum value of a is

(A) $\frac{1}{4}$ **(B)** $\frac{1}{16}$
(C) $-\frac{1}{4}$ **(D)** None of these

19. If the equations $x^2 + ax + b = 0$ and $x^2 + a'x + b' = 0$ have a common root, then this common root is equal to

(A) $\frac{(b-b')}{(a-a')}$ **(B)** $\frac{(b+b')}{(a'-a)}$
(C) $\frac{(b-b')}{(a+a')}$ **(D)** $\frac{(b-b')}{(a'-a)}$

20. If p and q are the roots of the equation $x^2 + px + q = 0$, then

(A) $p = 1, q = -2$ **(B)** $p = 0, q = 1$
(C) $p = -2, q = 0$ **(D)** $p = -2, q = 1$

21. If a and b are non-zero distinct roots of $x^2 + ax + b = 0$, then the least value of $x^2 + ax + b$ is

(A) $\frac{2}{3}$ **(B)** $\frac{9}{4}$

(C) $-\dfrac{9}{4}$ **(D)** 1

22. The number of solutions of the equation $x^3+2x^2+5x+2\cos x$ in $[0, 2\pi]$ is
(A) 0 **(B)** 1
(C) 2 **(D)** 3

23. The equation $(\cos p-1)x^2+\cos px+\sin p=0$ in x has real roots. Then the set of values of p is
(A) $[0, 2\pi]$ **(B)** $[-\pi, 0]$
(C) $\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$ **(D)** $[0, \pi]$

24. Let x^2+x+1 be divisible by 3. If x is divided by 3, the remainder will be
(A) 2 **(B)** 1
(C) 0 **(D)** None of these

25. If $a+b+c>\dfrac{9c}{4}$ and equation $ax^2+2bx-5c=0$ has non-real complex roots, then
(A) $a>0, c>0$ **(B)** $a>0, c<0$
(C) $a<0, c<0$ **(D)** $a<0, c>0$

26. If $a, b, c \in R, a \neq 0$ and $(b-1)^2<4ac$, then the number of real roots of the system of equation (in three unknowns x_1, x_2, x_3) $ax_1^2+bx_1+c=x_2, ax_2^2+bx_2+c=x_3, ax_3^2+bx_3+c=x_1$ is
(A) 0 **(B)** 1
(C) 2 **(D)** 3

27. For the equation $3x^2+px+3=0, p>0$, if one of the roots is the square of the other, then p is equal to
(A) 1/3 **(B)** 1
(C) 3 **(D)** 2/3

28. If the roots of equation $ax^2+bx+10=0$ are not real and distinct, where $a, b \in R$ and m and n are values of a and b, respectively, for which $5a+b$ is minimum, then the family of lines $(4x+2y+3)+n(x-y-1)=0$ are concurrent at
(A) $(1, -1)$ **(B)** $(-1/6, -7/6)$
(C) $(1, 1)$ **(D)** None of these

29. The values of a and b so that $x^4+12x^3+46x^2+ax+b$ is square of quadratic expression are, respectively,
(A) 60, 25 **(B)** 45, 25
(C) 60, 30 **(D)** 25, 45

30. If equation $x^2+5bx+8c=0$ does not have two distinct real roots, then minimum value of $5b+8c$ is
(A) 1 **(B)** 2
(C) -2 **(D)** -1

31. The number of real solutions of the equation $-x^2+x-1=\sin^4 x$ is
(A) 1 **(B)** 2
(C) 0 **(D)** 4

32. The number of real roots of equation $x^8-x^5+x^2-x+2=0$ is
(A) 2 **(B)** 4
(C) 6 **(D)** 0

33. If x^2+ax+b is an integer for every integer x then
(A) a is always an integer but b need not be an integer.
(B) b is always an integer but a need not be an integer.
(C) $a+b$ is always an integer.
(D) a and b are always integers.

34. If the roots of the equation $x^2-2ax+a^2+a-3=0$ are less than 3, then
(A) $a<2$ **(B)** $2 \le a \le 3$
(C) $3<a\le 4$ **(D)** $a>4$

35. If p, q, r are positive and are in AP, the roots of the quadratic equation $px^2+qx+r=0$ are real for
(A) $\left|\dfrac{r}{p}-7\right|\ge 4\sqrt{3}$ **(B)** $\left|\dfrac{p}{r}-7\right|<4\sqrt{3}$
(C) All p and all r **(D)** No p and no r

36. If the roots of $x^2+bx+c=0$ are both real and greater than unity, then $(b+c+1)$
(A) May be less than zero **(B)** May be equal to zero
(C) Must be greater than zero **(D)** Must be less than zero

37. If a, b, c are the sides of ΔABC and equations $ax^2+bx+c=0$ and $5x^2+12x+13=0$ have a common root, then $\angle C$ is
(A) 60° **(B)** 90°
(C) 120° **(D)** 45°

38. If $f(x)=(x-a_1)^2+(x-a_2)^2+\cdots+(x-a_n)^2$, find x where $f(x)$ is minimum.
(A) ∞ **(B)** $\dfrac{a_1+a_2+\cdots+a_n}{n}$
(C) $\dfrac{a_1-a_2-\cdots-a_n}{n}$ **(D)** $-\infty$

39. The values of a for which both the roots of the equation $(1-a^2)x^2+2ax-1=0$ lie between 0 and 1 are given by
(A) $a>2$ **(B)** $1<a<2$
(C) $-\infty<a<\infty$ **(D)** None of these

40. The least value of $|a|$ for which $\sin\theta$ and $\operatorname{cosec}\theta$ are the roots of the equation $x^2+ax+b=0$ is
(A) 2 **(B)** 1
(C) 1/2 **(D)** 0

41. The sum of all the values of m for which the roots x_1 and x_2 of the quadratic equation $x^2-2mx+m=0$ satisfy the condition $x_1^3+x_2^3=x_1^2+x_2^2$ is
(A) $\dfrac{3}{4}$ **(B)** 1
(C) $\dfrac{9}{4}$ **(D)** $\dfrac{5}{4}$

42. If $x^2-x+a-3<0$ for at least one negative value of x, then complete set of values of a is
(A) $(-\infty, 4)$ **(B)** $(-\infty, 2)$
(C) $(-\infty, 3)$ **(D)** $(-\infty, 1)$

43. If the quadratic equations $3x^2+ax+1=0$ and $2x^2+bx+1=0$ have a common root then the value of $5ab-2a^2-3b^2$, where $a, b \in R$, is equal to
(A) 0 **(B)** 1
(C) -1 **(D)** None of these

44. If $x^2+5=2x-4\cos(a+bx)$, where $a, b \in (0, 5)$, is satisfied for at least one real x, then the maximum value of $a+b$ is equal to
(A) 3π **(B)** 2π
(C) π **(D)** None of these

45. The set of values of a for which 1 lies between the roots of equation $x^2-ax-a+3=0$ is
(A) $(-\infty, -6)$ **(B)** $(-\infty, -6]$
(C) $(-\infty, -6) \cup (2, \infty)$ **(D)** $(2, \infty)$

46. The quadratic equation $(2x-a)(2x-c)+\lambda(x-2b)(x-2d)=0$, (where $0<4a<4b<c<4d$) has
(A) A root between b and d for all λ
(B) A root between b and d for all $-$ve λ

(C) A root between b and d for all +ve λ
(D) None of these

47. If α and β are the roots of the equation $ax^2 + bx + c = 0$, then the quadratic equation whose roots are $\frac{\alpha}{1+\alpha}$ and $\frac{\beta}{1+\beta}$ is
(A) $ax^2 - b(1 - x) + c(1 - x)^2 = 0$
(B) $ax^2 - bx(x - 1) + c(x - 1)^2 = 0$
(C) $ax^2 + b(1 - x) + c(1 - x)^2 = 0$
(D) $ax^2 + b(x + 1) + c(1 + x)^2 = 0$

48. If $a, b, c \in R$ and $x^2 + (a + b)x + c = 0$ has no real roots, then
(A) $c(a+b+c) > 0$ (B) $c + c(a+b+c) > 0$
(C) $c + c(a+b-c) > 0$ (D) $c(a+b-c) > 0$

49. If the roots of the equation $x^3 - 12x^2 + 39x - 28 = 0$ are in AP, then their common difference will be
(A) ± 1 (B) ± 2
(C) ± 3 (D) ± 4

50. The values of a for which the equation $2x^2 - 2(2a+1)x + a(a-1) = 0$ has roots α and β where $\alpha < a < \beta$ are such that
(A) $a > -3$ (B) $a < 0$
(C) $a > 0$ or $a < -3$ (D) None of these

51. The number of real solutions of the equation $|x|^2 - 4|x| + 3 = 0$ is
(A) 4 (B) 2
(C) 1 (D) 3

52. If the roots of $x^2 - ax + b = 0$ differ by unity, then
(A) $b^2 = 1 + 4a$ (B) $a^2 = 1 + 4b$
(C) $b^2 + 4a = 1$ (D) $a^2 + 4b = 1$

53. The equation $\tan^4 x - 2\sec^2 x + a^2 = 0$ will have at least one solution if
(A) $|a| \le 4$ (B) $|a| \le 2$
(C) $|a| \le \sqrt{3}$ (D) None of these

54. If α, β are the roots of $2x^2 + 6x + b = 0$ and $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} < 2$, then b lies in the interval
(A) $(0, \infty)$ (B) $(-1, \infty)$
(C) $(-\infty, 0)$ (D) $(-\infty, 0]$

55. If $p, q \in \{1, 2, 3, 4\}$, the number of equations of the form $px^2 + qx + 1 = 0$ having real roots is
(A) 15 (B) 9
(C) 7 (D) 8

56. If r be the ratio of the roots of the equation $ax^2 + bx + c = 0$ and $r^2 + r + 1 = 0$, then a, b, c are in
(A) AP (B) GP
(C) HP (D) None of these

57. If $x^2 - 2x + c = 0$ and $x^2 - ax + b = 0$ have a root in common and the second equation has equal roots, then $b + c$ is equal to
(A) a (B) $2a$
(C) $3a$ (D) None of these

58. If x be real and the roots of the equation $ax^2 + bx + c = 0$ are imaginary, then the expression $a^2x^2 + abx + ac$ is always
(A) Positive (B) Negative
(C) Non-positive (D) Non-negative

59. If 1 lies between the roots of the equation $ax^2 + bx - \sin\theta = 0$, $a > 0$, then $a + b$ is always less than
(A) -1 (B) 1
(C) $-\frac{3}{2}$ (D) None of these

60. If 2, 3 are roots of $2x^3 + mx^2 - 13x + n = 0$, then the values of m and n
(A) $-5, -30$ (B) $-8, 80$
(C) $5, 30$ (D) $-5, 30$

Practice Exercise 2

Single/Multiple Correct Choice Type Questions

1. Solve $\sqrt{x+3-4\sqrt{x-1}} + \sqrt{x+8-6\sqrt{x-1}} = 1$ for x.
(A) $\{5,10\}$ (B) $[1,\infty)$
(C) $[5,10]$ (D) None of these

2. If equation $x^3 - x + a^2 - a + \frac{2}{3\sqrt{3}} < 0$ is true for at least one positive x, then a belongs to
(A) $(0, 1)$ (B) $(1, \infty)$
(C) $(2, 3\sqrt{3})$ (D) None of these

3. Let α be k times repeated root of the equation
$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0 = 0$$
where $a_i \ne 0$; $0 \le i \le n$. If α satisfies the equation $n^2 a_n \alpha^{n-1} + (n-1)^2 a_{n-1}\alpha^{n-2} + \cdots + 4a_2\alpha + a_1 = 0$, then the least value of k is __________.
(A) 2 (B) 3
(C) 4 (D) 5

4. If the equation $x^3 - 3ax^2 + 3bx - c = 0$ has positive and distinct roots, then
(A) $a^2 > b$ (B) $ab > c$
(C) $a^3 > c$ (D) $a^3 > b^2 > c$

5. $x^4 - 4x - 1 = 0$ has
(A) at most one positive real root
(B) at most one negative real root
(C) at most two real roots
(D) none of these

6. If α and β are the roots of the equation $2x^2 - 3x - 6 = 0$, then equation whose roots are $\alpha^2 + 2, \beta^2 + 2$ is
(A) $4x^2 + 49x + 118 = 0$ (B) $4x^2 - 49x + 118 = 0$
(C) $4x^2 - 49x - 118 = 0$ (D) $x^2 - 49x + 118 = 0$

7. Given that $\tan A$ and $\tan B$ are the roots of $x^2 - px + q = 0$. Then the value of $\sin^2(A + B)$ is
(A) $\frac{p^2}{p^2 + (1-q)^2}$ (B) $\frac{p^2}{p^2 + q^2}$
(C) $\frac{q^2}{p^2 + (1-q)^2}$ (D) $\frac{p^2}{(p+q)^2}$

8. If $0 < c < b < a$ and the roots α, β of the equation $cx^2 + bx + a = 0$ are imaginary, then
(A) $\frac{|\alpha| + |\beta|}{2} = |\alpha|\,|\beta|$ (B) $\frac{1}{|\alpha|} = \frac{1}{|\beta|}$

(C) $\frac{1}{|\alpha|}+\frac{1}{|\beta|}<2$ **(D)** None of these

9. If $x^2 + ax + b$ is an integer for every integer x then
(A) a is always an integer but b need not be an integer.
(B) b is always an integer but a need not be an integer.
(C) $a + b$ is always an integer.
(D) a and b are always integers.

10. If $y = \log_{7-a}(2x^2 + 2x + a + 3)$ is defined $\forall\, x \in R$, then possible integral value(s) of a is (are)
(A) -3 **(B)** -2
(C) 4 **(D)** 5

11. Two integers m, n are such that $m = n^2 - n$
Statement 1: Then $m^2 - 2m$ is divisible by 24.
because
Statement 2: Product of r successive integers is divisible by r!
(A) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1.
(B) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1.
(C) Statement-1 is True, Statement-2 is False.
(D) Statement-1 is False, Statement-2 is True.

12. If a, b and c are the rational numbers ($a > b > c > 0$) and the quadratic equation $(a + b - 2c)x^2 + (b + c - 2a)x + (c + a - 2b) = 0$ has a root in the interval $(-1, 0)$, then
(A) $c + a < 2b$.
(B) Both roots are rational.
(C) The equation $ax^2 + 2bx + c = 0$ has both negative real roots.
(D) The equation $cx^2 + 2ax + b = 0$ has both negative real roots.

Comprehension Type Questions

Paragraph for Questions 13 to 15: Let m, n be the roots of the equation $x^2 + qx + r = 0$ and let s, t be the roots of the equation $x^2 + bx + c = 0$.

13. If $\frac{m}{n}=\frac{s}{t}$, then
(A) $r^2c = qb^2$ **(B)** $r^2b = qc^2$
(C) $rb^2 = cq^2$ **(D)** $rc^2 = bq^2$

14. If $mn = st$, then $q^2 - b^2$ is equal to
(A) $[(m + t) + (n + s)][(m + s) - (n + t)]$
(B) $[(m + t) + (n + s)][(m + s) + (n + t)]$
(C) $[(m + t) - (n + s)][(m + s) + (n + t)]$
(D) $[(m + t) - (n + s)][(m + s) - (n + t)]$

15. If $m = s$ and $rq = bc$, then n and t are the roots of the equation
(A) $x^2 - (b + q)x + bq = 0$ **(B)** $x^2 - (b + r)x + rb = 0$
(C) $x^2 - (c + q)x + cq = 0$ **(D)** $x^2 - (c + r)x + rc = 0$

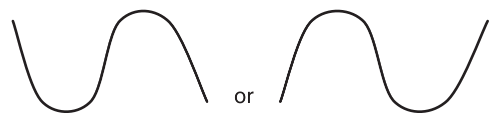
Paragraph for Questions 16 to 18: Let the equation $ax^2 + 2bx + c = 0$ and $a_1x^2 + 2b_1x + c_1 = 0$ have a common root and ($a, b, c, a_1, b_1, c_1 \in R$). If $\frac{a}{a_1},\frac{b}{b_1},\frac{c}{c_1}$ are in AP, then

16. a_1, b_1, c_1 are in
(A) AP **(B)** GP
(C) HP **(D)** None of these

17. The value of the common root is
(A) $-\frac{c_1}{a_1}$ **(B)** $-\frac{b_1}{a_1}$
(C) $-\frac{c_1}{b_1}$ **(D)** $-\frac{b_1}{c_1}$

18. If $\frac{b}{b_1},\frac{a}{a_1},\frac{c}{c_1}$ are in HP, then
(A) $a, -4b, -2c$ are in GP **(B)** $2a, -4b, -c$ are in GP
(C) $a, c, -b$ are in GP **(D)** None of these

Paragraph for Questions 19 to 21: While drawing the graph of $y = f(x)$, $f(x)$ is polynomial function, coefficient of highest degree term dominates, for example, if $f(x) = ax^2 + bx + c$ then if a is +ve, its graph will be upward parabola, otherwise it will be downward. In the same way if $f(x) = ax^3 + bx^2 + cx + d$, the graph of $f(x)$ will be either



as a is –ve or +ve, respectively. The point where $f(x)$ cuts the x-axis is called its roots.

The maximum number of +ve real roots which $f(x) = 0$ can have, are equal to the change of sign in the expression $f(x)$. The maximum number of –ve real roots which $f(x) = 0$ can have, are equal to the change of sign in the expression $f(-x)$. For example, $f(x) = x^3 - x + 1$ have two change of sign this implies $f(x) = 0$ can have maximum of two positive real roots.

19. If both the critical points of $f(x) = ax^3 + bx^2 + cx + d$ are –ve, then
(A) $ac < 0$ **(B)** $bc > 0$
(C) $ab < 0$ **(D)** None of these

20. The least number of imaginary roots of $f(x) = ax^6 + bx^4 - cx^2 + dx - c$, where a, b, c and d are same as in Question 19.
(A) Four **(B)** Two
(C) Six **(D)** 0

21. If critical points are c_1 and c_2 and $f(c_1)\, f(c_2) < 0$, then $f(x) = 0$ have (here $f(x)$ is same as in Question 19)
(A) Only one real root which is positive
(B) Three real roots in which at least two are negative
(C) Only one real root which is negative
(D) None of these

Answer Key

Practice Exercise 1

1. (A)	**2.** (D)	**3.** (B)	**4.** (B)	**5.** (A)	**6.** (D)
7. (A)	**8.** (A)	**9.** (A)	**10.** (B)	**11.** (B)	**12.** (D)
13. (A)	**14.** (D)	**15.** (A)	**16.** (C)	**17.** (A)	**18.** (B)
19. (D)	**20.** (A)	**21.** (C)	**22.** (A)	**23.** (D)	**24.** (B)

25. (B)	**26.** (A)	**27.** (C)	**28.** (B)	**29.** (A)	**30.** (D)
31. (C)	**32.** (D)	**33.** (C) and (D)	**34.** (A)	**35.** (A)	**36.** (C)
37. (B)	**38.** (B)	**39.** (A)	**40.** (A)	**41.** (D)	**42.** (D)
43. (B)	**44.** (A)	**45.** (D)	**46.** (A)	**47.** (C)	**48.** (C)
49. (C)	**50.** (C)	**51.** (A)	**52.** (B)	**53.** (C)	**54.** (C)
55. (C)	**56.** (B)	**57.** (A)	**58.** (A)	**59.** (B)	**60.** (D)

Practice Exercise 2

1. (C)	**2.** (A)	**3.** (B)	**4.** (A)	**5.** (C)	**6.** (B)
7. (A)	**8.** (A), (B), (C)	**9.** (C), (D)	**10.** (B), (C), (D)	**11.** (A)	**12.** (A), (B), (C), (D)
13. (C)	**14.** (D)	**15.** (A)	**16.** (B)	**17.** (C)	**18.** (A)
19. (B)	**20.** (B)	**21.** (B)			

Solutions

Practice Exercise 1

1. $f(x) = ax^2 - bx + 1$

$f(0) = 1$ (positive)

That is,

$f(x) = ax^2 - bx + 1 > 0$ ($\because ax^2 - bx + 1 = 0$ has imaginary roots)

$f(-1) = a + b + 1 > 0$

2. $\sum_{k=1}^{n} (x-k)^2 = 0$

Therefore, number of real root is zero.

3. $f(x) = x^3 - 3x + a$

$f'(x) = 3x^2 - 3$

$\Rightarrow f'(x) = 0$ at $x = \pm 1$

Coefficient of x^3 is positive.

$f(x)$ has three distinct real roots.

That is, $f(-1) > 0$ and $f(1) < 0$. (See Fig. 6.37.) So we have

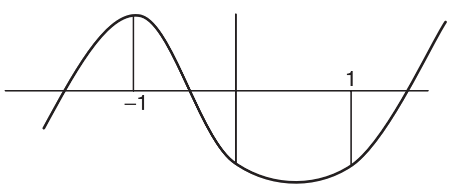


Figure 6.37

$\Rightarrow -1 + 3 + a > 0$ and $1 - 3 + a < 0$

$\Rightarrow a > -2$ and $a < 2$

$\Rightarrow a \in (-2, 2)$

4. $2x^2 - 4x + 2\sin\theta - 1 = 0$

Coefficient of x^2 is positive.

$f(0) < 0 \Rightarrow 2\sin\theta - 1 < 0 \Rightarrow \sin\theta < \frac{1}{2}$

$\Rightarrow \theta \in \left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, \pi\right)$

5. $(x^2 - 4x + 3) + \lambda(x^2 - 6x + 8) = 0$

$\Rightarrow x^2(1+\lambda) - 2x(2+3\lambda) + (3+8\lambda) = 0$

Discriminant,

$D = 4(2+3\lambda)^2 - 4(1+\lambda)(3+8\lambda)$

$\Rightarrow D = 4(\lambda^2 + \lambda + 1)$

If $\lambda \in R$, then $D > 0$. So root of given quadratic is always real.

6. $(k-2)x^2 + 8x + (k+4) > 0 \quad \forall x \in R$

$D = 64 - 4(k-2)(k+4) < 0$ and $k - 2 > 0$

$\Rightarrow (k+56)(k-4) > 0 \quad k > 2$

$\Rightarrow k < -6$ or $k > 4$

$\Rightarrow k > 4$

$\Rightarrow k = 5$

7. $4^x - (a-3)2^x + (a-4) = 0, x \leq 0$

Let $y = 2^x$. Then

$y^2 - (a-3)y + (a-4) = 0$

The roots of quadratic equation must lie between (0, 1].

(a) $(a-3)^2 - 4(a-4) \geq 0$

$\Rightarrow a^2 + 9 - 6a - 4a + 16 \geq 0$

$\Rightarrow a^2 - 10a + 25 \geq 0$

$\Rightarrow a \in R$

(b) $0 < \frac{a-3}{2} \leq 1$

$\Rightarrow 0 < a - 3 \leq 2$

$\Rightarrow 0 < a \leq 5$

(c) $f(0) = a - 4 > 0$ or $a > 4$

(d) $f(1) = 1 - a + 3 + a - 4 \geq 0 \qquad a \in (4, 5]$

8. $x^2 + 2ax + b \geq c \quad \forall x \in R$

$\Rightarrow x^2 + 2ax + b - c \geq 0$

$D = 4a^2 - 4(b-c) \leq 0 \Rightarrow a^2 \leq b - c$

9. $5x^2 - 10x + \log_{1/5} a = 0$ has real roots.

$\Rightarrow 100 - 20\log_{1/5} a \geq 0$

$\Rightarrow \log_{1/5} a \leq 5 = \log_{1/5}\left(\frac{1}{5}\right)^5$

$\Rightarrow a > 0$ and $a \geq \frac{1}{5^5}$

Therefore, minimum value of $a = \frac{1}{5^5}$

10. $x^2 + 2(a+1)x + a^2 - 6a + 8 = 0$

Root are a positive sign if $f(0) < 0$. (See Fig. 6.38.)

$a^2 - 6a + 8 < 0$

$\Rightarrow (a-4)(a-2) < 0$

$\Rightarrow a \in (2, 4)$

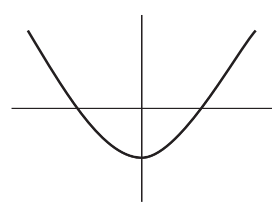


Figure 6.38

11. $a^2x^2+(2a-1)x+1\geq 0 \quad \forall x\in R$

$$\Rightarrow (2a-1)^2-4a^2\leq 0$$
$$\Rightarrow 4a^2+1-4a-4a^2\leq 0$$
$$\Rightarrow 4a\geq 1$$
$$\Rightarrow a\geq \frac{1}{4}$$

Therefore, the minimum value of a is $\frac{1}{4}$.

12. $(a^2+4a+3)x^2+(a^2-a-2)x+(a+1)a=0$

$$\Rightarrow (a+1)(a+3)x^2+(a+1)(a-2)x+(a+1)a=0$$
$$\Rightarrow (a+1)[(a+3)x^2+(a-2)x+a]=0$$

If $a=-1$, then quadratic equation has more than two roots.

13. See Fig. 6.39.

$x^2+(a+2)x-(a+3)=0$

As $f(2)<0$, we have

$\Rightarrow 4+2(a+2)-(a+3)<0$

$\Rightarrow a+5<0$

$\Rightarrow a<-5$

$\Rightarrow a\in(-\infty,-5)$

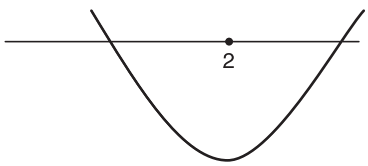


Figure 6.39

14. $f(x)=ax^2+bx+6=0$

$f(0)=6$ is positive. Therefore,

$$f(x)=ax^2+bx+6\geq 0 \quad \forall x\in R$$
$$\Rightarrow f(3)=9a+3b+6\geq 0$$
$$\Rightarrow 3(3a+b)\geq -6$$
$$\Rightarrow 3a+b\geq -2$$

15. $\dfrac{(x^2-1)(x+2)(x+1)^2}{(x-2)}<0$

$$\Rightarrow \frac{(x^2-1)(x^2-4)(x+1)^2}{(x-2)^2}<0$$
$$\Rightarrow (x^2-1)(x^2-4)<0$$
$$\Rightarrow 1<x^2<4$$
$$\Rightarrow -2<x<-1 \text{ or } 1<x<2$$

16. $x=7^{1/3}+7^{2/3}$

$$\Rightarrow x^3=7+49+3\cdot 7(7^{1/3}+2^{2/3})$$
$$\Rightarrow x^3=56+21x$$
$$\Rightarrow x^3-21x-56=0$$

Therefore, product of roots = 56.

17.
$$f(x)=ax^2-bx+c=0$$
$$f(0)=c>0$$
$$f(2)=4a-2b+c<0$$

So, the equation has a root which lies in the range (0, 2).

18. $x^2-4x+\log_{1/2}a=0$

$$\Rightarrow 16-4\log_{1/2}a\leq 0$$
$$\Rightarrow 4\leq \log_{1/2}a$$
$$\Rightarrow \log_{1/2}a\geq \log_{1/2}\left(\frac{1}{2}\right)^4$$
$$\Rightarrow a\leq \frac{1}{16} \text{ and } a>0$$
$$\Rightarrow a\in\left(0,\frac{1}{16}\right]$$

Therefore, maximum value of $a=\frac{1}{16}$.

19. Let the common root be α. Then,

$$\alpha^2+a\alpha+b=0$$
$$\Rightarrow \alpha^2+a'\alpha+b'=0$$
$$\Rightarrow (a-a')\alpha+(b-b')=0$$
$$\Rightarrow \alpha=\frac{b'-b}{a-a'}$$

20. $x^2+px+q=0$ has roots p and q. Therefore, sum of roots

$$p+q=-p$$
$$\Rightarrow 2p+q=0 \quad (1)$$

and product of roots

$$pq=q$$
$$\Rightarrow q(p-1)=0 \quad (2)$$

Solving Eqs. (1) and (2), we get

$p=1, q=-2$ or $q=0, p=0$

$p=0, q=0$ are not available in question. So, only solution is $p=1$, $q=-2$.

21. a and b are roots of $x^2+ax+b=0$.

Sum of roots: $a+b=-a$, $2a+b=0$

Product of roots: $ab=b$, $b(a-1)=0$

$$=\begin{cases} a=1 \\ b=-2 \end{cases} \quad b\neq 0$$

Minimum value of x^2+ax+b at $x=\dfrac{-a}{2}=\dfrac{-1}{2}$

$$=\frac{1}{4}+\left(-\frac{1}{2}\right)-2=\frac{-9}{4}$$

22. $x^3+2x^2+5x+2\cos x=0$

$$\Rightarrow x^3+2x^2+5x=-2\cos x$$

Now,

$$y=x(x^2+2x+5)=-2\cos x$$
$$\Rightarrow y=x(x^2+2x+5)$$

If $x\in[0,2\pi]$, then number of solutions is zero.

23. $(\cos p-1)x^2+\cos p\cdot x+\sin p=0$

The discriminant is given by

$$D=\cos^2 p-4\sin p(\cos p-1)\geq 0$$

Since roots are real, so $\sin p$ must be positive. Hence

$$0\leq p\leq \pi$$

24. **Case I:** $x=3m \Rightarrow 9m^2+3m+1 \Rightarrow$ not divisible by 3

Case II: $x=3m+1 \Rightarrow 9m^2+9m+3 \Rightarrow$ divisible by 3

Case III: $x=3m+2 \Rightarrow 9m^2+15m+7 \Rightarrow$ not divisible by 3

If x^2+x+1 is divisible by 3, $x=3m+1$.

Therefore, remainder = 1

25. Given, $4a + 4b - 5c > 0$
Let $f(x) = ax^2 + 2bx - 5c$. Then
$$f(2) = 4a + 4b - 5c > 0$$
Since equation $f(x) = 0$ has imaginary roots, therefore $f(x)$ will have same sign as that of a for all $x \in R$. Since $f(2) > 0$.

26. Let $f(x) = ax^2 + (b-1)x + c$
Given system of equations is equivalent to
$$\left.\begin{aligned} f(x_1) &= x_2 - x_1 \\ f(x_2) &= x_3 - x_2 \\ f(x_3) &= x_1 - x_3 \end{aligned}\right\} \Rightarrow f(x_1) + f(x_2) + f(x_3) = 0$$
Therefore,
$$af(x_1) + af(x_2) + af(x_3) = 0 \qquad \text{(not possible)}$$
As $(b-1)^2 - 4ac < 0$. We have
$$af(x_1), af(x_2), af(x_3) > 0$$
Hence, given system of equations has no real root.

27. $\alpha + \alpha^2 = -\dfrac{p}{3}$ (1)
$\Rightarrow \alpha^3 = 1 \Rightarrow \alpha = 1, \omega, \omega^2$
From Eq. (1), $p = -3(\alpha + \alpha^2) = -6, 3, 3$

28. Let $f(x) = ax^2 + bx + 10$.
Since equation $f(x) = 0$ has no real and distinct roots, therefore, $f(x)$ will have same sign for all real x.
But $f(0) = 10 > 0$
Hence, $f(x) \geq 0\ \forall\ x \in R$. This given
$$f(5) \geq 0 \Rightarrow 5(5a + b) + 10 \geq 0$$
$$\Rightarrow 5a + b \geq -2$$
Minimum value of $5a + b = -2$
According to question,
$$5m + n = -2 \Rightarrow n = -5m - 2$$
Given family of lines is
$$m(4x + 2y + 3) - (5m + 2)(x - y - 1) = 0$$
$$\Rightarrow 2(x - y - 1) + m(-x + 7y + 8) = 0$$
Clearly, this family of lines passes through the fixed point $\left(-\dfrac{1}{6}, -\dfrac{7}{6}\right)$.

29. Given $x^4 + 12x^3 + 46x^2 + ax + b = (x^2 + cx + d)^2$.
Comparing coefficient of x on both the sides, we get $12 = 2c$, $46 = 2d + c^2$, $a = 2cd$ and $b = d^2$.
Solving these, we get $c = 6$, $d = 5$, $a = 60$ and $b = 25$.

30. Since the equation $x^2 + 5bx + 8c = 0$ does not have two distinct real roots and coefficient of x^2 is positive, hence $x^2 + 5bx + 8c \geq 0$ $\Rightarrow 5b + 8c \geq -1$. Hence, the minimum value is -1.

31. Given equation is $-\left(x - \dfrac{1}{2}\right)^2 - \dfrac{3}{4} = \sin^4 x$.
LHS < 0 while RHS > 0, $\forall\ x$
Hence, the given equation has no solution.

32. Given equation is $x^8 - x^5 + x^2 - x + 2 = 0$. Clearly, given equation will have no negative root. Now given equation can be written as $x^5(x^3 - 1) + x(x - 1) + 2 = 0$.
Clearly, no value of x will satisfy the given equation.

33. Let $f(x) = x^2 + ax + b$
Clearly, $f(0) = b \Rightarrow b$ is an integer
Now, $f(1) = 1 + a + b \Rightarrow a$ is an integer.

34. Since roots are less than a real number, roots must be real
$$4a^2 - 4(a^2 + a - 3) \geq 0$$
$$\Rightarrow a \leq 3 \qquad (1)$$
Let $f(x) = x^2 - 2ax + a^2 + a - 3$. Since 3 lies outside the roots
$$f(3) > 0 \Rightarrow a < 2 \text{ or } a > 3 \qquad (2)$$
Sum of the roots must be less than 6
$$2a < 6 \Rightarrow a < 3 \qquad (3)$$
From Eqs. (1)–(3), we have
$$a < 2$$

35. Since p, q and r are in AP, we have $2q = p + r$.
The roots of $px^2 + qx + r = 0$ are real if
$$q^2 - 4pr \geq 0 \Rightarrow \left(\frac{p+r}{2}\right)^2 - 4pr \geq 0$$
$$\Rightarrow p^2 + r^2 - 14pr \geq 0 \Rightarrow \left(\frac{r}{p}\right)^2 - 14\left(\frac{r}{p}\right) + 1 \geq 0$$
Now,
$$\left(\frac{r}{p}\right)^2 - 14\left(\frac{r}{p}\right) + 1 = 0 \Rightarrow \left(\frac{r}{p}\right) = 7 \pm 4\sqrt{3}$$
So,
$$\frac{r}{p} - 7 \leq -4\sqrt{3} \text{ or } \frac{r}{p} - 7 \geq 4\sqrt{3} \Rightarrow \left|\frac{r}{p} - 7\right| \geq 4\sqrt{3}.$$

36. Let α and β be the roots of $x^2 + bx + c = 0$.
$$\alpha > 1 \text{ and } \beta > 1 \Rightarrow (\alpha - 1)(\beta - 1) > 0$$
$$\Rightarrow \alpha\beta - (\alpha + \beta) + 1 > 0 \Rightarrow c + b + 1 > 0$$
Graphically, as $f(x) = x^2 + bx + c$ is an upward parabola with 1 lying outside (Fig. 6.40), hence, $f(1) > 0 \Rightarrow c + b + 1 > 0$.

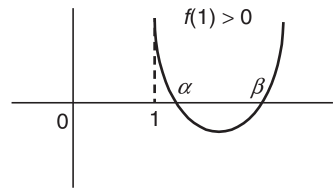


Figure 6.40

37. Since $5x^2 + 12x + 13 = 0$ has imaginary roots as $D = 144 - 4 \times 5 \times 13 < 0$. So, both roots of $ax^2 + bx + c = 0$ and $5x^2 + 12x + 13 = 0$ will be common.
Hence,
$$\frac{a}{5} = \frac{b}{12} = \frac{c}{13} \Rightarrow a^2 + b^2 = c^2 \Rightarrow \angle C = 90°$$

38. Minimum value of $f(x) = nx^2 - 2[a_1 + a_2 + \cdots + a_n]x + a_1^2 + a_2^2 + \cdots + a_n^2$ exists at $x = -\dfrac{B}{2A} = \dfrac{a_1 + a_2 + \cdots + a_n}{n}$.

39. Given equation is $(1 - a^2)x^2 + 2ax - 1 = 0$
Its discriminant is $D = 4$ and roots are $\dfrac{1}{a-1}, \dfrac{1}{a+1}$.
Given,
$$0 < \frac{1}{a-1} < 1,\ 0 < \frac{1}{a+1} < 1$$
Now,
$$\frac{1}{a-1} > 0 \Rightarrow a > 1$$
$$\frac{1}{a-1} < 1 \Rightarrow \frac{1}{a-1} - 1 < 0$$
$$\Rightarrow \frac{2-a}{a-1} < 0 \Rightarrow a < 1 \text{ or } a > 2$$
Hence, $\qquad a > 2 \qquad (1)$
$$\frac{1}{a+1} < 1 \Rightarrow a > -1$$
and
$$\frac{1}{a+1} < 1 \Rightarrow -\frac{a}{a+1} < 0$$

$$\Rightarrow a < -1 \quad \text{or } a > 0$$

Hence, $a > 0$. From Eqs. (1) and (2), $a > 2$. (2)

40. $\sin\theta + \text{cosec}\,\theta = -a$

$$\Rightarrow |a| = |\sin\theta + \text{cosec}\,\theta|$$

$$\Rightarrow \frac{\sin^2\theta + 1}{|\sin\theta|} = \frac{|\sin\theta|^2 + 1}{|\sin\theta|}$$

$$= |\sin\theta| + \frac{1}{|\sin\theta|} \geq 2$$

Hence, least value of $|a| = 2$.

41. Given

$$x_1 + x_2 = 2m, \quad x_1x_2 = m$$

According to given condition

$$x_1^3 + x_2^3 = x_1^2 + x_2^2$$

$$\Rightarrow (x_1 + x_2)(x_1^2 + x_2^2 - x_1x_2) = x_1^2 + x_2^2$$

$$\Rightarrow (x_1 + x_2)[(x_1 + x_2)^2 - 3x_1x_2] = (x_1 + x_2)^2 - 2x_1x_2$$

$$\Rightarrow 2m(4m^2 - 3m) = 4m^2 - 2m$$

Clearly sum is $\frac{5}{4}$.

42. The equation $x^2 - x + a - 3 = 0$ must have at least one negative root.

For real roots, $D \geq 0$

$$\Rightarrow 1 - 4(a-3) \geq 0$$

$$\Rightarrow a \leq \frac{13}{4}$$

Both roots will be non-negative if

$$D \geq 0,\ a - 3 \geq 0,\ 1 \geq 0$$

$$\Rightarrow a \leq \frac{13}{4},\ a \geq 3$$

$$\Rightarrow a \in \left[3, \frac{13}{4}\right]$$

Thus equation will at least one negative root if

$$a \in \left(-\infty, \frac{13}{4}\right] \sim \left[3, \frac{13}{4}\right] \Rightarrow a \in (-\infty, 3)$$

43. $3x^2 + ax + 1 = 0$, $2x^2 + bx + 1 = 0$ have a common root.

Subtracting these equations, we get

$$x^2 + (a - b)x = 0$$

$$\Rightarrow x = 0,\ x = (b - a)$$

Clearly the common root is $(b - a)$, So

$$3(b-a)^2 + a(b-a) + 1 = 0$$

$$\Rightarrow 3b^2 + 3a^2 - 6ab + ab - a^2 + 1 = 0$$

$$\Rightarrow 5ab - 2a^2 - 3b^2 = 1$$

44.

$$x^2 + 5 = 2x - 4\cos(a + bx)$$

$$\Rightarrow x^2 - 2x + 1 + 4 = -4\cos(a + bx)$$

$$\Rightarrow (x-1)^2 + 4[1 + \cos(a + bx)] = 0$$

$$\Rightarrow x = 1 \text{ and } 1 + \cos(a + bx) = 0$$

$$\Rightarrow \cos(a + b) = -1$$

$$\Rightarrow a + b = \pi, 3\pi, \ldots$$

45. Let $f(x) = x^2 - ax - (a - 3) = 0$.

Let α and β be the roots of equation $f(x) = 0$.

Since 1 lies between α and β, hence

$$f(1) < 0 \Rightarrow 1 - a - a + 3 < 0 \Rightarrow a > 2$$

46. Let $f(x) = (2x - a)(2x - c) + \lambda(x - 2b)(x - 2d)$.

Clearly $f(b)\cdot f(d) < 0\ \forall\ \lambda$.

Hence, a root of given equation will lie between b and d.

47. Since roots of $ax^2 + bx + c = 0$ are α and β, hence, roots of

$$cx^2 + bx + a = 0$$

will be $\frac{1}{\alpha}$ and $\frac{1}{\beta}$. Now replacing $x \to x - 1$, then roots of

$$c(x-1)^2 + b(x-1) + a = 0$$

will be $1 + \frac{1}{\alpha}$ and $1 + \frac{1}{\beta}$. Now replacing $x \to \frac{1}{x}$ we will get

$$c(1-x)^2 + b(1-x) + ax^2 = 0$$

whose roots are $\frac{\alpha}{1+\alpha}$ and $\frac{\beta}{1+\beta}$.

48. Since the roots of $ax^2 + bx + c = 0$ are non-real, thus $f(x) = ax^2 + bx + c$ will have same sign for every value of x.

$$f(0) = c$$

$$f(1) = a + b + c$$

$$f(-1) = a - b + c$$

$$f(2) = 4a - 2b + c$$

$$\Rightarrow c\cdot(a + b + c) > 0,\ c(a - b + c) > 0$$

$$\Rightarrow c(4a - 2b + c) > 0$$

49. Let roots be $a - d, a, a + d$ which are in AP. The common difference is d.

$$x^3 - 12x^2 + 39x - 28 = 0$$

Adding the three roots we have

$$a - d + a + a + d = 12$$

$$\Rightarrow 3a = 12 \Rightarrow a = 4$$

Product of the three roots is

$$a(a^2 - d^2) = 28 \Rightarrow d^2 = 9 \Rightarrow d = \pm 3$$

50. $2x^2 - 2(2a+1)x + a(a-1) = 0$

If $f(a) < 0$,

$$2a^2 - 2a(2a+1) + a(a-1) < 0$$

$$\Rightarrow a^2 + 3a > 0$$

$$\Rightarrow a < -3 \text{ or } a > 0$$

51.

$$|x|^2 - 4|x| + 3 = 0$$

$$\Rightarrow (|x| - 3)(|x| - 1) = 0$$

$$\Rightarrow |x| = 1 \text{ or } |x| = 3$$

$$\Rightarrow x = \pm 1 \text{ or } x = \pm 3a$$

So, the number of solutions is 4.

52. $x^2 - ax + b = 0$

Let α, β be the roots of quadratic. So

$$\alpha + \beta = a \text{ and } \alpha\beta = b \quad (1)$$

Now,

$$|\alpha - \beta| = 1$$

$$\sqrt{(\alpha - \beta)^2} = 1$$

[Using $(a - b)2 = (a + b)^2 - 4ab$]

$$\Rightarrow \sqrt{(\alpha + \beta)^2 - 4\alpha\beta} = 1 \quad (2)$$

From, Eqs. (1) and (2), we have

$$\sqrt{a^2 - 4b} = 1$$

$$\Rightarrow a^2 = 1 + 4b$$

53. $$\tan^4 x - 2\sec^2 x + a^2 = 0$$
$$\Rightarrow \tan^4 x - 2 - 2\tan^2 x + a^2 = 0$$
$$\Rightarrow (\tan^2 x)^2 - 2\tan^2 x + 1 = 3 - a^2$$
$$\Rightarrow (\tan^2 x - 1)^2 = 3 - a^2$$
$$\Rightarrow 3 - a^2 \geq 0$$
$$\Rightarrow a^2 \leq 3$$
$$\Rightarrow |a| \leq \sqrt{3}$$

54. α and β are roots of $2x^2 + 6x + b = 0$. Therefore
$$\alpha + \beta = -3 \quad (1)$$
$$\alpha\beta = \frac{b}{2} \quad (2)$$
Now,
$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} < 2$$
$$\Rightarrow \frac{\alpha^2 + \beta^2}{\alpha\beta} < 2$$
$$\Rightarrow \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} < 2$$
$$\Rightarrow \frac{(\alpha+\beta)^2}{\alpha\beta} < 4 \quad (3)$$
From Eqs. (1), (2) and (3), we have
$$\frac{a}{(b/2)} < 4$$
$$\Rightarrow \frac{1}{b} < \frac{2}{a}$$
$$\Rightarrow b \in (-\infty,) \cup \left(0, \frac{a}{2}\right)$$

55. $px^2 + qx + 1 = 0$
$$D = q^2 - 4p$$
We have to check for which pair of (h, k) discriminant is greater or equal to zero.
$q = 1, p =$ No value
$q = 2, p = 1$
$q = 3, p = 1, 2$
$q = 4, p = 1, 2, 3, 4$
So the number of pairs is equal to 7.

56. Let α, β be roots of the equation $ax^2 + bx + c = 0$.
$$r = \frac{\alpha}{\beta} \text{ (say)}$$
$$\frac{\alpha^2}{\beta^2} + \frac{\alpha}{\beta} + 1 = 0$$
$$\Rightarrow \alpha^2 + \beta^2 + \alpha\beta = 0$$
$$\Rightarrow (\alpha+\beta)^2 - 2\alpha\beta + \alpha\beta = 0$$
$$\Rightarrow (\alpha+\beta)^2 = \alpha\beta$$
$$\Rightarrow b^2 = ac$$
Therefore, a, b, c are in GP.

57. $x^2 - ax + b = 0$ has equal roots, that is, discriminant is equal to zero.
$$a^2 = 4b \quad (1)$$
$x^2 - 2x + c = 0$ and $x^2 - ax + b = 0$ have a common root and second equation have identical root.
$$(b - c)^2 = (ac - 2b)(2 - a)$$
$$\Rightarrow c = \frac{a}{2} \quad \text{[condition for 1 common root]}$$
Now,
$$b = c = \frac{a}{2}$$
So,
$$b + c = \frac{a}{2} + \frac{a}{2} = a$$

58. $ax^2 + bx + c = 0$ has imaginary roots. That is, $b^2 - 4ac < 0$
The discriminate of the equation $f(x) = a^2x^2 + abx + ac$ is
$$D = a^2b^2 - 4a^2ac = a^2(b^2 - 4ac) < 0$$
The coefficient of x^2 in quadratic equation is $(a^2 > 0)$ positive.
That is, $a^2x^2 + abx + ac > 0$.

59. $ax^2 + bx - \sin\theta = 0,\ a > 0$
$f(1) < 0$
$\Rightarrow a + b - \sin\theta < 0$
$\Rightarrow a + b < \sin\theta \leq 1$
$\Rightarrow a + b < 1$

60. $2x^3 + mx^2 - 13x + n = 0$ has three roots. We have two roots; let the third root be α. Product of the three root is
$$2\cdot 3 + 2\cdot\alpha + 3\cdot\alpha = \frac{-13}{2}$$
$$\Rightarrow 5\alpha + 6 = \frac{-13}{2}$$
$$\Rightarrow 5\alpha = \frac{-13}{2} - 6 = \frac{-25}{2}$$
$$\Rightarrow \alpha = \frac{-5}{2}$$
This gives the third root as $\frac{-5}{2}$.
Therefore,
$$2 + 3\frac{-5}{2} = \frac{-m}{2}$$
$$\Rightarrow \frac{5}{2} = \frac{-m}{2}$$
$$\Rightarrow m = -5$$
Also,
$$2\cdot 3\cdot\left(\frac{-5}{2}\right) = \frac{-n}{2}$$
$$\Rightarrow n = 30$$

Practice Exercise 2

1. Putting $x - 1 = t^2$ in the given equation, we get
$$\sqrt{t^2 + 1 + 3 - 4t} + \sqrt{t^2 + 1 + 8 - 6t} = 1$$
$\Rightarrow |t - 2| + |t - 3| = 1$
$\Rightarrow t \in [2, 3]$
$\Rightarrow x \in [5, 10]$

2. $x^3 - x < -a^2 + a - \dfrac{2}{3\sqrt{3}}$
$f(x) = x^3 - x$
$f'(x) = 3x^2 - 1 = 0$
$$\Rightarrow x = \pm\frac{1}{\sqrt{3}}$$

In positive region, minimum value of $f(x) = \frac{1}{3\sqrt{3}} - \frac{1}{\sqrt{3}} = -\frac{2}{3\sqrt{3}}$.

So,

$$-a^2 + a - \frac{2}{3\sqrt{3}} > -\frac{2}{3\sqrt{3}}$$
$$\Rightarrow a^2 - a < 0$$
$$\Rightarrow a \in (0, 1)$$

3. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0$
Differentiating w.r.t. x twice, we get

$$xf'(x) = na_nx^n + (n-1)a_{n-1}x^{n-1} + \cdots + a_1x$$

and

$$xf''(x) + f'(x) = n^2a_nx^{n-1} + (n-1)^2a_{n-1}x^{n-2} + \cdots + a_1$$
$$\Rightarrow \alpha f''(\alpha) + f'(\alpha) = 0$$
$$\Rightarrow f(\alpha) = 0; f'(\alpha) = 0; f''(\alpha) = 0$$

Hence, minimum value of k is 3.

4. Let α, β, γ be the roots. Then $\alpha + \beta + \gamma = 3a$, $\alpha\beta + \beta\gamma + \gamma\alpha = 3b$, $\alpha\beta\gamma = c$.
We have $(\alpha + \beta + \gamma)^3 > 27\,\alpha\beta\gamma \Rightarrow a^3 > c$
Also,

$$\left(\frac{\alpha+\beta+\gamma}{3}\right)\left(\frac{\alpha\beta+\beta\gamma+\gamma\alpha}{3}\right) > (\alpha\beta\gamma)^{1/3}(\alpha^2\beta^2\gamma^2)^{1/3} = \alpha\beta\gamma$$

$\Rightarrow ab > c$
Again

$$\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha > 0$$
$$\Rightarrow (\alpha + \beta + \gamma)^2 - 3(\alpha\beta + \beta\gamma + \gamma\alpha) > 0 \Rightarrow a^2 > b$$

5. Let $f(x) = x^4 - 4x - 1$
+ − −
Therefore, at most one positive real root.

$$f(-x) = x^4 + 4x - 1$$
+ + −

Therefore, at most one negative real root.
Hence, at most two real roots.

6. $\alpha + \beta = 3/2$, $\alpha\beta = -6/2 = -3$
$S = \alpha^2 + \beta^2 + 4 = (\alpha + \beta)^2 - 2\alpha\beta + 4 = \frac{49}{4}$
$P = \alpha^2\beta^2 + 2(\alpha^2 + \beta^2) + 4 = \alpha^2\beta^2 + 4 + 2[(\alpha + \beta)^2 - 2\alpha\beta] = \frac{118}{4}$
Therefore, the equation is

$$x^2 - \left(\frac{49}{4}\right)x + \frac{118}{4} = 0 \Rightarrow 4x^2 - 49x + 118 = 0$$

7. We have

$$\tan A + \tan B = p \text{ and } \tan A \tan B = q$$

Therefore,

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{p}{1-q}$$

Now,

$$\sin^2(A+B) = \frac{1}{2}[1 - \cos 2(A+B)] = \frac{1}{2}\left[1 - \frac{1-\tan^2(A+B)}{1+\tan^2(A+B)}\right]$$
$$= \frac{\tan^2(A+B)}{1+\tan^2(A+B)} = \frac{p^2}{p^2 + (1-q)^2}$$

8. Since roots are imaginary, so, discriminant < 0.

$$\alpha = \frac{-b + i\sqrt{4ac - b^2}}{2c}$$
$$\beta = \frac{-b - i\sqrt{4ac - b^2}}{2c}$$
$$|\alpha| = |\beta| = \sqrt{\frac{b^2}{4c^2} + \frac{4ac - b^2}{4c^2}} = \sqrt{\frac{a}{c}} > 1$$

9. Let $f(x) = x^2 + ax + b$
Clearly, $f(0) = b \Rightarrow b$ is an integer.
Now, $f(1) = 1 + a + b \Rightarrow a$ is an integer.

10. $2x^2 + 2x + a + 37 > 0$, $\forall x \in R$
As $D < 0$, we have

$$4 - 4(a+3)4 < 0$$
$$\Rightarrow 1 - 4(a+3) < 0$$
$$\Rightarrow 1 < 4a + 12$$
$$\Rightarrow -11 < 4a, a > \frac{-11}{4}$$
$$\Rightarrow 7 - a > 0, a < 7, \quad 7 - a \neq 1, 6 \neq a$$

11. $m^2 - 2m = (n^2 - n)^2 - 2(n^2 - n)$
$= (n-1)n(n+1)(n-2)$

12.

$$f(-1)\,f(0) < 0$$
$$\Rightarrow (2a - b - c)(c + a - 2b) < 0$$
$$\Rightarrow (a - b + a - c)(c + a - 2b) < 0$$
$$\Rightarrow c + a < 2b$$

One root is 1 and other is $\frac{c + a - 2b}{a + b - 2c}$
Therefore, both roots are rational.
Now, discriminant of $ax^2 + 2bx + c = 0$ is $4b^2 - 4ac$.
Using $(c + a) < 2b$, we have $D > 0$.
Also, a, b and c are +ve. Therefore, both the roots are real and −ve.

13. We have

$$\frac{m}{n} = \frac{s}{t}, \frac{m+n}{m-n} = \frac{s+t}{s-t}$$
$$\Rightarrow \frac{m+n}{\sqrt{(m+n)^2 - 4mn}} = \frac{s+t}{\sqrt{(s+t)^2 - 4st}}$$
$$\Rightarrow \frac{-q}{\sqrt{q^2 - 4r}} = \frac{-b}{\sqrt{b^2 - 4c}}$$
$$\Rightarrow q^2(b^2 - 4c) = b^2(q^2 - 4r)$$
$$\Rightarrow q^2c = b^2r$$

14. For $mn = st$,
$q^2 - b^2 = (m+n)^2 - (s+t) = (m-n)^2 - (s-t)^2$
$= [(m+t) - (n+s)][(m+s) - (n+t)]$

15. With $m = s$, $rq = bc$, $s + n = -q$, $sn = r$ and $s + t = -b$, $st = c$, we have

$$n - t = b - q \text{ and } \frac{n}{t} = \frac{r}{c}$$
$$\Rightarrow \frac{rt}{c} - t = b - q$$
$$\Rightarrow t(r - c) = cb - cq = q(r - c) \Rightarrow t = q \Rightarrow n = b$$

Hence, n and t are the roots of the equation $x^2 - (b + q)x + bq = 0$.

16.

$$a\alpha^2 + 2b\alpha + c = 0$$
$$a_1\alpha^2 + 2b_1\alpha + c_1 = 0 \Rightarrow \alpha = \frac{ac_1 - a_1c}{2(a_1b - ab_1)}$$

Since,

$$\frac{a}{a_1}, \frac{b}{b_1}, \frac{c}{c_1} \text{ are in AP.}$$

Hence, a_1, b_1, c_1 are in GP.

17. $$\frac{\frac{c}{c_1}-\frac{a}{a_1}}{\frac{b}{b_1}-\frac{a}{a_1}}=2$$

$$\Rightarrow \frac{a_1c-a_1c}{(a_1b-ab_1)}=-2\frac{c_1}{b_1}$$

$$\Rightarrow \alpha=-\frac{c_1}{b_1}$$

18. $$a_1\left(-\frac{c_1}{b_1}\right)^2+2b_1\left(-\frac{c_1}{b_1}\right)+c_1=0$$

$$\Rightarrow b_1^2=a_1c_1$$

$$\frac{a}{a_1}=\frac{2bc}{bc_1+b_1c}\Rightarrow\frac{ac_1-a_1c}{(a_1b-ab_1)}=\frac{c}{b}$$

$$\Rightarrow \alpha=\frac{c}{2b}$$

Therefore,

$$a\left(\frac{c}{2b}\right)^2+2b\cdot\frac{c}{2b}+c=0$$

$$\Rightarrow b^2=-\frac{1}{8}ac$$

Hence, $-\frac{a}{4}, b, \frac{c}{2}$ are in GP or $a, -4b, -2c$ are in GP.

19. $f'(x)=3ax^2+2bx+c$

As both the roots are negative, we have

$$-\frac{2b}{3a}<0 \text{ and } \frac{c}{3a}<0$$

$\Rightarrow$ a and b have same signs.

$\Rightarrow$ a and c have different signs.

$\Rightarrow bc<0$

20. **Case I:** a is positive

Case I(a): $d>0$

$f(x)$ have 3 signs of change and $f(-x)$ have 1 sign of change.

Case I(b): $d<0$

$f(x)$ have 1 sign of changes and $f(-x)$ have 3 signs of change.

Case II: $a<0$

Case II(a): $d>0$

$f(x)$ have 1 sign of changes and $f(-x)$ have 3 signs of change.

Case II(b): $d<0$

$f(x)$ have 3 sign of change and $f(-x)$ have 1 signs of change.

So,

$\Rightarrow$ Maximum number of real roots $=4$

$\Rightarrow$ Least number of imaginary root $=2$

21. There are only two cases (Fig. 6.41):

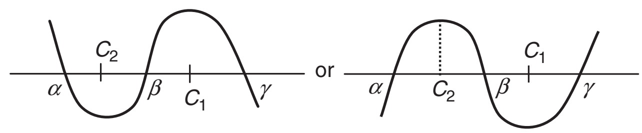


Figure 6.41

Here, C_1 and C_2 are negative. Hence, α and β are negative.

Solved JEE 2017 Questions

JEE Main 2017

1. Let $p(x)$ be a quadratic polynomial such that $p(0) = (1)$. If $p(x)$ leaves remainder 4 when divided by $x - 1$ and it leaves remainder 6 when divided by $x + 1$, then:

(A) $p(-2) = 11$ **(B)** $p(2) = 11$
(C) $p(2) = 19$ **(D)** $p(-2) = 19$

(ONLINE)

Solution: Let us consider

$$p(x) = Ax^2 + Bx + C \quad \text{(I)}$$

Substituting $x = 0$ in this equation, we get

$$p(0) = 0 + 0 + C$$
$$1 = C \quad (1)$$

Also, when $p(x)$ leaves remainder 4 when divided by $x - 1$, we get

$$x - 1 = 0 \Rightarrow x = 1$$

Therefore,

$$p(1) = A + B + C \Rightarrow 4 = A + B + C \quad (2)$$

Substituting $C = 1$ in Eq. (2), we get

$$4 = A + B + 1 \Rightarrow A + B = 3 \quad (3)$$

Also, when $p(x)$ leaves remainder 6 when divided by $x + 1$, we get

$$x + 1 = 0 \Rightarrow x = -1$$
$$\Rightarrow p(-1) = A - B + C \Rightarrow 6 = A - B + C \quad [C = 1 \text{ from Eq. (1)}]$$
$$6 = A - B + 1 \Rightarrow A - B = 5 \quad (4)$$

Solving Eqs. (3) and (4), we get

$$A + B = 3$$
$$A - B = 5$$
$$\text{Eq. (3)} + \text{Eq. (4)}: 2A = 8 \Rightarrow A = 4 \quad (5)$$
$$\text{Eq. (3)} - \text{Eq. (4)}: 2B = -2 \Rightarrow B = 1 \quad (6)$$

Substituting $C = 1$, $A = 4$, $B = -1$ from equation (1), (5) and (6) in Eq. (I), we get

$$p(x) = 4x^2 - x + 1$$
$$p(-2) = 4(-2)^2 - (-2) + 1$$
$$= 4 \times 4 + 2 + 1 = 16 + 2 + 1 \Rightarrow p(-2) = 19$$

Hence, the correct answer is option (D).

2. The sum of all the real values of x satisfying the equation $2^{(x+1)(x^2+5x-50)} = 1$ is

(A) -5 **(B)** 14
(C) -4 **(D)** 16

(ONLINE)

Solution: For $2^{(x-1)(x^2+5x-50)} = (1)^0$, we have

$$(x - 1)(x^2 + 5x - 50) = 0$$

Therefore,

$$x - 1 = 0, x^2 + 5x - 50 = 0$$
$$\Rightarrow x = 1, x^2 + 10x - 5x - 50 = 0$$
$$\Rightarrow x(x + 10) - 5(x + 10) = 0$$
$$\Rightarrow (x + 10)(x - 5) = 0 \Rightarrow x = -10, 5$$

Therefore, the sum of all the real values of x, we get

$$-10 + 5 + 1 = -4$$

Hence, the correct answer is option (C).

JEE Advanced 2017

Paragraph for Questions 1 and 2: Let p, q be integers and let α, β be the roots of the equation, $x^2 - x - 1 = 0$, where $\alpha \neq \beta$. For $n = 0, 1, 2, \ldots$, let $a_n = p\alpha^n + q\beta^n$.
FACT: If a and b are rational numbers and $a + b\sqrt{5} = 0$, then $a = 0 = b$.

1. $a_{12} =$ ____.

(A) $a_{11} - a_{10}$ **(B)** $a_{11} + a_{10}$
(C) $2a_{11} + a_{10}$ **(D)** $a_{11} + 2a_{10}$

Solution: It is given that $x^2 - x - 1 = 0$.
Also, α and β are roots of equation and $\alpha \neq \beta$.
Let p and q be integers and $p\alpha^n + q\beta^n = a_n$.
Since α and β are the roots of $x^2 = x + 1$, we get

$$\alpha^2 = \alpha + 1 \text{ and } \beta^2 = \beta + 1$$

Therefore,

$$a_{11} + a_{10} = p\alpha^{11} + q\beta^{11} + p\alpha^{10} + q\beta^{10}$$
$$= p\alpha^{11} + p\alpha^{10} + q\beta^{11} + q\beta^{10}$$
$$= p\alpha^{10}(\alpha + 1) + q\beta^{10}(\beta + 1)$$
$$= p\alpha^{10}\alpha^2 + q\beta^{10}\beta^2$$
$$= p\alpha^{12} + q\beta^{12} = a_{12}$$

That is, $a_{11} + a_{10} = a_{12}$.

Hence, the correct answer is option (B).

2. If $a_4 = 28$, then $p + 2q =$ ____.

(A) 21 **(B)** 14
(C) 7 **(D)** 12

Solution: It is given that $a_4 = 28$. Using $a_n = p\alpha^n + q\beta^n$, we get

$$a_n - a_{n-1} = p\alpha^n + q\beta^n - (p\alpha^{n-1} + q\beta^{n-1})$$
$$= p\alpha^n - p\alpha^{n-1} + q\beta^n - q\beta^{n-1}$$
$$= p\alpha^{n-2}(\alpha^2 - \alpha) + q\beta^{n-2}(\beta^2 - \beta)$$
$$= p\alpha^{n-2} + q\beta^{n-2}$$

Therefore,

$$a_n - a_{n-1} = a_{n-2}$$
$$\Rightarrow a_n = a_{n-1} + a_{n-2}$$
$$\Rightarrow a_4 = a_3 + a_2 = (a_2 + a_1) + (a_1 + a_0) = a_2 + 2a_1 + a_0$$
$$\Rightarrow a_4 = (a_1 + a_0) + 2a_1 + a_0 = 2a_0 + 3a_1$$
$$\Rightarrow a_4 = 2(p\alpha^\circ + q\beta^\circ) + 3(p\alpha^1 + q\beta^1)$$
$$\Rightarrow a_4 = 2p + 2q + 3(p\alpha + q\beta)$$

Now, from $x^2 - x - 1 = 0$, the roots of the equation are

$$x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$\Rightarrow \alpha = \frac{1+\sqrt{5}}{2} \text{ and } \beta = \frac{1-\sqrt{5}}{2}$$

$$\Rightarrow a_4 = 2p + 2q + 3\left(p\left(\frac{1+\sqrt{5}}{2}\right) + q\left(\frac{1-\sqrt{5}}{2}\right)\right)$$

$$= 2p + 2q + \frac{3}{2}p + \frac{3}{2}p\sqrt{5} + \frac{3}{2}q - \frac{3}{2}q\sqrt{5}$$

$$= 2p + 2q + p\left(\frac{3}{2} + \frac{3}{2}\sqrt{5}\right) + q\left(\frac{3}{2} - \frac{3}{2}\sqrt{5}\right)$$

It is given that if a and b are rational numbers and $a + b\sqrt{5} = 0$; then, $a = 0 = 6$. Therefore,

$$a_4 = 2p + 2q + \frac{3}{2}p + \frac{3}{2}q$$

and

$$\frac{3}{2}p = \frac{3}{2}q \Rightarrow p = q$$

$$\Rightarrow a_4 = 2p + 2p + \frac{3}{2}p + \frac{3}{2}p = 7p$$

$$\Rightarrow 28 = 7p \Rightarrow p = 4 \Rightarrow q = 4$$

$$\Rightarrow p + 2q = 4 + 2 \times 4 = 12$$

Hence, the correct answer is option (D).

7 Permutation and Combination

7.1 Introduction

A branch of mathematics where we count number of objects or number of ways of doing a particular job without actually counting them is known as combinatorics. In this chapter, we will deal with elementary combinatorics.

Consider the following example as an illustration. If in a room there are five rows of chairs and each row contains five chairs, then without counting them we can say, total number of chairs is 25.

We start this chapter with fundamental principles of counting.

7.2 Fundamental Principles of Counting

There are two fundamental counting principles:

1. Addition principle
2. Multiplication principle

7.2.1 Addition Principle

Suppose that A and B are two disjoint events (mutually exclusive), that is, they never occur together. Further, suppose that A occurs in m ways and B in n ways. Then, A or B can occur in $m + n$ ways. This rule can also be applied to more than two mutually exclusive events.

In other words, if a job can be done by n different methods and for the first method there are a_1 ways, for the second method there are a_2 ways, $\cdots$, for the n^{th} method, a_n ways, then the number of ways to get the job done is $(a_1 + a_2 + \cdots + a_n)$.

Illustration 7.1 A college offers 7 courses in the morning and 5 in the evening. Find the number of ways a student can select exactly one course, either in the morning or in the evening.

Solution: The student has 7 choices from the morning courses out of which he can select one course in 7 ways.

For the evening course, he has 5 choices out of which he can select one course in 5 ways.

Hence, he has total number of $7 + 5 = 12$ choices.

Illustration 7.2 How many three-digit numbers xyz with $x < y$ and $z < y$ can be formed?

Solution: Obviously, $2 \le y \le 9$ (y cannot be 1). If $y = k$, then x can take values from 1 to $k - 1$ and z can take values from 0 to $k - 1$. Thus, the number of three-digit numbers formed are

$$\sum_{k=2}^{9}(k-1)(k) = 1.2 + 2.3 + \cdots + 8.9 = 240$$

7.2.2 Multiplication Principle

Suppose that an event X can be decomposed into two stages A and B. Let stage A occur in m ways and suppose that these stages are unrelated, in the sense that stage B occurs in n ways regardless of the outcome of stage A. Then event X occurs in mn ways. This rule is applicable even if event X is decomposed in more than two stages.

In other words, if a job can be done in m ways, and when it is done in any one of these ways, and another job can be done in n ways, then both the jobs together can be done in mn ways. The rule can be extended to include any number of jobs.

7.2.2.1 Extended Rule

If one operation can be performed independently in m different ways, second operation can be performed independently in n different ways, a third operation can be performed independently in p different ways and so on, then the total number of ways in which all the operations can be performed in the stated order is $(m \times n \times p \times \cdots)$ ways.

Remark:

The rule of product is applicable only when the number of ways of doing each part is independent of each other, that is, corresponding to any method of doing the first part, the other part can be done by any method.

Illustration 7.3 A college offers 7 courses in the morning and 5 in the evening. Find the possible number of choices with a student if he wants to study one course in the morning and one in the evening.

Solution: The student has 7 choices from the morning courses, out of which he can select one course in 7 ways.

For the evening course, he has 5 choices out of which he can select one in 5 ways.

Hence, the total number of ways in which he can choice of one course in the morning and one in the evening is $7 \times 5 = 35$.

Illustration 7.4 In a monthly test, the teacher decides that there will be three questions, one from each of exercise 7, 8 and 9 of the text book. If there are 12 questions in exercise 7, 18 in exercise 8 and 9 in exercise 9, in how many ways can three questions be selected?

Solution: There are 12 questions in exercise 7. So, one question from exercise 7 can be selected in 12 ways.

Exercise 8 contains 18 questions. So, second question can be selected in 18 ways.

There are 9 questions in exercise 9. So, third question can be selected in 9 ways.

Hence, three questions can be selected in $12 \times 18 \times 9 = 1944$ ways.

Illustration 7.5 A person wants to go from station A to station C via station B. There are three routes from A to B and four routes from B to C. In how many ways can he travel from A to C?

Solution: There are three routes from station A to station B. So, one route from A to B can be selected in 3 ways.

There are four routes from station B to station C. So, one route can be selected in 4 ways.

Hence, the person can travel in $3 \times 4 = 12$ ways.

Illustration 7.6 Find the number of three-digit numbers in which all the digits are distinct, odd and the number is a multiple of 5.

Solution: Here it is equivalent to completing three jobs of filling units, tens and hundreds place. Number of ways of filling units place is only one, that is, 5.

Now, four odd digits are left, hence *ten's* place can be filled in four ways and *hundred's* place in three ways.

Therefore, the number of required three-digit natural numbers is $1 \times 4 \times 3 = 12$.

Illustration 7.7 How many different 7-digit numbers are there whose sum of digits is even?

Solution: Let us consider 10 successive seven-digit numbers

$$a_1a_2a_3a_4a_5a_6\,0$$
$$a_1a_2a_3a_4a_5a_6\,1$$
$$a_1a_2a_3a_4a_5a_6\,2$$
$$\cdots\cdots\cdots\cdots$$
$$a_1a_2a_3a_4a_5a_6\,9$$

where $a_1, a_2, a_3, a_4, a_5, a_6$ are some digits. We see that half of these 10 numbers, that is, 5 numbers have an even sum of digits.

The first digit a_1 can assume 9 different values and each of the digits a_2, a_3, a_4, a_5, a_6 can assume 10 different values.

The last digit a_7 can assume only five different values of which the sum of all digits is even.

Therefore, there are $9 \times 10^5 \times 5 = 45 \times 10^5$ seven-digit numbers, the sum of whose digits is even.

7.2.2.2 Factorial Notation

Let n be a positive integer. Then, the continued product of first n natural numbers is called factorial n, to be denoted by $n!$ Also, we define $0! = 1$.

When n is negative or a fraction, $n!$ is not defined.

Thus, $n! = n(n-1)(n-2) \cdots 3 \times 2 \times 1$.

Deduction: $n! = n(n-1)(n-2)(n-3) \cdots 3 \times 2 \times 1$

$$= n[(n-1)(n-2)(n-3)\cdots 3\times 2\times 1] = n[(n-1)!]$$

Thus, $5! = 5\times(4!)$, $3! = 3\times(2!)$ and $2! = 2(1!)$

Also, $1! = 1\times(0!) \Rightarrow 0! = 1$

7.2.2.3 Exponent of Prime *p* in *n*!

Let p be a prime number and n be a positive integer. Then the last integer amongst $1, 2, 3, \ldots, (n-1), n$ which is divisible by p is $\left[\frac{n}{p}\right]p$, where $\left[\frac{n}{p}\right]$ denote the greatest integer less than or equal to $\frac{n}{p}$.

For example: $\left[\frac{10}{3}\right] = 3$, $\left[\frac{12}{5}\right] = 2$, $\left[\frac{15}{3}\right] = 5$, etc.

Let $E_p(n)$ denote the exponent of the prime p in the positive integer n. Then

$$E_p(n!) = E_p(1\times 2\times 3\cdots(n-1)n) = E_p\left(p\times 2p\times 3p\cdots\left[\frac{n}{p}\right]p\right)$$
$$= \left[\frac{n}{p}\right] + E_p\left(1\times 2\times 3\cdots\left[\frac{n}{p}\right]\right)$$

[since, remaining integers between 1 and n are not divisible by p]

Now the last integer among $1, 2, 3, \ldots, \left[\frac{n}{p}\right]$ which is divisible by p is $\left[\frac{n/p}{p}\right] = \left[\frac{n}{p^2}\right] = \left[\frac{n}{p}\right] + E_p\left(p\times 2p\times 3p\cdots\left[\frac{n}{p^2}\right]p\right)$ because the remaining natural numbers from 1 to $\left[\frac{n}{p}\right]$ are not divisible by $p = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + E_p\left(1\times 2\times 3\cdots\left[\frac{n}{p^2}\right]\right)$.

Similarly, we get

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots + \left[\frac{n}{p^S}\right]$$

where S is the largest natural number, such that $p^S \le n \le p^{S+1}$.

Illustration 7.8 Find the number of zeros at the end of 100!.

Solution: In terms of prime factors, 100! can be written as $2^a \cdot 3^b \cdot 5^c \cdot 7^d \cdot \ldots$ Now

$$E_2(100!) = \left[\frac{100}{2}\right] + \left[\frac{100}{2^2}\right] + \left[\frac{100}{2^3}\right] + \left[\frac{100}{2^4}\right] + \left[\frac{100}{2^5}\right] + \left[\frac{100}{2^6}\right]$$
$$= 50 + 25 + 12 + 6 + 3 + 1 = 97$$

and

$$E_5(100!) = \left[\frac{100}{5}\right] + \left[\frac{100}{5^2}\right] = 20 + 4 = 24$$

Therefore,

$$100! = 2^{97} \cdot 3^b \cdot 5^{24} \cdot 7^d \cdot \cdots$$
$$= 2^{73} \cdot 3^b \cdot (2 \times 5)^{24} \cdot 7^d \cdot \cdots$$
$$= 2^{73} \cdot 3^b \cdot (10)^{24} \cdot 7^d \cdot \cdots$$

Hence, number of zeros at the end of 100! is 24.

So, exponent of 10 in 100! = min (97, 24) = 24.

7.3 Permutations (Arrangement of Objects)

The ways of arranging or selecting a smaller or an equal number of persons or objects at a time from a given group of persons or objects with due regard being paid to the order of arrangement or selection are called (different) *permutations*.

For example, three different things *a*, *b* and *c* are given. Then different arrangements which can be made by taking two things from the given three things are *ab*, *ac*, *bc*, *ba*, *ca* and *cb*.

Therefore, the number of permutations will be 6.

7.3.1 Number of Permutations

The number of permutations of *n* objects, taking *r* at a time, is the total number of arrangements of *r* objects, selected from *n* objects, where the order of the arrangement is important.

7.3.1.1 Without Repetition

1. Arranging *n* objects, taking *r* at a time, is equivalent to filling *r* places from *n* things.

***r*–Places:**	1	2	3	4	⋯	*r*
Number of Choices:	n	$n-1$	$n-2$	$n-3$		$n-r+1$

The number of ways of arranging = The number of ways of filling *r* places

$$= n(n-1)(n-2)\cdots(n-r+1)$$
$$= \frac{n(n-1)(n-2)\cdots(n-r+1)[(n-r)!]}{(n-r)!}$$
$$\Rightarrow \frac{n!}{(n-r)!} = {}^nP_r$$

2. The number of arrangements of *n* different objects taking all at a time $\Rightarrow {}^nP_n = n!$

Illustration 7.9 If ${}^nP_4 : {}^nP_5 = 1:2$, then find the value of *n*.

Solution:

$$\frac{{}^nP_4}{{}^nP_5} = \frac{1}{2} \Rightarrow \frac{n!}{(n-4)!} \times \frac{(n-5)!}{n!} = \frac{1}{2} \Rightarrow n-4 = 2 \Rightarrow n = 6$$

Illustration 7.10 In a train 5 seats are vacant. Then in how many ways can three passengers sit on 5 seats?

(A) 20 **(B)** 30 **(C)** 60 **(D)** 10

Solution:

Number of ways = ${}^5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{120}{2} = 60$

Illustration 7.11 How many words can be formed consisting of any three letters of the word 'UNIVERSAL'?

Solution:

Required numbers of words = ${}^9P_3 = \frac{9!}{(9-3)!} = \frac{9!}{6!} = 504$

Illustration 7.12 How many five-digit numbers can be formed from the numbers 0, 2, 4, 3 and 8 where repetition of digits is not allowed?

Solution: Given numbers are 0, 2, 4, 3 and 8

Numbers that can be formed = {Total – those beginning with 0}

= {5! – 4!} = 120 – 24 = 96

7.3.1.2 With Repetition

1. The number of permutations (arrangements) of *n* different objects, taking *r* at a time, when each object may occur once, twice, thrice, ⋯ up to *r* times in any arrangement.

   Therefore, the number of ways of filling *r* places where each place can be filled by any one of *n* objects.

***r*–Places:**	1	2	3	4	⋯	*r*
Number of Choices:	N	n	n	n		n

   The number of permutations = The number of ways of filling *r* places = $(n)^r$

2. The number of arrangements that can be formed using *n* objects out of which *p* are identical (and of one kind), *q* are identical (and of another kind), *r* are identical (and of another kind) and the rest are distinct is $\frac{n!}{p!q!r!}$.

Illustration 7.13 Find the number of arrangements that can be formed from the letters of the word 'CALCUTTA'.

Solution:

Required number of ways $= \frac{8!}{2!2!2!} = 5040$

[since, here we have 2*C*'s, 2*T*'s and 2*A*'s]

Illustration 7.14 Find the number of 5-digit telephone numbers having at least one of their digits repeated.

Solution: Using the digits 0, 1, 2, ⋯, 9 the number of five-digit telephone numbers which can be formed is 10^5 (since repetition is allowed).

The number of five-digit telephone numbers which have none of the digits repeated = ${}^{10}P_5 = 30240$.

Therefore, the required number of telephone numbers is

$$10^5 - 30240 = 69760$$

7.4 Conditional Permutation

1. Number of permutations of n dissimilar things taken r at a time when p particular things always occur = ${}^{n-p}C_{r-p}r!$
2. Number of permutations of n dissimilar things taken r at a time when p particular things never occur = ${}^{n-p}C_r r!$
3. Total number of permutations of n different things taken not more than r at a time, when each thing may be repeated any number of times, is $\dfrac{n(n^r-1)}{n-1}$.
4. Number of permutations of n different things, taken all at a time, when m specified things always come together is $m!\times(n-m+1)!$
5. Number of permutations of n different things, taken all at a time, when m specified things never come together is $n! - m!\times(n-m+1)!$
6. Let there be n objects, of which m objects are alike of one kind, and the remaining $(n-m)$ objects are alike of another kind. Then, the total number of mutually distinguishable permutations that can be formed from these objects is

$$\frac{n!}{(m!)\times(n-m)!}$$

Note:
The above theorem can be extended further, that is, if there are n objects, of which p_1 are alike of one kind; p_2 are alike of another kind; p_3 are alike of 3rd kind; $\cdots$ p_r are alike of rth kind, such that $p_1+p_2+\cdots+p_r=n$ then the number of permutations of these n objects is

$$\frac{n!}{(p_1!)\times(p_2!)\times\cdots\times(p_r!)}$$

Key Points

1. **Gap method:** Suppose 5 males A, B, C, D and E are arranged in a row as × A × B × C × D × E ×. There will be six gaps between these five. Four in between and two on either end. Now, if three females P, Q and R are to be arranged so that no two are together we shall use gap method, that is, arrange them in between these 6 gaps. Hence, the answer will be ${}^6P_3$.
2. **Together:** Suppose we have to arrange 5 persons in a row which can be done in 5! = 120 ways. But if two particular persons are to be together always, then we tie these two particular persons with a string. Thus, we have 5 – 2 + 1 (1 corresponding to these two together) = 3 + 1 = 4 units, which can be arranged in 4! ways. Now we loosen the string and these two particular people can be arranged in 2! ways. Thus, total number of arrangements = 24 × 2 = 48.

   Therefore,

   Never together = Total – Together = 120 – 48 = 72

Illustration 7.15 Prove from the definition that ${}^nP_r = n\,{}^{n-1}P_{r-1}$ and hence, deduce the value of ${}^nP_r$.

Solution: Suppose that there are n different letters and a row of r blank spaces, each of which has to be filled up with one letter. The number of ways of filling up the blank spaces is the number of ways of arranging n things, r at a time, that is, ${}^nP_r$.

The first space can be filled in n ways. Having filled it, there are $n-1$ letters left and $r-1$ spaces to be filled. By definition, the number of ways of filling up the $r-1$ spaces with $n-1$ letters is the number of ways of arranging $n-1$ things, $r-1$ at a time, that is, ${}^{n-1}P_{r-1}$.

Therefore, the number of ways of filling up the r blank spaces with n different letters is $n\cdot{}^{n-1}P_{r-1}$. So

$${}^nP_r = n\cdot{}^{(n-1)}P_{(r-1)}$$

Similarly,

$${}^{n-1}P_{r-1} = (n-1)\cdot{}^{(n-2)}P_{(r-2)}$$

$${}^nP_r = n(n-1)\cdot{}^{(n-1)}P_{(r-2)} = n(n-1)(n-2)\cdot{}^{(n-3)}P_{(r-3)}$$

Proceeding like this, we have

$${}^nP_r = n(n-1)(n-2)(n-3)\cdots(n-r+2)\cdot{}^{(n-r+1)}P_1$$

Note that the last term is formed from ${}^{n-(r-1)}P_{r-(r-1)} = {}^{(n-r+1)}P_1$

Evidently, ${}^{(n-r+1)}P_1 = n-r+1$

Therefore,

$${}^nP_r = n(n-1)(n-2)(n-3)\cdots(n-r+2)(n-r+1)$$

In particular,

$${}^nP_0 = 1,\ {}^nP_1 = n$$

$${}^nP_n = n(n-1)\cdots 1 = n!$$

That is, the number of permutations of n things, taken all at a time, is $n!$.

Illustration 7.16 Show that ${}^nP_r = {}^{(n-1)}P_r + r\cdot{}^{(n-1)}P_{(r-1)}$.

Solution: We take the RHS.

$${}^{(n-1)}P_r + r\cdot{}^{(n-1)}P_{(r-1)} = \frac{(n-1)!}{(n-1-r)!} + \frac{r(n-1)!}{(n-r)!}$$

$$= (n-1)!\left[\frac{1}{(n-1-r)!} + \frac{r}{(n-r)!}\right]$$

$$= (n-1)!\frac{(n-r+r)}{(n-r)!}\quad (\text{since, } (n-r)! = (n-r)(n-r-1)!)$$

$$= \frac{n(n-1)!}{(n-r)!} = \frac{n!}{(n-r)!} = {}^nP_r$$

A common sense interpretation of the identity above is possible. From the number of permutations of r things which may be made from n things, $r\,{}^{(n-1)}P_{r-1}$, contain one specified thing and ${}^{(n-1)}P_r$ do not contain that specified thing and these two together give ${}^nP_r$.

Illustration 7.17 All the letters of the word 'EAMCET' are arranged in all possible ways. Find the number of such arrangements in which two vowels are not adjacent to each other.

Solution: First we arrange 3 consonants in 3! ways and then at four places (two places between them and two places on two sides) 3 vowels can be placed in ${}^4P_3 \times \frac{1}{2!}$ ways.

Hence, the required ways $= 3! \times {}^4P_3 \times \frac{1}{2!} = 72.$

Illustration 7.18 The number of words which can be made out of the letters of the word 'MOBILE' when consonants always occupy odd places, is _____.

(A) 20 **(B)** 36 **(C)** 30 **(D)** 720

Solution: The word 'MOBILE' has three even places and three odd places. It has 3 consonants and 3 vowels. In three odd places we have to fix up 3 consonants, which can be done in ${}^3P_3$ ways. Now in the remaining three places we have to fix up the remaining three vowels, which can be done in ${}^3P_3$ ways.

Threrfore, the total number of ways = ${}^3P_3 \times {}^3P_3 = 36$.

Illustration 7.19 m Men and n women are to be seated in a row, so that no two women sit together. If $m > n$, then find the number of ways in which they can be seated.

Solution: First arrange m men, in a row in m! ways. Since $n < m$ and no two women can sit together, in any one of the m! arrangement, there are (m + 1) places in which n women can be arranged in ${}^{m+1}P_n$ ways.

Therefore, by the fundamental theorem, the required number of arrangement is

$$m!\,{}^{m+1}P_n = \frac{m!(m+1)!}{(m-n+1)!}$$

Illustration 7.20 If the letters of the word 'KRISNA' are arranged in all possible ways and these words are written out as in a dictionary, then find the rank of the word 'KRISNA'.

Solution:

Words starting from *A* are 5! = 120; Words starting from *I* are 5! = 120

Words starting from *KA* are 4! = 24; Words starting from *KI* are 4! = 24

Words starting from *KN* are 4! = 24; Words starting from *KRA* are 3! = 6

Words starting from *KRIA* are 2! = 2; Words starting from *KRIN* are 2! = 2

Words starting from *KRISAN* are 1! = 1; Words starting from *KRISNA* are 1! = 1

Hence, the rank of the word, KRISNA is

$$120 + 120 + 24 + 24 + 24 + 6 + 2 + 2 + 1 + 1 = 324$$

7.5 Circular Permutation (Arrangement of Object)

So far we have been considering the arrangement of objects in a line. Such permutations are known as linear permutations.

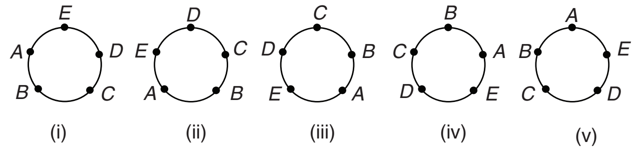


Figure 7.1

Instead of arranging the objects in a line, if we arrange them in the form of a circle, we call them circular permutations.

In circular permutations, what really matters is the position of an object relative to others.

Thus, in circular permutations, we fix the position of one of the object and then arrange the other objects in all possible ways.

There are two types of circular permutations:

1. The circular permutations in which clockwise and anticlockwise arrangements give rise to different permutations, for example, seating arrangements of persons around a table.

   Consider five persons *A*, *B*, *C*, *D* and *E* be seated on the circumference of a circular table in an order which has no head now. By shifting *A*, *B*, *C*, *D* and *E* one position in the anticlockwise direction we will get arrangements as shown in Fig. 7.1.

   We observe that arrangements in all the figures are different.

   Thus, the number of circular permutations of *n* different things taking all at a time is (n – 1)! if clockwise and anticlockwise orders are taken as different.

Illustration 7.21 20 persons were invited to a party. In how many ways can they and the host be seated around a circular table? In how many of these ways will two particular persons be seated on either side of the host?

Solution: See Fig. 7.2.

1st part: Total persons at the circular table = 20 guests + 1 host = 21

They can be seated in (21 – 1)!, that is, 20! ways.

2nd part: After fixing the places of three persons (1 host + 2 persons), treating (1 host + 2 person) as 1 unit, we have now 19 {(remaining 18 persons + 1 unit) = 19} and the number of arrangement will be (19 – 1)! = 18!. Also, these two particular persons can be seated on either side of the host in 2! ways.

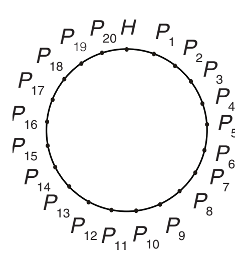


Figure 7.2

Hence, the number of ways of seating 21 persons at the circular table such that two particular persons be seated on either side of the host is 18! × 2! = 2 × 18! ways.

2. The circular permutations in which clockwise and anticlockwise arrangements give rise to same permutations, for example, arranging some beads to form a necklace.

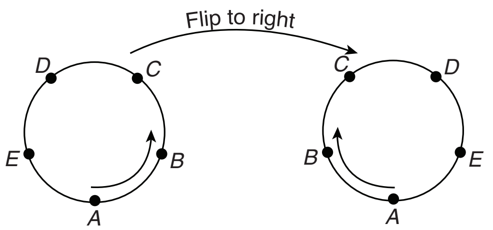


Figure 7.3

Consider five beads *A*, *B*, *C*, *D* and *E* in a necklace or five flowers *A*, *B*, *C*, *D* and *E* in a garland, etc. If the necklace or the garland on the left is turned over we obtain the arrangement on the right, that is, anticlockwise and clockwise order of arrangements are not different. We will get arrangements as follows:

We can see that the arrangements are not different.

Then the number of circular permutations of *n* different things taken all at a time is $\frac{1}{2}(n-1)!$, if clockwise and anticlockwise orders are not taken as different (Fig. 7.3).

Key Points

1. When the positions are numbered, the circular arrangement is treated as a linear arrangement.
2. In a linear arrangement, it does not make difference whether the positions are numbered or not.

Illustration 7.22 Consider 23 different coloured beads in a necklace. In how many ways can the beads be placed in the necklace so that 3 specific beads always remain together?

Solution: By theory, let us consider 3 beads as one. Hence we have, in effect, 21 beads, $n = 21$.

The number of arrangements = $\frac{1}{2}(n-1)! = \frac{1}{2}20!$

Also, the number of ways in which 3 beads can be arranged between themselves is $3! = 3 \times 2 \times 1 = 6$.

Thus, the total number of arrangements = (1/2) · 20! · 3!.

7.5.1 Number of Circular Permutations of *n* Different Things Taken *r* at a Time

1. **Case I:** If clockwise and anticlockwise orders are taken as different, then the number of circular permutations = $\frac{^nP_r}{r}$.
2. **Case II:** If clockwise and anticlockwise orders are not taken as different, then the number of circular permutations = $\frac{^nP_r}{2r}$.

Illustration 7.23 In how many ways can 24 persons be seated around a circular table if there are 13 seats?

Solution: In case of circular table, the clockwise and anticlockwise orders are different. Then the required number of circular permutations is

$$\frac{^{24}P_{13}}{13} = \frac{24!}{13 \times 11!}$$

Illustration 7.24 How many necklaces of 12 beads each can be made from 18 beads of various colours?

Solution: In the case of a necklace there is no distinction between the clockwise and anticlockwise arrangements. Then the required number of circular permutations is

$$\frac{^{18}P_{12}}{2 \times 12} = \frac{18!}{6! \times 24} = \frac{18 \times 17 \times 16 \times 15 \times 14 \times 13!}{6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 24} = \frac{119 \times 13!}{2}$$

Illustration 7.25 Consider 21 different pearls on a necklace. In how many ways can the pearls be placed on this necklace such that 3 specific pearls always remain together?

Solution: After fixing the places of three pearls, let us consider 3 specific pearls as one unit.

So, we have now 18 pearls + 1 unit = 19 and the number of arrangement will be (19 – 1)! = 18!.

Also, the number of ways in which 3 pearls can be arranged between themselves is 3! = 6. There is no distinction between the clockwise and anticlockwise arrangements.

So, the required number of arrangements = $\frac{1}{2}18! \cdot 6 = 3\,(18!)$.

Illustration 7.26 In how many ways can 10 boys and 5 girls sit around a circular table so that no two girls sit together?

Solution: 10 boys can be seated in a circle in 9! ways. There are 10 spaces in between the boys, which can be occupied by 5 girls in $^{10}P_5$ ways. Hence, the total numbers of ways is 9! $^{10}P_5$.

Illustration 7.27 The number of ways in which 5 male and 2 female members of a committee can be seated around a round table so that the two females are not seated together is

(A) 480 **(B)** 600
(C) 720 **(D)** 840

Solution: Fix up a male and the remaining 4 males can be seated in 4! ways. Now, no two females are to sit together and the 2 females are to be arranged in five empty seats between two consecutive males.

So, the number of arrangement will be $^5P_2$. Hence, by fundamental theorem, the total number of ways is

$$4! \times {}^5P_2 = 24 \times 20 = 480 \text{ ways}$$

Your Turn 1

1. How many numbers can be made with the help of the digits 0, 1, 2, 3, 4 and 5 which are greater than 3000? (Repetition is not allowed) **[IIT 1976]**

   (A) 180 **(B)** 360
   (C) 1380 **(D)** 1500 **Ans.** (C)

2. How many words can be made from the letters of the word 'COMMITTEE'?

Ans. $\dfrac{9!}{2!2!2!}=\dfrac{9!}{(2!)^3}$

3. Find the number of 4-digit numbers that can be formed from the digits 0, 1, 2, 3, 4, 5, 6 and 7, so that each number contains digit 1, and repetition of digits is not allowed

Ans. 750

4. We are to form different words with the letters of the word 'INTEGER'. Let m_1 be the number of words in which I and N are never together, and m_2 be the number of words which begin with I and end with R. Then find m_1/m_2.

Ans. 30

5. An n-digit number is a positive number with exactly n digits. Nine hundred distinct n-digit numbers are to be formed using only the three digits, 2, 5 and 7. Then find the smallest value of n for which this is possible.

Ans. $3^n \geq 900 \Rightarrow n=7$

6. Find the number of numbers that can be formed with the help of the digits 1, 2, 3, 4, 3, 2 and 1 so that odd digits always occupy odd places.

Ans. $6 \times 3 = 18$

7. In how many ways a garland can be made from exactly 10 flowers?

Ans. $\dfrac{1}{2}(n-1)!=\dfrac{9!}{2}$

8. In how many ways can 5 boys and 5 girls sit in a circle so that no boys sit together?

Ans. $4! \times 5!$

9. The number of ways in which 5 beads of different colours form a necklace is

(A) 12 **(B)** 24

(C) 120 **(D)** 60

Ans. (A)

7.6 Combination (Selection of Object)

Each of the different groups or selections which can be formed by taking some or all of a number of objects, irrespective of their arrangements, is called a combination.

Suppose we want to select two out of three persons A, B and C. We may choose AB or BC or AC. Clearly, AB and BA represent the same selection or group but they give rise to different arrangements. So, in a group or selection, the order in which the objects are arranged is immaterial.

Notation: The number of all combinations of n things, taken r at a time is denoted by

$$C(n,r) \text{ or } {}^nC_r \text{ or } \binom{n}{r}$$

7.6.1 Difference between Permutation and Combination

1. In a combination, only selection is made whereas in a permutation, not only a selection is made, but also an arrangement in a definite order is considered.
2. In a combination, the ordering of the selected objects is immaterial whereas in a permutation, the ordering is essential. For example, AB and BA are same combination, but different permutations.
3. Practically to find the permutation of n different items, taken r at a time, we first select r items from n items and then arrange them. So, usually the number of permutations exceeds the number of combinations.
4. Each combination corresponds to many permutations. For example, the six permutations ABC, ACB, BCA, BAC, CBA and CAB correspond to the same combination ABC.

Generally, we use the word 'arrangements' for permutations and the word 'selection' for combinations.

7.6.2 Selection of Object Without Repetition

The number of combinations (selections or groupings) that can be formed from n different objects taken r at a time is denoted by ${}^nC_r$ and its value is

$${}^nC_r=\frac{n!}{(n-r)!r!} \quad (0 \leq r \leq n)$$

where ${}^nC_r=\dfrac{{}^nP_r}{r!}$

as in a permutation, the arrangement of r selected objects out of n is done in $r!$ ways, and in combination, arrangement in a group is not considered.

In particular,

1. ${}^nC_0 = {}^nC_n = 1$, that is, there is only one way to select none or to select all objects out of n distinct objects.
2. ${}^nC_1 = n$, there are n ways to select one thing out of n distinct things.
3. ${}^nC_r = {}^nC_{n-r}$. Therefore, ${}^nC_x = {}^nC_y \Leftrightarrow x = y$ or $x + y = n$.
4. If n is odd, then the greatest value of ${}^nC_r$ is ${}^nC_{\frac{n+1}{2}}$ or ${}^nC_{\frac{n-1}{2}}$.
5. If n is even, then the greatest value of ${}^nC_r$ is ${}^nC_{n/2}$.

7.6.3 Selection of Object With Repetition

The number of combinations of n distinct objects taken r at a time when each may occur once, twice, thrice, $\cdots$ up to r times in any combination is given by ${}^{n+r-1}C_r$.

Or

The number of combinations of n distinct objects taken r at a time when any object may be repeated any number of times

$= \text{Coefficient of } x^r \text{ in } (1+x+x^2+\cdots+x^r)^n$

$= \text{Coefficient of } x^r \text{ in } (1-x)^{-n} = {}^{n+r-1}C_r$

Illustration 7.28 Calculate the following:

(a) If ${}^{15}C_{3r} = {}^{15}C_{r+3}$, then find r.

(b) If ${}^{n+1}C_3 = 2\,{}^nC_2$, then find n.

(c) If ${}^nC_{r-1} = 36$, ${}^nC_r = 84$ and ${}^nC_{r+1} = 126$, then find r.

Solution:

(a) ${}^{15}C_{3r} = {}^{15}C_{r+3} \Rightarrow {}^{15}C_{15-3r} = {}^{15}C_{r+3} \Rightarrow 15-3r = r+3 \Rightarrow r = 3$

(b) ${}^{n+1}C_3 = 2\cdot{}^nC_2 \Rightarrow \dfrac{(n+1)!}{3!(n-2)!} = 2\cdot\dfrac{n!}{2!(n-2)!}$

$$\Rightarrow \frac{n+1}{3\cdot 2!} = \frac{2}{2!} \Rightarrow n+1 = 6 \Rightarrow n = 5$$

(c) $\dfrac{{}^nC_{r-1}}{{}^nC_r} = \dfrac{36}{84}$ and $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{84}{126}$

$$3n - 10r = -3 \text{ and } 4n - 10r = 6$$

On solving, we get $n = 9$ and $r = 3$.

Illustration 7.29 In a conference of 8 persons, if each person shakes hand with others only once, then find the total number of handshakes.

Solution: Total number of handshakes when each person shakes hand with others only once is

$${}^8C_2 = 28 \text{ handshakes}$$

Illustration 7.30 How many words of 4 consonants and 3 vowels can be formed from 6 consonants and 5 vowels?

Solution:

Required number of words $= {}^6C_4 \times {}^5C_3 \times 7! = 756{,}000$

[selection can be made in ${}^6C_4 \times {}^5C_3$, while the 7 letters can be arranged in 7! ways]

Illustration 7.31 To fill 12 vacancies there are 25 candidates of which five are from scheduled caste. If 3 of the vacancies are reserved for scheduled caste candidates while the rest are open to all, then find the number of ways in which the selection can be made.

Solution: The selection can be made in ${}^5C_3 \times {}^{22}C_9$, [since 3 vacancies can be filled from 5 candidates in ${}^5C_3$ ways and now remaining candidates are 22 and remaining seats are 9, then remaining vacancies can be filled by ${}^{22}C_9$ ways. Hence, total number of ways is ${}^5C_3 \times {}^{22}C_9$.

Illustration 7.32 Let 15 toys be distributed among 3 children subject to the condition that any child can take any number of toys. Find the required number of ways to do this if

(a) toys are distinct. **(b)** toys are identical.

Solution:

(a) Toys are distinct: Here we have 3 children and we want the 15 toys to be distributed to the 3 children with repetition. In other words, it is same as selecting and arranging children 15 times out of 3 children with the condition that any child can be selected any number of times, which can be done in 3^{15} ways ($n = 3, r = 15$).

(b) Toys are identical: Here we only have to select children 15 times out of 3 children with the condition that any child can be selected any number of times. This can be done in

$${}^{3+15-1}C_{15} = {}^{17}C_2 \text{ ways } (n = 3, r = 5)$$

7.6.4 All Possible Selections

7.6.4.1 Selection from Distinct Objects

The number of ways (or combinations) of selection from n distinct objects, taking at least one of them is

$${}^nC_1 + {}^nC_2 + {}^nC_3 + \cdots + {}^nC_n = 2^n - 1$$

Logically, it can be explained in two ways, as one can be selected in ${}^nC_1$ ways, two in ${}^nC_2$ ways and so on, and by addition principle of counting, the total number of ways of doing either of the jobs is

$${}^nC_1 + {}^nC_2 + \dots + {}^nC_n$$

Also, for every object, there are two choices, either selection or non-selection. Hence, total choices are 2^n. But this also includes the case when none of them is selected.

Therefore, the number of selections when at least one is selected is $2^n - 1$.

Illustration 7.33 Find the number of ways in which we can put n distinct objects into two identical boxes so that no box remains empty.

Solution: Let us first label the boxes 1 and 2. We can select at least one or at most $(n - 1)$ objects for box 1 in

${}^nC_1 + {}^nC_2 + \dots + {}^nC_{n-1}$ ways
$= {}^nC_0 + {}^nC_1 + \dots + {}^nC_n - {}^nC_0 - {}^nC_n$
[since ${}^nC_0 + {}^nC_1 + \dots + {}^nC_n = 2^n$, ${}^nC_0 = 1$, and ${}^nC_n = 1$]
$= 2^n - 2$
$= 2(2^{n-1} - 1)$ ways

In this way, box 2 is not empty. But since the boxes are identical, the number of ways that no box remains empty is

$$\frac{1}{2} \times 2(2^{n-1}) = 2^{n-1} - 1$$

Alternative Solution: Let us first label the boxes 1 and 2. There are then two choices for each of the n objects; we can put it in the first box or in the second box. Therefore, the number of choices for n distinct objects is

$$\underbrace{2 \times 2 \times \cdots \times 2}_{n \text{ times}} = 2^n$$

Two of these choices correspond to either the first or the second box being empty. Thus, there are $2^n - 2$ ways in which neither box is empty. If we now remove the labels from the boxes so that they become identical, this number must be divided by 2, yielding the answer $(1/2)(2^n - 2) = 2^{n-1} - 1$.

Illustration 7.34 There are five different green dyes, four different blue dyes and three different red dyes. How many combinations of dyes can be chosen taking at least one green and one blue dye?

Solution: Any one dye of a particular colour can be either chosen or not; and, thus there are 2 ways in which each one may be dealt with.

Number of ways of selection so that at least one green dye is included is

$$2^5 - 1 = 31$$

(1 is subtracted to correspond to the case when none of the green dyes is chosen.)

A similar argument may be advanced in respect of other two colours also.

$$\text{Number of combinations} = (2^5 - 1)(2^4 - 1)(2^3)$$
$$= 31 \times 15 \times 8 = 3720$$

7.6.4.2 Selection from Identical Objects

1. The number of selections of r ($r \le n$) objects out of n identical objects is 1.
2. The number of ways of selections of at least one object out of n identical object is n.
3. The number of ways of selections of at least one out of $a_1 + a_2 + \cdots + a_n$ objects, where a_1 are alike of one kind, a_2 are alike of second kind, and a_n are alike of n^{th} kind, is
$$(a_1 + 1)(a_2 + 1)\cdots(a_n + 1) - 1$$
4. The number of ways of selections of at least one out of $a_1 + a_2 + a_3 + \cdots + a_n + k$ objects, where a_1 are alike of one kind, $\cdots$ a_n are alike of n^{th} kind and k is distinct is
$$\left[(a_1+1)(a_2+1)\cdots(a_n+1)\right]2^k - 1$$

Illustration 7.35 Find the number of combinations that can be formed with 5 oranges, 4 mangoes and 3 bananas when it is essential to take

(a) at least one fruit
(b) one fruit of each kind.

Solution: Here, 5 oranges are alike of one kind, 4 mangoes are alike of second kind and 3 bananas are alike of third kind.

(a) The required number of combinations (with at least one fruit) is

$$(5 + 1)(4 + 1)(3 + 1) - 1$$
$$= 120 - 1 = 119$$

(b) The required number of combinations (with one fruit of each kind) is

$${}^5C_1 \times {}^4C_1 \times {}^3C_1 = 5 \times 4 \times 3 = 60$$

Illustration 7.36 There are 10 lamps in a hall. Each one of them can be switched on independently. Find the number of ways in which the hall can be illuminated.

Solution:

Number of ways = $2^{10} - 1 = 1023$

[1 is subtracted to correspond to the case when none of the lamps is switched on]

7.6.5 Conditional Combination

1. The number of ways in which r objects can be selected from n different objects if k particular objects are
   (a) Always included = ${}^{n-k}C_{r-k}$ **(b)** Never included = ${}^{n-k}C_r$
2. The number of combinations of n objects, of which p are identical, taken r at a time is
$${}^{n-p}C_r + {}^{n-p}C_{r-1} + {}^{n-p}C_{r-2} + \cdots + {}^{n-p}C_0, \text{ if } r \le p$$
and
$${}^{n-p}C_r + {}^{n-p}C_{r-1} + {}^{n-p}C_{r-2} + \cdots + {}^{n-p}C_{r-p}, \text{ if } r > p$$

Illustration 7.37 A lady desires to give a dinner party for 8 guests. In how many ways can the lady select guests for the dinner from her 12 friends, if two of the guests will not attend the party together?

Solution: The following three methods of approach are indicated.

(a) Number of ways of forming the party = ${}^{12}C_8 - {}^{10}C_6$

[since ${}^{10}C_6$ is the number of ways of making up the party with both the specified guests included]

$$= 495 - 210 = 285$$

(OR)

(b) Number of ways of forming the party

= Number of ways of forming without both of them

+ Number of ways of forming with one of them and without the other

$$= {}^{10}C_8 + 2 \cdot {}^{10}C_7 = 45 + 240 = 285$$

(OR)

(c) Split the number of ways of forming the party

= Those with one of the two (say A) + those without A

$$= {}^{10}C_7 + {}^{11}C_8 = 120 + 165 = 285$$

Illustration 7.38 A bag contains 23 balls in which 7 are identical. Find the number of ways of selecting 12 balls from the bag.

Solution: Here, $n = 23$, $p = 7$, $r = 12$ ($r > p$).

Hence, the required number of selections is

$${}^{n-p}C_r + {}^{n-p}C_{r-1} + \cdots + {}^{n-p}C_{r-p} = \sum_{r=5}^{12} {}^{16}C_r$$
$$= {}^{16}C_5 + {}^{16}C_6 + {}^{16}C_7 + {}^{16}C_8 + {}^{16}C_9 + {}^{16}C_{10} + {}^{16}C_{11} + {}^{16}C_{12}$$
$$= ({}^{16}C_5 + {}^{16}C_6) + ({}^{16}C_7 + {}^{16}C_8) + ({}^{16}C_9 + {}^{16}C_{10}) + ({}^{16}C_{11} + {}^{16}C_{12})$$
$$= {}^{17}C_6 + {}^{17}C_8 + {}^{17}C_{10} + {}^{17}C_{12} \text{ (since } {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r)$$
$$= {}^{17}C_{11} + {}^{17}C_9 + {}^{17}C_{10} + {}^{17}C_{12} \text{ (since } {}^nC_r = {}^nC_{n-r})$$

$= ({}^{17}C_{11} + {}^{17}C_{12}) + ({}^{17}C_9 + {}^{17}C_{10})$
$= {}^{18}C_{12} + {}^{18}C_{10} = {}^{18}C_6 + {}^{18}C_8$

Illustration 7.39 Among the 13 cricket players 4 are bowlers, then how many ways can a cricket team be formed of 11 players in which at least 2 bowlers are included?

Solution: The number of ways can be given as follows:

2 bowlers and 9 other players = ${}^4C_2 \times {}^9C_9$

3 bowlers and 8 other players = ${}^4C_3 \times {}^9C_8$

4 bowlers and 7 other players = ${}^4C_4 \times {}^9C_7$

Hence, required number of ways is ${}^4C_2 \times {}^9C_9 + {}^4C_3 \times {}^9C_8 + {}^4C_4 \times {}^9C_7 = 6 \times 1 + 4 \times 9 + 1 \times 36 = 78$

Illustration 7.40 In how many ways a team of 10 players out of 22 players can be made if 6 particular players are always to be included and 4 particular players are always excluded?

Solution: 6 particular players are always to be included and 4 are always excluded, so total number of selection now is 4 players out of 12 players.

Hence, the number of ways = ${}^{12}C_4$.

Illustration 7.41 A delegation of four students is to be selected from a total of 12 students. In how many ways can the delegation be selected,

(a) if all the students are equally willing?
(b) if two particular students have to be included in the delegation?
(c) if two particular students do not wish to be together in the delegation?
(d) if two particular students wish to be included together only?
(e) if two particular students refuse to be together and two other students wish to be together only in the delegation?

Solution:

(a) Formation of delegation means selection of 4 out of 12.

Hence, the number of ways = ${}^{12}C_4 = 495$.

(b) Two particular students are already selected. Here we need to select only 2 out of the remaining 10.

Hence, the number of ways = ${}^{10}C_2 = 45$.

(c) The number of ways in which both are selected = 45. Hence, the number of ways in which the two are not included together = 495 – 45 = 450.

(d) There are two possible cases:

(i) Either both are selected. In this case, the number of ways in which the selection can be made = 45.

(ii) Or both are not selected. In this case, all the four students are selected from the remaining 10 students.

This can be done in ${}^{10}C_4 = 210$ ways.
Hence, the total number of ways of selection is

$$45 + 210$$
$$= 255$$

(e) We assume that students A and B wish to be selected together and students C and D do not wish to be together. Now there are following 6 cases:

(i) (A, B, C) selected, (D) not selected
(ii) (A, B, D) selected, (C) not selected
(iii) (A, B) selected, (C, D) not selected
(iv) (C) selected, (A, B, D) not selected
(v) (D) selected, (A, B, C) not selected
(vi) A, B, C, D not selected

For (i), the number of ways of selection = ${}^8C_1 = 8$
For (ii), the number of ways of selection = ${}^8C_1 = 8$
For (iii), the number of ways of selection = ${}^8C_2 = 28$
For (iv), the number of ways of selection = ${}^8C_3 = 56$
For (v), the number of ways of selection = ${}^8C_3 = 56$
For (vi), the number of ways of selection = ${}^8C_4 = 70$
Hence, total number of ways = 8 + 8 + 28 + 56 + 56 + 70 = 226

7.7 Divisors of a Given Natural Number

Let $n \in N$ and $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \cdots \cdot p_k^{\alpha_k}$, where $p_1, p_2, p_3, \cdots, p_k$ are different prime numbers and $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_k$ are natural numbers. Then

1. The total number of divisors of n including 1 and n is

$$(\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1)\cdots(\alpha_k + 1)$$

2. The total number of divisors of n excluding 1 and n is

$$(\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1)\cdots(\alpha_k + 1) - 2$$

3. The total number of divisors of n excluding exactly one out of 1 and n is

$$(\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1)\cdots(\alpha_k + 1) - 1$$

4. The sum of these divisors is

$$(p_1^0 + p_1^1 + p_1^2 + \cdots + p_1^{\alpha_1})(p_2^0 + p_2^1 + p_2^2 + \cdots + p_2^{\alpha_2})\cdots$$
$$(p_k^0 + p_k^1 + p_k^2 + \cdots + p_k^{\alpha_k})$$
$$= \left[\frac{p_1^{\alpha_1+1} - 1}{p_1 - 1}\right]\left[\frac{p_2^{\alpha_2+1} - 1}{p_2 - 1}\right]\cdots\left[\frac{p_k^{\alpha_k+1} - 1}{p_k - 1}\right]$$

(using sum of GP in each bracket)

5. The number of ways in which n can be resolved as a product of two factors is

$\frac{1}{2}(\alpha_1 + 1)(\alpha_2 + 1)\cdots(\alpha_k + 1)$, if n is not a perfect square

$\frac{1}{2}[(\alpha_1 + 1)(\alpha_2 + 1)\cdots(\alpha_k + 1) + 1]$, if n is a perfect square

6. The number of ways in which the composite number n can be resolved into two factors which are relatively prime (or coprime) to each other is equal to 2^{k-1}, where k is the number of different factors (or different primes) in n.

Illustration 7.42 If $n = 10800$, then find the

(a) total number of divisors of n
(b) number of even divisors
(c) number of divisors of the form $4m + 2$
(d) number of divisors which are multiples of 15.

Solution:

$$n = 10800 = 2^4 \times 3^3 \times 5^2$$

Any divisor of n will be of the form $2^a \times 3^b \times 5^c$, where $0 \le a \le 4$, $0 \le b \le 3$, $0 \le c \le 2$.

For any distinct choices of a, b and c, we get a divisor of n:

(a) Total number of divisors = (4 + 1) (3 + 1) (2 + 1) = 60

(b) For a divisor to be even, a should be at least one. So, the total number of even divisors = 4(3 + 1) (2 + 1) = 48.

(c) $4m + 2 = 2(2m + 1)$. In any divisor of the form $4m + 2$, a should be exactly 1. So the number of divisors of the form $4m + 2$ = 1 (3 + 1) (2 + 1) = 12.

(d) A divisor of n will be a multiple of 15 if b is at least one and c is at least one. So the number of such divisors = $(4 + 1) \times 3 \times 2 = 30$.

Illustration 7.43 Find the number of divisors of 428652000 excluding the number and unity. Also, find the sum of the divisors.

Solution:

$$428652000 = 2^5 \times 3^7 \times 5^3 \times 7^2$$

Any divisor of the given number has to be a combination of the 2's (five), 3's (seven), 5's (three), and 7's (two).

There are 5 + 1 = 6 ways of selecting none or one or two, etc., Similar argument repeats for the other numbers.

The number of divisors = $6 \times 8 \times 4 \times 3 = 576$.

This includes 1 and the given number also.

Excluding these two, the number of divisors = 574, with regard to the sum of the divisors.

Any divisor is of the form $2^p 3^q 5^s 7^t$, where $0 \le p \le 5$; $0 \le q \le 7$; $0 \le s \le 3$ and $0 \le t \le 2$

Thus, the sum of the divisors is

$$(1+2+\cdots+2^5)(1+3+\cdots+3^7)(1+5+\cdots+5^3)(1+7+7^2)$$

$$= (2^6 - 1)\left(\frac{3^8-1}{2}\right)\left(\frac{5^4-1}{4}\right)\left(\frac{7^3-1}{6}\right)$$

$$= (2^6 - 1)(3^8 - 1)(5^4 - 1)(7^3 - 1)/48$$

Your Turn 2

1. A man has 10 friends. In how many ways can he invite one or more of them to a party?

(A) $10!$ **(B)** 2^{10}
(C) $10! - 1$ **(D)** $2^{10} - 1$

Ans. (D)

2. Numbers greater than 1000 but not greater than 4000 which can be formed with the digits 0, 1, 2, 3 and 4 (repetition of digits is allowed) are **[AIEEE 2002; IIT 1976]**

(A) 350 **(B)** 375
(C) 450 **(D)** 576

Ans. (B)

3. Ten different letters of English alphabet are given. Out of these letters, words of five letters are formed. How many words are formed where at least one letter is repeated?

Ans. 99748

4. In how many ways can 6 persons be selected from 4 officers and 8 constables, if at least one officer is to be included?

Ans. 896

5. Find the number of divisors of 9600 including 1 and 9600.

Ans. 48

6. Find the number of divisors of $n = 38808$ (except 1 and n).

Ans. 70

7. A train going from Cambridge to London stops at nine intermediate stations. Six persons enter the train during the journey with six different tickets. How many different sets of tickets they have had?

Ans. ${}^{45}C_6$

8. Find the number of ways in which we can arrange four letters of the word 'mathematics'.

Ans. 2454

7.8 Division of Object into Groups

7.8.1 Division of Distinct Object into Groups

In the case of grouping, we have the following.

If $m + n + p$ things are divided into 3 groups first containing m, second containing n and third containing p things, then number of groupings is

$${}^{(m+n+p)}C_m \cdot {}^{(n+p)}C_n \cdot {}^{p}C_p = \frac{(m+n+p)!}{m!\,n!\,p!}$$

where m, n, p are distinct natural numbers.

If $m = n = p$ (say), then the number of groupings (unmindful of the order of grouping) is

$$\frac{3m!}{(m!)^3\,3!}$$

Thus, if 52 cards have to be divided into four groups of 13 each, the number of grouping is

$$\frac{52!}{(13!)^4\,4!}$$

On the other hand, when 52 cards are dealt 13 each to four persons, the number of ways in which this can be done is

$$\frac{52!}{(13!)^4}$$

As another example, if we consider the division of 52 cards into four groups, three groups containing 16 and each the fourth cards, the number of ways in which this can be done is

$$\frac{52!}{3!(16!)^3\,4!}$$

Note the 3! factorial in the denominator. This is for the reason that there are only 3 equal groups.

In general, the number of ways in which mn different things can be divided equally into m distinct groups is $\frac{(mn)!}{(n!)^m}$ when order of groups is important.

On the other hand, when the order of groups is not important, then division into m equal groups is done in $\frac{(mn)!}{m!(n!)^m}$ ways.

In general, we can write formula for grouping as a factorial of total number of elements divided by the product of factorial of number of elements in each group and product of factorial of number of groups having same number of elements, if any. Also, formula for number of distribution is the number of grouping multiplied with factorial of number of persons in which objects are distributed, divided by factorial of number of persons who got nothing, if any.

Illustration 7.44 A city has 'p' parallel roads running east-west and 'q' parallel roads running north-south. How many rectangles are formed with their sides along these roads? If the distance between every consecutive parallel road is the same, how many shortest possible routes are there to go from one corner of the city to the diagonally opposite corner (Fig. 7.4)?

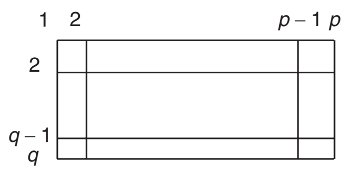


Figure 7.4

Solution: To form a rectangle one needs to take two roads from the 'p' parallel roads and two roads from the 'q' parallel roads. The number of rectangles thus formed is

$$^pC_2 \cdot {^qC_2} = \frac{pq(p-1)(q-1)}{4}$$

Let the distance between any two parallel roads be one unit. On going from one corner to the diagonally opposite corner, one has to travel $(p - 1)$ units in the north-south direction and $(q - 1)$ units in the east-west direction.

Totally, therefore, one has to travel a distance of $p + q - 2$ units of which $(p - 1)$ units are in one direction and $(q - 1)$ units are in the other direction. These displacements can be taken in any order. As such, it is a problem of arranging $p + q - 2$ units of which $(p - 1)$ are of one kind and $(q - 1)$ are of second kind. Hence, the number of ways in which this may be done is equal to division of $p + q - 2$ into two groups of $(p - 1)$ and $(q - 1)$, which is equal to

$$\frac{(p+q-2)!}{(p-1)!(q-1)!}$$

7.8.2 Division of Identical Objects into Groups

The number of ways of division or distribution of n identical things into r different groups is $^{n+r-1}C_{r-1}$ or $^{n-1}C_{r-1}$, according as empty groups are allowed or not allowed.

Illustration 7.45 Find the number of ways in which 11 identical apples can be distributed among 6 children, so that each child receives at least one apple.

Solution: First give one apple each to each child. There are remaining 5 apples that are to be distributed among 6 children (so that each may receive any number of apples not exceeding five).

The number of ways required is

$$^{(11-1)}C_{6-1} = {^{10}C_5} = 252$$

Alternatively, the number of ways of distribution is given by the coefficient of x^{11} in $(x + x^2 + \cdots + x^{11})^6$.

7.8.3 Arrangement in Groups

The number of ways of distribution and arrangement of n distinct things into r different groups is $n!\,^{n+r-1}C_{r-1}$ or $n!\,^{n-1}C_{r-1}$, according as empty groups are allowed or not allowed.

Illustration 7.46 In how many ways can three balls of different colours be put in 4 glass cylinders of equal width such that any glass cylinder may have either 0, 1, 2 or 3 balls?

Solution: There are four glass cylinders. Consider additionally $(4 - 1) = 3$ things; and, the number of ways is $^6C_3$ (corresponding to $^{(n+r-1)}C_{r-1}$) multiplied by $3! = 120$.

Illustration 7.47 Find the number of ways in which 9 persons can be divided into three equal groups.

Solution:

$$\text{Total ways} = \frac{9!}{(3!)^3} = \frac{9\times8\times7\times6\times5\times4}{3\times2\times3\times2\times3\times2} = 280$$

Illustration 7.48 Find the number of ways of dividing 52 cards amongst four players equally.

Solution:

$$\text{Required number of ways} = {^{52}C_{13}} \times {^{39}C_{13}} \times {^{26}C_{13}} \times {^{13}C_{13}}$$

$$= \frac{52!}{39!13!} \times \frac{39!}{26!13!} \times \frac{26!}{13!13!} \times \frac{13!}{13!} = \frac{52!}{(13!)^4}$$

7.9 Method of Inclusion and Exclusion

If $A_1, A_2, \ldots, A_m$ are finite sets and $A = A_1 \cup A_2 \cup \cdots \cup A_m$, then

$$n(A) = a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{m+1} a_m$$

where

$a_1 = n(A_1) + n(A_2) + \cdots + n(A_m)$

$a_2 = \sum_{1\le i<j\le m} n(A_i \cap A_j)$, $a_3 = \sum_{1\le i<j<k\le m} n(A_i \cap A_j \cap A_k)$ and so on.

Corollary (Sieve-Formula): If $A_1, A_2, \ldots, A_m$ are m subsets of a set A containing N elements, then

$$n(A_1' \cap A_2' \cap \cdots \cap A_m')$$

$$= N - \sum_i n(A_i) + \sum_{1\le i<j\le m} n(A_i \cap A_j) - \sum_{1\le i<j<k\le m} n(A_i \cap A_j \cap A_k)$$
$$+ \cdots + (-1)^m n(A_1 \cap A_2 \cap \cdots \cap A_m)$$

Derangements: It is rearrangement of objects such that no one goes to its original place. OR Any change in the given order of the things is called a derangement.

If n things are arranged in a row, the number of ways in which they can be rearranged so that none of them occupies its original place is

$$n!\left(1 - \frac{1}{1!} + \frac{1}{2!} + \cdots + (-1)^n \frac{1}{n!}\right)$$

It is denoted by

$$D_n = n!\sum_{r=0}^{n}(-1)^r \frac{1}{r!}$$

Illustration 7.49 There are four balls of different colours and four boxes of colours same as those of the balls. Find the number of ways in which the balls, one in each box, could be placed such that a ball does not go to box of its own colours.

Solution: Number of derangement is

$$4!\left\{\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right\} = 12 - 4 + 1 = 9$$

Illustration 7.50 A person writes letters to six friends and addresses the corresponding envelopes. In how many ways can the letters be placed in the envelopes so that

(a) At least two of them are in the wrong envelopes?

(b) All the letters are in the wrong envelopes?

Solution:

(a) The number of ways in which at least two of them are in the wrong envelopes is

$$\sum_{r=2}^{6} {}^nC_{n-r}D_r = {}^nC_{n-2}D_2 + {}^nC_{n-3}D_3 + {}^nC_{n-4}D_4 + {}^nC_{n-5}D_5 + {}^nC_{n-6}D_6$$

Here, $n = 6$. So we have

$$\sum_{r=2}^{6} {}^6C_{6-r}D_r \; {}^6C_4 \cdot 2!\left(1 - \frac{1}{1!} + \frac{1}{2!}\right) + {}^6C_3 \cdot 3!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right)$$
$$+ {}^6C_2 \cdot 4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)$$
$$+ {}^6C_1 \cdot 5!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right)$$
$$+ {}^6C_0 \cdot 6!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right)$$
$$= 15 + 40 + 135 + 264 + 265 = 719.$$

(b) The number of ways in which all letters are placed in wrong envelopes is

$$6!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right)$$
$$= 720\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720}\right)$$
$$= 360 - 120 + 30 - 6 + 1 = 265$$

7.10 Use of Multinomial

1. If there are l objects of one kind m objects of second kind, n objects of third kind, and so on, then the number of ways of choosing r objects out of these objects, that is, $l + m + n + \cdots$ is the coefficient of x^r in the expansion of

   $$(1 + x + x^2 + x^3 + \cdots + x^l)(1 + x + x^2 + \cdots + x^m)(1 + x + x^2 + \cdots + x^n)\cdots$$

   Further, if one object of each kind is to be included, then the number of ways of choosing r objects out of these objects, that is, $l + m + n + \cdots$ is the coefficient of x^r in the expansion of

   $$(x + x^2 + x^3 + \cdots + x^l)(x + x^2 + x^3 + \cdots + x^m)(x + x^2 + x^3 + \cdots + x^n)\cdots$$

2. If there are l objects of one kind, m object of second kind, n object of third kind and so on; then the number of possible arrangements/permutations of r objects out of these objects (that is, $l + m + n + \cdots$) is the coefficient of x^r in the expansion of $(1 - x)^{-p}$

   Note: For use in problems of the above type the following Binomial expansions may be noted

   $$\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + \cdots + x^r + \cdots$$
   $$\frac{1}{(1-x)^2} = (1-x)^{-2} = 1 + 2x + 3x^2 + \cdots + (r+1)x^r + \cdots$$
   $$\frac{1}{(1-x)^3} = (1-x)^{-3} = 1 + 3x + 6x^2 + \cdots + \frac{(r+1)(r+2)}{1\cdot 2}x^r + \cdots$$
   $$\frac{1}{(1-x)^4} = (1-x)^{-4} = 1 + 4x + 10x^2 + \cdots + \frac{(r+1)(r+2)(r+3)}{1\cdot 2\cdot 3}x^r + \cdots$$

   and more generally

   $$\frac{1}{(1-x)^p} = (1-x)^{-p}$$
   $$= 1 + px + \cdots + \frac{(r+1)(r+2)\cdots(r+p-1)}{1\cdot 2\cdot 3\cdots(p-1)}x^r + \cdots$$

   and the coefficient of x^r in this general case is easily seen to be ${}^{(r+p-1)}C_{p-1}$.

Illustration 7.51 Let us consider the more general problem of distributing n identical things given among r persons, each one whom can receive 0, 1, 2 or more things ($\le n$).

Solution: Consider r brackets corresponding to the r persons.

In each bracket, take an expression given by $1 + x + x^2 + \cdots + x^n$ (the various powers of x namely, 0, 1, 2, $\ldots$, n correspond to the number of things each person can have in the distribution).

In the continued product $(1 + x + x^2 + \cdots + x^n)()()\cdots$ repeated r times, collect the coefficient of x^n. This coefficient gives the required number of ways of distribution.

Therefore, the number of ways is

Coefficient of x^n in $(1 + x + x^2 + \cdots + x^4)\,(\,)\,(\,)\cdots$ repeated r times

$$= \text{Coefficient of } x^n \text{ in } \left(\frac{1-x^{n+1}}{1-x}\right)^r$$

$$= \text{Coefficient of } x^n \text{ in } (1 - x^{n+1})^r (1 - x)^{-r}$$

$$= \text{Coefficient of } x^n \text{ in } (1 - x)^{-r}$$

$$= \frac{(r+1)(r+2)\cdots(r+n-1)}{1\cdot 2\cdots(r-1)}$$

$$= {}^{(n+r-1)}C_{r-1}$$

7.10.1 An Alternative Method for the General Problem

In this case, we consider additionally $(r - 1)$ things (this number is one less than r, the number of persons). It may be seen that the number of ways of dividing the n things among r persons as per the condition of the problem is ${}^{(n+r-1)}C_{(r-1)}$.

Illustration 7.52 An unlimited number of red, white, blue and green balls are given. Ten balls are drawn. Find the number of ways of selection.

Solution: In this, the expression in x to be considered is $1 + x + x^2 + \cdots$ to correspond to the unlimited number available in each colour. There are four such colours.

The required number of ways is

Coefficient of x^{10} in $[1 + x + x^2 + \cdots]^4$

$$= \text{Coefficient of } x^{10} \text{ in } \left(\frac{1}{1-x}\right)^4$$

$$= \frac{11 \times 12 \times 13}{6}$$

$$= 286$$

7.10.2 Use of Multinomial Theorem in Solving Linear Equation

Let $x_1, x_2, \ldots, x_m$ be integers. Then number of solutions to the equation

$$x_1 + x_2 + \cdots + x_m = n \tag{1}$$

subject to the condition

$$a_1 \le x_1 \le b_1, a_2 \le x_2 \le b_2, \ldots, a_m \le x_m \le b_m \tag{2}$$

is equal to the coefficient of x^n in

$$(x^{a_1} + x^{a_1+1} + \cdots + x^{b_1})(x^{a_2} + x^{a_2+1} + \cdots + x^{b_2})\cdots$$
$$(x^{a_m} + x^{a_{m+1}} + \cdots + x^{b_m}) \tag{3}$$

This is because the number of ways in which sum of m integers in Eq. (1) equals n is the same as the number of times x^n comes in Eq. (3).

7.10.3 Use of Solution of Linear Equation and Coefficient of a Power in Expansions to Find the Number of ways of Distribution

1. The number of integral solutions of $x_1 + x_2 + x_3 + \cdots + x_r = n$ where, $x_1 \ge 0, x_2 \ge 0, \cdots, x_r \ge 0$ is the same as the number of ways to distribute n identical things among r persons.

This is also equal to the coefficient of x^n in the expansion of $(x^0 + x^1 + x^2 + x^3 + \cdots)^r$

$$= \text{coefficient of } x^n \text{ in } \left(\frac{1}{1-x}\right)^r$$

$$= \text{coefficient of } x^n \text{ in } (1-x)^{-r}$$

$$= \text{coefficient of } x^n \text{ in}$$

$$\left\{1 + rx + \frac{r(r+1)}{2!}x^2 + \cdots + \frac{r(r+1)(r+2)\cdots(r+n-1)}{n!}x^n + \cdots\right\}$$

$$= \frac{r(r+1)(r+2)\cdots(r+n-1)}{n!} = \frac{(r+n-1)!}{n!(r-1)!} = {}^{n+r-1}C_{r-1}$$

2. The number of integral solutions of $x_1 + x_2 + x_3 + \cdots + x_r = n$ where $x_1 \ge 1, x_2 \ge 1, \cdots, x_r \ge 1$ is same as the number of ways to distribute n identical things among r persons each getting at least 1. This also equal to the coefficient of x^n in the expansion of $(x^1 + x^2 + x^3 + \cdots)^r$

$$= \text{coefficient of } x^n \text{ in } \left(\frac{x}{1-x}\right)^r$$

$$= \text{coefficient of } x^n \text{ in } x^r(1-x)^{-r}$$

$$= \text{coefficient of } x^n \text{ in}$$

$$x^r\left\{1 + rx + \frac{r(r+1)}{2!}x^2 + \cdots + \frac{r(r+1)(r+2)\cdots\cdots(r+n-1)}{n!}x^n + \cdots\right\}$$

$$= \text{coefficient of } x^{n-r} \text{ in}$$

$$\left\{1 + rx + \frac{r(r+1)}{2!}x^2 + \cdots + \frac{r(r+1)(r+2)\cdots\cdots(r+n-1)}{n!}x^n + \cdots\right\}$$

$$= \frac{r(r+1)(r+2)\cdots(r+n-r-1)}{(n-r)!} = \frac{r(r+1)(r+2)\cdots(n-1)}{(n-r)!}$$

$$= \frac{(n-1)!}{(n-r)!(r-1)!} = {}^{n-1}C_{r-1}$$

Illustration 7.53 A student is allowed to select utmost n books from a collection of $(2n+1)$ books. If the total number of ways in which he can select one book is 63, then find the value of n.

(a) 2 **(b)** 3

(c) 4 **(d)** None of these

Solution: The student is allowed to select utmost n books out of $(2n+1)$ books. Therefore, in order to select one book he has the choice to select one, two, three, $\cdots$, n books.

Thus, if T is the total number of ways of selecting one book, then

$$T = {}^{2n+1}C_1 + {}^{2n+1}C_2 + \cdots + {}^{2n+1}C_n = 63$$

Again, the sum of binomial coefficients,

$$\begin{aligned}&{}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + \cdots + {}^{2n+1}C_n + {}^{2n+1}C_{n+1}\\ &+ {}^{2n+1}C_{n+2} + \cdots + {}^{2n+1}C_{2n+1} = (1+1)^{2n+1} = 2^{2n+1}\end{aligned}$$

or

$${}^{2n+1}C_0 + 2({}^{2n-1}C_1 + {}^{2n+1}C_2 + \cdots + {}^{2n+1}C_n) + {}^{2n+1}C_{2n+1} = 2^{2n+1}$$

$$\Rightarrow 1 + 2(T) + 1 = 2^{2n+1} \quad \Rightarrow 1 + T = \frac{2^{2n+1}}{2} = 2^{2n}$$

$$\Rightarrow 1 + 63 = 2^{2n} \quad \Rightarrow 2^6 = 2^{2n} \quad \Rightarrow n = 3$$

Illustration 7.54 Find the number of non-negative integral solutions of $x + y + z + w = 20$.

Solution: Any one of the four variables can take values from zero to 20 and hence we construct a polynomial in a variable (say x) with x raised to different powers which would constitute the values that any one variable can take when the equation is solved in the manner indicated. We, thus, consider the product expression

$$\begin{aligned}&(1 + x + x^2 + \cdots + x^{20})(1 + x + \cdots + x^{20})\\ &(1 + x + \cdots + x^{20})(1 + x + \cdots + x^{20})\end{aligned}$$

There are four factors since there are four variables. If we take x^4 in the first factor, x^5 in the second, x^8 in the third, then we take the term x^3 in the fourth so that the sum of the powers $(4 + 5 + 8 + 3 = 20)$ is 20. It is then we say that there is a solution corresponding to $x = 4$, $y = 5$, $z = 8$, $w = 3$. Hence, the number of solutions in the manner required is

Coefficient of x^{20} in $(1 + x + \cdots + x^{20})^4$

$= $ Coefficient of x^{20} in $\left(\dfrac{1 - x^{21}}{1 - x}\right)^4$

$= $ Coefficient of x^{20} in $(1 - x^{21})^4 (1 - x)^{-4}$

$= $ Coefficient of x^{20} in $(1 - x)^{-4}$

$= {}^{23}C_3$

[**Note:** In $(1 - x)^{-4}$ coefficient of x^n is ${}^{(n+3)}C_3$]

Illustration 7.55 Let n and k be positive integers such that $n \geq \dfrac{k(k+1)}{2}$. Find the number of solutions $(x_1, x_2, \ldots, x_k)$, $x_1 \geq 1$, $x_2 \geq 2, \ldots, x_k \geq k$, all integers satisfying $x_1 + x_2 + \cdots + x_k = n$.

Solution: The number of solutions is Coefficient of x^n in

$$(x + x^2 + \cdots + x^n)(x^2 + x^3 + \cdots + x^n)\cdots(x^k + x^{k+1} + \cdots + x^n)$$

$= $ Coefficient of x^n in $x^{\frac{k(k+1)}{2}} \dfrac{(1 - x^n)}{1 - x}\dfrac{(1 - x^{n-1})}{1 - x}\cdots\left(\dfrac{1 - x^{n-k+1}}{1 - x}\right)$

$= $ Coefficient of x^n in $x^{\frac{k(k+1)}{2}} (1 - x^n)(1 - x^{n-1})\cdots(1 - x^{n-k+1})(1 - x)^{-k}$

$= $ Coefficient of $x^{n - \frac{k(k+1)}{2}}$ in $(1 - x)^{-k}$

$$= {}^{\left(n - \frac{k(k+1)}{2} + k - 1\right)}C_{k-1}$$

[**Note:** Coefficient of x^n in $(1 - x)^{-k}$ is ${}^{(n+k-1)}C_{k-1}$]

Illustration 7.56 How many non-negative integral solutions are there in the equation $x + y + z + w = 29$ given $x > 0$, $y > 1$, $z > 2$ and $w \geq 0$?

Solution: The number of solutions is the coefficient of x^{29} in

$$(x + x^2 + \cdots + x^{29})(x^2 + \cdots + x^{29})(x^3 + \cdots x^{29})(1 + x + \cdots + x^{29})$$

$= $ Coefficient of x^{29} in $x^6 \dfrac{1 - x^{29}}{1 - x}\dfrac{1 - x^{28}}{1 - x}\dfrac{1 - x^{27}}{1 - x}\dfrac{1 - x^{30}}{1 - x}$

$= $ Coefficient of x^{23} in $(1 - x^{27})(1 - x^{28})(1 - x^{29})(1 - x^{30})(1 - x)^{-4}$

$= $ Coefficient of x^{23} in $(1 - x)^{-4}$

$$= {}^{26}C_3 = \frac{26 \cdot 25 \cdot 24}{1 \cdot 2 \cdot 3} = 2600$$

Illustration 7.57 How many positive integral solutions are there for the equation $x + y + z + w = 20$?

Solution: The number of positive integral solutions of the given equation is equal to number of ways to divide 20 identical objects among 4 persons such that each gets one or more.

Therefore, the total number of solutions is

$${}^{19}C_3 \; ({}^{n-1}C_{r-1}) = 969$$

7.11 Some Important Points for Solving Geometrical Problems

1. Number of total different straight lines formed by joining n points on a plane of which m ($< n$) are collinear is ${}^nC_2 - {}^mC_2 + 1$.
2. Number of total triangles formed by joining n points on a plane of which m ($< n$) are collinear is ${}^nC_3 - {}^mC_3$.
3. Number of diagonals in a polygon of n sides is ${}^nC_2 - n$.
4. If m parallel lines in a plane are intersected by a family of other n parallel lines, then total number of parallelograms so formed is

$${}^mC_2 \times {}^nC_2 = \frac{mn(m-1)(n-1)}{4}$$

5. Given n points on the circumference of a circle. Then
   (a) Number of straight lines $= {}^nC_2$
   (b) Number of triangles $= {}^nC_3$
   (c) Number of quadrilaterals $= {}^nC_4$

6. If n straight lines are drawn in the plane such that no two lines are parallel and no three lines are concurrent, then the number of parts into which these lines divide the plane is $= 1+\sum n$.
7. Number of rectangles of any size in a square of $n\times n$ is $\sum_{r=1}^{n} r^3$ and number of squares of any size is $\sum_{r=1}^{n} r^2$.
8. In a rectangle of $n\times p\ (n<p)$ number of rectangles of any size is $\frac{np}{4}(n+1)(p+1)$ and number of squares of any size is $\sum_{r=1}^{n}(n+1-r)(p+1-r)$.

Illustration 7.58

(a) How many diagonals are there in an n-sided polygon ($n > 3$)?

(b) How many triangles can be formed by joining the vertices of an n-sided polygon? How many of these triangles have

(i) exactly one side common with that of the polygon?

(ii) exactly two sides common with that of the polygon?

(iii) no sides common with that of the polygon?

Solution:

(a) The number of lines formed by joining the n vertices of a polygon

= Number of selections of 2 points from the given n points

$= {}^nC_2 = \frac{n(n-1)}{2}$

Out of ${}^nC_2$ lines, n lines are the sides of the polygon. Hence

Number of diagonals $= {}^nC_2 - n$

$$= \frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}$$

(b) Number of triangles formed by joining the vertices of the polygon = Number of selections of 3 points from n points

$$= {}^nC_3 = \frac{n(n-1)(n-2)}{3\cdot 2\cdot 1}$$

Let the vertices of the polygon be marked as $A_1, A_2, A_3, \cdots, A_n$.

(i) Select two consecutive vertices A_1, A_2 of the polygon. For the required triangle, we can select the third vertex from the points $A_4, A_5, \cdots, A_{n-1}$. This can be done in ${}^{n-4}C_1$ ways. Also two consecutive points (end points of a side of polygon) can be selected in n ways. Hence, the total number of required triangles $= n\,{}^{n-4}C_1 = n(n-4)$.

(ii) For the required triangle, we have to select three consecutive vertices of the polygon, that is, $(A_1\ A_2\ A_3)$, $(A_2\,A_3\,A_4)$, $(A_3\,A_4\,A_5)$, $\cdots$, $(A_n\,A_1\,A_2)$. This can be done in n ways.

(iii) Triangles having no side common + Triangles having exactly one side common + Triangles having exactly two sides common (with those of the polygon)

= Total number of triangles formed

$\Rightarrow$ Triangles having no side common with those of the polygon

$$= {}^nC_3 - n(n-4) - n = \frac{n(n-1)(n-2)}{3\cdot 2\cdot 1} - n(n-4) - n$$

$$= \frac{n}{6}[n^2 - 3n + 2 - 6n + 24 - 6]$$

$$= \frac{n}{6}[n^2 - 9n + 20] = \frac{n(n-4)(n-5)}{6}$$

Illustration 7.59 Out of 18 points in a plane, no three are in the same straight line except five points, which are collinear. Find the number of

(a) straight lines

(b) triangles which can be formed by joining them.

Solution: Out of 18 points, 5 are collinear.

(a) Number of straight lines $= {}^{18}C_2 - {}^5C_2 + 1 = 153 - 10 + 1 = 144$

(b) Number of triangles $= {}^{18}C_3 - {}^5C_3 = 816 - 10 = 806$

7.12 Problems on Formation of Numbers

Formation of numbers using 10 digits with or without repetition is also one of the counting techniques or combinatorics.
Some illustrations are as follows:

Illustration 7.60 How many numbers are there of nine digits with all different digits? What is their sum?

Solution: In forming the nine-digit numbers, the first digit, excluding zero, can be one of the 9 and the remaining 8 digits, together with zero, making up 9 digits may be used in forming the next 8 digits in ${}^9P_8 = 9!$ ways.

Hence, the required number of 9-digit numbers = 9(9!)
Now regarding their sum:
With 1 in the units digit there are $8 \times {}^8P_7 = 8(8!)$ numbers; and the 1 in all these numbers added up make up a sum 8(8!). The same is true of numbers with 2 in the units digit, but their sum is 8(8!) 2. This way the sum of all the numbers in the unit digit is 8(8!) (1 + 2 + ⋯ + 9) = 8(8!) 45. If any digit, instead of being in the unit place, is in the tenths place the value will be 10.

Proceeding in the same way, sum of all nine-digit number with different digits is

$$8 \times 8! \times 45(1 + 10 + 10^2 + \cdots + 10^8)$$

Illustration 7.61 Given the digits 0, 1, 2, 3, 4 and 5.

(a) How many five-digit numbers can be formed?

(b) How many of the five-digit numbers ending with zero?

(c) How many of the five-digit numbers with the odd digits in odd places?

Solution:

(a) Five-digit numbers (number of) = 5(5!) = 600

(b) Number of numbers ending with zero $= {}^5P_4 = 120$

(c) The 3 odd numbers may form the 3 odd places in 3! ways and the 3 even numbers may form the two even places in 3! ways. Thus, the total number of number is 3! 3! = 36.

Illustration 7.62 How many four-digit numbers are there which contain not more than two different digits?

Solution: Such four-digit numbers that can

(a) contain only one digit, numbers like 1111, 2222, …, and their number is 9

(b) contain only two different digits of which

those with zero, ${}^9C_1\,({}^3C_1 + {}^3C_2 + {}^3C_3)$	63
those without zero, ${}^9C_2\,({}^4C_1 + {}^4C_2 + {}^4C_3)$	504
Total	576

Your Turn 3

1. In how many ways can 5 prizes be distributed among four students when every student can take one or more prizes?

Ans. $4^5 = 1024$

2. A question paper is divided into two parts A and B and each part contains 5 questions. Find the number of ways in which a candidate can answer 6 questions selecting at least two questions from each part.

Ans. 200

3. The number of diagonals in an octagon will be

(A) 28 **(B)** 20
(C) 10 **(D)** 16

Ans. (B)

4. The number of straight lines joining 8 points on a circle is **[MP PET 1984]**

(A) 8 **(B)** 16
(C) 24 **(D)** 28

Ans. (D)

5. The number of triangles that can be formed by choosing the vertices from a set of 12 points, seven of which lie on the same straight line, is

(A) 185 **(B)** 175
(C) 115 **(D)** 105

Ans. (A)

6. How many numbers greater than 7×10^5 can be formed from the digits 0, 5, 6, 7, 8 and 9 if

(A) repetitions are allowed? **(B)** repetitions are not allowed?

Ans. (A) 2×6^5, (B) $2 \times 6!$

7. How many seven-digit numbers are there which read the same way from either side?

Ans. 9×10^3

Additional Solved Examples

1. In a chess tournament, all participants were to play one game with the other. Two players fell ill after having played 3 games each. If total number of games played in the tournament is equal to 84, then total number of participants in the beginning was

(A) 10 **(B)** 15
(C) 12 **(D)** 14

Solution: Let there were 'n' players in the beginning. Total number of games played in the tournament is ${}^nC_2$ and each player would have played $(n - 1)$ games. Thus,

$$
\begin{aligned}
&{}^nC_2 - ((n-1) + (n-1) - 1) + 6 = 84 \\
&\Rightarrow n^2 - 5n - 150 = 0 \\
&\Rightarrow n = 15
\end{aligned}
$$

Hence, the correct answer is option (B).

2. Number of ways in which four letters of the word 'DEGREE' can be selected is

(A) 7 **(B)** 6
(C) $\dfrac{6!}{3!}$ **(D)** None of these

Solution: In 'DEGREE' we have three E's and D, G, R.

Four letters can be selected in following ways:

(i) Three alike, one different letter ${}^3C_1 \times {}^3C_3$
(ii) Two alike, two different letters ${}^3C_2$
(iii) All different letters ${}^3C_3$
Total number of ways $= {}^3C_1 + {}^3C_2 + {}^3C_3 = 7$

Hence, the correct answer is option (A).

3. If letters of the word 'KUBER' are written in all possible orders and arranged as in a dictionary, then rank of the word 'KUBER' will be

(A) 67 **(B)** 68
(C) 65 **(D)** 69

Solution: Alphabetical order of these letters is B, E, K, R, U.

Total words starting with B = 4! = 24
Total words starting with E = 4! = 24
Total words starting with KB = 3! = 6
Total words starting with KE = 3! = 6
Total words starting with KR = 3! = 6
Next word will be KUBER.
Thus rank of the word KUBER = 24 + 24 + 18 + 1 = 67
Hence, the correct answer is option (A).

4. A is a set containing n elements. A subset P of A is chosen. The set A is reconstructed by replacing the elements of P. A subset Q of A is again chosen. The number of ways of chosen P and Q so that $P \cap Q = \phi$ is

(A) $2^{2n} - {}^{2n}C_n$ **(B)** 2^n
(C) $2^n - 1$ **(D)** 3^n

Solution: Let $A = \{a_1, a_2, a_3, \ldots, a_n\}$. For $a_i \in A$, we have the following choices:

(i) $a_i \in P$ and $a_i \in Q$ **(ii)** $a_i \in P$ and $a_i \notin Q$
(iii) $a_i \notin P$ and $a_i \in Q$ **(iv)** $a_i \notin P$ and $a_i \notin Q$

Out of these, only (ii), (iii) and (iv) imply $a_i \notin P \cap Q$. Therefore, the number of ways in which none of $a_1, a_2, \ldots, a_n$ belong to $P \cap Q$ is 3^n.

Hence, the correct answer is option (D).

5. Total number of three letter words that can be formed from the letters of the word 'SAHARANPUR' is equal to

(A) 210 **(B)** 237
(C) 247 **(D)** 227

Solution: The word 'SAHARANPUR' has 1S, 3A, 1H, 2R, 1N, 1P, 1U.

When all letters are different,

$$\text{Corresponding ways} = {}^7C_3 .\, 3! = {}^7P_3 = 210$$

When two letters are of one kind and the other is different,

$$\text{Corresponding ways} = {}^2C_1 .{}^6C_1 .\frac{3!}{2!} = 36$$

When all letters are alike, corresponding ways = 1.

Thus, total number of words that can be formed is

$$210 + 36 + 1 = 247$$

Hence, the correct answer is option (C).

6. If n objects are arranged in a row, then the number of ways of selecting three objects so that no two of them are next to each other is

(A) $\frac{(n-2)(n-3)(n-4)}{6}$ **(B)** ${}^{n-2}C_3$
(C) ${}^{n-3}C_3 + {}^{n-3}C_2$ **(D)** All of these

Solution: Let x_0 be the number of objects to the left of the first object chosen, x_1 the number of objects between the first and the second, x_2 the number of objects between the second and the third and x_3 the number of objects to the right of the third object. We have

$$x_0, x_3 \geq 0\,,\, x_1, x_2 \geq 1 \text{ and } x_0 + x_1 + x_2 + x_3 = n - 3 \qquad (1)$$

The number of solutions of Eq. (1)

$= \text{Coefficient of } y^{n-3} \text{ in } (1 + y + y^2 + \cdots)(1 + y + y^2 + \cdots)(y + y^2 + y^3 + \cdots)(y + y^2 + y^3 + \cdots)$

$= \text{Coefficient of } y^{n-3} \text{ in } y^2(1 + y + y^2 + y^3 + \cdots)^4$

$= \text{Coefficient of } y^{n-5} \text{ in } (1 - y)^{-4}$

$= \text{Coefficient of } y^{n-5} \text{ in } (1 + {}^4C_1\, y + {}^5C_2\, y^2 + {}^6C_3\, y^3 + \cdots)$

$= {}^{n-5+3}C_{n-5} = {}^{n-2}C_3$

$= \frac{(n-2)(n-3)(n-4)}{6}$

Also, ${}^{n-3}C_3 + {}^{n-3}C_2 = {}^{n-2}C_3$.

Hence, the correct answer is option (D).

7. Ten different letters of English alphabet are given. Words with five letters are formed from these given letters. Then, the number of words which have at least one letter repeated is

(A) 69760 **(B)** 30240
(C) 99748 **(D)** None of these

Solution: Suppose we have 5 places, each of which is to be filled by one letter from the 10 letters. The first place may be filled in 10 ways. When repetitions of the letters are allowed, the second place may also be filled in 10 ways.

Proceeding in this way, it is clear that words with five letters are formed in 10^5 ways. These 10^5 ways also include the number of ways of forming words with all different letters without repetition. These are ${}^{10}P_5$ in number.

Therefore, the number of words which have at least one letter repeated is

$$10^5 - {}^{10}P_5 = 100000 - 30240 = 69760$$

Hence, the correct answer is option (A).

8. The value of the expression ${}^{47}C_4 + \sum_{j=1}^{5} {}^{(52-j)}C_3$ is equal to

(A) ${}^{47}C_3$ **(B)** ${}^{52}C_5$
(C) ${}^{52}C_4$ **(D)** None of these

Solution: The given expression is

$({}^{47}C_4 + {}^{47}C_3) + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3$

$= ({}^{48}C_4 + {}^{48}C_3) + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3$

Using the formula ${}^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$ repeatedly we get

$= ({}^{49}C_4 + {}^{49}C_3) + {}^{50}C_3 + {}^{51}C_3 = ({}^{50}C_4 + {}^{50}C_3) + {}^{51}C_3$

$= {}^{51}C_4 + {}^{51}C_3 = {}^{52}C_4$

Hence, the correct answer is option (C).

9. Ten persons are arranged in a row. The number of ways of selecting four persons so that no two persons sitting next to each other are selected is

(A) 34 **(B)** 36 **(C)** 35 **(D)** None of these

Solution: To each selection of 4 persons we associate binary sequence of the form 1001001010 where 1(0) at i^{th} place means the i^{th} person is selected (not selected).

There exists one-to-one correspondence between the set of selections of 4 persons and set of binary sequence containing 6 zeros and 4 ones.

We are interested in the binary sequences in which no 2 ones are consecutive. We first arrange 6 zeros:

0 0 0 0 0 0

This can be done in just one way.

Now, 4 ones can be arranged at any of the 4 places marked with a cross in the following arrangement:

$$\times 0 \times 0 \times 0 \times 0 \times 0 \times 0 \times$$

We can arrange 4 ones at 7 places in ${}^7C_4 = 35$ ways.

Hence, the correct answer is option (C).

10. Fifteen persons, amongst whom are A, B and C, are to speak at a function. Find in how many ways can the speech be done if A wants to speak before B and B is to speak before C?

Solution: We can select three positions out of 15 positions by ${}^{15}C_3$ ways.

We can provide these positions to A, B, C in only one way.

Other 12 persons can speak in 12! ways.

Hence, total number of ways will be ${}^{15}C_3 \times 1 \times 12\,!$

11. A committee of 12 is to be formed from 9 women and 8 men. In how many ways can this be done if at least five women have to be included in the committee? In how many of these committees

(a) The women are in a majority?

(b) The men are in a majority?

Solution: The possible ways of formation of the committee are listed as follows:

Constitution of the committee			Number of ways of formation
(9) Women	(8) Men		
5	7	$\longrightarrow$	${}^9C_5 \cdot {}^8C_7 = 1008$
6	6	$\longrightarrow$	${}^9C_6 \cdot {}^8C_6 = 2352$
7	5	$\longrightarrow$	${}^9C_7 \cdot {}^8C_5 = 2016$
8	4	$\longrightarrow$	${}^9C_8 \cdot {}^8C_4 = 630$
9	3	$\longrightarrow$	${}^9C_9 \cdot {}^8C_3 = 56$
	Total number of ways		6062

(a) Number of committees with women in majority = 2016 + 630 + 56 = 2702.

(b) Number of committees with men in majority = 1008.

12. How many seven-digit numbers can be formed using only the three digits 1, 2 and 3 with the digit 2 occurring only twice in each number.

Solution: Any two of the seven digits can be chosen, and in these places 2 is filled and rest five are filled with 1 or 3.

Therefore, the required number is ${}^7C_2 \cdot 2^5 = 672$

13. Suppose $A_1, A_2, \cdots, A_{30}$ are 30 sets each with five elements and $B_1, B_2, \cdots, B_n$ are n sets each with three elements. Let $\bigcup_{i=1}^{30} A_i = \bigcup_{j=1}^{n} B_j = S$. Assume that each element of S belongs to exactly 10 of the Ai's and to exactly nine of the Bj's. Find n.

Solution:

(a) $\bigcup_{i=1}^{30} A_i = \bigcup_{j=1}^{n} B_j = 5$

Since each of the 30 sets contains 5 elements, therefore

$$\text{Total number of elements in sets } A = 150$$

Since each element of S belongs to exactly 10 of the A's, therefore

$$\text{Number of elements in } S = \frac{150}{10} = 15$$

Also, $\bigcup_{j=1}^{n} B_j = S$.

Each of these n sets of B contains only 3 elements, therefore

$$\text{Number of elements in sets } B = 3n \quad (1)$$

Also, since each elements of S belongs to exactly 9 of the B's, therefore

$$\text{Number of elements of } B\text{'s} = 15 \times 9 = 135 \quad (2)$$

From Eqs. (1) and (2), we get

$$3n = 135 \Rightarrow n = 45$$

Hence, the correct answer is (45).

14. Five balls of different colours are to be placed in three boxes of different sizes. Each box can hold all five. In how many different ways can we place the balls so that no box remains empty?

Solution: Let us find out the various possibilities in which five balls of different colours can be placed in the three boxes.

First box	Second box	Third box	Numbers of ways
1	1	3	${}^5C_1 \times {}^4C_1 \times {}^3C_3 = 20$
1	3	1	${}^5C_1 \times {}^4C_3 \times {}^1C_1 = 20$
3	1	1	${}^5C_3 \times {}^2C_1 \times {}^1C_1 = 20$
2	1	2	${}^5C_2 \times {}^3C_1 \times {}^2C_2 = 30$
2	2	1	${}^5C_2 \times {}^3C_2 \times {}^1C_1 = 30$
1	2	2	${}^5C_1 \times {}^4C_2 \times {}^2C_2 = 30$

Total number of ways = 20 + 20 + 20 + 30 + 30 + 30 = 150

Hence, the correct answer is (150).

15. In how many ways can you divide 52 cards in 4 sets, three of them having 17 cards each and the fourth one just 1 card?

Solution: In the first set, 17 cards out of 52 can be put in ${}^{52}C_{17}$ ways.

In the second set, 17 cards out of the remaining can be put in ${}^{35}C_{17}$ ways.

In the third set, 17 cards out of the remaining 18 in ${}^{18}C_{17}$ ways.

In the last set, 1 card can be put only in 1 way.

Total number of ways in which 52 cards can be divided such that first 3 sets contain 17 cards and fourth set only one card is

$$\frac{52!}{35!17!} \times \frac{35!}{18!17!} \times \frac{18!}{17!1!} \times 1 = \frac{52!}{(17!)^3}.$$

The first three sets containing 17 cards each can be interchanged among themselves in 3! ways.

Therefore, total number of ways in the given problem is

$$\frac{52!}{(17!)^3 3!}$$

Hence, the correct answer is $\left(\frac{52!}{(17!)^3 3!}\right)$.

16. If n distinct objects are arranged in a circle, show that the number of ways of selecting three of these n things so that no two of them is next to each other is $\frac{n(n-4)(n-5)}{6}$.

Solution: Let the n things be $x_1, x_2, \cdots, x_n$.

The first choice may be one of these n things, and this is done in ${}^nC_1$ ways.

Suppose x_1 is the one chosen. The next two may be chosen – excluding x_1 and, the two next to x_1, namely, x_2, x_n from the remaining $(n-3)$ in ${}^{n-3}C_2$ ways.

Of these ${}^{n-3}C_2$ there are $(n-4)$ selections when the second two chosen are next to each other, like $x_3x_4, x_4x_5, \cdots, x_{n-2}x_{n-1}$.

Therefore, the number of ways of selecting the second two after x_1 is chosen, so that the two are not next to each other is

$$ {}^{n-3}C_2 - (n-4) = \frac{(n-3)(n-4)}{1\cdot 2} - (n-4) = \frac{(n-4)(n-5)}{2} $$

The two objects can be relatively interchanged in two ways. Further, the order of the choice of the three is not to be considered. Hence, the number of ways of choice of the three is

$$ \frac{n(n-4)(n-5)}{2}\frac{2!}{3!} = \frac{n(n-4)(n-5)}{6} $$

17. $2n$ persons are to be seated n on each side of a long table. r ($<n$) particular persons desire to sit on one side; and s ($<n$) other persons desire to sit on the other side. In how many ways can the persons be seated?

Solution: For the side where r persons desire to sit, we need $(n-r)$ more persons. This $(n-r)$ may be chosen from $(2n-r-s)$ in ${}^{(2n-r-s)}C_{n-r}$ ways. Automatically, the remaining $(n-s)$ persons go to the other side where already there are s desirous of seating. Thus, there are ${}^{(2n-r-s)}C_{n-r}$ ways of distributing n persons for each side provided with the restriction of r on one side and s on the other side. n persons on each side can be permuted in n seats in $n!$ ways. The number of ways of seating the $2n$ persons, n on each side, is therefore, ${}^{(2n-r-s)}C_{n-r}(n!)^2$.

Previous Years' Solved JEE Main/AIEEE Questions

1. The set $S = \{1, 2, 3, \ldots, 12\}$ is to be partitioned into three sets A, B and C of equal size. Thus, $A \cup B \cup C = S, A \cap B = B \cap C = A \cap C = \phi$. The number of ways to partition S is

(A) $\dfrac{12!}{3!(4!)^3}$ **(B)** $\dfrac{12!}{3!(3!)^4}$

(C) $\dfrac{12!}{(4!)^3}$ **(D)** $\dfrac{12!}{(3!)^4}$

[AIEEE 2007]

Solution: The total number of ways is

$$ {}^{12}C_4 \times {}^{12-4}C_4 \times {}^{12-4-4}C_4 = {}^{12}C_4 \times {}^8C_4 \times {}^4C_4 = \frac{12!}{8!4!} \times \frac{8!}{4!4!} \times 1 = \frac{12!}{(4!)^3} $$

Hence, the correct answer is option (C).

2. In a shop there are five types of ice creams available. A child buys six ice creams.

Statement 1: The number of different ways the child can buy the six ice creams is ${}^{10}C_5$.

Statement 2: The number of different ways the child can buy the six ice creams is equal to the number of different ways of arranging 6 A's and 4 B's in a row.

(A) Statement 1 is false, Statement 2 is true

(B) Statement 1 is true, Statement 2 is true; Statement 2 is a correct explanation for Statement 1

(C) Statement 1 is true, Statement 2 is true; Statement 2 is not a correct explanation for Statement 1

(D) Statement 1 is true, Statement 2 is false

[AIEEE 2008]

Solution: If $x_1 + x_2 + x_3 + x_4 + x_5 = 6$, we need to find the number of integral solutions, which is given by

$$ {}^{5+6-1}C_{5-1} = {}^{10}C_4 $$

Therefore, Statement 1 is false.

The total number of different ways of arranging the 6 A's and 4 B's in a row is $\dfrac{10!}{6!\times 4!} = {}^{10}C_4$, which is equal to the total number of ways the child can get six ice creams.

Therefore, Statement 2 is true.

Hence, the correct answer is option (A).

3. How many different words can be formed by jumbling the letters in the word MISSISSIPPI in which no two S are adjacent?

(A) $8 \cdot {}^6C_4 \cdot {}^7C_4$ **(B)** $6 \cdot 7 \cdot {}^8C_4$

(C) $6 \cdot 8 \cdot {}^7C_4$ **(D)** $7 \cdot {}^6C_4 \cdot {}^8C_4$

[AIEEE 2008]

Solution: Other than the letter S, the seven letters M, I, I, I, P, P and I can be arranged in $\dfrac{7!}{2!4!} = 7\cdot 5\cdot 3$

Now, four S can be placed in eight spaces ways between the 7 letters in ${}^8C_4$ ways. Therefore, the required number of ways is

$$ 7.5.3.{}^8C_4 = 7.{}^6C_4.{}^8C_4 $$

Hence, the correct answer is option (D).

4. From 6 different novels and 3 different dictionaries, 4 novels and 1 dictionary are to be selected and arranged in a row on a shelf so that the dictionary is always in the middle. Then the number of such arrangements is

(A) less than 500.

(B) at least 500 but less than 750.

(C) at least 750 but less than 1000.

(D) at least 1000.

[AIEEE 2009]

Solution: The four novels can be selected from six novels in ${}^6C_4$ ways.

One dictionary can be selected from three dictionaries in ${}^3C_1$ ways.

As the dictionary selected is fixed in the middle, the remaining four novels can be arranged in 4! ways.

Therefore, the required number of ways of arrangement = ${}^6C_4 \times {}^3C_1 \times 4! = 1080$

Hence, the correct answer is option (D).

5. There are two urns. Urn A has 3 distinct red balls and urn B has 9 distinct blue balls. From each urn two balls are taken out at random and then transferred to the other. The number of ways in which this can be done is

(A) 36 **(B)** 66
(C) 108 **(D)** 3

[AIEEE 2010]

Solution: From Urn A 2 balls are taken out in ${}^3C_2$ ways,

From Urn B 2 balls are taken out in ${}^9C_2$ ways.

So, total number of ways is

$${}^3C_2 \times {}^9C_2 = 3 \times \frac{9 \times 8}{2} = 3 \times 36 = 108$$

Hence, the correct answer is option (C).

6. Statement 1: The number of ways of distributing 10 identical balls in 4 distinct boxes such that no box is empty is ${}^9C_3$.

Statement 2: The number of ways of choosing any 3 places from 9 different places is ${}^9C_3$.

(A) Statement 1 is true, Statement 2 is true; Statement 2 is not a correct explanation for Statement 1
(B) Statement 1 is true, Statement 2 is false
(C) Statement 1 is false, Statement 2 is true
(D) Statement 1 is true, Statement 2 is true; Statement 2 is a correct explanation for Statement 1

[AIEEE 2011]

Solution: Since we want at least one ball in each box, therefore formula for no box empty is,

$${}^{(n-1)}C_{(r-1)} = {}^{(10-1)}C_{(4-1)} = {}^9C_3$$

where n is the number of identical balls and r is the number of distinct boxes.

To partition 10 identical balls, we put 4 identical partitions (Fig. 7.5).

Now, when empty boxes are allowed, the number of ways are ${}^{10+4-1}C_{4-1}$.

But here at least one ball is in each partition, so we apply the same formula that is, ${}^{n+r-1}C_{r-1}$ but now n is made $(n-4)$.

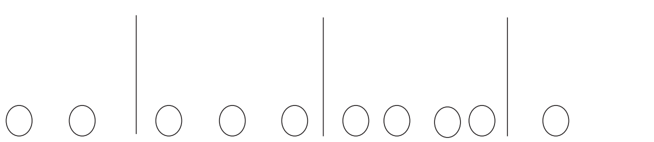


Figure 7.5

Therefore, ${}^{n-4+4-1}C_{4-1} = {}^{n-1}C_3 = {}^{10-1}C_3 = {}^9C_3$.

Therefore, Statement-1 is correct. The number of ways of choosing any 3 places from 9 different places is ${}^9C_3$, which is also correct. From 9, we can select 3 in ${}^9C_3$ ways. But Statement-2 is not the correct explanation of statement-1.

Hence, the correct answer is option (A).

7. Assuming the balls to be identical except for difference in colours, the number of ways in which one or more balls can be selected from 10 white, 9 green and 7 black balls is

(A) 880 **(B)** 629
(C) 630 **(D)** 879

[AIEEE 2012]

Solution: Number of ways of selecting one or more balls from 10 white, 9 green and 7 black balls is

$$(10 + 1)(9 + 1)(7 + 1) - 1 = 11 \times 10 \times 8 - 1 = 879$$

Hence, the correct answer is option (D).

8. Let T_n be the number of all possible triangles formed by joining vertices of an n-sided regular polygon. If $T_{n+1} - T_n = 10$, then the value of n is

(A) 5 **(B)** 10
(C) 8 **(D)** 7

[JEE 2013]

Solution: If $T_{n+1} - T_n = 10$, then the value of n is obtained as follows:

$${}^{n+1}C_3 - {}^nC_3 = 10 \Rightarrow \frac{(n+1)(n)(n-1)}{6} - \frac{n(n-1)(n-2)}{6} = 10$$
$$\Rightarrow n(n-1)(\cancel{n} + 1 - \cancel{n} + 2) = 60$$
$$\Rightarrow n(n-1) = 20 \Rightarrow n(n-1) = 5 \times 4$$

Therefore, $n = 5$.

Hence, the correct answer is option (A).

9. Let A and B be two sets containing 2 elements and 4 elements, respectively. The number of subsets of $A \times B$ having 3 or more elements is

(A) 220 **(B)** 219
(C) 211 **(D)** 256

[JEE MAIN 2013]

Solution: We know that $A \times B$ will have eight elements. Out of these 8 elements, the total number of subsets containing 3 or more elements is

$${}^8C_3 + {}^8C_4 + {}^8C_5 + {}^8C_6 + {}^8C_7 + {}^8C_8$$
$$= {}^8C_0 + {}^8C_1 + \ldots + {}^8C_8 - {}^8C_0 - {}^8C_1 - {}^8C_2$$
$$= 2^8 - {}^8C_0 - {}^8C_1 - {}^8C_2 = 256 - 1 - 8 - 28 = 219$$

Hence, the correct answer is option (B).

10. The sum of the digits in the unit's place of all the 4-digit numbers formed by using the numbers 3, 4, 5 and 6, without repetition is

(A) 432 **(B)** 108
(C) 36 **(D)** 18

[JEE MAIN 2014 (ONLINE SET-1)]

Solution: If any number comes at unit place, then total ways in which number can be formed is 3! ways. So, sum of all numbers at unit place is given by

$$3!(6+5+4+3)=6(18)=108$$

Since,

$$\underbrace{3\ 4\ 5}_{3!\text{ ways}}\ 6\quad \underbrace{3\ 4\ 6}_{3!\text{ ways}}\ 5\quad \underbrace{3\ 5\ 6}_{3!\text{ ways}}\ 4\quad \underbrace{4\ 5\ 6}_{3!\text{ ways}}\ 3$$

Hence, the correct answer is option (B).

11. An eight-digit number divisible by 9 is to be formed using digits from 0 to 9 without repeating the digits. The number of ways in which this can be done is

(A) 72 (7!) **(B)** 18 (7!)
(C) 40(7!) **(D)** 36 (7!)

[JEE MAIN 2014 (ONLINE SET-2)]

Solution: A number is divisible by 9 if sum of its digits is divisible by 9.

Therefore, the following cases are there to form such eight-digit number.

Case I: Digits used are 1, 2, 3, 4, 5, 6, 7, 8

Case II: Digits used are 0, 2, 3, 4, 5, 6, 7, 9

Case III: Digits used are 0, 1, 3, 4, 5, 6, 8, 9

Case IV: Digits used are 0, 1, 2, 4, 5, 7, 8, 9

Case V: Digits used are 0, 1, 2, 3, 6, 7, 8, 9

In case I, digits can be arranged in $7 \times 7!$ ways.

Therefore, total number of eight-digit numbers divisible by 9 is

$$8!+4\times(7\times7!)$$
$$=8\times7!+28\times7!$$
$$=36\times7!$$

Hence, the correct answer is option (D).

12. 8-digit numbers are formed using the digits 1, 1, 2, 2, 2, 3, 4, 4. The number of such numbers in which the odd digits do not occupy odd places is

(A) 160 **(B)** 120
(C) 60 **(D)** 48

[JEE MAIN 2014 (ONLINE SET-3)]

Solution:

1	2	3	4	5	6	7	8

Out of 4 even places, 3 can be selected in ${}^4C_3$ ways. The odd digits 1, 1, 3 at these places can be arranged in $\frac{3!}{2!}$ ways.

Also, at rest 5 places 5 even digits 2, 2, 2, 4, 4 can be arranged in $\frac{5!}{3!2!}$ ways.

Hence,

$$\text{Total number of ways } = {}^4C_3\times\frac{3!}{2!}\times\frac{5!}{3!2!}=120$$

Hence, the correct answer is option (B).

13. Two women and some men participated in a chess tournament in which every participant played two games with each of the other participants. If the number of games that the men played between themselves exceeds the number of games that the men played with the women by 66, then the number of men who participated in the tournament lies in the interval:

(A) [8, 9] **(B)** [10, 12)
(C) (11, 13] **(D)** (14, 17)

[JEE MAIN 2014 (ONLINE SET-4)]

Solution: Let the number of men who participated be n. See Fig. 7.6.

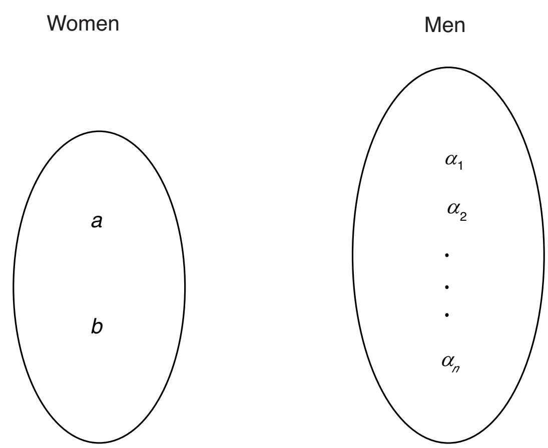


Figure 7.6

Number of matches men played with women = $2(n+n)=4n$

Number of matches men played among themselves = $2\,{}^nC_2$

Now, according to the question,

$$2\,{}^nC_2=4n+66 \Rightarrow \frac{n(n-1)}{2\times1}=2n+33$$
$$\Rightarrow n^2-n=4n+66$$
$$\Rightarrow n^2-5n-66=0$$

Therefore,

$$n=\frac{5\pm\sqrt{25+264}}{2}=\frac{5\pm\sqrt{289}}{2}=\frac{5\pm17}{2}=11 \text{ or } -6$$

Therefore, the number of men is 11, which lies in the interval (10, 12).

Hence, the correct answer is option (B).

14. The number of integers greater than 6000 that can be formed using the digits 3, 5, 6, 7 and 8 without repetition is

(A) 192 **(B)** 120
(C) 72 **(D)** 216

[JEE MAIN 2015 (OFFLINE)]

Solution: Any number greater than 6000 but less than 10,000 that can be formed using the digits 3, 5, 6, 7 and 8 without repetition has its thousand place digit 6, 7 or 8.

Therefore, for the first left place, number of choices = 3
For second left place, number of choices = 4
For third left place, number of choices = 3
For fourth left place number of choices = 2

Therefore, the number of 4-digit numbers greater than 6000 = 72

Now, if we use all the 5 integers the number obtained is definitely greater than 6000; number of such numbers = 5! = 120

Therefore, total numbers formed = 72 + 120 = 192

Hence, the correct answer is option (A).

15. The number of ways of selecting 15 teams from 15 men and 15 women, such that each team consists of a man and a woman, is

(A) 1120 **(B)** 1240
(C) 1880 **(D)** 1960

[JEE MAIN 2015 (ONLINE SET-1)]

Solution: Male members of 1st team can be selected in 15 ways and female member of 1st team can be selected in 15 ways.

Therefore, 1st team can be selected out in 15 × 15 = $(15)^2$ ways.

Now, we are left with 14 men and 14 women. 2nd team can be selected out in $(14)^2$ ways.

Similarly, 3rd team can be selected out in $(13)^2$ ways and so on.

Therefore, the total number of ways of selecting 15 teams is

$$(15)^2+(14)^2+(13)^2+\cdots+(2)^2+(1)^2=\sum_{n=1}^{15} n^2=\frac{15(16)(31)}{6}=1240 \text{ ways.}$$

Hence, the correct answer is option (B).

16. If all the words (with or without meaning) having five letters formed by using the letters of the word 'SMALL' are arranged as in a dictionary, then the position of the word SMALL is

(A) 58th **(B)** 46th
(C) 59th **(D)** 52nd

[JEE MAIN 2016 (OFFLINE)]

Solution: For the word SMALL

The number of word starting from A $=\frac{4!}{2!}=12$

The number of words starting from L = 4! = 24

The number of words starting from M $=\frac{4!}{2!}=12$

The number of words starting with SA $=\frac{3!}{2!}=3$

The number of words starting with SL = 3! = 6

The number of words starting with SMALL is 1.

Therefore, the position of the word SMALL to occur is 12 + 24 + 12 + 3 + 6 + 1 = 58th position.

Hence, the correct answer is option (A).

17. The value of $\sum_{r=1}^{15} r^2\left(\frac{^{15}C_r}{^{15}C_{r-1}}\right)$ is equal to

(A) 1240 **(B)** 560
(C) 1085 **(D)** 680

[JEE MAIN 2016 (ONLINE SET-1)]

Solution: We know that

$$\frac{^nC_r}{^nC_{r-1}}=\frac{n-r+1}{r},$$

$$\sum_{r=1}^{n} r^2\frac{^nC_r}{^nC_{r-1}}=\sum_{r=1}^{n} r^2\cdot\frac{(n-r+1)}{r}$$

$$=\sum_{r=1}^{n} r(n-r+1)$$

Now,

$$\sum_{r=1}^{n}(n+1)r-r^2=\frac{(n+1)n(n+1)}{2}-\frac{n(n+1)(2n+1)}{6}$$

Substituting $n = 15$, we get

$$\frac{15\times16\times16}{2}-\frac{15\times16\times31}{2.3}=\frac{15\times16}{2}\left(16-\frac{31}{3}\right)$$

$$=\frac{15\times16\times17}{2\times3}=680$$

Hence, the correct answer is option (D).

18. If the four letter words (need not be meaningful) are to be formed using the letters from the word 'MEDITERRANEAN' such that the first letter is R and the fourth letter is E, then the total number of all such words is

(A) 110 **(B)** 59
(C) $\frac{11!}{(2!)^3}$ **(D)** 56

[JEE MAIN 2016 (ONLINE SET-1)]

Solution: If the first letter is R and the last letter is E, then the total number of all such words is obtained as follows:

M(1)
E(3)
D(1)
I(1)
T(1)
R(2)
A(2)
N(2)

Both letters to be distinct = $^8C_2 \times 2! = 56$

Both letters to be identical = $^3C_1 \cdot 1 = 3$

Total number of words = 59

Hence, the correct answer is option (B).

19. If $\frac{^{n+2}C_6}{^{n-2}P_2} = 11$, then n satisfies the equation:

(A) $n^2 + n - 110 = 0$ **(B)** $n^2 + 2n - 80 = 0$
(C) $n^2 + 3n - 108 = 0$ **(D)** $n^2 + 5n - 84 = 0$

[JEE MAIN 2016 (ONLINE SET-2)]

Solution: We have

$$\frac{^{n+2}C_6}{^{n-2}P_2} = 11$$

$$\Rightarrow \frac{(n+2)!(n-2-2)!}{6!(n+2-6)!(n-2)!} = 11$$

$$\Rightarrow \frac{(n+2)!}{6!(n-2)!} = 11$$

$$\Rightarrow \frac{(n+2)(n+1)n(n-1)}{6!} = 11$$

$$\Rightarrow (n^2 + n)(n^2 + n - 2) = 7920 = 90 \times 88$$

Therefore, we get

$$n^2 + n = 90$$
$$n^2 + n - 90 = 0$$

and

$$n^2 + n - 2 = 88$$
$$n^2 + n - 90 = 0$$

So,

$$(n + 10)(n - 9) = 0 \Rightarrow n = 9, -10 \Rightarrow n = 9$$

(As n cannot be negative)

So, the equation given in option (C) satisfies the value $n = 9$.

Hence, the correct answer is option (C).

Previous Years' Solved JEE Advanced/ IIT-JEE Questions

1. The letters of the word 'COCHIN' are permuted and all the permutations are arranged in an alphabetical order as in an English dictionary. The number of words that appear before the word 'COCHIN' is

(A) 360 **(B)** 192
(C) 96 **(D)** 48

[IIT-JEE 2007]

Solution: The given word is 'COCHIN'. The alphabets comprising it are C, C, H, I, N, O.

The number of words starting with $CC = 4! = 24$.

The number of words starting with $CH = 4! = 24$.

The number of words starting with $CI = 4! = 24$.

The number of words starting with $CN = 4! = 24$.

Now, the first word of the series CO is COCHIN.

Therefore, the number of words that appear before the word 'COCHIN' is 96.

Hence, the correct answer is option (C).

2. Consider all possible permutations of the letters of the word 'ENDEANOEL'. Match the statements/expressions in **Column I** with the values given in **Column II**.

Column I	Column II
(A) The number of permutations containing the word ENDEA is	**(P)** 5!
(B) The number of permutations in which the letter E occurs in the first and the last position is	**(Q)** $2\times5!$
(C) The number of permutations in which none of the letters D, L, N occurs in the last five positions is	**(R)** $7\times5!$
(D) The number of permutations in which the letters A, E, O occur only in odd positions is	**(S)** $21\times5!$

[IIT-JEE 2008]

Solution:

ENDEANOEL

(A) → (P)

ENDEA, N, O, E, L are five different letters, then permutation = 5!

(B) → (S)

If E is in the first and last position,

$$\frac{(9-2)!}{2!} = \frac{7\times6\times5!}{2} = 7\times3\times5! = 21\times5!$$

(C) → (Q)

Arrangements of last five letters $= \frac{5!}{3!} = 20$

$\Rightarrow$ Number of permutation $= 12\times20 = 240 = 2\times5!$

(D) → (Q)

Arrangements of O, E and A $= \frac{5!}{3!}$ and that of other letters $= \frac{4!}{2!}$

$\Rightarrow$ Number of permutation $= \frac{5!}{3!}\times\frac{4!}{2!} = 2\times5!$

Hence, the correct matches are (A)→(P); (B)→(S); (C)→(Q); (D)→(Q).

3. The number of seven-digit integers with sum of the digits equal to 10 and formed by using the digits 1, 2 and 3 only is

(A) 55 **(B)** 66
(C) 77 **(D)** 88

[IIT-JEE 2009]

Solution:

Coefficient of x^{10} in $(x + x^2 + x^3)^7$ = Coefficient of x^3 in $(1 + x + x^2)^7$

= Coefficient of x^3 in $(1 - x^3)^7(1 - x)^{-7}$

$$= {^{7+3-1}C_3} - 7$$
$$= {^9C_3} - 7$$
$$= \frac{9\times8\times7}{6} - 7 = 77$$

Alternate Solution:

The digits are 1, 1, 1, 1, 1, 2, 3 or 1, 1, 1, 1, 2, 2, 2

Hence, the number of seven-digit numbers formed is

$$\frac{7!}{5!}+\frac{7!}{4!3!}=77$$

Hence, the correct answer is option (C).

4. The total number of ways in which 5 balls of different colours can be distributed among 3 persons so that each person gets at least one ball is

(A) 75 **(B)** 150
(C) 210 **(D)** 243

[IIT-JEE 2012]

Solution:

$$\text{Number of ways } = 3^5 - {}^3C_1 \cdot 2^5 + {}^3C_2 \cdot 1^5$$
$$= 243 - 96 + 3 = 150$$

Hence, the correct answer is option (B).

5. Let $n_1 < n_2 < n_3 < n_4 < n_5$ be positive integers such that $n_1 + n_2 + n_3 + n_4 + n_5 = 20$. Then the number of such distinct arrangements $(n_1, n_2, n_3, n_4, n_5)$ is ______.

[JEE ADVANCED 2014]

Solution: $n_1 < n_2 < n_3 < n_4 < n_5$

$$n_1 = n_1 \quad (1)$$
$$n_2 = \underbrace{n_1 + \alpha_1 + 1} \quad (2)$$
$$n_3 = n_2 + \alpha_2 + 1 = \overbrace{(n_1 + \alpha_1 + 1)} + \alpha_2 + 1 = \underbrace{n_1 + \alpha_1 + \alpha_2 + 2} \quad (3)$$
$$n_4 = n_3 + \alpha_3 + 1 = \overbrace{n_1 + \alpha_1 + \alpha_2 + 2} + \alpha_3 + 1 = \underbrace{n_1 + \alpha_1 + \alpha_2 + \alpha_3 + 3} \quad (4)$$
$$n_5 = n_4 + \alpha_4 + 1 = \overbrace{n_1 + \alpha_1 + \alpha_2 + \alpha_3 + 3} + \alpha_4 + 1$$
$$= n_1 + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) + 4 \quad (5)$$

Adding Eqs. (1), (2), (3), (4) and (5) we get

$$20 = 5n_1 + 4\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4 + 10$$

Let $n_1 = \alpha_0 + 1$. Therefore,

$$20 = 5\,(\alpha_0 + 1) + 4\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4 + 10$$
$$\Rightarrow 5\alpha_0 + 4\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4 = 5 \quad (6)$$

where $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \geq 0$ and integers are values which are introduced to adjust the gaps in the numbers n_1, n_2, n_3, n_4 and n_5.

Now, the problem transforms into finding non-negative integral solutions of Eq. (6).

Therefore, numbers of solutions is equal finding the

Coefficient of x^5 in $\left(\frac{1}{1-x^5}\right)\left(\frac{1}{1-x^4}\right)\left(\frac{1}{1-x^3}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^2}\right)$

= Coeff. of x^5 in $(1 - x)^{-1}(1 - x^2)^{-1}(1 - x^3)^{-1}(1 - x^4)^{-1}(1 - x^5)^{-1}$

= Coeff. of x^5 in $\underbrace{(1 + x + x^2 + x^3 + x^4 + x^5 + \cdots)(1 + x^2 + x^4 + x^6 + \cdots)}$

= Coeff. of x^5 in $\underbrace{(1 + x^3 + x^6 + x^9 + \cdots)(1 + x^4 + x^8 + \cdots)}(1 + x^5 + \cdots)$

= Coeff. of x^5 in $(1 + x^2 + x^4 + x + x^3 + x^5 + x^2 + x^4 + x^3 + x^5 + x^4 + x^5)$ $(1 + x^4 + x^3)(1 + x^5)$

= Coeff. of x^5 in $(1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5)(1 + x^5 + x^4 + x^3)$

= 1 + 1 + 2 + 3 = 7

Note: {Terms having powers more than 5 are not required.}

Note: Finding the number of solutions in Eq. (6) can be understood as making a sum 5 by choosing tickets numbered 1, 2, 3, 4, 5 taking none or more tickets.

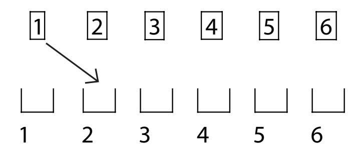
Hence, the correct answer is (7).

6. Six cards and six envelopes are numbered 1, 2, 3, 4, 5, 6 and cards are to be placed in envelopes so that each envelope contains exactly one card and no card is placed in the envelope bearing the same number and moreover the card numbered 1 is always placed in envelope numbered 2. Then the number of ways it can be done is

(A) 264 **(B)** 265
(C) 53 **(D)** 67

[JEE ADVANCED 2014]

Solution:



This is problem of derangement. Number of derangements of n things is

$$n!\sum_{k=0}^{n}\frac{(-1)^k}{k!} = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots \frac{(-1)^n}{n!}\right]$$

In the equation, two possibilities are there;

(i) When card two goes to envelope 1.

Derangement of 3, 4, 5, 6 cards is

$$4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)$$
$$= 24\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right) = 12 - 4 + 1$$
$$= 9 \text{ ways}$$

(ii) When card two does not go to envelope 1.

Now, derangement of 2, 3, 4, 5, 6 in envelopes 1, 3, 4, 5, 6 is

$$5!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right)$$
$$= 120\left(1 - \frac{1}{1} + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right)$$
$$= 60 - 20 + 5 - 1 = 44 \text{ ways}$$

Therefore, total number of ways = 9 + 44 = 53

Hence, the correct answer is option (C).

7. Let n be the number of ways in which 5 boys and 5 girls can stand in a queue in such a way that all the girls stand consecutively in the queue. Let m be the number of ways in which 5 boys and 5 girls can stand in a queue in such a way that exactly four girls stand consecutively in the queue. Then the value of m/n is ______.

[JEE ADVANCED 2015]

Solution: Let

n = (5 girls are consecutive) = $6!\cdot 5!$

m = (4 girls are consecutive)

= Arrange 5 boys in a queue in 5! ways, arrange 4 girls out of 5 together in $({}^5C_4\cdot 4!)$ ways.

Put a girl and group of 4 girls (together) in any two places out of 6 between the 5 boys in ${}^6P_2$ ways. Therefore

$$m = 5!\,({}^5C_4\cdot 4!)\cdot {}^6P_2$$

So,

$$\frac{m}{n} = \frac{5!\cdot {}^5C_4\cdot 4!\cdot {}^6P_2}{6!\cdot 5!} = 5$$

Hence, the correct answer is (5).

8. A debate club consists of 6 girls and 4 boys. A team of 4 members is to be selected from this club including the selection of a captain (from among these 4 members) for the team. If the team has to include at most one boy, then the number of ways of selecting the team is

(A) 380 **(B)** 320
(C) 260 **(D)** 95

[JEE ADVANCED 2016]

Solution: The club consists of 6 girls and 4 boys. If a team of 4 members is to be selected which consists at most 1 boy (including 1 captain), then the number of ways of selecting the team is obtained as follows:

$${}^4C_1({}^4C_1\cdot{}^6C_3 + {}^6C_4) = 4(80 + 15) = 380 \text{ ways}$$

Hence, the correct answer is option (A).

Practice Exercise 1

1. A convex polygon has 44 diagonals. The number of its sides is

(A) 9 **(B)** 10
(C) 11 **(D)** 12

2. A polygon has 65 diagonals. The number of its sides is

(A) 8 **(B)** 10
(C) 11 **(D)** 13

3. Everybody in a room shakes hand with everybody else. The total number of handshakes is equal to 153. The total number of persons in the room is equal to

(A) 18 **(B)** 19
(C) 17 **(D)** 16

4. Number of triangles that can be formed joining the angular points of decagon is

(A) 30 **(B)** 20
(C) 90 **(D)** 120

5. A class contains three girls and four boys. Every Saturday, five students go on a picnic, a different group of students is being sent each week. During the picnic, each girl in the group is given a doll by the accompanying teacher. All possible groups of five have gone once. The total number of dolls the girls have got is

(A) 21 **(B)** 45
(C) 27 **(D)** 24

6. The number of all the odd divisors of 3600 is

(A) 45 **(B)** 4
(C) 18 **(D)** 9

7. There are 'n' numbered seats around a round table. Total number of ways in which $n_1 (n_1 < n)$ persons can sit around the round table is equal to

(A) ${}^nC_{n_1}$ **(B)** ${}^nP_{n_1}$
(C) ${}^nC_{n_1-1}$ **(D)** ${}^nP_{n_1-1}$

8. Total number of words that can be formed using the alphabets of the word 'KUBER', so that no alphabet is repeated in any of the formed word, is equal to

(A) 325 **(B)** 320
(C) 240 **(D)** 365

9. The number of words from the letters of the word 'BHARAT' in which B and H never come together is

(A) 360 **(B)** 240
(C) 120 **(D)** None of these

10. The total number of three-digit numbers, the sum of whose digits is even, is equal to

(A) 450 **(B)** 350
(C) 250 **(D)** 325

11. The number of ways 6 different flowers can be given to 10 girls, if each can receive any number of flowers is

(A) 6^{10} **(B)** 10^6
(C) 60 **(D)** ${}^{10}C_6$

12. Number of permutations of n different objects taken $r\,(\geq 3)$ at a time which includes 3 particular objects is

(A) ${}^nP_r \times 3!$ **(B)** ${}^nP_{r-3} \times 3!$
(C) ${}^{n-3}P_{r-3}$ **(D)** ${}^rP_3 \times {}^{n-3}P_{r-3}$

13. A teacher takes three children from her class to the zoo at a time as often as she can, but she doesn't take the same set of three children more than once. She finds out that she goes to the zoo 84 times more than a particular child goes to the zoo. Total number of students in her class is equal to

(A) 12 **(B)** 14
(C) 10 **(D)** 11

14. Five balls of different colours are to be placed in three boxes of different sizes. Each box can hold all five balls. The number of ways in which we can place the balls in the boxes (order is not considered in the box) so that no box remains empty is

(A) 150 **(B)** 300
(C) 200 **(D)** None of these

15. Total number of ways of selecting two numbers from the set {1, 2, 3, 4, …, 3*n*}, so that their sum is divisible by 3 is equal to

(A) $\frac{2n^2-n}{2}$ **(B)** $\frac{3n^2-n}{2}$
(C) $2n^2-n$ **(D)** $3n^2-n$

16. There are 20 persons among whom two are brothers. The number of ways in which we can arrange them around a circle so that there is exactly one person between the brothers is

(A) 19! **(B)** $2\times 18!$
(C) 2! 17! **(D)** None of these

17. The total number of 4-digit numbers that are greater than 3000, that can be formed using the digits 1, 2, 3, 4, 5, 6 (no digit is being repeated in any number) is equal to

(A) 120 **(B)** 240
(C) 480 **(D)** 80

18. Let *A* be the set of 4-digit numbers $a_1a_2a_3a_4$ where $a_1 > a_2 > a_3 > a_4$. Then $n(A)$ is equal to

(A) 126 **(B)** 84
(C) 210 **(D)** None of these

19. The total number of flags with three horizontal strips, in order, that can be formed using 2 identical red, 2 identical green and 2 identical white strips is equal to

(A) 4! **(B)** 3 (4!)
(C) 2 (4!) **(D)** None of these

20. Two teams are to play a series of 5 matches between them. A match ends in a win or loss or draw for a team. A number of people forecast the result of each match and no two people make the same forecast for the series of matches. The smallest group of people in which one person forecasts correctly for all the matches will contain *n* people, where *n* is

(A) 81 **(B)** 243
(C) 486 **(D)** None of these

21. The total number of four-digit numbers having all different digits is equal to

(A) 4536 **(B)** 504
(C) 5040 **(D)** 720

22. The number of ways in which 6 men can be arranged in a row so that three particular men are consecutive is

(A) ${}^4P_4$ **(B)** ${}^4P_4\times{}^3P_3$
(C) ${}^6P_6\times{}^3P_3$ **(D)** ${}^3P_3\times{}^3P_3$

23. The total number of ways in which a person can put 8 different rings in the fingers of his right hand is equal to

(A) ${}^{16}P_8$ **(B)** ${}^{11}P_8$
(C) ${}^{16}C_8$ **(D)** ${}^{11}C_8$

24. The number of solutions of $x_1+x_2+x_3=51$ (x_1, x_2, x_3 being odd natural numbers) is

(A) 300 **(B)** 325 **(C)** 330 **(D)** 350

25. In a certain test, there are *n* questions. In this test, 2^{n-i} students gave wrong answers to at least *i* questions, where $i = 1, 2, \cdots, n$. If the total number of wrong answers given is 2047, then *n* is equal to

(A) 10 **(B)** 11 **(C)** 12 **(D)** 13

26. The number of times of the digits 3 will be written when listing the integers from 1 to 1000 is

(A) 269 **(B)** 300 **(C)** 271 **(D)** 302

27. In a plane, there are two families of lines $y = x + r$, $y = -x + r$, where $r \in \{0, 1, 2, 3, 4\}$. The number of squares of diagonals of length 2 formed by the lines is

(A) 9 **(B)** 16 **(C)** 25 **(D)** None of these

28. The number of ways of arranging six persons (having A, B, C and D among them) in a row so that A, B, C and D are always in order ABCD (not necessarily together) is

(A) 4 **(B)** 10 **(C)** 30 **(D)** 720

29. Let $A = \{x \mid x$ is a prime number and $x < 30\}$. The number of different rational numbers whose numerator and denominator belong to *A* is

(A) 90 **(B)** 180 **(C)** 91 **(D)** None of these

30. Let *S* be the set of all functions from the set *A* to the set *A*. If $n(A) = k$, then $n(S)$ is

(A) $k!$ **(B)** k^k **(C)** $2^k - 1$ **(D)** 2^k

31. In a plane, there are 37 straight lines, of which 13 pass through the point *A* and 11 pass through the point *B*. Besides, no three lines pass through one point, no line passes through both points *A* and *B*, and no two are parallel. Then the number of intersection points the lines have is equal to

(A) 535 **(B)** 601
(C) 728 **(D)** None of these

32. A set contains $(2n + 1)$ elements. The number of subsets of the set which contain at most *n* elements is

(A) 2^n **(B)** 2^{n+1} **(C)** 2^{n-1} **(D)** 2^{2n}

33. Let *p* be a prime number such that $p \geq 3$. Let $n = p! + 1$. The number of primes in the list $n + 1, n + 2, n + 3, \cdots, n + p - 1$ is

(A) $p - 1$ **(B)** 2 **(C)** 1 **(D)** None of these

34. The number of ordered pairs of non-negative integers {*x*, *y*} having sum 7596 without any carries is equal to

(A) 3300 **(B)** 3360
(C) 270 **(D)** 3320

35. The smallest possible value of $S = a_1\cdot a_2\cdot a_3 + b_1\cdot b_2\cdot b_3 + c_1\cdot c_2\cdot c_3$, where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ is a permutation of the number 1, 2, 3, 4, 5, 6, 7, 8, and 9 is

(A) 213 **(B)** 216
(C) 324 **(D)** 214

36. Two teachers are interviewing top 6 students in CAT exam, in two different subjects starting at the same time. Each teacher interviews for 15 minutes. The number of ways in which interview can be scheduled is

(A) 6! **(B)** 5!
(C) 44 x 6! **(D)** 265 x 6!

37. The total number of 4-digit numbers which have digits in decreasing order (from left to right) such that the number is divisible by 4 is

(A) 110 **(B)** 112
(C) 118 **(D)** 111

38. Total number of even divisors of 1323000 which are divisible by 105 is

(A) 36 **(B)** 48
(C) 64 **(D)** 54

39. The number of ways in which a necklace can be formed using 6 identical diamonds and 3 identical pearls is

(A) $\frac{8!}{6!2!}\times\frac{1}{2}$ **(B)** $\frac{8!}{6!2!}$
(C) 8 **(D)** None of these

40. How many odd perfect squares are divisors of the product $1!2!3!\cdots 9!$?

(A) 22 **(B)** 42
(C) 30 **(D)** None of these

41. A is a set containing n_1 elements and B is another set containing n_2 elements. The number of non-decreasing functions from $A\to B$ is

(A) ${}^{n_1+n_2-1}C_{n_1}$ **(B)** ${}^{n_1+n_2-1}C_{n_2}$
(C) ${}^{n_1}C_{n_2}$ **(D)** ${}^{n_2}C_{n_1}$

42. The number of ways in which 420 can be resolved into three positive factors other than unity and having LCM 420 is

(A) 6 **(B)** 12
(C) 18 **(D)** None of these

43. The number of ways in which we can select three numbers from the set $\{10,11,\cdots,100\}$, such that they form a GP with a common ratio greater than 1 is

(A) 18 **(B)** 19
(C) 20 **(D)** None of these

44. 8 different persons are sitting around a table. Numbers of ways in which we can select 3 of them so that no two are consecutive is

(A) 12 **(B)** 16
(C) 15 **(D)** 22

45. A rectangle of dimension $m\times n$ (m and n being odd) is divided into square of unit length by drawing horizontal and vertical lines. The number of rectangles having side of odd unit length is

(A) m^2n^2 **(B)** $\left(\frac{(m+1)(n+1)}{4}\right)^2$
(C) $\left(\frac{(m-1)(n-1)}{2}\right)^2$ **(D)** $(2m-1)(2n-1)$

46. A man has 7 relatives in which 4 are ladies and 3 are gentlemen. His wife also has 7 relatives in which 3 are ladies and 4 are gentlemen. In how many ways can the couple invite the relative for a dinner party, 3 ladies and 3 gentlemen, so that 3 of them are man's relatives and 3 of them are his wife's relatives?

47. There are n points in a plane, no three of which are collinear except 'p' points all of which are on a line. How many (a) straight lines can be formed and (b) triangles can be formed out of these n points?

48. Given the digits 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, how many four-digit numbers can be formed?

49. If the letters of the word 'MOTHER' be permuted among themselves and the words so formed are arranged as in a dictionary, what is the rank of the word 'MOTHER'?

50. Six boys and six girls sit in a row. Find the number of ways in which they can be seated

(a) when the girls are all together.

(b) when the boys and girls are seated alternately.

(c) when the girls are separated.

Discuss the above problem in the case when the boys and girls are seated in a circle.

51. Three boys picked five apples. In how many ways can these five apples be distributed among the three boys so that each can have any number, of course, not exceeding five? (All apples are considered the same.)

52. There are n points in a plane which are joined in all possible ways by indefinite straight lines, and no two of these joining lines are parallel and no three of them meet in a point. Find the number of points of intersection, excluding the n given points.

53. All possible two-factor products are formed from the numbers $1, 2, 3, \ldots, 100$. How many numbers out of the total obtained are multiples of 3?

54. Five persons are to address a meeting. If a specified speaker is to speak before another specified speaker, find the number of ways in which this could be arranged. In how many of these arrangements will the first speaker come immediately before the second?

55. How many even numbers lying between 200 and 500 can be formed from the figures 1, 2, 3, 4, 5 and 6 if no figure is to appear more than once in any number.

56. Find the number of positive integers which can be formed by any number of digits 0, 1, 2, 3, 4 and 5 but using each digit not more than once in each number. How many of these integers are greater than 3000?

57. How many different numbers can be formed to satisfy all the conditions given below:

(a) The number is less than 2×10^8.
(b) The number is formed from the digits 0, 1 and 2.
(c) The number is divisible by 3.

Practice Exercise 2

Single/Multiple Correct Choice Type Questions

1. The number of 4-digit natural numbers in which digit at 1000^{th} places are 1 and exactly one of the digit comes twice are

(A) 108 **(B)** 216
(C) 432 **(D)** 864

2. Consider 26 tangent lines to an ellipse. The lines separate the plane into several regions, some enclosed and others unbounded. Then the number of unbounded regions is

(A) 50 **(B)** 52
(C) ${}^{26}C_2$ **(D)** None of these

3. Given 6 different toys of green colour, 5 different toys of blue colour and 4 different toys of red colour. Combination of toys that can be chosen taking at least one green and one blue toys are

(A) 31258 **(B)** 31248
(C) 31 **(D)** 63

4. Given distinct lines $L_1, L_2, \cdots, L_{1000}$ in which all lines of the form L_{4n}, where n is a positive integer, are parallel to each other. All lines L_{4n-3}, are concurrent at a point. The maximum number of the points of intersection of pairs of line from the complete set $(L_1, L_2, \cdots, L_{1000})$ is

(A) 437251 **(B)** 437250
(C) 437252 **(D)** 437200

5. The number of integral points on the hyperbola $x^2 - y^2 = (2000)^2$ is (an integral point is a point both of whose co-ordinates are integer)

(A) 98 **(B)** 96
(C) 48 **(D)** 24

6. The number of three-digit number abc formed such that $a \neq b \neq c$ and $2b = a + c$ is

(A) 12 **(B)** 256
(C) 32 **(D)** 36

7. The number of three-digit natural numbers divisible by 3 in which at least one of the digit is repeated is

(A) 64 **(B)** 66
(C) 68 **(D)** 72

8. The number of ways of choosing triplet (x, y, z) such that $z \geq \max\{x, y\}$ and $x, y, z \in \{1, 2, \cdots, n, n + 1\}$ is

(A) ${}^{n+1}C_3 + {}^{n+2}C_3$ **(B)** $\frac{1}{6}n(n + 1)(2n + 1)$
(C) $1^2 + 2^2 + \ldots + n^2$ **(D)** $2({}^{n+2}C_3) - {}^{n+1}C_2$

Comprehension Type Questions

Paragraph for Questions 9–11: The integers a, b and c are selected from $3n$ consecutive integers $\{1, 2, 3, \cdots, 3n\}$. Then in how many ways can these integers be selected such that,

9. $(a^2 - b^2)$ is divisible by 3

(A) $n^2 + {}^nC_2$ **(B)** $n^2 + {}^{n+1}C_2$
(C) $n^2 + 3{}^nC_2$ **(D)** None of these

10. $(a^3 + b^3)$ is divisible by 3

(A) $\frac{3n^2 - n}{2}$ **(B)** $\frac{3n^2 + n}{2}$
(C) $\frac{n(n+1)}{2}$ **(D)** None of these

11. Their sum is divisible by 3

(A) $\frac{n}{2}(3n^2 - 3n + 2)$ **(B)** $3n^2 - 3n + 2$
(C) $\frac{n}{2}$ **(D)** None of these

Paragraph for Questions 12–14: If a cricket team of 11 players is to be selected from 8 batsmen, 6 bowlers, 4 all-rounder and 2 wicket keepers, then

12. The number of selections when at most 1 all-rounder and 1 wicket keeper will play is

(A) ${}^4C_1 \cdot {}^{14}C_{10} + {}^2C_1 \cdot {}^{14}C_{10} + {}^4C_1 \cdot {}^2C_1\ {}^2C_1 \cdot {}^{14}C_9 + {}^{14}C_{11}$
(B) ${}^4C_1 \cdot {}^{15}C_{11} + {}^{15}C_{11}$
(C) ${}^4C_1 \cdot {}^{15}C_{10} + {}^{15}C_{11}$
(D) None of these

13. Number of selection when 2 particular batsmen do not want to play when a particular bowler will play is

(A) ${}^{17}C_{10} + {}^{19}C_{11}$ **(B)** ${}^{17}C_{10} + {}^{19}C_{11} + {}^{17}C_{11}$
(C) ${}^{17}C_{10} + {}^{20}C_{11}$ **(D)** ${}^{19}C_{10} + {}^{19}C_{11}$

14. The number of selections when a particular batsman and a particular wicket keeper do not want to play together is

(A) $2 \cdot {}^{18}C_{10}$ **(B)** ${}^{19}C_{11} + {}^{18}C_{10}$
(C) ${}^{19}C_{10} + {}^{19}C_{11}$ **(D)** None of these

Matrix Match Type Questions

15. Match the following:

Column I	Column II
(A) Let y be a element of the set $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$ and x_1, x_2, x_3 be integers such that $x_1x_2x_3 = y$. If λ be the number of integral solutions of $x_1x_2x_3 = y$, then λ is divisible by	(p) 4
(B) If λ is the number of ways in which 6 boys and 5 girls can be seated around a round table and if all the 6 girls do not sit together, then λ is divisible by	(q) 6
(C) If λ is the number of solutions of $x + y + z = 15$ such that $x \geq 1$, $y \geq 2$ and $z \geq 3$, then λ is divisible by	(r) 8
(D) If n is the number of ways in which 12 different things can be distributed in 5 sets of 2, 2, 3, 3 things, then $\frac{(3!)^3(2!)^4\, 5!}{12!} \times n$ is divisible by	(s) 5

16. Consider a three-digit number $x_1x_2x_3$ such that $x_1x_2x_3 \in N$. Match the following:

Column I	Column II
(A) Numbers possible, if $x_1 < x_2 \le x_3$	(*p*) 66
(B) Numbers possible, if $x_1 = x_2 \ge x_3$	(*q*) 189
(C) Numbers possible, if $x_1 x_2 x_3 = 480$	(*r*) 45
(D) Number of positive integral solution of $x_1 + x_2 + x_3 = 10$	(*s*) 120

17. Match the following:

List I	List II
(A) The number of integral solutions of $x + y + z = 1$, $x \ge -4, y \ge -4, z \ge -4$, is	(i) 40
(B) The number of arrangements of the word BANANA in which two N's do not appear together is	(ii) 99
(C) If $a_1, a_2, a_3, \dots, a_{100}$ are in HP then value of $\sum_{i=1}^{99} \frac{a_i a_{i+1}}{a_1 a_{100}}$ is	(iii) 105
(D) If 8 points out of 11 points are in same straight line then number of triangles formed is	(iv) 109

Integer Type Questions

18. The number of ordered triplet (*x*, *y*, *z*) such that LCM (*x*, *y*) = 3375, LCM (*y*, *z*) = 1125 and LCM (*z*, *x*) = 3375 is __________.

19. The number of ordered pairs (*a*, *b*) such that the equations $ax + by = 1$ and $x^2 + y^2 = 50$ have all solutions integral is __________.

20. Find the sum of the integers lying between 1 and 100 and divisible by 3 or 5 or 7.

Answer Key

Practice Exercise 1

1. (C)	**2.** (D)	**3.** (A)	**4.** (D)	**5.** (B)
6. (D)	**7.** (B)	**8.** (A)	**9.** (B)	**10.** (A)
11. (B)	**12.** (D)	**13.** (C)	**14.** (A)	**15.** (B)
16. (B)	**17.** (C)	**18.** (C)	**19.** (A)	**20.** (B)
21. (A)	**22.** (B)	**23.** (B)	**24.** (B)	**25.** (B)
26. (B)	**27.** (B)	**28.** (C)	**29.** (C)	**30.** (B)
31. (A)	**32.** (D)	**33.** (D)	**34.** (B)	**35.** (D)
36. (D)	**37.** (C)	**38.** (D)	**39.** (D)	**40.** (C)
41. (A)	**42.** (A)	**43.** (A)	**44.** (B)	**45.** (B)
46. 485	**47.** **(i)** ${}^nC_2 - {}^pC_2 + 1$, **(ii)** ${}^nC_3 - {}^pC_3$		**48.** 540	**49.** 309

50. **(i)** $7!6!$, **(ii)** $2(6!)^2$, **(iii)** ${}^7P_6(7!)$; For circular arrangement: **(i)** 6! 6! = 518400, **(ii)** 5! 6! = 86400, **(iii)** 5! 6! = 86400

51. ${}^7C_2$	**52.** $\frac{1}{8} \cdot n(n-1)(n-2)(n-3)$		**53.** 2739	**54.** 60, 24
55. 28	**56.** 1380	**57.** 4373		

Practice Exercise 2

1. (C)	**2.** (B)	**3.** (B)	**4.** (A)	**5.** (A)
6. (D)	**7.** (D)	**8.** (B)	**9.** (C)	**10.** (A)
11. (A)	**12.** (A)	**13.** (B)	**14.** (B)	**15.** (A) → (*p*, *r*); (B) → (*p*, *q*, *s*); (C) → (*s*); (D) → (*p*, *q*, *r*, *s*)
16. (A) → (*s*); (B) → (*r*); (C) → (*q*); (D) → (*p*)	**17.** (A) → (iii); (B) → (i); (C) → (ii); (D) → (iv)	**18.** 50	**19.** 66	**20.** 2838

Solutions

Practice Exercise 1

1. If number of sides is n, then

Total number of diagonals of a convex polygon is

$$^nC_2 - n = 44 \text{ (given)}$$

$$\Rightarrow \frac{n!}{(n-2)!2!} - n = 44$$

$$\Rightarrow \frac{n(n-1)}{2} - n = 44$$

$$\Rightarrow n(n-1) - 2n = 88$$

$$\Rightarrow n^2 - 3n - 88 = 0$$

$$\Rightarrow (n-11)(n+8) = 0$$

$$\Rightarrow n = 11$$

2. Let number of sides be n, then

$$^nC_2 - n = 65$$

$$\Rightarrow n(n-3) = 130 \Rightarrow n = 13$$

3. If there are n persons, then the total number of handshakes = $^nC_2 = 153$ (given)

$$\Rightarrow \frac{n!}{(n-2)!2!} = 153$$

$$\Rightarrow \frac{n(n-1)}{2} = 153$$

$$\Rightarrow n(n-1) = 306$$

$$\Rightarrow n^2 - n - 306 = 0$$

$$\Rightarrow (n-18)(n+17) = 0$$

$$\Rightarrow n = 18$$

4. The total number of points in a decagon = 10

So, the number of triangles formed joining these 10 points is

$$^{10}C_3 = 120$$

5. The number of times, a particular girl goes on picnic = $^6C_4$

Now, each time she goes, she gets a doll.

Thus, total number of dolls given to the girls is

$$^6C_4\, 3 = \frac{6!}{4!2!}\cdot 3 = 45$$

6. $3600 = 2^4 \times 3^2 \times 5^2$

For odd divisors, only odd prime numbers are to be included.

So, number of odd divisors = $(2 + 1)(2 + 1) = 9$

7. When seats are numbered, circular permutation is same as linear permutation.

Thus, total number of sitting arrangements is equal to $^nP_{n_1}$.

8. Total words having exactly one alphabet = 5

Total words having exactly two alphabets = $5 \cdot 4 = 20$

Total words having exactly three alphabets = $5\cdot4\cdot3 = 60$

Total words having exactly 4 alphabets = $5\cdot4\cdot3\cdot2 = 120$

Total words having exactly 4 alphabets = $5\cdot4\cdot3\cdot2\cdot1 = 120$

Thus, total words that can be formed = 5 + 20 + 60 + 120 + 120 = 325

9. The number of words in which B and H will come together is

$$\frac{5!}{2!}\times 2! = 120$$

Number of total words having all the alphabets is

$$\frac{6!}{2!} = 360$$

So, the number of words in which B and H will never come together = 360 – 120 = 240

10. Let the three-digits of number $n = x_1, x_2, x_3$.

Since, $x_1 + x_2 + x_3$ is even, there are following cases:

(a) x_1, x_2, x_3 all are even. So

$4\cdot5\cdot5 = 100$ ways

(b) x_1 even and x_2, x_3 are odd. So

$4\cdot5\cdot5 = 100$ ways

(c) x_1 odd, x_2 even, x_3 odd. So

$5\cdot5\cdot5 = 125$ ways

(d) x_1 odd, x_2 odd, x_3 even. So

$5\cdot5\cdot5 = 125$ ways

11. Number of ways is 10^6.

12. Number of ways of selecting $(r-3)$ objects out of $(n-3)$ objects

$$= {}^{n-3}C_{r-3}$$

Number of ways of arranging these r objects = $r!$

Total number of ways = $^{n-3}C_{r-3} \times r! = {}^rP_3 \times {}^{n-3}P_{r-3}$

13. Let the number of students be n.

Then total number of times the teacher goes to zoo is equal to $^nC_3$ and total number of times, a particular student goes to the zoo is equal to $^{n-1}C_2$. Thus

$$^nC_3 - {}^{n-1}C_2 = 84$$

$$\Rightarrow \frac{n!}{(n-3)!3!} - \frac{(n-1)!}{(n-3)!2!} = 84$$

$$\Rightarrow \frac{n(n-1)(n-2)}{3!} - \frac{(n-1)(n-2)}{2} = 84$$

$$\Rightarrow n(n-1)(n-2) - 3(n-1)(n-2) = 504$$

$$\Rightarrow (n-1)(n-2)(n-3) = 504$$

$$\Rightarrow (n-1)(n-2)(n-3) = 9\cdot8\cdot7$$

$$\Rightarrow n = 10$$

14. One possible arrangement is

2	2	1

Three such arrangements are possible.

Therefore, the number of ways is

$$(^5C_2)(^3C_2)(^1C_1)(3) = 90$$

The other possible arrangement,

1	1	3

Three such arrangements are possible.

In this case, the number of ways is $({}^5C_1)({}^4C_1)({}^3C_3)(3) = 60$

Hence, the total number of ways is 90 + 60 = 150.

15. Given numbers can be rearranged as

$1\ 4\ 7 \cdots 3n-2 \rightarrow 3\lambda - 2$ type

$2\ 5\ 8 \cdots 3n-1 \rightarrow 3\lambda - 1$ type

$3\ 6\ 9 \cdots 3n \rightarrow 3\lambda$ type

This means we must take two numbers from last row or one number each from first and second row.

$$\text{Total ways} = {}^nC_2 + {}^nC_1\,{}^nC_1 = \frac{n(n-1)}{2} + n^2$$

$$= \frac{3n^2 - n}{2}$$

16. We can arrange 18 persons around a circle in (18 − 1)! = 17! ways.

Now, there are exactly 18 places where we can arrange the two brothers. Also, the two brothers can be arranged in 2! Ways.

Thus, the number of ways of arranging the persons subject to the given condition is (17!)(18)(2!) = 2(18!).

17. Let the formed number is $x_1 x_2 x_3 x_4$. Clearly, $x_1 \geq 3$.

Thus, the total number of such numbers is

$$4{\cdot}5{\cdot}4{\cdot}3 = 240$$

18. Any selection of four digits from the ten digits 0, 1, 2, 3, ⋯, 9 gives one such number.

So, the required number of numbers is

$${}^{10}C_4 = 210$$

19. All strips are of different colours, then number of flags 3! = 6

When two strips are of same colour, then

$$\text{Number of flags} = {}^3C_1 \frac{3!}{2} \cdot {}^2C_1 = 18$$

Therefore, total flags = 6 + 18 = 24 = 4!

20. The smallest number of people = Total number of possible forecasts

Total number of possible results = 3 × 3 × 3 × 3 × 3 = 243

21. Let the number be $x, x_2 x_3 x_4$.

Then x_1 can be chosen in 9 ways. x_2 can be chosen in 9 ways. Similarly, x_3 and x_4 can be chosen in 8 and 7 ways, respectively.

Therefore, total number of such numbers = 9·9·8·7 = 4536

22. Considering three particular persons as a single group.

Number of ways in which these four can be arranged in a row is ${}^4P_4$. Those three can arrange themselves in ${}^3P_3$ ways. So, total number of ways = ${}^4P_4 \times {}^3P_3$.

23. $x_1 + x_2 + x_3 + x_4 = 8$

Non-negative integral solution = ${}^{11}C_3 = {}^{11}C_8$

Total number of ways = ${}^{11}C_8 \times 8! = {}^{11}P_8$

24. Let odd natural numbers be $2a - 1, 2b - 1, 2c - 1$, where a, b, c are natural numbers

$$2a - 1 + 2b - 1 + 2c - 1 = 51$$

$$2a + 2b + 2c = 48$$

$$a + b + c = 24$$

$$a \geq 1, b \geq 1, c \geq 1$$

Number of solutions is the coefficient of x^{24} in $(1 - x)^{-3}$ is (1)

$${}^{26}C_2 = 13 \times 25 = 325$$

25. The number of students answering exactly i $(1 \leq i \leq n - 1)$ questions wrongly is $2^{n-i} - 2^{n-i-1}$.

The number of students answering all n questions wrongly is 2^0. Thus, the total number of wrong answers is

$1(2^{n-1} - 2^{n-2}) + 2(2^{n-2} - 2^{n-3}) + \dots + (n-1)(2^1 - 2^0) + n(2^0) + \dots$

$= 2^{n-1} + 2^{n-2} + 2^{n-3} + \dots + 2^0 = 2^n - 1$

Thus,

$$2^n - 1 = 2047 \Rightarrow 2^n = 2048 = 2^{11} \Rightarrow n = 11$$

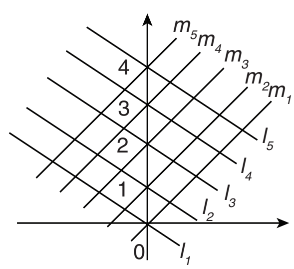
26. Since 3 do not occur in 1000, we have to count the number of times 3 occurs when we list the integers from 1 to 999. Any number between 1 and 999 is of the form xyz where $0 \leq x, y, z \leq 9$.

Let us first count the numbers in which 3 occurs exactly once. Since, 3 can occur at one place in ${}^3C_1$ ways, there are ${}^3C_1\,(9 \times 9) = 3 \times 9^2$ such numbers.

Next, 3 can occur exactly at two places in $({}^3C_2)(9) = 3 \times 9$ such numbers.

Lastly, 3 can occur in all three digits in one number only. Hence, the number of times 3 occurs is $1 \times (3 \times 9^2) + 2 \times (3 \times 9) + 3 \times 1 = 300$

27. There are two sets of five parallel lines at equal distances. Clearly, lines like l_1, l_3, m_1, m_3 form a square whose length of the diagonal is 2.



So, the number of required squares = 3 × 3

[Since, choices are $(l_1, l_3), (l_2, l_4)$ (l_3, l_5) for one set, etc.]

Hence, the correct answer is option (B).

28. The number of ways of arranging $ABCD$ is 4! For each arrangement of $ABCD$, the number of ways of arranging six persons is same.

Hence, the required number is $\dfrac{6!}{4!} = 30$

29. $A = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$.

A rational number is made by taking any two in any order.

So, the required number of rational numbers is (including 1)

$${}^{10}P_2 + 1 = \frac{10!}{8!} + 1 = 90 + 1 = 91$$

30. Each element of the set A can be given the image in the set A in k ways.

So, the required number of functions,

$$n(S) = k \times k \times \cdots (k \text{ times}) = k^k$$

31. In the general position, 37 straight lines have $^{37}C_2$ points of intersection. But 13 straight lines passing through the point A yield one intersection point instead of $^{13}C_2$ and 11 straight lines passing through the point B yield one intersection point instead of $^{11}C_2$.

Therefore, the lines have $^{37}C_2 - {}^{13}C_2 - {}^{11}C_2 + 2$ points of intersection. This gives

$$666 - 78 - 55 + 2 = 535$$

32. The number of subsets of the set which contain at most n elements is

$$^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + \cdots + {}^{2n+1}C_n = N \text{ (say)}$$

We have

$$2N = 2({}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + \cdots + {}^{2n+1}C_n)$$

$$= ({}^{2n+1}C_0 + {}^{2n+1}C_{2n+1}) + ({}^{2n+1}C_1 + {}^{2n+1}C_{2n}) + \cdots + ({}^{2n+1}C_n + {}^{2n+1}C_{n+1})$$

(Since, $^nC_r = {}^nC_{n-r}$)

$$= {}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + \cdots + {}^{2n+1}C_{2n+1} = 2^{2n+1} \Rightarrow N = 2^{2n}$$

33. For $1 \le k \le p - 1$, $n + k = p! + k + 1$, is clearly divisible by $k + 1$. Therefore, there is no prime number in the given list.

Hence, the correct answer is option (D).

34. Let the numbers be $(a\ b\ c\ d)$ and $(A\ B\ C\ D)$. Then

$$D + d = 6, C + c = 9, B + b = 5 \text{ and } A + a = 7$$

Number of such pairs is $(6+1)(9+1)(5+1)(7+1) = 3360$.

Hence, the correct answer is option (B).

35. The idea is to get 3 terms as close as possible. We have

$$214 = 70 + 72 + 72$$

$$= 2{\cdot}5{\cdot}7 + 1{\cdot}8{\cdot}9 + 3{\cdot}4{\cdot}6 \text{ by } AM \ge GM$$

$$S \ge 3\cdot(9!)^{1/3}\left(\frac{S}{3} \ge (a_1a_2a_3\cdot b_1b_2b_3\cdot c_1c_2c_3)^{1/3}\right)$$

$$\text{Since}, 9! = 70\cdot 72\cdot 72 > 71^3$$

$$= 214$$

36. Number of ways = (Number of one-one function from set A = $\{s_1, s_2 \cdots s_6\}$ to A such that $f(s_i) \ne s_i \times 6! = D_6 \times 6!$

37. A number has integers in decreasing order (left to right) and divisible by 4, if the last two digits are 20, 32, 40, 52, 60, 64, 72 and 76.

Total favourable cases = 118

38. $1323000 = 2^3 \times 3^3 \times 5^3 \times 7^2$

For even divisors and divisible by $105; 2, 3, 5, 7$ must occur at least one time.

Therefore, the total number of required divisors are

$$3 \times 3 \times 3 \times 2 = 54$$

39. Between 6 identical diamonds we have to place 3 identical pearls which can be grouped as (3, 0), (2, 1) and (1, 1, 1) each having 1, 1 and 1 ways of doing it total ways = 3.

40. Exponent of 3 in $2!\cdot 3!\cdot 5!\cdots 9! = 1+1+1+1+1+1+3 = 9$

Exponent of 5 in $2!\cdot 3!\cdot 5!\cdots 9! = 1+1+1+1+1 = 5$

Exponent of 7 in $2!\cdot 3!\cdot 5!\cdots 9! = 1+1+1 = 3$

$\Rightarrow$ Number of odd perfect squares $= 5\times 3\times 2 = 30$

41. Number of functions (non-decreasing) is number of non-negative integral solution of $x_1 + x_2 + \cdots + x_{n_2} = n_1$

$$= {}^{n_1+n_2-1}C_{n_1}$$

42. $420 = 3\times 4\times 5\times 7$

$$\Rightarrow \text{Number of ways } = {}^4C_2 = 6$$

43. Let the common ratio of GP be r and first term is a.

We have

$$a \ge 10$$

$$ar^2 \le 100 \Rightarrow r^2 \le 10 \Rightarrow r = 2, 3$$

$$\text{For } r = 2,\ a \le \frac{100}{4} = 25$$

Also,

$$10 \le a \le 15 \qquad \text{(16 values)}$$

For $r = 3$,

$$a \le \frac{100}{9} \Rightarrow a \le 11$$

$$10 \le a \le 11 \qquad \text{(2 values)}$$

So, there are 18 such terms.

44. For n persons sitting around a table, number of ways of selecting 3, of which no two are consecutive is equal to no side of triangle is common with n sided regular polygon is given by

$$\frac{1}{6}n(n-4)(n-5).$$

Put $n = 8$.

Hence, the correct answer is option (B).

45. Number of ways of selecting a side of odd length along the side of m units

$$m + (m-2) + (m-4) + \cdots + 3 + 1 = \frac{1}{2}\left(\frac{m+1}{2}\right)(m+1) = \frac{(m+1)^2}{4}$$

Similarly, number of ways of selecting a side of odd length along the side of n units $= \dfrac{(n+1)^2}{4}$

Hence, the total number of ways $= \dfrac{(m+1)^2}{4}\dfrac{(n+1)^2}{4}$.

46. The possible number of ways can be listed as follows.

Man		Wife		Number of ways
Ladies (4)	Gentlemen (3)	Ladies (3)	Gentlemen (4)	
3	0	0	3	$^4C_3 \cdot {}^4C_3 = 16$
2	1	1	2	$({}^4C_2 \cdot {}^3C_1)^2 = 324$
1	2	2	1	$({}^4C_1 \cdot {}^3C_2)^2 = 144$
0	3	3	0	$({}^3C_3 \cdot {}^3C_3) = 1$
			Total	485

Therefore, total number of ways of inviting relatives for a dinner party of six = 485.

47. (a) To form a line we need two points; and these two points may be chosen in ${}^nC_2$ ways; but, it happens that '*p*' of the '*n*' points are on a line; consequently, these points would form only one line instead of ${}^pC_2$.

Therefore, number of lines = ${}^nC_2 - {}^pC_2 + 1$

(b) Number of triangles = ${}^nC_3 - {}^pC_3$

48. Such four-digit numbers can

(a) Contain all different digits, any four chosen from 1, 2, 3, 4 and 5 and arranged to form four digit number and this is done in ${}^5P_4 = 120$ ways.

(b) Contain one repeated pair, the other two different, numbers like 1123, 3345, 3435, ⋯ and this is done in

$${}^5C_1 \times {}^4C_2 \times \frac{4!}{2!} = 360 \text{ ways.}$$

(c) Contain two repeat pairs, numbers like 1122, 1212, 3113, …, and this is done in ${}^5C_2 \dfrac{4!}{2!2!} = 60$ ways.

Therefore, total number of numbers = 540.

49. The letters, arranged alphabetically, are

E, H, M, O, R, T

Number of words beginning with E	5! = 120
Number of words beginning with H	5! = 120
Number of words beginning with ME	4! = 24
Number of words beginning with MH	4! = 24
Number of words beginning with MOE	3! = 6
Number of words beginning with MOH	3! = 6
Number of words beginning with MOR	3! = 6
Number of words beginning with MOTE	2! = 2
Number of words beginning with MOTHER	1
	Total = 309

The position (rank) of the word 'MOTHER' is 309.

50. (a) Considering the girls as one unit, and the six boys make up 7. These 7 can be seated in 7! ways. The girls among themselves may be relatively interchanged in 6! ways.

The number of arrangements is $7! \cdot 6! = 3628800$

(b) In this case first arrange the six boys in 6! ways.

Suppose,

• • • • • •
B_1 B_2 B_3 B_4 B_5 B_6 is one such arrangement.

Then the six girls can have their positions between any two of these boys, the arrangement starting with a girl.

G_1 • G_2 • G_3 • G_4 • G_5 • G_6 •
B_1 B_2 B_3 B_4 B_5 B_6

and the number of arrangements is $(6!)^2$ (or) the positions of the six girls can be (arrangement now starting with a boy)

• G_1 • G_2 • G_3 • G_4 • G_5 • G_6
B_1 B_2 B_3 B_4 B_5 B_6

and the number of arrangements of six boys and six girls (seated alternately) is $2(6!)^2 = 1036800$.

(c) In this case, the girls are separated, not necessarily by only one boy, between any two girls. First the boys are arranged in 6! ways.

•	•	•	•	•	•
B_1	B_2	B_3	B_4	B_5	B_6

1 2 3 4 5 6 7

The positions available for the six girls can be chosen from the seven (as indicated). The girls are arranged in ${}^7P_6 6!$ ways.

Therefore, the total number of arrangement is 6! 7! = 3628800

In the case of circular permutation, the arrangements, correspondingly, are

(i) 6! 6! = 518400 **(ii)** 5! 6! = 86400 **(iii)** 5! 6! = 86400

Note: In this case, whether they sit alternately, or one group (of boys) is separated by the other (of girls) the effect is the same. Hence, in (ii) and (iii) cases, number of arrangements are equal.

51. Let us consider, in addition to 5 apples, 2 new things say oranges (this number is one less than the number of boys). Then the number of ways of distribution will correspond to the number of arrangements of five A's and two O's. For example, consider the following arrangements:

(a) I OAAA I O AA; draw line just before O's to partition this arrangement. This arrangement corresponds to 0 + 3 + 2, (the numbers corresponding to the number of A's before the line of demarcation, between consecutive lines and after the last line).

(b) I O A I O AAAA corresponds to 0 + 1 + 4

(c) A I OA I O AAA corresponds to 1 + 1 + 3

$$\text{The number of permutations} = \frac{7!}{5!2!} \text{ or } {}^7C_2$$

Alternative Solution: If *n* apples are distributed among *r* boys, the number of ways of distribution would be ${}^{n+(r-1)}C_{r-1}$. Hence, in this case number of ways are ${}^7C_2$.

52. Since two points are required to determine a straight line, we have $N = {}^nC_2$ straight lines by joining the *n* given points in all possible ways.

Since no two are parallel and no three are concurrent, we have ${}^NC_2$ points of intersection. But each of the given *n* points is counted as a point of intersection ${}^{n-1}C_2$ times, since $n - 1$ straight lines pass through each point. Therefore, Required number = ${}^{N}C_2 - n \cdot {}^{n-1}C_2$

$$= \frac{{}^{n}C_2\,({}^{n}C_2 - 1)}{2} - \frac{n\,(n-1)\,(n-2)}{2}$$

$$= \frac{\frac{n\,(n-1)}{2}\left[\frac{n\,(n-1)}{2} - 1\right]}{2} - \frac{n\,(n-1)\,(n-2)}{2}$$

$$= \frac{n\,(n-1)}{2}\left[\frac{n\,(n-1)-2}{4} - (n-2)\right]$$

$$= \frac{n\,(n-1)}{2} \cdot \left(\frac{n^2 - 5n + 6}{4}\right)$$

$$= \frac{1}{8} \cdot n\,(n-1)\,(n-2)\,(n-3)$$

53. The total number of two-factor products = ${}^{100}C_2$

Out of the numbers 1, 2, 3, ⋯, 100; the multiples of 3 are 3, 6, 9, ⋯, 99; that is, there are 33 multiples of 3, and therefore there are 67 non-multiples of 3.

Therefore, the number of two-factor products which are not multiples of 3 = ${}^{67}C_2$

The required number = ${}^{100}C_2 - {}^{67}C_2$

= 4950 – 2211 = 2739

Alternatively, the number of two-factor products formed when both factors are multiples of 3 = ${}^{33}C_2$ and the number of two-factor products formed when one is a multiple of 3 and the other a non-multiple of 3 = ${}^{33}C_1 \times {}^{67}C_1$

It either case the product is a multiple of 3.

Therefore, the required number = 528 + 2211 = 2739

54. Let A, B be the corresponding specified speakers.

(a) Without any restriction the five persons can be arranged among themselves in 5! ways; but the number of ways in which A speaks before B and the number of ways in which B speaks before A together make up 5!.

Also, the number of ways in which A speaks before B is exactly equal to the number of ways in which B speaks before A.

Therefore, the required number of ways = $\frac{1}{2} \cdot 5! = 60$.

(b) Regarding AB in that order as a single person, we can arrange them with the remaining three in 4 ways. Each of these arrangements corresponds to a way in which A speaks immediately before B.

Therefore, the required number of ways in this case = 4! = 24.

55. Each even number is to lie between 200 and 500.

Hence, the position on the left in $x\,x\,x$ is to be filled with 2, 3 or 4.

When 2 or 4 is filled on the left, the position on the right can be filled with the remaining 2 even figures in 2 ways and the middle position can be filled with the remaining 4 figures in 4 ways.

Therefore, the total number of ways of forming the even numbers in this case = 2 · 4 · 2 = 16.

When 3 is filled on the left, the position on the right can be filled with the 3 even figures in 3 ways, and the middle position can be filled with the remaining 4 figures in 4 ways.

Therefore, the total number of ways of forming the even numbers in this case = 1 · 4 · 3 = 12.

Therefore, the required number = 16 + 12 = 28.

56. Zero cannot be a starting digit for any number. Therefore, while forming a 6-digit number, we can fill up the first place in 5 ways. The restriction on zero ends with the starting place. Having filled it, there are 5 figures left, and therefore, the remaining places can be filled in ${}^{5}P_5$ ways.

Therefore, the number of 6-digit numbers = $5 \times {}^{5}P_5 = 600$

Similarly,

Number of 5-digit numbers = $5 \cdot {}^{5}P_4 = 600$

Number of 4-digit numbers = $5 \cdot {}^{5}P_3 = 300$

Number of 3-digit numbers = $5 \cdot {}^{5}P_2 = 100$

Number of 2-digit numbers = $5 \cdot {}^{5}P_1 = 25$

Number of single digit numbers = 5

Therefore, the total number of positive integers = 1630.

For finding the numbers greater than 3000, we take all the 6- and 5-digit numbers together with 4-digit numbers, starting with 3, 4 or 5. The number of 4-digit numbers starting with 3, 4 or 5 is $3 \cdot {}^{5}P_3 = 180$.

Therefore, the total number of integers greater than 3000 is

600 + 600 + 180 = 1380.

57. Now $2 \times 10^8 = 200000000$

Numbers less than 2×10^8 are of the form $a_1\,a_2\,a_3\,a_4\,a_5\,a_6\,a_7\,a_8\,a_9$.

For all the nine-digit numbers $a_1 = 1$ and $a_2, a_3, \cdots$, or a_8 (at present a_9 is deliberately not considered) can be one of 0, 1 or 2.

For all the 8-digit numbers, $a_1 = 0, a_2 \neq 0$; for all the 7-digit numbers $a_1 = 0, a_2 = 0, a_3 \neq 0$.

Thus, all numbers of 9-digits and less than 9-digits are included when a_1 is chosen in one of two ways (0 or 1); and $a_2, a_3, \cdots, a_8$ are each chosen in one of 3 ways (0, 1 or 2).

Therefore, the choice of $a_1, a_2, \cdots, a_8$ may be done 2×3^7 ways. Suppose one such choice is $a_1, a_2, \cdots, a_8$. If $a_1 + a_2 + \cdots + a_8$ is already divisible by $3, 3, a_9$ (which was not considered before).

If $a_1 + a_2 + \cdots + a_8$ is of the form 3p + 1, leaving a remainder 1 on division by 3, then $a_9 = 2$.

If $a_1 + a_2 + \cdots + a_8$ is of the form 3p + 2, leaving a remainder 2 on division by 3, then $a_9 = 1$. Thus, in any case, there is only one way of choice for a_9. The number of numbers, is therefore, $2 \times 3^7 \times 1$ and this includes 000000000, that is, zero. The required number of numbers is $2 \times 3^7 - 1 = 4373$.

Practice Exercise 2

1.

1	1	x	y
1	x	1	y
1	x	y	1

$3 \times {}^9C_2 \times 2 = 108 \times 2 = 216$

1	x	x	y
1	x	y	x
1	y	x	x

$3 \times {}^9C_2 \times 2 = 216$

Total number of 4-digit number = 216 + 216 = 432
Hence, (C) is the correct answer.

2. For every tangent line introduced there are two unbounded regions formed, so for 26 tangents 2 × 26 = 52 unbounded regions are formed.
Hence, (B) is the correct answer.

3. At least one green toy can be selected out of 6 different toys in

$${}^6C_1 + {}^6C_2 + \dots + {}^6C_6 = 63 \text{ ways}$$

After selecting one or more green toys we can select at least one blue toy out of 5 different blue toys in

$${}^5C_1 + {}^5C_2 + \dots + {}^5C_5 = 31 \text{ ways}$$

After selecting at least one green toy and one blue toy, selection of red toys (no restriction) can be made in

$${}^4C_0 + {}^4C_1 + \dots + {}^4C_4 = 16 \text{ ways}$$

Therefore, the total number of selections = 63 × 31 × 16 = 31248.
Hence, (B) is the correct answer.

4. 1000 lines intersects at ${}^{1000}C_2 = 499500$ points but 250 lines are parallel here ${}^{250}C_2 = 31125$ intersections are lost. Also, 250 lines are concurrent so ${}^{250}C_2 - 1 = 31124$ more intersection lost.

So,

$$499500 - 31125 - 31124 = 437251$$

Hence, (A) is the correct answer.

5. $x^2 - y^2 = 2000^2 = (2^4 \times 5^3)^2$

$$\Rightarrow (x + y)(x - y) = 2^8 \times 5^6$$

For $y = 0$, $x = \pm 2000$

Let $(x > 0, y > 0)$ $x + y$ and $x - y$ are both even. Then

the number of integral values of $x, y = \dfrac{7 \times 7 - 1}{2} = 24$

$\Rightarrow$ the number of integral points = 24 × 4 + 2 = 98.

6. $2b = a + c \Rightarrow$ either both odd or both even

when both odd, the number of ways = ${}^5C_2 \times 2$

when both even, the number of ways = ${}^4C_2 \times 2 + {}^4C_1$

Therefore, the total numbers of ways = 36.

Hence, (D) is the correct answer.

7. Total 3-digit natural number divisible by 3 = 300

3-digit natural number in which digits not repeated

$= {}^3C_2 \times 2\ 2! + {}^3C_3 \times 3!\ 3 + {}^3C_1\ {}^3C_1 \times 2\ 2! + {}^3C_1\ {}^3C_1\ {}^3C_1\ 3! = 228$

$\Rightarrow 300 - 228 = 72$

8. When $z = n + 1$ we can choose x, y from $\{1, 2, \cdots, n\}$

Therefore, when $z = n + 1$; x, y can be chosen in n^2 ways and $z = n$; x, y can be chosen in $(n - 1)^2$ ways and so on

Therefore,
$n^2 + (n - 1)^2 + \cdots + 1^2 = \dfrac{1}{6} n(n + 1)(2n + 1)$ ways of choosing triplets.

Hence, (B) is the correct answer.

9. $a^2 - b^2$ is divisible by 3, if either $a + b$, is divisible by 3 or $a - b$ is divisible by 3 or both.

$G_1 \to a + b$ is 3λ type

$G_2 \to a + b$ is $3\lambda - 1$ type

$G_3 \to a + b$ is $3\lambda - 2$ type

Clearly, this is possible if either a and b are chosen from same group or one of them is chosen from G_2 and other from G_3.

Therefore, the number of ways = ${}^nC_2 + {}^nC_2 + {}^nC_2 + {}^nC_1 \cdot {}^nC_1$

$= \dfrac{3n(n-1)}{2} + n^2$

Hence, (B) is the correct answer.

10. $(a^3 + b^3)$ is divisible by 3 if $a + b$ is divisible by 3.

Therefore, the required number of ways = ${}^nC_2 + {}^nC_1 \cdot {}^nC_1 = \dfrac{3n^2 - n}{2}$

11. $a + b + c$ is divisible by 3 only when a, b and c are selected from same group or when one integer is selected from each group.

The required number of ways = ${}^3C_1 \cdot {}^nC_3 + {}^nC_1 \cdot {}^nC_1 \cdot {}^nC_1$

$= \dfrac{n}{2}(3n^2 - 3n + 2)$

Hence, (B) is the correct answer.

12. When 1 all-rounder and 10 players from bowlers and batsman play number of ways = ${}^4C_1 \cdot {}^{14}C_{10}$

When 1 wicket keeper and 10 players from bowlers and batsman play number of ways = ${}^2C_1 \cdot {}^{14}C_{10}$

When 1 all-rounder 1 wicket keeper and 9 from batsmen and bowlers play number of ways = ${}^4C_1 \cdot {}^2C_1 \cdot {}^{14}C_9$

When all 11 players play from bowlers and batsmen then the number of ways = ${}^{14}C_{11}$

Therefore, the total number of selections = ${}^{4}C_1 \cdot {}^{14}C_{10} + {}^{2}C_1 \cdot {}^{14}C_{10} + {}^{4}C_1 \cdot {}^{2}C_{10} \cdot {}^{14}C_9 + {}^{14}C_{11}$

Hence, (B) is the correct answer.

13. If 2 batsmen do not want to play then the rest of 10 players can be selected from 17 other players, number of selection = ${}^{17}C_{10}$.

If the particular bowler does not play then number of selection = ${}^{19}C_{11}$

If all the three do not play, number of selection = ${}^{17}C_{10}$

Therefore, the total number of selections = ${}^{17}C_{10} + {}^{19}C_{11} + {}^{17}C_{11}$

Hence, (B) is the correct answer.

14. If the particular batsman is selected then the rest of 10 players can be selected in ${}^{18}C_{10}$ ways.

If particular wicket keeper is selected then rest of 10 players can be selected in ${}^{18}C_{10}$ ways.

If both are not selected the number of ways = ${}^{18}C_{11}$

Therefore the total number of ways = $2 \cdot {}^{18}C_{10} + {}^{18}C_{11}$

$= {}^{19}C_{11} + {}^{18}C_{10}$

Hence, (B) is the correct answer.

15. (A) The number of solutions is same as number of solutions of $x_1x_2x_3 = 30 = 2 \times 3 \times 5$

Therefore, the number of solutions = $\lambda = {}^{1+4-1}C_{4-1} \times {}^{1+4-1}C_{4-1} \times {}^{1+4-1}C_{4-1} = 64$

Therefore, (A) $\rightarrow$ (*p*, *r*).

(B) The required number of ways = $10! - 6!\,5! = 41 \times 5! \times 6!$

Therefore, (B) $\rightarrow$ (*p*, *q*, *r*, *s*).

(C) Given $x \geq 1, y \geq 2, z \geq 3$ and $x + y + z = 15$

The required number of solutions = coefficient of a^{15} in $(a + a^2 + \dots)(a^2 + a^3 + \dots)(a^3 + a^4 + \dots)$

$= {}^{12-1}C_{3-1} = 55$

Therefore, (C) $\rightarrow$ (*s*).

(D) Required total ways is similar to division of 12 objects into group size of 2,3

The required number of solutions = $\dfrac{12!}{(2!)^3(3!)^2 \times 3! \times 2!}$

Therefore, (D) $\rightarrow$ (*p*, *q*, *r*, *s*).

Hence, (B) is the correct answer.

16. (A)

$x_1 < x_2 = x_3 \Rightarrow {}^9C_2$

$x_1 < x_2 < x_3 \Rightarrow {}^9C_3$

$\Rightarrow {}^9C_2 + {}^9C_3 = 84 + 36 = 120$

Therefore, (A) $\rightarrow$ (*s*).

(B)

$x_1 = x_2 = x_3 \Rightarrow {}^9C_1$

$x_1 = x_2 > x_3 \Rightarrow {}^9C_2$

$\Rightarrow {}^9C_2 + {}^9C_1 = 45$

Therefore, (B) $\rightarrow$ (*r*).

(C) Given, $x_1 \cdot x_2 \cdot x_3 = 480 = 2^5 \cdot 3 \cdot 5$

Distribution of 2^5 in x_1, x_2, x_3 is done by 3 ways.

Distribution of $2^3 \cdot 2 \cdot 2$ in x_1, x_2, x_3 is done by 3 ways.

Distribution of $2^2 \cdot 2^2 \cdot 2$ in x_1, x_2, x_3 is done by 3 ways.

Distribution of $2^4 \cdot 2$ in x_1, x_2, x_3 is done by 6 ways.

Distribution of $2^3 \cdot 2^2$ in x_1, x_2, x_3 is done by 6 ways.

Therefore, total number of ways of distribution of 2 is 21.

Distribution of 3 in x_1, x_2, x_3 is done by 3 ways.

Distribution of 5 in x_1, x_2, x_3 is done by 3 ways.

Hence, total possible numbers = $21 \cdot 3 \cdot 3 = 189$

Therefore, (C)$\rightarrow$(*q*).

(D) Coefficient of x^{10} in $(1 - x)^{-3}$ is ${}^{12}C_{10} = 66$

Therefore, (D) $\rightarrow$ (*p*).

17. (A) Put $x + 4 = x_1, y + 4 = y_1, z + 4 = z_1$. Then

$$x_1 + y_1 + z_1 = 13$$
$$x_1, y_1, z_1 \geq 0$$

The number of solutions = coefficient T^{13} in $(1 - t)^{-3} = 105$

Therefore, A $\rightarrow$ (iii).

(B) Required number of arrangements = Total arrangements – Total arrangements 2*N's* appear together

$$= \frac{6!}{2!\,3!} - 20 = 60 - 20 = 40$$

Therefore, B $\rightarrow$ (i).

(C)

$$a_2 - a_1 = a_1a_2d$$
$$a_3 - a_2 = a_2a_3d$$
$$\vdots \qquad \vdots$$
$$a_{100} - a_{99} = a_{99}a_{100}d$$

$$a_{100} - a_1 = d\sum_{i=1}^{99} a_ia_{i+1} = 99\, a_1a_{100}d$$

$$\Rightarrow \sum_{i=1}^{99} \frac{a_ia_{i+1}}{a_1a_{100}} = 99$$

Therefore, C $\rightarrow$ (ii).

(D) Number of triangles = ${}^{11}C_3 - {}^{8}C_3$

$$= \frac{11 \times 10 \times 9}{6} - \frac{8 \times 7 \times 6}{6}$$

$$= 11 \times 15 - 56 = 109$$

Therefore, D $\rightarrow$ (iv).

18. $1125 = 5^3 \cdot 3^2$, $3375 = 5^3 \cdot 3^3$

Clearly, 3^3 is a factor of *x* and 3^2 is a factor of at least one of *y* and *z*. This can be done in 5 ways.

Also, 5^3 is a factor of at least two of the numbers *x*, *y*, *z* which can be done in $({}^3C_2 \times 4 - 2) = 10$

Number of ordered pair = $10 \times 5 = 50$.

19. $x^2 + y^2 = 50$ and $ax + by = 1$

$x^2 + y^2 = 50$ has 12 integral points given by $(\pm 1, \pm 7)$, $(\pm 7, \pm 1)$ and $(\pm 5, \pm 5)$.

If line $ax + by = 1$ intersects the circle at all integral points, then the number of such lines are ${}^{12}C_1 + {}^{12}C_2 - 6 = 66$.

20. The integers divisible by 3 are 33 in numbers.

The integers divisible by 5 are 20 in numbers.

The integers divisible by 7 are 14 in numbers.

The integers divisible by both 3 and 5 are 6 in numbers.

The integers divisible by both 3 and 7 are 4 in numbers.

The integers divisible by both 5 and 7 are 2 in numbers.

There are no integers divisible by all three.

Hence, the sum of numbers divisible by 3 or 5 or 7,

$$\frac{33}{2}(3+99)+\frac{20}{2}(5+100)+\frac{14}{2}(7+98)-$$
$$\frac{6}{2}(15+90)-\frac{4}{2}(21+84)-(35+70)=2838$$

Solved JEE 2017 Questions

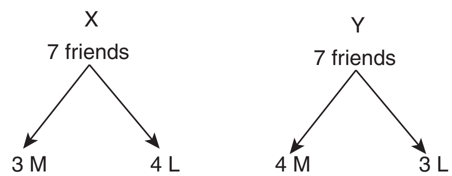
JEE Main 2017

1. A man X has 7 friends, 4 of them are ladies and 3 are men. His wife Y also has 7 friends, 3 of them are ladies and 4 are men. Assume X and Y have no common friends. Then, the total number of ways in which X and Y together can throw a party inviting 3 ladies and 3 men, so that 3 friends of each of X and Y are in this party, is

(A) 468 **(B)** 469
(C) 484 **(D)** 485

(OFFLINE)

Solution: The situation is depicted as in the following figure:



- **Case I:** 3L from X side and 3M from Y side. Therefore,

$$^4C_3 \times {}^4C_3 = 4 \times 4 = 16.$$

- **Case II:** 3M from X side 3L from Y side. Therefore,

$$^3C_3 \times {}^3C_3 = 1 \times 1 = 1$$

- **Case III:** 2L and 1M from X side and 2M and 1L from Y side. Therefore,

$$({}^4C_2 \times {}^3C_1) \times ({}^4C_2 \times {}^3C_1) = (6 \times 3) \times (6 \times 3) = 18 \times 18 = 324$$

- **Case IV:** 2M and 1L from X side and 1M and 2L from Y side.

$$({}^3C_2 \times {}^4C_1) \times ({}^4C_1 \times {}^3C_2) = (3 \times 4) \times (4 \times 3) = 12 \times 12 = 144$$

Therefore, the total number of ways in which X and Y together can throw a party inviting 3 ladies and 3 men, so that 3 friends of each of X and Y are in this party, is

$$\text{Case I} + \text{Case II} + \text{Case III} + \text{Case IV} = 16 + 1 + 324 + 144 = 485$$

Hence, the correct answer is option (D).

2. The value of $({}^{21}C_1 - {}^{10}C_1) + ({}^{21}C_2 - {}^{10}C_2) + ({}^{21}C_3 - {}^{10}C_3) +$ $({}^{21}C_4 - {}^{10}C_4) + \cdots + ({}^{21}C_{10} - {}^{10}C_{10})$ is

(A) $2^{21} - 2^{10}$ **(B)** $2^{20} - 2^9$
(C) $2^{20} - 2^{10}$ **(D)** $2^{21} - 2^{11}$

(OFFLINE)

Solution: $({}^{21}C_1 + {}^{21}C_2 + \cdots + {}^{21}C_{10}) - ({}^{10}C_1 + {}^{10}C_2 + \cdots + {}^{10}C_{10})$

$$= \frac{1}{2}[2 \times {}^{21}C_1 + 2 \times {}^{21}C_2 + \cdots + 2 \times {}^{21}C_{10}] - (2^{10} - 1)$$

$$= \frac{1}{2}({}^{21}C_0 + {}^{21}C_1 + {}^{21}C_2 + \cdots + {}^{21}C_{10} + {}^{21}C_{11} + \cdots + {}^{21}C_{20} + {}^{21}C_{21} - ({}^{21}C_0 + {}^{21}C_{21})] - (2^{10} - 1)$$

$$= \frac{1}{2}(2^{21} - 2) - (2^{10} - 1)$$

$$= 2^{20} - 1 - 2^{10} + 1$$

$$= 2^{20} - 2^{10}$$

Hence, the correct answer is option (C).

3. If all the words, with or without meaning, are written using the letters of the word QUEEN and are arranged as in English dictionary, then the position of the word QUEEN is

(A) 47^{th} **(B)** 44^{th}
(C) 45^{th} **(D)** 46^{th}

(ONLINE)

Solution: To find the position of the word QUEEN:

- The number of words starting with E is $4! = 24$.
- The number of words starting with N is $\frac{4!}{2} = 12$.
- The number of words starting with QE is $3! = 6$.
- Number of words starting with QN is $\frac{3!}{2} = 3$.

Therefore, the position of the word QUEEN is next to the sum, $24 + 12 + 6 + 3 = 45$.
That is, the word 'QUEEN' will be on 46^{th} position.
Hence, the correct answer is option (D).

4. The number of ways, in which 5 boys and 3 girls can be seated on a round table if a particular boy B_1 and a particular girl G_1 never sit adjacent to each other, is

(A) 7! **(B)** $5 \times 6!$
(C) $6 \times 6!$ **(D)** $5 \times 7!$

(ONLINE)

Solution: The number of ways of arranging 5 boys and 3 girls (i.e. 8 people) on a round table would be 7!.

We subtract the number of way of arranging those people, where B_1 and G_1 are always together. When B_1 & G_1 are together, we get

$$4 \text{ Boys} + 2 \text{ Girls} + 1(B_1 + G_1)$$

That is, 7 people and since $B_1 + G_1$ be permitted in 2 ways, those can be arranged in $6! \times 2$ ways.
Subtracting, we have the required number of ways as follows:

$$7! - 6! \times 2 = 6!(7 - 2) = 5 \times 6! \text{ ways}$$

Hence, the correct answer is option (B).

JEE Advanced 2017

1. Words of length 10 are formed using the letters A, B, C, D, E, F, G, H, I, J. Let x be the number of such words where no letter is repeated; and let y be the number of such words where exactly one letter is repeated twice and no other letter is repeated. Then, $\frac{y}{9x} =$ ______.

Solution: The given, formed word is of length 10. It is given that x is the number of words where no letter is repeated.

Also, it is given that y is the number of words where exactly one letter is repeated twice and no other letter is repeated. Therefore,

$$x = 10!$$

and $$y = {}^{10}C_1 \times {}^{10}C_2 \times {}^{9}C_8 \times 8!$$

Thus,

$$\frac{y}{9x} = \frac{{}^{10}C_1 \times {}^{10}C_2 \times {}^{9}C_8 \times 8!}{9 \times 10!}$$

Using ${}^nC_r = \dfrac{n!}{r!(n-r)!}$, we get

$$\frac{y}{9x} = \frac{\frac{10!}{1!(10-1)!} \times \frac{10!}{2!(10-2)!} \times \frac{9!}{8!(9-8)!} \times 8!}{9 \times 10!}$$

$$= \frac{10!}{9 \times 2! \times 8!} = \frac{10 \times 9 \times 8!}{9 \times 2 \times 8!} = \frac{10}{2} = 5 \quad [\text{Using } n! = n(n-1)(n-2)\ldots.1!]$$

Hence, the correct answer is (5).

8 Binomial Theorem

8.1 Binomial Expression

Any algebraic expression consisting of two terms is known as binomial expression. The terms may consist of variables, x, y, etc., or constants or their mixed combinations.

For example, $2x+3y$, $4xy+5$, $\left(x+\frac{1}{y}\right)$, $\left(x+\frac{3}{x}\right)$, $\left(\frac{2}{x}-\frac{1}{x^2}\right)$, etc.

8.2 Binomial Theorem for Positive Integral Index

A formula by which any power of a binomial expression can be expanded in the form of a series is known as binomial theorem. For a positive integer n, the expansion is given by

$$(x+a)^n = {}^nC_0x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + \cdots + {}^nC_rx^{n-r}a^r + \cdots + {}^nC_na^n$$

$$= \sum_{r=0}^{n} {}^nC_rx^{n-r}a^r$$

where ${}^nC_r = \frac{n!}{(n-r)!r!}$ for r = 0, 1, 2, $\cdots$, n is called binomial coefficient.

8.2.1 Proof of Binomial Theorem

The binomial theorem can be proved by mathematical induction.

Let $P(n)$ stands for the mathematical statement

$$(x+a)^n = x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + \cdots + {}^nC_rx^{n-r}a^r + \cdots + a^n \qquad (1)$$

Note that there are $(n+1)$ terms in RHS and all the terms are of the same degree in x and a together.

When $n=1$, LHS $= x+a$ and RHS $= x+a$ (there are only 2 terms).

Therefore, $P(1)$ is verified to be true.

Assume $P(m)$ to be true, that is,

$$(x+a)^m = x^m + {}^mC_1x^{m-1}a + {}^mC_2x^{m-2}a^2 + \cdots + {}^mC_rx^{m-r}a^r + \cdots + a^m \qquad (2)$$

Multiplying Eq. (2) by $(x+a)$, we have,

$$(x+a)^m(x+a) = (x+a)(x^m + {}^mC_1x^{m-1}a + {}^mC_2x^{m-2}a^2 + \cdots + {}^mC_rx^{m-r}a^r + \cdots + a^m)$$

$$\Rightarrow (x+a)^{m+1} = x^{m+1} + ({}^mC_1+1)x^ma + ({}^mC_2+{}^mC_1)x^{m-1}a^2 + \cdots + \left({}^mC_r + {}^mC_{r-1}\right)x^{m-r+1}a^r + \cdots + a^{m+1}$$

(Using the formula, ${}^nC_r + {}^nC_{r-1} = {}^{(n+1)}C_r$)

$$= x^{m+1} + {}^{(m+1)}C_1x^ma + {}^{(m+1)}C_2x^{m-1}a^2 + \cdots + {}^{m+1}C_rx^{m+1-r}a^r + \cdots a^{m+1} \qquad (3)$$

Eq. (3) implies that $P(m+1)$ is true

Hence, by induction $P(n)$ is true.

8.2.2 Alternative Method

By choosing x from all the brackets, we get the term x^n.

Choosing x from $(n-1)$ factors and a from the remaining factor, we get, $x^{n-1}a$.

But the number of ways of doing this is equal to the number of ways of choosing one factor from n factors (i.e. ${}^nC_1$).

Choosing x from $(n-2)$ factor and a from the remaining two factors, we get $x^{n-2}a^2$.

But the number of ways of doing this is equal to the number of ways of choosing two factors from n factors, that is, ${}^nC_2$.

Finally, choosing a from all the factors we get the term a^n.

Therefore,

$$(x+a)^n = x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + \cdots + {}^nC_rx^{n-r}a^r + \cdots + a^n$$

Illustration 8.1 Expand $(x+2a)^5$

Solution:

$$(x+2a)^5 = {}^5C_0x^5 + {}^5C_1x^4(2a)^1 + {}^5C_2x^3(2a)^2 + {}^5C_3x^2(2a)^3 + {}^5C_4x^1(2a)^4 + {}^5C_5x^0(2a)^5$$

$$= x^5 + 10x^4a + 40x^3a^2 + 80x^2a^3 + 80xa^4 + 32a^5$$

Illustration 8.2 Expand $\left(x-\frac{1}{2x}\right)^6$.

Solution:

$$\left(x-\frac{1}{2x}\right)^6 = {}^6C_0x^6 + {}^6C_1x^5\left(\frac{-1}{2x}\right) + {}^6C_2x^4\left(\frac{-1}{2x}\right)^2 + {}^6C_3x^3\left(\frac{-1}{2x}\right)^3 + {}^6C_4x^2\left(\frac{-1}{2x}\right)^4 + {}^6C_5x\left(\frac{-1}{2x}\right)^5 + {}^6C_6x^0\left(\frac{-1}{2x}\right)^6$$

$$= x^6 - 3x^4 + \frac{15}{4}x^2 - \frac{5}{2} + \frac{15}{16x^2} - \frac{3}{16^4} + \frac{1}{64x^6}$$

8.2.2.1 Some More Expansions

$$(x+a)^n = x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + \cdots + {}^nC_rx^{n-r}a^r + \cdots + a^n \quad (1)$$

1. Replacing a by $-a$ in Eq. (1) we get,

$$(x-a)^n = {}^nC_0x^{n-0}\cdot a^0 - {}^nC_1x^{n-1}\cdot a^1 + {}^nC_2x^{n-2}\cdot a^2 + \cdots + (-1)^r\,{}^nC_rx^{n-r}\cdot a^r + \cdots + (-1)^n\,{}^nC_nx^0a^n$$

that is, $$(x-a)^n = \sum_{r=0}^{n}(-1)^r\,{}^nC_rx^{n-r}\cdot a^r \quad (2)$$

The terms in the expansion of $(x-a)^n$ are alternatively positive and negative, the last term is positive or negative depending whether n is even or odd.

2. Replacing x by 1 and a by x in Eq. (1) we get,

$$(1+x)^n = \sum_{r=0}^{n}{}^nC_rx^r$$

3. Replacing x by 1 and a by $-x$ in Eq. (1) we get,

$$(1-x)^n = \sum_{r=0}^{n}(-1)^r\,{}^nC_rx^r$$

4. $(x+a)^n + (x-a)^n = 2[{}^nC_0x^na^0 + {}^nC_2x^{n-2}a^2 + {}^nC_4x^{n-4}a^4 + \cdots]$

and

$(x+a)^n - (x-a)^n = 2[{}^nC_1x^{n-1}\cdot a^1 + {}^nC_3x^{n-3}\cdot a^3 + {}^nC_5x^{n-5}a^5 + \cdots]$

8.2.2.2 Observations

1. There are $(n+1)$ terms in the expansion of $(x+a)^n$.
2. Sum of powers of x and a in each term in the expansion of $(x+a)^n$ is constant and is equal to n.
3. The p^{th} term from the end $= (n-p+2)^{th}$ term from the beginning.
4. Coefficient of x^{n-r} in the expansion of $(x+a)^n$ is ${}^nC_rx^{n-r}a^r$.
5. ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$.
6. In the expansion of $(x+a)^n$ and $(x-a)^n$, x^r occurs in $(r+1)^{th}$ term.
7. If n is odd, then $(x+a)^n + (x-a)^n$ and $(x+a)^n - (x-a)^n$, both have the same number of terms, that is, $\left(\frac{n+1}{2}\right)$.
8. If n is even, then $(x+a)^n + (x-a)^n$ has $\left(\frac{n}{2}+1\right)$ terms and $(x+a)^n - (x-a)^n$ has $\frac{n}{2}$ terms.
9. The coefficient of $(r+1)^{th}$ term in the expansion of $(1+x)^n$ is ${}^nC_r$.
10. The coefficient of x^r in the expansion of $(1+x)^n$ is ${}^nC_r$.

Illustration 8.3 Find the value of $(\sqrt{2}+1)^6 + (\sqrt{2}-1)^6$.

Solution:

$$(x+a)^n + (x-a)^n = 2[{}^nC_0x^na^0 + {}^nC_2x^{n-2}a^2 + {}^nC_4x^{n-4}a^4 + \cdots]$$

$$(\sqrt{2}+1)^6 + (\sqrt{2}-1)^6$$

$$= 2[(\sqrt{2})^6 + {}^6C_2(\sqrt{2})^4(1)^2 + {}^6C_4(\sqrt{2})^2(1)^4 + {}^6C_6(\sqrt{2})^0(1)^6]$$

$$= 2[8 + 15\times4 + 30 + 1] = 198$$

Illustration 8.4 Find the largest of $99^{50} + 100^{50}$ and 101^{50}.

Solution:

$$101^{50} = (100+1)^{50} = 100^{50} + 50\cdot100^{49} + \frac{50\cdot49}{2\cdot1}100^{48} + \cdots \quad (1)$$

and

$$99^{50} = (100-1)^{50} = 100^{50} - 50\cdot100^{49} + \frac{50\cdot49}{2\cdot1}100^{48} - \cdots \quad (2)$$

Subtracting, Eq. (2) from Eq. (1)

$$101^{50} - 99^{50} = 100^{50} + 2\cdot\frac{50\cdot49\cdot48}{3\cdot2\cdot1}100^{47} + \cdots > 100^{50}$$

Hence, $101^{50} > 100^{50} + 99^{50}$.

Illustration 8.5 Sum of odd terms is A and sum of even terms is B in the expansion of $(x+a)^n$, then find the value of $4AB$.

Solution:

$$(x+a)^n = {}^nC_0x^na^0 + {}^nC_1x^{n-1}a^1 + {}^nC_2x^{n-2}a^2 + \cdots + {}^nC_nx^{n-n}\cdot a^n$$

$$= ({}^nC_0x^na^0 + {}^nC_2x^{n-2}a^2 + \cdots) + ({}^nC_1x^{n-1}a^1 + {}^nC_3x^{n-3}a^3 + \cdots) = A + B \quad (1)$$

Similarly,

$$(x-a)^n = A - B \quad (2)$$

From Eqs. (1) and (2), we get

$$(A+B)^2 - (A-B)^2 = 4AB$$

$$4AB = (x+a)^{2n} - (x-a)^{2n}$$

Illustration 8.6 If the coefficients of the second, third and fourth terms in the expansion of $(1+x)^n$ are in AP, show that $n = 7$.

Solution: According to the question, ${}^nC_1\cdot{}^nC_2\cdot{}^nC_3$ are in AP. So,

$${}^nC_2 - {}^nC_1 = {}^nC_3 - {}^nC_2$$

$$\Rightarrow 2\,{}^nC_2 = {}^nC_1 + {}^nC_3$$

$$\Rightarrow \frac{2n(n-1)}{2} = n + \frac{n(n-1)(n-2)}{6}$$

$$\Rightarrow n^2 - 9n + 14 = 0 \Rightarrow (n-2)(n-7) = 0$$

$$\Rightarrow n = 2 \text{ or } 7$$

Since, the symbol ${}^nC_3$ demands that n should be ≥ 3.
So, n cannot be 2.
Therefore, $n = 7$.

Illustration 8.7 Find the

1. last digit of 17^{256}.
2. last two digits of 17^{256}.
3. last three digits of 17^{256}.

Solution:

$$17^{256} = 289^{128} = (290-1)^{128}$$

$$= {}^{128}C_0(290)^{128} - {}^{128}C_1(290)^{127} + \cdots + {}^{128}C_{126}(290)^2 - {}^{128}C_{127}(290) + 1$$

$$= 1000m + {}^{128}C_2(290)^2 - {}^{128}C_1(290) + 1$$

$$= 1000m + \frac{128\times127}{2}\times(290)^2 - \frac{128\times290}{1} + 1$$
$$= 1000m + 683527680 + 1$$

Hence,

1. Last digit is 1.
2. Last two digits are 81.
3. Last three digits are 681.

Illustration 8.8 If the binomial coefficients of $(2r+4)^{\text{th}}$ and $(r-2)^{\text{th}}$ terms in the expansion of $(a+bx)^{18}$ are equal, find r.

Solution: This is possible only when either

$$2r+3 = r-3 \quad (1)$$

or

$$2r+3+r-3 = 18 \quad (2)$$

From Eq. (1), $r=-6$ is not possible.
But, from Eq. (2)

$$r = 6$$

Hence, $r = 6$ is the only solution.

8.3 General Term

$$(x+a)^n = {}^nC_0x^na^0 + {}^nC_1x^{n-1}a^1 + {}^nC_2x^{n-2}a^2 + \cdots + {}^nC_rx^{n-r}a^r + \cdots + {}^nC_nx^0a^n$$

The first term $= {}^nC_0x^na^0$
The second term $= {}^nC_1x^{n-1}a^1$
The third term $= {}^nC_2x^{n-2}a^2$ and so on.
The term ${}^nC_rx^{n-r}a^r$ is the $(r+1)^{\text{th}}$ term from beginning in the expansion of $(x+a)^n$.
Let T_{r+1} denote the $(r+1)^{\text{th}}$ term. Therefore,

$$T_{r+1} = {}^nC_rx^{n-r}a^r$$

This is called the general term because by giving different values to r, we can determine all terms of the expansion.
In the binomial expansion of $(x-a)^n$, $T_{r+1} = (-1)^r\,{}^nC_rx^{n-r}a^r$
In the binomial expansion of $(1+x)^n$, $T_{r+1} = {}^nC_rx^r$
In the binomial expansion of $(1-x)^n$, $T_{r+1} = (-1)^r\,{}^nC_rx^r$

8.4 Independent Term or Constant Term

Independent term or constant term of a binomial expansion is the term in which exponent of the variable is zero.

Illustration 8.9 Find the coefficient of

(i) x^7 in $\left(ax^2+\frac{1}{bx}\right)^{11}$ **(ii)** x^{-7} in $\left(ax-\frac{1}{bx^2}\right)^{11}$.

Find the relation between a and b, if these coefficients are equal.

Solution:

The general term in $\left(ax^2+\frac{1}{bx}\right)^{11} = {}^{11}C_r(ax^2)^{11-r}\left(\frac{1}{bx}\right)^r$

$$= {}^{11}C_r\frac{a^{11-r}}{b^r}x^{22-3r}$$

If in this term power of x is 7, then

$$22-3r = 7 \Rightarrow r = 5$$

Therefore,

$$\text{coefficient of } x^7 = {}^{11}C_5\frac{a^6}{b^5} \quad (1)$$

The general term in $\left(ax-\frac{1}{bx^2}\right)^{11} = (-1)^r\,{}^{11}C_r(ax)^{11-r}\frac{1}{(bx^2)^r}$

$$= (-1)^r\,{}^{11}C_r\frac{(a)^{11-r}}{(b)^r}(x)^{11-3r}$$

If in this term power of x is -7, then

$$11-3r = -7 \Rightarrow r = 6$$

Therefore,

$$\text{coefficient of } x^{-7} = (-1)^6\,{}^{11}C_6\frac{(a)^{11-6}}{(b)^6} = {}^{11}C_5\frac{(a)^5}{(b)^6}$$

If these two coefficient are equal, then

$${}^{11}C_5\frac{(a)^6}{(b)^5} = {}^{11}C_5\frac{(a)^5}{(b)^6}$$

$$\Rightarrow (ab)^6 = (ab)^5 \Rightarrow (ab)^5(ab-1) = 0 \Rightarrow ab = 1 (a, b \neq 0)$$

Illustration 8.10 If the 4^{th} term in the expansion of $(px+x^{-1})^m$ is 2.5 for all $x\in R$, then find p and m.

Solution: We have

$$T_4 = \frac{5}{2} \Rightarrow T_{3+1} = \frac{5}{2}$$

$$\Rightarrow {}^mC_3(px)^{m-3}\left(\frac{1}{x}\right)^3 = \frac{5}{2} \Rightarrow {}^mC_3p^{m-3}x^{m-6} = \frac{5}{2} \quad (1)$$

Clearly, RHS of the above equality is independent of x

Therefore,

$$m-6 = 0,\ m = 6$$

Putting $m = 6$ in Eq. (1) we get,

$${}^6C_3p^3 = \frac{5}{2} \Rightarrow p = \frac{1}{2}$$

Hence, $p = 1/2$ and $m = 6$.

Illustration 8.11 If the second, third and fourth terms in the expansion of $(x+a)^n$ are 240, 720 and 1080, respectively, then find the value of n.

Solution: It is given that

$$T_2 = 240,\ T_3 = 720 \text{ and } T_4 = 1080$$

Now,

$$T_2 = 240 \Rightarrow T_2 = {}^nC_1x^{n-1}a^1 = 240 \quad (1)$$

$$T_3 = 720 \Rightarrow T_3 = {}^nC_2x^{n-2}a^2 = 720 \quad (2)$$

and

$$T_4 = 1080 \Rightarrow T_4 = {}^nC_3x^{n-3}a^3 = 1080 \quad (3)$$

To eliminate x,

$$\frac{T_2 \cdot T_4}{T_3^2} = \frac{240 \cdot 1080}{720 \cdot 720} = \frac{1}{2}$$

$$\Rightarrow \frac{T_2}{T_3} \cdot \frac{T_4}{T_3} = \frac{1}{2}$$

Now,

$$\frac{T_{r+1}}{T_r} = \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n!(r-1)!(n-r+1)!}{r!(n-r)!n!} = \frac{n-r+1}{r}$$

Putting $r = 3$ and 2 in above expression, we get

$$\frac{n-2}{3} \cdot \frac{2}{n-1} = \frac{1}{2} \Rightarrow n = 5$$

Illustration 8.12 Find the term independent of x in the expansion of $\left(\sqrt{\frac{x}{3}} + \frac{3}{2x^2}\right)^{10}$.

Solution:

$$T_{r+1} = {}^{10}C_r \left(\sqrt{\frac{x}{3}}\right)^{10-r} \left(\frac{3}{2x^2}\right)^r = {}^{10}C_r \left(\frac{1}{\sqrt{3}}\right)^{10-r} x^{\frac{10-r}{2}-2r} \left(\frac{3}{2}\right)^r$$

Power of x equals to zero,

$$\frac{10-r}{2} - 2r = 0$$

$$\Rightarrow 5r = 10$$

$$r = 2$$

Therefore,

$$T_3 = {}^{10}C_2 \left(\frac{1}{3}\right)^{8/2} \left(\frac{3}{2}\right)^2 = \frac{5}{4}$$

Illustration 8.13 Find the coefficient of x^5 in the expansion of $(1+x^2)^5(1+x)^4$.

Solution: We have

$$(1+x^2)^5(1+x)^4 = ({}^5C_0 + {}^5C_1x^2 + {}^5C_2x^4 + \cdots)({}^4C_0 + {}^4C_1x^1 + {}^4C_2x^2 + \cdots)$$

So, coefficient of x^5 in $[(1+x^2)^5(1+x)^4] = {}^5C_2 \cdot {}^4C_1 + {}^4C_3 \cdot {}^5C_1 = 60.$

Illustration 8.14 Find the term independent of x in the expansion of $(1+x)^n\left(1+\frac{1}{x}\right)^n$.

Solution: We know that,

$$(1+x)^n = {}^nC_0 + {}^nC_1x^1 + {}^nC_2x^2 + \cdots + {}^nC_nx^n$$

$$\left(1+\frac{1}{x}\right)^n = {}^nC_0 + {}^nC_1\frac{1}{x^1} + {}^nC_2\frac{1}{x^2} + \cdots + {}^nC_n\frac{1}{x^n}$$

So, the term independent of x will be

$${}^nC_0 \cdot {}^nC_0 + {}^nC_1{}^nC_1 + \cdots + {}^nC_n \cdot {}^nC_n = C_0^2 + C_1^2 + \cdots + C_n^2$$

Your Turn 1

1. The 5^{th} term from the end in the expansion of $\left(\frac{x^3}{2} - \frac{2}{x^3}\right)^9$ is
   (A) $63x^3$ **(B)** $-\frac{252}{x^3}$
   (C) $\frac{672}{x^{18}}$ **(D)** None of these **Ans.** (B)

2. If $\frac{T_2}{T_3}$ in the expansion of $(a+b)^n$ and $\frac{T_3}{T_4}$ in the expansion of $(a+b)^{n+3}$ are equal, then n is equal to
   (A) 3 **(B)** 4
   (C) 5 **(D)** 6 **Ans.** (C)

3. If the coefficients of second, third and fourth term in the expansion of $(1+x)^{2n}$ are in AP, then $2n^2 - 9n + 7$ is equal to
   (A) –1 **(B)** 0
   (C) 1 **(D)** 3/2 **Ans.** (B)

4. If A and B are the coefficients of x^n in the expansions of $(1+x)^{2n}$ and $(1+x)^{2n-1}$, respectively, then
   (A) $A = B$ **(B)** $A = 2B$
   (C) $2A = Bf$ **(D)** None of these **Ans.** (B)

5. The coefficient of x^n in the expansion of $(1+x)(1-x)^n$ is
   (A) $(-1)^{n-1}n$ **(B)** $(-1)^n(1-n)$
   (C) $(-1)^{n-1}(n-1)^2$ **(D)** $(n-1)$ **Ans.** (B)

8.5 Middle Term in the Binomial Expansion

There are two cases:

1. **When *n* is even:** Clearly, in this case we have only one middle term, that is, $T_{n/2+1}$. Thus, middle term in the expansion of $(a + x)^n$ will be $T_{n/2+1} = {}^nC_{n/2}\, a^{n/2}x^{n/2}$ term.
2. **When *n* is odd:** In this case, we have two middle terms, that is,
   $T_{\frac{n+1}{2}}$ and $T_{\frac{n+3}{2}}$. Thus, the middle terms in the expansion of $(a + x)^n$ are ${}^nC_{\frac{n-1}{2}} \cdot a^{\frac{n+1}{2}} \cdot x^{\frac{n-1}{2}}$ and ${}^nC_{\frac{n+1}{2}} \cdot a^{\frac{n-1}{2}} \cdot x^{\frac{n+1}{2}}$.

Illustration 8.15 Find the middle term in the expansion of $\left(\frac{a}{x} + bx\right)^{12}$.

Solution: As $n = 12$. So, 7^{th} term is the middle term.

$$T_{6+1} = {}^{12}C_6 \cdot \left(\frac{a}{x}\right)^6 \cdot (bx)^6 = {}^{12}C_6\, a^6 b^6$$

Illustration 8.16 Find the middle term in the expansion of $\left(3x-\frac{x^3}{6}\right)^9$.

Solution: There will be two middle terms as $n = 9$ is an odd number.

The middle terms will be $\left(\frac{9+1}{2}\right)^{\text{th}}$ and $\left(\frac{9+3}{2}\right)^{\text{th}}$ terms.

$$T_5 = {}^9C_4(3x)^5\left(-\frac{x^3}{6}\right)^4 = \frac{189}{8}x^{17}$$

$$T_6 = {}^9C_5(3x)^4\left(-\frac{x^3}{6}\right)^5 = -\frac{21}{16}x^{19}$$

Illustration 8.17 Find the middle term in the expansion of $\left(x+\frac{1}{x}\right)^{10}$.

Solution: As n is even. So, the middle term is

$$T_{\left(\frac{10}{2}+1\right)} = T_6 \Rightarrow T_6 = T_{5+1} = {}^{10}C_5x^5\cdot\frac{1}{x^5} = {}^{10}C_5$$

Illustration 8.18 Find the middle term in the expansion of $(1-2x+x^2)^n$.

Solution:

$$(1-2x+x^2)^n = [(1-x)^2]^n = (1-x)^{2n}$$

Here, $2n$ is an even integer

Therefore, $\left(\frac{2n}{2}+1\right)^{\text{th}}$ term, that is, $(n+1)^{\text{th}}$ term will be the middle term.

Now,

$$(n+1)^{\text{th}} \text{ term in } (1-x)^{2n} = {}^{2n}C_n\,(1)^{2n-n}\,(-x)^n$$
$$= {}^{2n}C_n(-x)^n = \frac{(2n)!}{n!n!}(-x)^n$$

Illustration 8.19 Prove that the middle term in the expansion of $\left(x+\frac{1}{x}\right)^{2n}$ is $\frac{1\cdot3\cdot5\cdots(2n-1)}{n!}\cdot2^n$.

Solution: Since, $2n$ is even.

Therefore, $\left(\frac{2n}{2}+1\right)^{\text{th}}$ term, that is, $(n+1)^{\text{th}}$ term will be the middle term.

Now, $(n+1)^{\text{th}}$ term, that is, the middle term in $\left(x+\frac{1}{x}\right)^{2n}$ is given by

$$T_{n+1} = {}^{2n}C_nx^{2n-n}\left(\frac{1}{x}\right)^n = {}^{2n}C_nx^n\frac{1}{x^n} = {}^{2n}C_n$$
$$= \frac{(2n)!}{n!n!} = \frac{2n(2n-1)(2n-3)\cdots4\cdot3\cdot2\cdot1}{n!n!}$$
$$= \frac{[1\cdot3\cdot5\cdots(2n-1)2^n n(n-1)(n-2)(n-3)\cdots2\cdot1]}{n!n!}$$
$$= \frac{1\cdot3\cdot5\cdots(2n-1)2^n n!}{n!n!} = \frac{1\cdot3\cdot5\cdots(2n-1)2^n}{n!}$$

8.6 Greatest Binomial Coefficient

In the binomial expansion of $(1+x)^n$, when n is even, the greatest binomial coefficient is given by ${}^nC_{n/2}$.

Similarly, if n be odd, the greatest binomial coefficient will be ${}^nC_{\frac{n+1}{2}}$ and ${}^nC_{\frac{n-1}{2}}$, both being equal.

8.7 Numerically Greatest Term

If T_r and T_{r+1} be the r^{th} and $(r+1)^{\text{th}}$ term in the expansion of $(1+x)^n$, then

$$\frac{T_{r+1}}{T_r} = \frac{{}^nC_rx^r}{{}^nC_{r-1}x^{r-1}} = \frac{n-r+1}{r}x$$

Let numerically T_{r+1} be the greatest term in the above expansion. Then $T_{r+1} \ge T_r$

or
$$\frac{T_{r+1}}{T_r} \ge 1 \Rightarrow \frac{n-r+1}{r}|x| \ge 1$$
$$\Rightarrow r \le \frac{(n+1)|x|}{(1+|x|)} \quad (1)$$

Now putting values of n and x in Eq. (1), we get $r \le m+f$ or $r \le m$ where m is a positive integer, f is a fraction such that $0 \le f < 1$. Now, if $f = 0$, then T_{m+1} and T_m both the terms will be numerically equal and the greatest, while if $f \ne 0$, then T_{m+1} is the greatest term of the binomial expansion.

That is to find the greatest term (numerically) in the expansion of $(1+x)^n$.

1. Calculate $m = \frac{(n+1)|x|}{(1+|x|)}$.
2. If m is an integer, then T_m and T_{m+1} are equal and are the greatest terms.
3. If m is not an integer, then $T_{[m]+1}$ is the greatest term (where [.] denotes the greatest integer function).

Illustration 8.20 Find which term is/are the largest term in the expansion of $(3+2x)^{50}$, where $x = \frac{1}{5}$.

Solution: $(3+2x)^{50} = 3^{50}\left[1+\frac{2x}{3}\right]^{50}$, now greatest term in $\left(1+\frac{2x}{3}\right)^{50}$

$$r = \left|\frac{x(n+1)}{1+x}\right| = \left|\frac{\frac{2x}{3}(50+1)}{\frac{2x}{3}+1}\right| = \frac{\frac{2\cdot\frac{1}{5}}{3}(51)}{\frac{2}{15}+1} = 6 \text{ (an integer)}$$

Therefore, T_r and $T_{[r]+1}$, that is, T_6 and T_7 are numerically the greatest terms.

Illustration 8.21 Find the greatest term in the expansion of $(4+3x)^7$, when $x=\frac{2}{3}$.

Solution: Here, the greatest term means numerically the greatest term.

$$\left|\frac{T_{r+1}}{T_r}\right| = \frac{{}^7C_r 4^{7-r}(3x)^r}{{}^7C_{r-1}4^{8-r}(3x)^{r-1}} = \frac{8-r}{r}\cdot\frac{3x}{4} = \frac{8-r}{2r} \quad [\text{since } x = 2/3]$$

Now

$$|T_{r+1}| \geq |T_r| \text{ if } 8-r \geq 2r \text{ or } \frac{8}{3} \geq r$$

This inequality is valid only for $r = 1$ or 2. Thus,

$$\text{for } r = 1, 2;\ |T_{r+1}| > |T_r|,$$

and

$$\text{for } r = 3, 4;\ |T_{r+1}| < |T_r|$$

Therefore, $|T_1| < |T_2| < |T_3| > |T_4| > |T_5| > \ldots$

Greatest term $= |T_3| = {}^7C_2 4^5 \cdot (3x)^2$, where $x=\frac{2}{3}$

$$= 21 \times 4^5 \times 2^2 = 86016$$

Illustration 8.22 Find numerically the greatest term in the expansion of $(3-5x)^{11}$, when $x=\frac{1}{5}$.

Solution: Since,

$$(3-5x)^{11} = 3^{11}\left(1-\frac{5x}{3}\right)^{11}$$

Now in the expansion of $\left(1-\frac{5x}{3}\right)^{11}$, we have

$$\frac{T_{r+1}}{T_r} = \frac{(11-r+1)}{r}\left|-\frac{5x}{3}\right|$$

$$= \left(\frac{12-r}{r}\right)\left|-\frac{5}{3}\times\frac{1}{5}\right| \quad \left(x=\frac{1}{5}\right)$$

$$= \left(\frac{12-r}{r}\right)\left(\frac{1}{3}\right)$$

$$= \left(\frac{12-r}{3r}\right)$$

Therefore,

$$\frac{T_{r+1}}{T_r} \geq 1 \Rightarrow \frac{12-r}{3r} \geq 1 \Rightarrow 4r \leq 12 \Rightarrow r \leq 3$$

Thus, $r = 2, 3$

So, the greatest terms are T_{2+1} and T_{3+1}. Hence, greatest terms (where $r = 2$) $= 3^{11}|T_{2+1}|$

$$= 3^{11}\left|{}^{11}C_2\left(-\frac{5}{2}x\right)^2\right|$$

$$= 3^{11}\left|{}^{11}C_2\left(-\frac{5}{3}\times\frac{1}{5}\right)^2\right| \quad \left(x=\frac{1}{5}\right)$$

$$= 3^{11}\left|\frac{11\cdot 10}{1\cdot 2}\times\frac{1}{9}\right| = 55\times 3^9$$

and greatest term (where $r = 3$) $= 3^{11}|T_{3+1}|$

$$= 3^{11}\left|{}^{11}C_3\left(-\frac{5}{3}x\right)^3\right|$$

$$= 3^{11}\left|{}^{11}C_3\left(-\frac{5}{3}\times\frac{1}{5}\right)^3\right|$$

$$= 3^{11}\left|\frac{11\cdot 10\cdot 9}{1\cdot 2\cdot 3}\right|\times\frac{-1}{27} = 55\times 3^9$$

From the above, we say that the values of both greatest terms are equal.

Your Turn 2

1. Find the middle term in the expansion of $(1+x)^{2n}$.

Ans. $\frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}\cdot 2^n x^n$

2. Find the greatest term in the expansion of $(1+x)^{2n+2}$.

Ans. $\frac{(2n+2)!}{[(n+1)!]^2}$

3. The interval in which x must lie so that the greatest term in the expansion of $(1+x)^{2n}$ has the greatest coefficient is

(A) $\left(\frac{n-1}{n}, \frac{n}{n-1}\right)$ **(B)** $\left(\frac{n}{n+1}, \frac{n+1}{n}\right)$

(C) $\left(\frac{n}{n+2}, \frac{n+2}{n}\right)$ **(D)** None of these

Ans. (B)

4. Find the value of the greatest term in the expansion of $\sqrt{3}\left(1+\frac{1}{\sqrt{3}}\right)^{20}$.

Ans. 2871.11

8.8 Properties of Binomial Coefficient

In the binomial expansion of $(1+x)^n$,

$$(1+x)^n, (1+x) = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \cdots + {}^nC_rx^r + \cdots + {}^nC_nx^n$$

Where ${}^nC_0, {}^nC_1, {}^nC_2, \cdots, {}^nC_n$ are the coefficients of various powers of x and called binomial coefficients, and they are written as $C_0, C_1, C_2, \cdots, C_n$.

Hence, $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_rx^r + \cdots + C_nx^n$ (1)

1. ${}^nC_{r_1} = {}^nC_{r_2} \Rightarrow r_1 = r_2$ or $r_1 + r_2 = n$
2. ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
3. $r\,{}^nC_r = n\,{}^{n-1}C_{r-1}$
4. $\frac{{}^nC_r}{r+1} = \frac{{}^{n+1}C_{r+1}}{n+1}$

Putting $x = 1$ in Eq. (1), we get,

$$2^n = C_0 + C_1 + C_2 + \cdots + C_n$$

Therefore, the sum of binomial coefficients in the expansion of $(1+x)^n$ is 2^n.

Illustration 8.23 Prove that the sum of the coefficients in the expansion of $(1 + x - 3x^2)^{2163}$ is -1.

Solution: Putting $x = 1$ in $(1 + x - 3x^2)^{2163}$, the required sum of coefficients is

$$(1 + 1 - 3)^{2163} = (-1)^{2163} = -1$$

Hence, proved.

Illustration 8.24 If the sum of the coefficients in the expansion of $(\alpha x^2 - 2x + 1)^{35}$ is equal to the sum of the coefficients in the expansion of $(x - \alpha y)^{35}$, then find the value of α.

Solution:
Sum of the coefficients in the expansion of $(\alpha x^2 - 2x + 1)^{35}$
= Sum of the coefficients in the expansion of $(x - \alpha y)^{35}$
Putting $x = y = 1$. Therefore,

$$(\alpha - 1)^{35} = (1 - \alpha)^{35}$$
$$\Rightarrow (\alpha - 1)^{35} = -(\alpha - 1)^{35}$$
$$\Rightarrow 2(\alpha - 1)^{35} = 0$$
$$\Rightarrow \alpha - 1 = 0 \text{ or } \alpha = 1$$

The sum of the coefficients of the odd terms in the expansion of $(1 + x)^n$ is equal to the sum of the coefficients of the even terms and each is equal to 2^{n-1}.

Since,

$$(1 + x)^n = C_0 + C_1 + C_2x^2 + C_3x^3 + \cdots + C_nx^n$$

Putting $x = -1$,

$$0 = C_0 - C_1 + C_2 - C_3 + \cdots + (-1)^n C_n$$

and

$$2^n = C_0 + C_1 + C_2 + C_3 + \cdots + C_n \qquad \text{\{from Eq. (2)\}}$$

Adding and subtracting these two equations, we get

$$2^n = 2(C_0 + C_2 + C_4 + \cdots) \text{ and } 2^n = 2(C_1 + C_3 + C_5 + \cdots)$$

Therefore,

$$C_0 + C_2 + C_4 + \cdots = C_1 + C_3 + C_5 + \cdots = 2^{n-1}$$

Sum of coefficients of odd terms = sum of coefficients of even terms = 2^{n-1}

Illustration 8.25 Evaluate the sum: ${}^8C_1 + {}^8C_3 + {}^8C_5 + {}^8C_7$.

Solution: Since, ${}^8C_1 + {}^8C_3 + {}^8C_5 + {}^8C_7$ = sum of even term coefficients in the expansion of $(1 + x)^8 = 2^{8-1} = 2^7 = 128$.

Illustration 8.26 Find the value of $\sum_{r=0}^{n}\left(\frac{r+2}{r+1}\right){}^nC_r$.

Solution: The given value is

$$\sum_{r=0}^{n}\left(\frac{r+2}{r+1}\right){}^nC_r = \sum_{r=0}^{n}\left(1+\frac{1}{r+1}\right){}^nC_r$$

$$\left(\text{using, } \frac{{}^nC_r}{r+1} = \frac{{}^{n+1}C_{r+1}}{n+1}\right)$$

$$= \sum_{r=0}^{n}{}^nC_r + \frac{1}{n+1}\sum_{r=0}^{n}{}^{n+1}C_{r+1}$$

$$= 2^n + \frac{({}^{n+1}C_1 + {}^{n+1}C_2 + \cdots + {}^{n+1}C_{n+1})}{n+1}$$

$$= 2^n + \frac{2^{n+1} - 1^{(n+1)}}{n+1}$$

$$= \frac{2^n(n+3) - 1}{n+1}$$

Illustration 8.27 If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \cdots C_nx^n$, show that $(C_0 + C_1)(C_1 + C_2)(C_2 + C_3) \cdots (C_{n-1} + C_n) = \frac{(n+1)^n}{n!}C_0C_1\cdots C_{n-1}$

Solution: As we know,

$$T_r = C_{r-1} + C_r = {}^{n+1}C_r = \frac{(n+1)!}{r!(n+1-r)!}$$

$$= \frac{(n+1)}{r}\frac{n!}{(r-1)!(n-r+1)!} = \frac{n+1}{r}C_{r-1}$$

Hence,

$$C_0 + C_1 = \left(\frac{n+1}{1}\right)C_0$$

$$C_1 + C_2 = \left(\frac{n+1}{2}\right)C_1$$

$$\cdots$$

$$\cdots$$

$$C_{n-1} + C_n = \frac{n+1}{n}C_{n-1}$$

$$\Rightarrow (C_0 + C_1)(C_1 + C_2) \cdots (C_{n-1} + C_n) = \frac{(n+1)^n}{n!}C_0C_1\cdots C_{n-1}$$

8.9 Summation of Series Including Binomial Coefficient

1. Series involving binomial coefficients with alternate sign.

Illustration 8.28 Evaluate $C_0 - C_2 + C_4 - C_6 + \cdots$

Solution: Here, +ve and –ve signs occur alternately.
This can be obtained by putting $(-i)$ and (i) in place of x in

$$(1 + x)^n = C_0 + C_1x + \cdots + C_nx^n$$

we get,

$$C_0 + C_1 i + C_2 i^2 + \cdots + (i)^n C_n = (1 + i)^n$$

and $$C_0 - C_1 i + C_2 (-i)^2 + \cdots + (-i)^n C_n = (1 - i)^n$$

Now to obtain the sum, $C_0 - C_2 + C_4 + \cdots$, we add $(1 + i)^n$ and $(1 - i)^n$.

$$2(C_0 - C_2 + C_4 - C_6 + \cdots) = (1 - i)^n + (1 + i)^n$$

$$(1-i)^n + (1+i)^n = \left[\sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)\right]^n + \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^n$$

$$= 2^{n/2+1}\cdot\cos\frac{n\pi}{4} \qquad \text{(By Demoivre's theorem)}$$

$$\Rightarrow C_0 - C_2 + C_4 - C_6 + \cdots = 2^{\frac{n}{2}}\cos\frac{n\pi}{4}$$

Illustration 8.29 Evaluate $C_0 - C_3 + C_6 - C_9 + \cdots$

Solution: Here, +ve and –ve signs occur alternately.

This can be obtained by putting (–1), ($-w$) and ($-w^2$) in place of x in

$$(1+x)^n = C_0 + C_1 x + \cdots + {}^nC_n x^n$$

we get,

$$C_0 + C_1(-1) + C_2(-1)^2 + \cdots + (-1)^n C_n = (1+(-1))^n$$
$$C_0 + C_1(-w) + C_2(-w)^2 + \cdots + (-w)^n C_n = (1+(-w))^n$$
$$C_0 + C_1(-w^2) + C_2(-w^2)^2 + \cdots + (-w^2)^n C_n = (1+(-w^2))^n$$

Now to obtain the sum, $C_0 - C_3 + C_6 - C_9 \cdots$

we add $(1+(-1))^n$, $(1+(-w))^n$ and $(1+(-w^2))^n$

$$3(C_0 - C_3 + C_6 - C_9 + \cdots) = (1+(-1))^n + (1+(-w))^n + (1+(-w^2))^n$$

Note:
Similarly, the cube roots of unity may be used to evaluate

$$C_0 + C_3 + C_6 + \cdots \text{ or } C_1 + C_4 + \cdots \text{ or } C_2 + C_5 + \cdots$$

put $x = 1$, $x = w$, $x = w^2$ in $(1+x)^n = C_0 + C_1x + \cdots + C_nx^n$ and add to get $C_0 + C_3 + C_6 + \cdots$ the other two may be obtained by suitably multiplying $(1+w)^n$ and $(1+w^2)^n$ by w and w^2, respectively.

2. Series involving binomial coefficients in which each term is a product of an integer and a binomial coefficient that is in the form $k\,{}^nC_r$.

(a) By Algebra: Write down the general term and use

$${}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1}$$

Illustration 8.30 Show that $C_1 + 2\cdot C_2 + 3\cdot C_3 + \cdots + n\cdot C_n = n\cdot 2^{n-1}$.

Solution: The numbers multiplying binomial coefficients are 1, 2, 3, $\cdots$, n and these are in arithmetic progression.

Let $S = C_1 + 2\cdot C_2 + 3\cdot C_3 + \cdots + (n-1)\cdot C_{n-1} + n\cdot C_n$

Also,

$$S = n\cdot C_0 + (n-1)\cdot C_1 + (n-2)\cdot C_2 + (n-3)\cdot C_3 + \cdots + 1\cdot C_{n-1}$$

(writing the terms in the reverse order and using $C_r = C_{n-r}$), and adding

$$2S = n\cdot C_0 + n\cdot C_1 + n\cdot C_2 + \cdots + n\cdot C_{n-1} + n\cdot C_n$$
$$= n\cdot[C_0 + C_1 + C_2 + \cdots + C_n] = n\cdot 2^n$$

Therefore, $S = n\cdot 2^{n-1}$.

Alternative method:

$$S = \sum_{r=1}^{n} r\cdot {}^nC_r = \sum_{r=1}^{n} r\cdot\frac{n}{r}\cdot {}^{n-1}C_{r-1} = \sum_{r=1}^{n} n\cdot {}^{n-1}C_{r-1}$$
$$= n\sum_{r=1}^{n} {}^{n-1}C_{r-1} = n\left\{{}^{(n-1)}C_0 + {}^{(n-1)}C_1 + {}^{(n-1)}C_2 + \cdots + {}^{(n-1)}C_{n-1}\right\}$$
$$= n\cdot(1+1)^{n-1} = n\cdot 2^{n-1}$$

Illustration 8.31 Show that $C_1 - 2\cdot C_2 + 3\cdot C_3 - 4\cdot C_4 + \cdots + (-1)^{n-1}\, n\cdot C_n = 0$.

Solution:

$$S = \sum_{r=1}^{n}(-1)^{r-1}\cdot r\cdot {}^nC_r = \sum_{r=1}^{n}(-1)^{r-1}\cdot r\cdot\frac{n}{r}\cdot {}^{n-1}C_{r-1} = \sum_{r=1}^{n}(-1)^{r-1}\cdot n\cdot {}^{n-1}C_{r-1}$$
$$= n\sum_{r=1}^{n}(-1)^{r-1}\cdot {}^{n-1}C_{r-1} = n\left\{{}^{(n-1)}C_0 - {}^{(n-1)}C_1 + {}^{(n-1)}C_2 + \cdots + (-1)^{n-1}\,{}^{(n-1)}C_{n-1}\right\}$$
$$= n\cdot(1-1)^{n-1} = 0$$

Illustration 8.32 Show that $2\cdot C_0 + 7\cdot C_1 + 12\cdot C_2 + \cdots + (5n+2)\,C_n = (5n+4)\,2^{n-1}$.

Solution: We have

$$2\cdot C_0 + 7\cdot C_1 + 12\cdot C_2 + \cdots + (5n+2)\,C_n.$$
$$= \sum_{r=0}^{n}(5r+2)C_r = \sum_{r=0}^{n}(5r+2)\,{}^nC_r = 5\sum_{r=0}^{n} r.{}^nC_r + 2\sum_{r=0}^{n}{}^nC_r$$
$$= 5\sum_{r=1}^{n} r\cdot\frac{n}{r}\cdot {}^{n-1}C_{r-1} + 2\sum_{r=0}^{n}{}^nC_r = 5n\sum_{r=1}^{n}{}^{n-1}C_{r-1} + 2\sum_{r=0}^{n}{}^nC_r$$
$$= 5n[(1+1)^{n-1}] + 2(1+1)^n = 5n\cdot 2^{n-1} + 2.2^n = 2^{n-1}\cdot(5n+4).$$

Illustration 8.33 Find the value of $C_0 + 4C_1 + 7C_2 + \cdots + (3n+1)C_n$.

Solution: We have

$$C_0 + 4C_1 + 7C_2 + \cdots + (3n+1)C_n$$
$$= \sum_{r=0}^{n}(3r+1)C_r = \sum_{r=0}^{n}(3r+1)\,{}^nC_r = 3\sum_{r=0}^{n} r.{}^nC_r + \sum_{r=0}^{n}{}^nC_r$$
$$= 3\sum_{r=1}^{n} r\cdot\frac{n}{r}\cdot {}^{n-1}C_{r-1} + \sum_{r=0}^{n}{}^nC_r = 3n\sum_{r=1}^{n}{}^{n-1}C_{r-1} + \sum_{r=0}^{n}{}^nC_r$$
$$= 3n[(1+1)^{n-1}] + (1+1)^n = 3n\cdot 2^{n-1} + 2^n = 2^{n-1}\cdot(3n+2).$$

(b) By Calculus: Use of differentiation

This method is applied only when the numericals occur as the product of binomial coefficients.

Solution process:

1. If the last term of the series, leaving plus or minus sign, be m, then divide m by n, if q be the quotient and r be the remainder, that is, $m = nq + r$.
   Then replace x by x^q in the given series and multiply both sides of expansion by x^r.
2. After processing Eq. (1), differentiate both sides w.r.t. x and put $x = 1$ or -1 or i or $-i$, according to the given series.
3. If the product of two binomial coefficient (or square of binomial coefficient) or three binomial coefficient (or cube of binomial coefficient) then differentiate twice or thrice.

Illustration 8.34 Evaluate (where n is an integer greater than 1)

1. $C_1 + 2\cdot C_2 + 3\cdot C_3 + \cdots + n\cdot C_n$
2. $a - {}^nC_1(a-1) + {}^nC_2(a-2) - \cdots + (-1)^n(a-n)$
3. $3\cdot C_0 + 7\cdot C_1 + 11\cdot C_2 + \cdots + (4n+3)\,C_n$
4. $C_0 + 3C_1 + 5C_2 + \cdots + (2n+1)C_n$

Solution: We know that,

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$$

1. Differentiating both sides w.r.t. x, we get,

$$n(1+x)^{n-1} = 0 + C_1 + 2C_2x + 3C_3x^2 + \cdots + nC_nx^{n-1}$$

Putting $x = 1$, we get

$$n\cdot 2^{n-1} = C_1 + 2C_2 + 3C_3 + \cdots + nC_n$$

2. We have

$$a[C_0 - C_1 + C_2 - \cdots] + [C_1 - 2C_2 + 3C_3 - \cdots]$$
$$= a[C_0 - C_1 + C_2 - \cdots] - [-C_1 + 2C_2 - 3C_3 + \cdots]$$

We know that $(1-x)^n = C_0 - C_1x + C_2x^2 - \cdots + (-1)^nC_nx^n$;

Put $x = 1$, $0 = C_0 - C_1 + C_2 \cdots$

Then differentiating both sides w.r.t. to x, we get

$$n(1-x)^{n-1} = 0 - C_1 + 2C_2x - 3C_3x^2 + \ldots\ldots$$

Put $x = 1$, $0 = -C_1 + 2C_2 - 3C_3 + \cdots = a[0] - [0] = 0$.

3. This problem can be solved by differentiating the expansion of $x^3(1+x^4)^n$ and putting $x = 1$.

$$x^3(1+x^4)^n = x^3(C_0 + C_1x^4 + C_2x^8 + \cdots + C_nx^{4n})$$
$$= C_0x^3 + C_1x^7 + C_2x^{11} + \cdots + C_nx^{4n+3}$$

Differentiating we get,

$$3x^2(1+x^4)^n + x^3n(1+x^4)^{n-1}4x^3$$
$$= 3x^2C_0 + 7x^6C_1 + 11x^{10}C_2 + \cdots + (4n+3)x^{4n+2}C_n$$

Now substituting $x = 1$ in both sides.

$$3C_0 + 7C_1 + 11C_2 + \cdots + (4n+3)C_n = 3(2^n) + 4n(2)^{n-1}$$
$$= (3+2n)2^n$$

4. This problem can be solved by differentiating the expansion of $x(1+x^2)^n$ and putting $x = 1$.

$$x(1+x^2)^n = x(C_0 + C_1x^2 + C_2x^4 + \cdots + C_nx^{2n})$$
$$= C_0x + C_1x^3 + C_2x^5 + \cdots + C_nx^{2n+1}$$

Differentiating we get,

$$(1+x^2)^n + xn(1+x^2)^{n-1}2x = C_0 + 3x^2C_1 + 5x^4C_2 + \cdots + (2n+1)x^{2n}C_n$$

Now substituting $x = 1$ in both sides.

$$C_0 + 3C_1 + 5C_2 + \cdots + (2n+1)C_n = (2^n) + 2n(2)^{n-1} = (1+n)2^n$$

3. Series involving binomial coefficients in which each term is a binomial coefficient divided by an integer that is in the form, $\frac{{}^nC_r}{k}$.

(a) By Algebra: Write down the general term and use

$$\frac{{}^nC_r}{n} = \frac{{}^{n-1}C_{r-1}}{r}$$

Illustration 8.35 Show that $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \cdots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$

Solution:

$$\text{LHS} = 1 + \frac{{}^nC_1}{2} + \frac{{}^nC_2}{3} + \frac{{}^nC_3}{4} + \cdots + \frac{{}^nC_n}{n+1}$$

$$= 1 + \frac{n}{2} + \frac{n(n-1)}{1\cdot2\cdot3} + \frac{n(n-1)(n-2)}{1\cdot2\cdot3\cdot4} + \cdots + \frac{n(n-1)(n-2)\cdots3\cdot2\cdot1}{1\cdot2\cdot3\cdots n\cdot(n+1)}$$

$$= \frac{1}{n+1}\left\{(n+1) + \frac{(n+1)n}{1\cdot2} + \frac{(n+1)n(n-1)}{1\cdot2\cdot3} + \cdots + \frac{(n+1)n(n-1)\cdots3\cdot2\cdot1}{1\cdot2\cdot3\cdots(n+1)}\right\}$$

$$= \frac{1}{(n+1)}\left\{{}^{(n+1)}C_1 + {}^{(n+1)}C_2 + {}^{(n+1)}C_3 + \cdots + {}^{(n+1)}C_{n+1}\right\}$$

$$= \frac{1}{(n+1)}\{2^{n+1} - 1\} \text{ (from the expansion of } (1+1)^{n+1})$$

Alternative method:

$$\sum_{r=0}^{n}\frac{{}^nC_r}{r+1} = \sum_{r=0}^{n}\frac{1}{n+1}\cdot\frac{n+1}{r+1}\cdot{}^nC_r = \sum_{r=0}^{n}\frac{1}{n+1}\cdot{}^{n+1}C_{r+1}$$

$$= \frac{1}{(n+1)}\left\{{}^{(n+1)}C_1 + {}^{(n+1)}C_2 + {}^{(n+1)}C_3 + \cdots + {}^{(n+1)}C_{n+1}\right\}$$

$$= \frac{1}{(n+1)}\{2^{n+1} - 1\} \text{ (from the expansion of } (1+1)^{n+1})$$

Illustration 8.36 Show that

$$2\cdot C_0 + 2^2\cdot\frac{C_1}{2} + 2^3\cdot\frac{C_2}{3} + \cdots + 2^{n+1}\frac{C_n}{n+1} = \frac{3^{n+1}-1}{n+1}$$

Solution:

$$\sum_{r=0}^{n}2^{r+1}\frac{{}^nC_r}{r+1} = \sum_{r=0}^{n}\frac{2^{r+1}}{n+1}\cdot\frac{n+1}{r+1}\cdot{}^nC_r = \sum_{r=0}^{n}\frac{2^{r+1}}{n+1}\cdot{}^{n+1}C_{r+1}$$

$$= \frac{1}{n+1}\left\{{}^{(n+1)}C_1\cdot2 + {}^{(n+1)}C_2\cdot2^2 + {}^{(n+1)}C_3\cdot2^3 + \cdots + {}^{(n+1)}C_{n+1}\cdot2^{(n+1)}\right\}$$

$$= \frac{1}{n+1}\left\{1 + {}^{(n+1)}C_1\cdot2 + {}^{(n+1)}C_2\cdot2^2 + \cdots + {}^{(n+1)}C_{n+1}\cdot2^{(n+1)} - 1\right\}$$

$$= \frac{1}{(n+1)}\{(1+2)^{n+1} - 1\} = \frac{3^{n+1}-1}{n+1}$$

(b) By Calculus: Use of integration

This method is applied only when the numericals occur as the denominator of the binomial coefficients.

Solution process: If $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$, then we integrate both sides between the suitable limits which gives the required series.

1. If the sum contains $C_0, C_1, C_2, \cdots, C_n$ with all positive signs, then integrate between limit 0 to 1.
2. If the sum contains alternate signs (that is +, –) then integrate between limit –1 to 0.
3. If the sum contains odd coefficients (that is C_0, C_2, C_4, $\cdots$) then integrate between –1 to 1.
4. If the sum contains even coefficients (that is $C_1, C_3, C_5, \cdots$) then subtracting process (2) from process (1) and then dividing by 2.
5. If in denominator of binomial coefficients, the product of two numericals is present, then integrate two times, first taking limit between 0 to x and second time take suitable limits.

Illustration 8.37 Show that

1. $C_0+\frac{C_1}{2}+\frac{C_2}{3}+\cdots+\frac{C_n}{n+1}=\frac{2^{n+1}-1}{n+1}$

2. $3\cdot C_0+3^2\cdot\frac{C_1}{2}+3^3\cdot\frac{C_2}{3}+\cdots+3^{n+1}\frac{C_n}{n+1}=\frac{4^{(n+1)}-1}{n+1}$

3. $\frac{C_0}{2}-\frac{C_1}{3}+\frac{C_2}{4}-\frac{C_3}{5}+\cdots=\frac{1}{(n+1)(n+2)}$

Solution:

1. Integrating the expansion of $(1+x)^n$ between the limits 0 to 1.

$$\int_0^1(1+x)^n dx=\int_0^1(C_0+C_1x+\cdots+C_nx^n)dx$$

$$\Rightarrow \left.\frac{(1+x)^{n+1}}{n+1}\right|_0^1=\left.C_0x+C_1\frac{x^2}{2}+\cdots+C_n\frac{x^{n+1}}{n+1}\right|_0^1$$

$$\Rightarrow C_0+\frac{C_1}{2}+\frac{C_2}{3}+\cdots+\frac{C_n}{n+1}=\frac{2^{(n+1)}-1}{n+1}$$

2. Integrating the expansion of $(1+x)^n$ between the limits 0 to 3.

$$\int_0^3(1+x)^n dx=\int_0^3(C_0+C_1x+\cdots+C_nx^n)dx$$

$$\Rightarrow \left.\frac{(1+x)^{n+1}}{n+1}\right|_0^3=\left.C_0x+C_1\frac{x^2}{2}+\cdots+C_n\frac{x^{n+1}}{n+1}\right|_0^3$$

$$\Rightarrow 3\cdot C_0+3^2\cdot\frac{C_1}{2}+3^3\cdot\frac{C_2}{3}+\cdots+3^{n+1}\cdot\frac{C_n}{n+1}=\frac{4^{(n+1)}-1}{n+1}$$

3. Integrating the expansion of $x(1-x)^n$ between the limits 0 to 1.

$$\int_0^1 x(1-x)^n dx=\int_0^1 x(C_0-C_1x+C_2x^2-\cdots+(-1)^nC_nx^n)dx$$

$$\int_0^1 x(1-x)^n dx=C_0\left[\frac{x^2}{2}\right]_0^1-C_1\left[\frac{x^3}{3}\right]_0^1+C_2\left[\frac{x^4}{4}\right]_0^1-\cdots \quad (1)$$

The integral on LHS of Eq. (1)

$$\int_1^0(1-t)t^n(-dt) \text{ by putting } 1-x=t,$$

$$\Rightarrow \int_1^0(t^n-t^{n+1})dt=\frac{1}{n+1}-\frac{1}{n+2}$$

Whereas the integral on the RHS of Eq. (1)

$$\frac{C_0}{2}-\frac{C_1}{3}+\frac{C_2}{4}-\cdots \text{ to } (n+1) \text{ terms}=\frac{1}{n+1}-\frac{1}{n+2}=\frac{1}{(n+1)(n+2)}$$

4. Product of two expansions can be used to solve some problems related to series of binomial coefficients in which each term is a product of two binomial coefficients.

Illustration 8.38 Show that $C_0^2+C_1^2+C_2^2+\cdots+C_n^2=\frac{(2n)!}{n!n!}$

Solution: This example can be solved by considering two binomial expansions $(1+x)^n$ and $\left(1+\frac{1}{x}\right)^n$ in which the coefficients of x^n and $\frac{1}{x^n}$ are equal and in the product of these expansions, the constant term will contain the square of binomial coefficients. Consider,

$$(1+x)^n=C_0+C_1x+C_2x^2+\cdots+C_nx^n$$

$$\left(1+\frac{1}{x}\right)^n=C_0+\frac{C_1}{x}+\frac{C_2}{x^2}+\cdots+\frac{C_n}{x^n}$$

Taking the product of these two expansions and collecting the constant term in the product.

Constant term in RHS $=C_0^2+C_1^2+C_2^2+\cdots C_n^2$

$=$ constant term in LHS $=$ constant term in $(1+x)^n\left(1+\frac{1}{x}\right)^n$

$=$ constant term in $\frac{(1+x)^{2n}}{x^n}=$ coefficient of x^n in $(1+x)^{2n}$

$$={}^{2n}C_n=\frac{(2n)!}{(n!)(n!)}$$

Illustration 8.39 Show that

$$C_0^2-C_1^2+C_2^2-C_3^2+\cdots+(-1)^nC_n^2=\begin{cases}0, & \text{if } n \text{ is odd}\\ \dfrac{(-1)^{n/2}n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}, & \text{if } n \text{ is even}\end{cases}$$

Solution: Consider the product of the expansion of $(1+x)^n$ and $\left(1-\frac{1}{x}\right)^n$ and compare the constant term.

$$C_0^2-C_1^2+C_2^2+\cdots+(-1)^nC_n^2=\text{constant term in } (1+x)^n\left(1-\frac{1}{x}\right)^n$$

$=$ constant term in $\frac{(1+x)^n(x-1)^n}{x^n}$

$=$ constant term in $\frac{(-1)^n(1-x^2)^n}{x^n}$

$=$ coefficient of x^n in $(-1)^n(1-x^2)^n$

$=0$, if n is odd since all the terms in $(1-x^2)^n$ contain only even power of x, or

$=$ coefficient of x^{2m} in $(-1)^{2m}(1-x^2)^{2m}$, if n is even $=2m$

$$=(-1)^m\cdot{}^{2m}C_m=(-1)^m\frac{(2m)!}{m!m!}=\frac{n!(-1)^{n/2}}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}$$

Illustration 8.40 Show that

$$C_0C_r+C_1C_{r+1}+C_2C_{r+2}+\cdots+C_{n-r}C_n=\frac{(2n)!}{(n-r)!(n+r)!}$$

Solution: Consider

$$(1+x)^n=C_0+C_1x+C_2x^2+\cdots+C_rx^r+C_{r+1}x^{r+1}+\cdots+C_nx^n$$

and $$\left(1+\frac{1}{x}\right)^n = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \cdots + \frac{C_r}{x^r} + \frac{C_{r+1}}{x^{r+1}} + \cdots + \frac{C_n}{x^n}$$

In the product of these two expansions, collecting the coefficient of x^r

$$C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \cdots + C_{n-r}C_n = \text{coefficient of } x^r \text{ in } \frac{(1+x)^{2n}}{x^n}$$

$$= \text{coefficient of } x^{n+1} \text{ in } (1+x)^{2n}$$

$$= {}^{2n}C_{n+r} = \frac{(2n)!}{(n+r)!(n-r)!}$$

8.10 An Important Theorem

1. If $(\sqrt{P}+Q)^n = I + f$, where I and n are positive integers, n being odd, and $0 \le f < 1$, then show that $(I + f)\, f = k^n$, where $P - Q^2 = k > 0$ and $\sqrt{P} - Q < 1$.

Proof: Given

$$\sqrt{P} - Q < 1$$

Therefore,

$$0 < (\sqrt{P} - Q)^n < 1$$

Now let $(\sqrt{P} - Q)^n = f'$, where $0 < f' < 1$ (1)

Thus, $I + f - f' = (\sqrt{P}+Q)^n - (\sqrt{P}-Q)^n$.

Since, RHS contains even powers of $\sqrt{P}$ (since n is odd), so RHS is an integer.

Thus, RHS and I are integers

So,

$$f - f' \text{ is also an integer}$$
$$\Rightarrow f - f' = 0$$

Then,

$$-1 < f - f' < 1$$

or

$$f = f'$$

Hence,

$$(I+f)f = (I+f)f' = (\sqrt{P}+Q)^n(\sqrt{P}-Q)^n$$
$$= (P - Q^2)^n = k^n$$

2. If $(\sqrt{P}+Q)^n = I + f$, where I and n are positive integers, n being even, and $0 \le f < 1$, then show that $(I + f)\,(1 - f) = k^n$, where $P - Q^2 = k > 0$ and $\sqrt{P} - Q < 1$.

Proof: If n is an even integer then,

$$(\sqrt{P}+Q)^n + (\sqrt{P}-Q)^n = I + f + f' \quad \text{[using, Eq.(1)]}$$

Thus, LHS and I are integers.

Therefore, $f + f'$ is also an integer, that is,

$$f + f' = 1 \text{ since, } 0 < f + f' < 2$$
$$f' = (1 - f)$$

Hence,

$$(I+f)(1-f) = (I+f)f' = (\sqrt{P}+Q)^n(\sqrt{P}-Q)^n$$
$$= (P - Q^2)^n = k^n$$

Illustration 8.41 If $(2+\sqrt{3})^n = I + f$, where I and n are positive integers and $0 < f < 1$, show that

1. I is an odd integer, and
2. $(I+f)(1-f) = 1$.

Solution:

1. Now,

$$0 < 2 - \sqrt{3} < 1, \text{ since } 2 - \sqrt{3} = 0.268 \text{ (approx.)}$$

Therefore, $0 < (2-\sqrt{3})^n < 1$; we can take $(2-\sqrt{3})^n$ as f'.

Now,

$$(2+\sqrt{3})^n + (2-\sqrt{3})^n = I + f + f'$$

But,

$$\text{LHS} = 2\{2^n + {}^nC_2 2^{n-2}(\sqrt{3})^2 + {}^nC_4 2^{n-4}(\sqrt{3})^4 + \cdots\}$$
$$= \text{an integer (in fact an even integer)}$$

Thus,

$$\text{RHS} = I + f + f' = \text{an even integer}$$

Also, $f + f' = 1$, since f and f' are both positive proper fractions. Hence, I = an even integer – 1 = an odd integer.

2. $(I+f)(1-f) = (I+f)(f') = (2+\sqrt{3})^n \cdot (2-\sqrt{3})^n = (4-3)^n = 1^n = 1$

Illustration 8.42 Let $R = (5\sqrt{5}+11)^{2n+1}$ and $f = R - [R]$, where [ ] denotes the greatest integer function. Prove that $Rf = 4^{2n+1}$.

Solution: Greatest integer function is defined as follows:

$$[x] = \text{greatest integer} \le x$$

In the case of positive number x

$$[x] = \text{integral part of } x$$

Therefore, $f = R - [R]$ implies that f is the fractional part of R.

Thus, $0 < f < 1$.

Since, $144 > 125 > 121$, $\sqrt{125} = 5\sqrt{5}$ lies between 11 and 12.

So, $0 < 5\sqrt{5} - 11 < 1$ and hence $(5\sqrt{5}-11)^{2n+1}$ will also be a proper fraction.

Let $g = (5\sqrt{5}-11)^{2n+1}$. Then

$$[R] + f - g = R - g$$
$$= (5\sqrt{5}+11)^{2n+1} - (5\sqrt{5}-11)^{2n+1}$$
$$= 2\left\{{}^{(2n+1)}C_1(5\sqrt{5})^{2n}\cdot 11^1 + {}^{(2n+1)}C_3(5\sqrt{5})^{2n-2}\cdot 11^2 + \cdots\right\}$$
$$= \text{an even integer}$$

Since, $[R]$ is an integer, the above implies $f - g = 0$, that is, $f = g$.

Hence,

$$Rf = Rg = (5\sqrt{5}+11)^{2n+1}\cdot(5\sqrt{5}-11)^{2n+1}$$
$$= (125-121)^{2n+1} = 4^{2n+1}$$

Your Turn 3

1. If $S_n = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$ and $T_n = \sum_{r=0}^{n} \frac{r}{{}^nC_r}$. Then $\frac{T_n}{S_n}$ is equal to

Ans. $n/2$

2. If $(1-x+x^2)^n = a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n}$, then value of $a_0 + a_2 + a_4 + \cdots + a_{2n}$ is equal to ______.

Ans. $\frac{3^n+1}{2}$

3. In the expansion of $(1 + x)^5$, find the sum of the coefficient of the terms.

Ans. 32

4. If the sum of coefficient in the expansion of $(\alpha^2 x^2 - 2\alpha x + 1)^{51}$ vanishes, then find the value of α.

Ans. 1

5. If C_r stands for ${}^nC_r$, the sum of given series

$$\frac{2(n/2)!(n/2)!}{n!}\left[C_0^2 - 2C_1^2 + 3C_2^2 - \ldots + (-1)^n(n+1)C_n^2\right]$$

where n is an even positive integer, is

(A) 0 **(B)** $(-1)^{n/2}(n+1)$

(C) $(-1)^n(n+2)$ **(D)** $(-1)^{n/2}(n+2)$

Ans. (D)

8.10.1 Multinomial Theorem

If n is positive integer and $a_1, a_2, a_3, \ldots, a_n \in C$ then,

$$(a_1 + a_2 + a_3 + \cdots + a_m)^n = \sum \frac{n!}{n_1!n_2!n_3!\ldots n_m!} a_1^{n_1} a_2^{n_2} \ldots a_m^{n_m}$$

where $n_1, n_2, n_3, \ldots, n_m$ are all non-negative integers subject to the condition, $n_1 + n_2 + n_3 + \ldots + n_m = n$.

1. The coefficient of $a_1^{n_1} \cdot a_2^{n_2} \ldots a_m^{n_m}$ in the expansion of

$$(a_1 + a_2 + a_3 + \ldots + a_m)^n \text{ is } \frac{n!}{n_1!n_2!n_3!\ldots n_m!}$$

2. The greatest coefficient in the expansion of

$$(a_1 + a_2 + a_3 + \ldots a_m)^n \text{ is } \frac{n!}{(q!)^{m-r}[(q+1)!]^r}$$

where q is the quotient and r is the remainder when n is divided by m.

3. If n is a +ve integer and $a_1, a_2, \ldots, a_m \in C$, $a_1^{n_1} \cdot a_2^{n_2} \ldots a_m^{n_m}$, then coefficient of x^r in the expansion of

$$(a_1 + a_2 x + \ldots + a_m x^{m-1})^n \text{ is } \sum \frac{n!}{n_1!n_2!n_3!\ldots n_m!}$$

where, $n_1, n_2, \ldots + n_m$ are all non-negative integers subject to the condition, $n_1 + n_2 + \ldots + n_m = n$ and $n_2 + 2n_3 + 3n_4 + \cdots + (m-1)n_m = r$.

4. The number of distinct or dissimilar terms in the multinomial expansion $(a_1 + a_2 + a_3 + \ldots + a_m)^n$ is ${}^{n+m-1}C_{m-1}$.

5. Sum of all the coefficients is obtained by putting all the variables, a_i equal to 1 and sum is equal to m^n.

Illustration 8.43 If $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ and $x_1 + x_2 = 5$, $(x_1, x_2, x_3, x_4, x_5 \geq 0)$ then find the number of non-negative integral solutions of above equation.

Solution:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20,$$

$$x_1 + x_2 = 5 \quad (1)$$

$$\Rightarrow x_3 + x_4 + x_5 = 15 \quad (2)$$

Number of solutions are

Coefficient of x^5 in $(x^0 + x^1 + x^2 + x^3 + x^5)^2 \cdot$ Coefficient of x^{15} in $(x^0 + x^1 + x^2 + \cdots + x^{15})$

$$\Rightarrow \text{Coefficient of } x^5 \left(\frac{1-x^6}{1-x}\right)^2 . \text{ coefficient of } x^{15} \left(\frac{1-x^{16}}{1-x}\right)^3$$

$$\Rightarrow \text{Coefficient of } x^5 \text{ in } (1-x)^{-2} \times \text{coefficient of } x^{15}\ (1-x)^{-3}$$

$${}^{2+5-1}C_1 \times {}^{3+15-1}C_{3-1} = {}^6C_1 \times {}^{17}C_2 = \frac{6 \times 17 \times 16}{2} = 48 \times 17 = 816$$

Illustration 8.44 Find the coefficient of x^5 in the expansion of $(x^2 - x - 2)^5$.

Solution: Coefficient of x^5 in the expansion of $(x^2 - x - 2)^5$ is

$$\sum \frac{5!}{n_1!n_2!n_3!}(1)^{n_1}(-1)^{n_2}(-2)^{n_3}$$

where, $n_1 + n_2 + n_3 = 5$ and $n_2 + 2n_3 = 5$.

The possible values of n_1, n_2 and n_3 are shown in the margin

n_1	n_2	n_3
1	3	1
2	1	2
0	5	0

The coefficient of

$$x^5 = \frac{5!}{1!3!1!}(1)^1(-1)^3(-2)^1 + \frac{5!}{2!1!2!}(1)^2(-1)^1(-2)^2 + \frac{5!}{0!5!0!}(1)^0(-1)^5(-2)^0$$

$$= 40 - 120 - 1 = -81$$

Illustration 8.45 Find the coefficient of $a^3b^4c^5$ in the expansion of $(bc + ca + ab)^6$.

Solution: In this case,

$$a^3b^4c^5 = (ab)^x (bc)^y (ca)^z = a^{x+z} \cdot b^{x+y} \cdot c^{y+z}$$

$$z + x = 3,\ x + y = 4,\ y + z = 5;$$

$$2(x + y + z) = 12;\ x + y + z = 6$$

Then, $x = 1$, $y = 3$, $z = 2$.

Therefore, the coefficient of $a^3b^4c^5$ in the expansion of

$$(bc + ca + ab)^6 = \frac{6!}{1!3!2!} = 60$$

Illustration 8.46 Find the total number of terms in the expansion of $(x + y + z + w)^n$, $n \in N$.

Solution: The number of terms in the expansion of $(x + y + z + w)^n$ is

$${}^{n+4-1}C_{4-1} = {}^{n+3}C_3 = \frac{(n+3)(n+2)(n+1)}{6}$$

Alternative method:

We know that

$$(x + y + z + w)^n = \{(x + y) + (z + w)\}^n$$

$$= (x+y)^n + {}^nC_1(x+y)^{n-1}(z+w) + {}^nC_2(x+y)^{n-2}(z+w)^2 + \cdots + {}^nC_n(z+w)^n$$

Therefore, the number of terms in RHS

$$= (n+1) + n\cdot 2 + (n-1)\cdot 3 + \cdots + 1\cdot(n+1)$$

$$= \sum_{r=0}^{n}(n-r+1)(r+1)$$

$$= \sum_{r=0}^{n}((n+1)+nr-r^2) = (n+1)\sum_{r=0}^{n}1 + n\sum_{r=0}^{n}r - \sum_{r=0}^{n}r^2$$

$$= (n+1)\cdot(n) + n\cdot\frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)(n+2)(n+3)}{6}$$

8.10.2 Binomial Theorem for Any Index

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \text{terms upto } \infty$$

When n is a negative integer or a fraction, where $-1 < x < 1$, otherwise expansion will not be possible.

If $x < 1$, the terms of the above expansion go on decreasing and if x be very small, a stage may be reached when we may neglect the terms containing higher power of x in the expansion, then $(1+x)^n = 1 + nx$.

General term: $T_{r+1} = \dfrac{n(n-1)(n-2)\cdots(n-r+1)}{r!}x^r$

Some important expansions:

1. $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \cdots + \dfrac{n(n-1)(n-2)\cdots(n-r+1)}{r!}x^r + \cdots$

2. $(1-x)^n = 1 - nx + \dfrac{n(n-1)}{2!}x^2 - \cdots + \dfrac{n(n-1)(n-2)\cdots(n-r+1)}{r!}(-x)^r + \cdots$

3. $(1-x)^{-n} = 1 + nx + \dfrac{n(n+1)}{2!}x^2 + \dfrac{n(n+1)(n+2)}{3!}x^3 + \cdots + \dfrac{n(n+1)\cdots(n+r-1)}{r!}x^r + \cdots$

4. $(1+x)^{-n} = 1 - nx + \dfrac{n(n+1)}{2!}x^2 - \dfrac{n(n+1)(n+2)}{3!}x^3 + \cdots + \dfrac{n(n+1)\cdots(n+r-1)}{r!}(-x)^r + \cdots$

(a) Replace *n* by 1 in (3):

$(1-x)^{-1} = 1 + x + x^2 + \cdots + x^r + \cdots$

General term, $T_{r+1} = x^r$

(b) Replace *n* by 1 in (4):

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots (-x)^r + \cdots$

General term, $T_{r+1} = (-x)^r$

(c) Replace *n* by 2 in (3):

$$(1-x)^{-2} = 1 + 2x + 3x^2 + \cdots + (r+1)x^r + \cdots$$

General term, $T_{r+1} = (r+1)x^r$

(d) Replace *n* by 2 in (4):

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots + (r+1)(-x)^r + \cdots$$

General term, $T_{r+1} = (r+1)(-x)^r$

(e) Replace *n* by 3 in (3):

$$(1+x)^{-3} = 1 - 3x + 6x^2 + 10x^3 + \cdots + \frac{(r+1)(r+2)}{2!}x^r + \cdots$$

General term, $T_{r+1} = \dfrac{(r+1)(r+2)}{2!}x^r$

(f) Replace *n* by 3 in (4):

$$(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \cdots + \frac{(r+1)(r+2)}{2!}(-x)^r + \cdots$$

General term, $T_{r+1} = \dfrac{(r+1)(r+2)}{2!}\cdot(-x)^r$

Illustration 8.47 To expand $(1+2x)^{-1/2}$ as an infinite series, find the range of x.

Solution: $(1+2x)^{-1/2}$ can be expanded, if $|2x| < 1$, that is,

$$|x| < \frac{1}{2} \Rightarrow -\frac{1}{2} < x < \frac{1}{2}$$

So, $x \in \left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$.

Illustration 8.48 If the value of x is so small that x^2 and higher power can be neglected, then $\dfrac{\sqrt{1+x} + \sqrt[3]{(1-x)^2}}{1+x+\sqrt{1+x}}$ is equal to

Solution: Given expression can be written as

$$\frac{(1+x)^{1/2} + (1-x)^{2/3}}{1+x+(1+x)^{1/2}}$$

$$= \frac{\left[1 + \frac{1}{2}x + \left(-\frac{1}{8}\right)x^2\right] + \left(1 - \frac{2}{3}x - \frac{1}{9}x^2 - \cdots\right)}{1 + x + \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots\right)}$$

$$= \frac{1 - \frac{1}{12}x - \frac{1}{144}x^2 + \cdots}{1 + \frac{3}{4}x - \frac{1}{16}x^2 + \cdots} = 1 - \frac{5}{6}x + \cdots = 1 - \frac{5}{6}x$$

when $x^2, x^3 \cdots$ are neglected.

Illustration 8.49 If $(1+ax)^n = 1 + 8x + 24x^2 + \cdots$, then find the value of a and n.

Solution: We know that

$$(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \cdots$$

$$(1+ax)^n = 1+\frac{n(ax)}{1!}+\frac{n(n-1)(ax)^2}{2!}+\cdots$$

$$\Rightarrow 1+8x+24x^2+\cdots = 1+\frac{n(ax)}{1!}+\frac{n(n-1)(ax)^2}{2!}+\cdots$$

Comparing coefficients of both sides we get, $na = 8$ and

$$\frac{n(n-1)a^2}{2!} = 24$$

on solving, $a = 2$, $n = 4$.

Illustration 8.50 Find the square root of $(99)^{1/2}$ correct to 4 places of a decimal.

Solution:

$$(99)^{\frac{1}{2}} = (100-1)^{\frac{1}{2}} = \left[100\left(1-\frac{1}{100}\right)\right]^{1/2}$$

$$= \left[100\left(1-\frac{1}{100}\right)\right]^{1/2} = (100)^{\frac{1}{2}}(1-\cdot 01)^{\frac{1}{2}} = 10(1-\cdot 01)^{\frac{1}{2}}$$

$$= 10\left[1+\frac{\frac{1}{2}}{1!}(-\cdot 01)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-\cdot 01)^2+\cdots \text{to } \infty\right]$$

$= 10\ [1 - 0.005 - 0.0000125 + \cdots \text{ to } \infty]$
$= 10\ (.9949875) = 9.94987 = 9.9499$

Illustration 8.51 Find the coefficient of x^r in the expansion of $(1-2x)^{-1/2}$.

Solution: Coefficient of x^r

$$= \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\cdots\left(-\frac{1}{2}-r+1\right)}{r!}(-2)^r$$

$$= \frac{1\cdot 3\cdot 5\cdots(2r-1)\cdot(-1)^r\cdot(-1)^r\cdot 2^r}{2^r r!} = \frac{1\cdot 3\cdot 5\cdots(2r-1)}{r!}\cdot\frac{2.4.6\cdots 2r}{2.4.6\cdots 2r}$$

$$= \frac{(2r)!}{r!r!2^r}$$

Illustration 8.52 The coefficient of x^{25} in $(1+x+x^2+x^3+x^4)^{-1}$ is_____.

Solution: Coefficient of x^{25} in $(1+x+x^2+x^3+x^4)^{-1}$

$$= \text{Coefficient of } x^{25} \text{ in } \left[\frac{1(1-x^5)}{1-x}\right]^{-1}$$

$$= \text{Coefficient of } x^{25} \text{ in } (1-x^5)^{-1}\cdot(1-x)$$

$$= \text{Coefficient of } x^{25} \text{ in } [(1-x^5)^{-1} - x(1-x^5)^{-1}]$$

$$= [1+(x^5)^1+(x^5)^2+\cdots] - x[1+(x^5)^1+(x^5)^2+\cdots]$$

$= \text{Coefficient of } x^{25} \text{ in } [1+x^5+x^{10}+x^{15}+\cdots] -$ Coefficient of x^{25} in $[1+x^5+x^{10}+x^{15}+\cdots] = 1-0 = 1$

$= 1-0 = 1$

8.11 Some Important Results

1. Pascal's triangle:

1	$(x+y)^0$
1 1	$(x+y)^1$
1 2 1	$(x+y)^2$
1 3 3 1	$(x+y)^3$
1 4 6 4 1	$(x+y)^4$
1 5 10 10 5 1	$(x+y)^5$

Pascal's triangle gives the direct binomial coefficients.

For example, $(x+y)^4 = 1x^4+4x^3y+6x^2y^2+4xy^3+y^4$

2. Method for finding terms free from radical or rational terms in the expansion of $(a^{1/p}+b^{1/q})^N\ \forall a, b \in$ prime numbers:

Find the general term, $T_{r+1} = {}^NC_r(a^{1/p})^{N-r}(b^{1/q})^r = {}^NC_r a^{\frac{N-r}{p}}\cdot b^{\frac{r}{q}}$

Putting the values of $0 \le r \le N$, when indices of a and b are integers.

Key Points

Number of irrational terms = Total terms – Number of rational terms.

Illustration 8.53 Find the number of integral terms in the expansion of $(\sqrt{3}+\sqrt[8]{5})^{256}$.

Solution:

$$T_{r+1} = {}^{256}C_r\cdot 3^{\frac{256-r}{2}}\cdot 5^{\frac{r}{8}}$$

First term $= {}^{256}C_0 3^{128}5^0 =$ integer and after eight terms, that is,

9^{th} term $= {}^{256}C_8 3^{124}\cdot 5^1 =$ integer

Continuing like this, we get an AP, $1^{st}, 9^{th}, \cdots, 257^{th}$;

$$T_n = a+(n-1)d \Rightarrow 257 = 1+(n-1)8 \Rightarrow n = 33$$

Illustration 8.54 Find the number of irrational terms in the expansion of $(\sqrt[8]{5}+\sqrt[6]{2})^{100}$.

Solution:

$$T_{r+1} = {}^{100}C_r 5^{\frac{100-r}{8}}\cdot 2^{\frac{r}{6}}$$

As 2 and 5 are co-prime, T_{r+1} will be rational if $100-r$ is a multiple of 8 and r is a multiple of 6, also $0 \le r \le 100$.

Therefore, $r = 0, 6, 12, \cdots, 96$;

Thus, $100-r = 4, 10, 16 \cdots 100$ (1)

But $100-r$ is to be a multiple of 8.

So, $100-r = 0, 8, 16, 24, \cdots 96$ (2)

Common terms in Eqs. (1) and (2) are 16, 40, 64, 88.

Hence, $r = 84, 60, 36, 12$ give rational terms.

Therefore, the number of irrational terms = 101 – 4 = 97.

3. Three/four consecutive term or coefficients:

(a) If consecutive coefficients are given: In this case, divide consecutive coefficients pair wise. We get equations and then solve them.

(b) If consecutive terms are given: In this case, divide consecutive terms pair wise, that is, if four consecutive terms be

$T_r, T_{r+1}, T_{r+2}, T_{r+3}$ then find $\frac{T_r}{T_{r+1}}, \frac{T_{r+1}}{T_{r+2}}, \frac{T_{r+2}}{T_{r+3}}$

$\Rightarrow \lambda_1, \lambda_2, \lambda_3,$ (say) then divide λ_1 by λ_2 and λ_2 by λ_3 and solve.

Illustration 8.55 If a_1, a_2, a_3, a_4 are the coefficients of any four consecutive terms in the expansion of $(1+x)^n$, then find the value of $\frac{a_1}{a_1+a_2}+\frac{a_3}{a_3+a_4}$.

Solution:

Let a_1, a_2, a_3, a_4 respectively be the coefficients of $(r+1)^{th}$, $(r+2)^{th}$, $(r+3)^{th}$, $(r+4)^{th}$ terms in the expansion of $(1+x)^n$. Then

$$a_1 = {}^nC_r, a_2 = {}^nC_{r+1}, a_3 = {}^nC_{r+2}, a_4 = {}^nC_{r+3}$$

Now,

$$\frac{a_1}{a_1+a_2}+\frac{a_3}{a_3+a_4} = \frac{{}^nC_r}{{}^nC_r+{}^nC_{r+1}}+\frac{{}^nC_{r+2}}{{}^nC_{r+2}+{}^nC_{r+3}}$$

$$= \frac{{}^nC_r}{{}^{n+1}C_{r+1}}+\frac{{}^nC_{r+2}}{{}^{n+1}C_{r+3}} = \frac{{}^nC_r}{\frac{n+1}{r+1}\,{}^nC_r}+\frac{{}^nC_{r+2}}{\frac{n+1}{r+3}\,{}^nC_{r+2}}$$

$$= \frac{r+1}{n+1}+\frac{r+3}{n+1} = \frac{2(r+2)}{n+1}$$

$$= 2\cdot\frac{{}^nC_{r+1}}{{}^{n+1}C_{r+2}} = 2\cdot\frac{{}^nC_{r+1}}{{}^nC_{r+1}+{}^nC_{r+2}} = \frac{2a_2}{a_2+a_3}$$

Additional Solved Examples

1. The digit at units place in the number $17^{1995}+11^{1995}-7^{1995}$ is

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3

Solution: We have

$17^{1995}+11^{1995}-7^{1995} = (7+10)^{1995}+(1+10)^{1995}-7^{1995}$

$= [7^{1995}+{}^{1995}C_1 7^{1994}\cdot 10^1+{}^{1995}C_2 7^{1993}\cdot 10^2+\cdots+{}^{1995}C_{1995}\cdot 10^{1995}]$

$+[{}^{1995}C_0+{}^{1995}C_1\, 10^1+{}^{1995}C_2\, 10^2+\cdots+{}^{1995}C_{1995}\, 10^{1995}]-7^{1995}$

$= [{}^{1995}C_1 7^{1994}\cdot 10^1+\cdots+10^{1995}]+[{}^{1995}C_1 10^1+\cdots+{}^{1995}C_{1995}\, 10^{1995}]$
$+{}^{1995}C_0$

$= 10\{[{}^{1995}C_1 7^{1994}+\cdots+10^{1994}]+[{}^{1995}C_1+\cdots+{}^{1995}C_{1995}10^{1995}]\}+1$

$=$ (a multiple of 10) + 1

Thus, the digit in the units place is 1.

Hence, the correct answer is option (B).

2. If the sum of the coefficients in the expansion of $(1+2x)^n$ is 6561, the greatest term in the expansion at $x = 1/2$ is

(A) 4^{th} **(B)** 5^{th} **(C)** 6^{th} **(D)** None of these

Solution: Sum of the coefficients in the expansion of

$$(1+2x)^n = 6561$$
$$\Rightarrow (1+2x)^n = 6561, \text{ when } x = 1$$
$$\Rightarrow 3^n = 6561$$
$$\Rightarrow 3^n = 3^8 \Rightarrow n = 8$$

Now,

$$m = (n+1)\frac{|2x|}{1+|2x|} = \frac{9}{2} = 4.5$$

Since, m is not an integer.

Therefore, $T_{[m]+1}$ is the greatest term.

Hence, the 5^{th} term is the greatest term.

Hence, the correct answer is option (B).

3. The number of terms in the expansion of $(a+b+c)^n$, where $n \in N$, is

(A) $\frac{(n+1)(n+2)}{2}$ **(B)** $n+1$ **(C)** $n+2$ **(D)** $(n+1)n$

Solution:

$(a+(b+c))^n = a^n+{}^nC_1\, a^{n-1}(b+c)^1+{}^nC_2 a^{n-2}(b+c)^2+\cdots+{}^nC_n(b+c)^n$

Further expanding each term of RHS:

First term on expansion gives one term.

Second term on expansion gives two terms and so on.

Therefore, the total number of terms $= 1+2+3+\cdots+(n+1)$

$$= \frac{(n+1)(n+2)}{2}$$

Hence, the correct answer is option (A).

4. The number of terms which are free from fractional powers in the expansion of $(a^{1/5}+b^{2/3})^{45}$, $a \neq b$ is

(A) 9 **(B)** 15 **(C)** 4 **(D)** None of these

Solution: The general term in the expansion of $(a^{1/5}+b^{2/3})^{45}$ is

$$T_{r+1} = {}^{45}C_r\,(a^{1/5})^{45-r}(b^{2/3})^r$$
$$= {}^{45}C_r a^{9-(r/5)}\, b^{2r/3}$$

This will be free from fractional powers if both $r/5$ and $2r/3$ are whole numbers, that is, if $r = 0, 15, 30, 45$.

Hence, there are only four terms which are free from fractional powers.

Hence, the correct answer is option (C).

5. If n is an odd natural number, then $\sum_{r=0}^{n}\frac{(-1)^r}{{}^nC_r}$ equals

(A) 0 **(B)** $1/n$ **(C)** $n/2^n$ **(D)** None of these

Solution:

$$I = \sum_{r=0}^{n}\frac{(-1)^r}{{}^nC_r} = \sum_{r=0}^{n}\frac{(-1)^{n-r}}{{}^nC_{n-r}}$$

$$I = \frac{1}{2}\left(\sum_{r=0}^{n}\frac{(-1)^r}{{}^nC_r}+\sum_{r=0}^{n}\frac{(-1)^{n-r}}{{}^nC_{n-r}}\right)$$

(collecting the terms equidistant from the beginning and end in pairs)

$$= \sum_{r=0}^{(n+1)/2}(-1)^r\left\{\frac{1}{{}^nC_r}+\frac{-1}{{}^nC_r}\right\} \text{ [since, } (-1)^n = -1 \text{ as } n \text{ is odd]}$$

$= 0$

Hence, the correct answer is option (A).

6. The value of $\sum_{k=0}^{n} k^2 \; ({}^nC_k)$ is equal to

(A) $n(n+1)2^{n-2}$ **(B)** $n(n+1)2^{n-1}$

(C) $n(n+1)2^n$ **(D)** None of these

Solution: General term is

$$k^2 \; {}^nC_k = k^2 \frac{n(n-1)(n-2)...(n-k+1)}{1\cdot 2\cdot 3 ... (k-1)k}$$

$$= k\cdot \frac{n(n-1)(n-2)\cdots(n-k+1)}{1\cdot 2\cdot 3\cdots(k-1)}$$

$$= (k-1+1)\frac{n(n-1)(n-2)\cdots(n-k+1)}{1\cdot 2\cdot 3\cdots(k-2)(k-1)}$$

$$= \frac{n(n-1)(n-2)\cdots(n-k+1)}{1\cdot 2\cdot 3\cdots(k-2)} + \frac{n(n-1)(n-2)\cdots(n-k+1)}{1\cdot 2\cdots(k-2)(k-1)}$$

$$= n(n-1)\; {}^{(n-2)}C_{k-2} + n^{n-1}C_{k-1}$$

Therefore,

$$\sum_{k=0}^{n} k^2 \; {}^nC_k = n(n-1)\sum_{k=0}^{n} {}^{(n-2)}C_{k-2} + n\sum_{k=0}^{n-1} {}^{(n-1)}C_{k-1}$$

$$= n(n-1)\sum_{k=2}^{n} {}^{(n-2)}C_{k-2} + n\sum_{k=1}^{n-1} {}^{(n-1)}C_{k-1}$$

$$= n(n-1)\cdot 2^{(n-2)} + n\cdot 2^{n-1}$$

$$= 2^{n-2}\{n(n-1)+2n\} = n(n+1)\,2^{n-2}$$

Hence, the correct answer is option (A).

7. Find the coefficient of x^5 in the expansion of $(1+x+x^3)^9$.

Solution:

$$(1+x+x^3)^9 = [(1+x)+x^3]^9$$

$$= (1+x)^9 + {}^9C_1(1+x)^8x^3 + {}^9C_2(1+x)^7x^6 + \cdots \quad (1)$$

The coefficient of x^5 in $(1+x)^9$ is ${}^9C_5$, that is, ${}^9C_4$

The coefficient of x^5 in ${}^9C_1(1+x)^8x^3$ = coefficient of x^2 in ${}^9C_1(1+x)^8$
$= 9\cdot {}^8C_2$

All the remaining terms in Eq. (1) contain powers of x higher than the fifth.

Therefore, the required coefficient is

$${}^9C_4 + 9\cdot {}^8C_2 = \frac{9\cdot 8\cdot 7\cdot 6}{1\cdot 2\cdot 3\cdot 4} + \frac{9\cdot 8\cdot 7}{1\cdot 2} = 378$$

Alternatively,

$$[1+x(1+x^2)]^9 = 1 + {}^9C_1x(1+x^2) + {}^9C_2x^2(1+x^2)^2 + {}^9C_3x^3(1+x^2)^3 + {}^9C_4x^4(1+x^2)^4 + \cdots$$

x^5 occurs in 4th and 6th terms only and it is equal to

$$3\times {}^9C_3 + {}^9C_5 = 252 + 126 = 378$$

8. Show that $11^{n+2} + 12^{2n+1}$ is divisible by 133.

Solution:

$$11^{n+2} + 12^{2n+1} = 11^2\cdot 11^n + 12(144)^n$$

Now 144 and 121 should be expressed in terms of 133; 144 as (133 + 11) or 121 as (133 – 12)

$$= 121\cdot 11^n + 12(11+133)^n$$

$$= 121\cdot 11^n + 12[11^n + {}^nC_1\,11^{n-1}\cdot 133 + \cdots]$$

$$= 121\cdot 11^n + 12\cdot 11^n + \text{terms containing 133 as a factor}$$

$$= 11^n(121+12) + \text{terms containing 133 as a factor}$$

$$= 11^n\cdot 133 + \text{terms containing 133 as a factor}$$

Hence, the expression is divisible by (133).

9. Evaluate $\frac{C_1}{2} + \frac{C_2}{4} + \frac{C_5}{6} + \cdots$.

Solution: We have

$$\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \cdots$$

$$= \frac{n}{2\cdot 1} + \frac{n(n-1)(n-2)}{4\cdot 3\cdot 2\cdot 1} + \frac{n(n-1)(n-2)(n-3)(n-4)}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$$

$$+ \cdots \frac{1}{n+1}\left[\frac{(n+1)n}{2!} + \frac{(n+1)(n)(n-1)(n-2)}{4!}\cdots\right]$$

$$= \frac{2^{(n+1)-1}-1}{n+1} = \frac{2^n-1}{n+1}$$

10. The term independent of x in the expansion of

$$\left\{\frac{x+1}{(x^{2/3}-x^{1/3}+1)} - \frac{x-1}{x-\sqrt{x}}\right\}^{15} \text{ is}$$

(A) ${}^{15}C_7$ **(B)** ${}^{15}C_9$ **(C)** ${}^{15}C_5$ **(D)** None of these

Solution: Now

$$\frac{x+1}{(x^{2/3}-x^{1/3}+1)} = \frac{(x^{1/3})^3+1^3}{(x^{1/3})^2 - x^{1/3}+1^2} = x^{1/3}+1$$

since, $a^3+b^3 = (a+b)(a^2-ab+b^2)$

and $\frac{x-1}{x-\sqrt{x}} = \frac{(\sqrt{x}+1)(\sqrt{x}-1)}{\sqrt{x}(\sqrt{x}-1)} = \frac{\sqrt{x}+1}{\sqrt{x}} = 1+x^{-1/2}$

Hence,

$$\left\{\frac{x+1}{x^{2/3}-x^{1/3}+1} - \frac{x-1}{x-\sqrt{x}}\right\}^{15} = \{x^{1/3}+1-(1+x^{-1/2})\}^{15} = (x^{1/3}-x^{-1/2})^{15}$$

General term,

$$T_{r+1} = {}^{15}C_r(x^{1/3})^{15-r}(-x^{-1/2})^r$$

$$= {}^{15}C_r x^{(15-r)/3}(-1)^r x^{-r/2}$$

$$= {}^{15}C_r(-1)^r x^{(15-r)/3-(r/2)}$$

This will be independent of x if $\frac{15-r}{3} - \frac{r}{2} = 0$, that is, if

$$30 - 2r - 3r = 0$$
$$\Rightarrow r = 6$$

Therefore, the term independent of x is T_{6+1}, that is,

$$T_7 = {}^{15}C_6(-1)^6 = {}^{15}C_6 = {}^{15}C_9 \quad (\text{using, } {}^nC_r = {}^nC_{n-r})$$

Hence, the correct answer is option (B).

11. If a, b, c and d are any four consecutive coefficients of a binomial expansion, prove that $\frac{a}{a+b} + \frac{c}{c+d} = \frac{2b}{b+c}$ or $\frac{a+b}{a}$, $\frac{b+c}{b}$, $\frac{c+d}{c}$ are in HP.

Solution: Let a, b, c and d be the coefficients of $(r+1)^{th}$, $(r+2)^{th}$, $(r+3)^{th}$ and $(r+4)^{th}$ terms of $(1+x)^n$.
Therefore,

$$a={}^nC_r,\ b={}^nC_{r+1},\ c={}^nC_{r+2},\ d={}^nC_{r+3}$$

$$\frac{a}{a+b}=\frac{{}^nC_r}{{}^nC_r+{}^nC_{r+1}}=\frac{{}^nC_r}{{}^{(n+1)}C_{r+1}}=\frac{r+1}{n+1}$$

$$\frac{b}{b+c}=\frac{{}^nC_{r+1}}{{}^nC_{r+1}+{}^nC_{r+2}}=\frac{{}^nC_{r+1}}{{}^{(n+1)}C_{r+2}}=\frac{r+2}{n+1}$$

Similarly,

$$\frac{c}{c+d}=\frac{r+3}{n+1}$$

Hence,

$$\frac{a}{a+b}+\frac{c}{c+d}=\frac{r+1}{n+1}+\frac{r+3}{n+1}=\frac{2(r+2)}{n+1}=2\cdot\frac{b}{b+c}$$

Therefore, $\frac{a}{a+b},\frac{b}{b+c},\frac{c}{c+d}$ are in AP.

So, $\frac{a+b}{a},\frac{b+c}{b},\frac{c+d}{c}$ are in HP.

12. Find the coefficient of x^4 in the expansion of $(1+x+x^2+x^3)^{11}$.

Solution:

$$1+x+x^2+x^3=(1+x)+x^2(1+x)=(1+x)(1+x^2)$$

Therefore,

$$(1+x+x^2+x^3)^{11}=(1+x)^{11}(1+x^2)^{11}$$
$$=\left[\sum_{r=0}^{11}{}^{11}C_r\,x^r\right]\times\left[\sum_{s=0}^{11}{}^{11}C_s\,(x^2)^s\right]$$

The general term in the product of these two series is

$$^{11}C_r\times{}^{11}C_s\,x^{r+2s}$$

Now, $r+2s$ must be equal to 4 for values of r, s, $0\le r,s\le 11$.
The possible values of r and s are $r=0, s=2; r=2, s=1; r=4, s=0$

Therefore, coefficient of x^4 $={}^{11}C_0\times{}^{11}C_2+{}^{11}C_2\times{}^{11}C_1+{}^{11}C_4\times{}^{11}C_0$
$=55+605+330$
$=990$

Alternative method:

Coefficient of x^4 in $(1+x+x^2+x^3)^{11}$
$=$ Coefficient of x^4 in $(1-x^4)^{11}(1-x)^{-11}$
$=$ Coefficient of x^4 in $(1-11x^4)(1-x)^{-11}$
$={}^{11+4-1}C_{10}-11={}^{14}C_{10}-11=1001-11=990$

13. For what value of x is the fourth term in the expansion of

$$\left\{(\sqrt{x})^{\frac{1}{\log x+1}}+\sqrt[12]{x}\right\}^6$$ is equal to 200; $\log x=\log_{10}x$.

Solution:

The fourth term of the expansion $={}^6C_3\,(\sqrt{x})^{\frac{3}{\log x+1}}\,(x^{1/12})^3$
$=20\,x^{\frac{1}{4}+\frac{3}{2(1+\log x)}}=200$

Therefore,

$$x^{\frac{1}{4}+\frac{3}{2(1+\log x)}}=10$$

Taking logarithm on both sides,

$$\left\{\frac{1}{4}+\frac{3}{2(1+\log x)}\right\}\log x=1$$
$$\Rightarrow\{(1+\log x)+6\}\log x=4(1+\log x)$$
$$\Rightarrow(\log x)^2+3\log x-4=0\Rightarrow(\log x+4)(\log x-1)=0$$

Either $\log_{10}x=-4$ or $\log_{10}x=1$. So,
$x=10^{-4}$ or 10

14. Find the coefficient of x^{50} in the expansion

$$S=(1+x)^{1000}+2x(1+x)^{999}+3x^2(1+x)^{998}+\cdots+1001\,x^{1000}$$

Solution: Take $(1+x)^{1000}$ common, and let $\frac{x}{1+x}=r$

$$S=(1+x)^{1000}\left[1+2\frac{x}{1+x}+\frac{3x^2}{(1+x)^2}+\ldots+1001\left(\frac{x}{1+x}\right)^{1000}\right]$$
$$=(1+x)^{1000}[1+2r+3r^2+\cdots+1001r^{1000}]$$
$$=(1+x)^{1000}\left\{\left(\frac{1-r^{1001}}{(1-r)^2}\right)-\frac{1001r^{1000}}{1-r}\right\}$$, using the formula of GP
$$=(1+x)^{1000}\left\{\frac{1-\left(\frac{x}{1+x}\right)^{1001}}{\left(\frac{1}{1+x}\right)^2}-\frac{1001\left(\frac{x}{1+x}\right)^{1001}}{\frac{1}{1+x}}\right\}$$
$$=(1+x)^{1002}-x^{1001}(1+x)-1001\,x^{1001}$$

Therefore, coefficient of x^{50} in $S=$ coefficient of x^{50} in $(1+x)^{1002}$
$={}^{1002}C_{50}$

15. Find the sum of the series

$$\sum_{r=0}^{n}(-1)^r\,{}^nC_r\left[\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\ldots\text{upto } m \text{ terms}\right].$$

Solution: We have

$$\sum_{r=0}^{n}(-1)^r\,{}^nC_r\left[\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\ldots\text{upto } m \text{ terms}\right]$$
$$=\sum_{r=0}^{n}(-1)^r\,{}^nC_r\cdot\left(\frac{1}{2}\right)^r+\sum_{r=0}^{n}(-1)^r\,{}^nC_r\left(\frac{3}{4}\right)^r+\sum_{r=0}^{n}(-1)^r\,{}^nC_r\left(\frac{7}{8}\right)^r$$
$+\cdots$ upto m terms
$$=\left(1-\frac{1}{2}\right)^n+\left(1-\frac{3}{4}\right)^n+\left(1-\frac{7}{8}\right)^n+\cdots\text{ upto } m \text{ terms}$$
$$=\left(\frac{1}{2}\right)^n+\left(\frac{1}{4}\right)^n+\left(\frac{1}{8}\right)^n+\cdots\text{ upto } m \text{ terms}$$
$$=\frac{1}{2^n}+\frac{1}{(2^2)^n}+\frac{1}{(2^3)^n}+\cdots\text{ upto } m \text{ terms}$$
$$=\frac{1}{2^n}\left(\frac{1-\left(\frac{1}{2^n}\right)^m}{1-\frac{1}{2^n}}\right)\left(\text{being the sum of } m \text{ terms of a GP with } r=\frac{1}{2^n}\right)$$
$$=\frac{2^{mn}-1}{2^{mn}(2^n-1)}$$

Previous Years' Solved JEE Main/AIEEE Questions

1. In the binomial expansion of $(a-b)^n, n\geq 5$, the sum of 5th and 6th terms is zero, then $\frac{a}{b}$ equals:

(A) $\frac{5}{n-4}$ **(B)** $\frac{6}{n-5}$ **(C)** $\frac{n-5}{6}$ **(D)** $\frac{n-4}{5}$

[AIEEE 2007]

Solution:

$$T_5+T_6=0 \Rightarrow T_{4+1}+T_{5+1}=0 \Rightarrow {}^nC_4 a^{n-4}(-b)^4+{}^nC_5 a^{n-5}(-b)^5=0$$

$$\Rightarrow {}^nC_4 a^{n-4}b^4-{}^nC_5 a^{n-5}b^5=0$$

$$\Rightarrow \frac{a^{n-4}b^4}{a^{n-5}b^5}=\frac{{}^nC_5}{{}^nC_4}=\frac{n!}{5!\,n!-5}\times\frac{4!\,n!-4}{n!}\Rightarrow\left(\frac{a}{b}\right)=\frac{n-4}{5}$$

Hence, the correct answer is option (D).

2. The sum of the series ${}^{20}C_0-{}^{20}C_1+{}^{20}C_2-{}^{20}C_3+\cdots+{}^{20}C_{10}$ is

(A) $-{}^{20}C_{10}$ **(B)** $\frac{1}{2}\,{}^{20}C_{10}$ **(C)** 0 **(D)** ${}^{20}C_{10}$

[AIEEE 2007]

Solution: We have

$$(1+x)^{20}={}^{20}C_0+{}^{20}C_1x+\cdots+{}^{20}C_{10}x^{10}+\cdots+{}^{20}C_{20}x^{20}$$

Substituting $x=-1$, we get

$$0=({}^{20}C_0-{}^{20}C_1+\cdots-{}^{20}C_9)+{}^{20}C_{10}+(-{}^{20}C_{11}+\cdots+{}^{20}C_{20})$$

$$0=({}^{20}C_0-{}^{20}C_1+\cdots-{}^{20}C_9)+{}^{20}C_{10}+(-{}^{20}C_9+\cdots+{}^{20}C_0)$$

[since ${}^nC_r={}^nC_{n-r}$]

$$0=2({}^{20}C_0-{}^{20}C_1+\cdots-{}^{20}C_9)+{}^{20}C_{10}\Rightarrow$$

$$={}^{20}C_0-{}^{20}C_1+\cdots+{}^{20}C_{10}=\frac{1}{2}\,{}^{20}C_{10}$$

Hence, the correct answer is option (B).

3. Statement-1: $\sum_{r=0}^{n}(r+1)\,{}^nC_r=(n+2)2^{n-1}$

Statement-2: $\sum_{r=0}^{n}(r+1)\,{}^nC_r x^r=(1+x)^n+nx(1+x)^{n-1}$

(A) Statement-1 is false, Statement-2 is true
(B) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(C) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(D) Statement-1 is true, Statement-2 is false

[AIEEE 2008]

Solution:

$$\sum_{r=0}^{n}(r+1)\,{}^nC_r=\sum_{r=0}^{n}\left(r\,{}^nC_r+{}^nC_r\right)=\sum_{r=0}^{n}r\frac{n}{r}\,{}^{n-1}C_{r-1}+\sum_{r=0}^{n}{}^nC_r$$

$$=n2^{n-1}+2^n=2^{n-1}(n+2)$$

Hence, statement-1 is true.

$$\sum(r+1)\,{}^nC_r x^r=\sum r\,{}^nC_r x^r+\sum {}^nC_r x^r=n\sum_{r=0}^{n}{}^{n-1}C_{r-1}x^r+\sum_{r=0}^{n}{}^nC_r x^r$$

$$=nx(1+x)^{n-1}+(1+x)^n$$

Substituting $x=1$

$$\sum(r+1)\,{}^nC_r=n2^{n-1}+2^n$$

Hence, the correct answer is option (B).

4. The remainder left out when $8^{2n}-(62)^{2n+1}$ is divided by 9 is

(A) 0 **(B)** 2 **(C)** 7 **(D)** 8

[AIEEE 2009]

Solution:

$$8^{2n}-(62)^{2n+1}=(64)^n-(62)^{2n+1}=(1+63)^n-(63-1)^{2n+1}$$

$$=(1+63)^n+(1-63)^{2n+1}=(1+{}^nC_1 63+{}^nC_2(63)^2+\cdots+$$

$$(63)^n)+(1-{}^{(2n+1)}C_1 63+{}^{(2n+1)}C_2(63)^2+\cdots+(-1)(63)^{(2n+1)})$$

$$=2+63\{{}^nC_1+{}^nC_2(63)+\cdots+(63)^{n-1}-{}^{(2n+1)}C_1+{}^{(2n+1)}$$

$$C_2(63)+\cdots-(63)^{(2n)}\}$$

Therefore, the remainder is 2.

Hence, the correct answer is option (B).

5. Let $S_1=\sum_{j=1}^{10}j(j-1)\,{}^{10}C_j$, $S_2=\sum_{j=1}^{10}j\,{}^{10}C_j$ and $S_3=\sum_{j=1}^{10}j^2\,{}^{10}C_j$.

Statement-1: $S_3=55\times 2^9$

Statement-2: $S_1=90\times 2^8$ and $S_2=10\times 2^8$.

(A) Statement-1 is true, Statement-2 is true; Statement-2 is not the correct explanation for Statement-1
(B) Statement-1 is true, Statement-2 is false
(C) Statement-1 is false, Statement-2 is true
(D) Statement-1 is true, Statement-2 is true; Statement-2 is the correct explanation for Statement-1

[AIEEE 2010]

Solution:

$$S_1=\sum_{j=1}^{10}j(j-1)\frac{10!}{j(j-1)(j-2)!(10-j)!}=90\sum_{j=2}^{10}\frac{8!}{(j-2)!(8-(j-2))!}$$

$$=90\cdot 2^8$$

$$S_2=\sum_{j=1}^{10}j\frac{10!}{j(j-1)!(9-(j-1))!}=10\sum_{j=1}^{10}\frac{9!}{(j-1)!(9-(j-1))!}=10\cdot 2^9$$

$$S_3=\sum_{j=1}^{10}[j(j-1)+j]\frac{10!}{j!(10-j)!}=\sum_{j=1}^{10}j(j-1)\,{}^{10}C_j+\sum_{j=1}^{10}j\,{}^{10}C_j$$

$$=90\cdot 2^8+10\cdot 2^9=90\cdot 2^8+20\cdot 2^8=110\cdot 2^8=55\cdot 2^9$$

Hence, the correct answer is option (B).

6. The coefficient of x^7 in the expansion of $(1-x-x^2+x^3)^6$ is

(A) –132 **(B)** –144 **(C)** 132 **(D)** 144

[AIEEE 2011]

Solution: We have

$$[1-x-x^2(1-x)]^6=(1-x)^6(1-x^2)^6$$

$$=[{}^6C_0-{}^6C_1x+{}^6C_2x^2-{}^6C_3x^3+{}^6C_4x^4-{}^6C_5x^5$$

$$+{}^6C_6x^6]\times[{}^6C_0-{}^6C_1x^2+{}^6C_2x^4-{}^6C_3x^6+\cdots]$$

Coefficient of x^7 is

$${}^6C_1\,{}^6C_3-{}^6C_3\,{}^6C_2+{}^6C_5\,{}^6C_1=120-300+36=-144$$

Hence, the correct answer is option (B).

7. If n is a positive integer, then $(\sqrt{3}+1)^{2n}-(\sqrt{3}-1)^{2n}$ is
(A) an irrational number.
(B) an odd positive integer.
(C) an even positive integer.
(D) a rational number other than positive integers.

[AIEEE 2012]

Solution:

$$(\sqrt{3}+1)^{2n}-(\sqrt{3}-1)^{2n}=[(\sqrt{3}+1)^2]^n-[(\sqrt{3}-1)^2]^n$$
$$=(4+2\sqrt{3})^n-(4-2\sqrt{3})^n$$
$$=2^n[(2+\sqrt{3})^n-(2-\sqrt{3})^n]$$
$$=2^n\{[{}^nC_0 2^n+{}^nC_1 2^{n-1}\sqrt{3}+{}^nC_2 2^{n-2}3+\cdots]-[{}^nC_0 2^n$$
$$-{}^nC_1 2^{n-1}\sqrt{3}+{}^nC_2 2^{n-2}3-\cdots]\}$$
$$=2^{n+1}[{}^nC_1 2^{n-1}\sqrt{3}+{}^nC_3 2^{n-3}3\sqrt{3}+\cdots]=2^{n+1}\sqrt{3}\times\text{(some integer)}$$

Hence, the correct answer is option (A).

8. The term independent of x in expansion of
$$\left(\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right)^{10} \text{ is}$$
(A) 120 **(B)** 210 **(C)** 310 **(D)** 4

[JEE MAIN 2013]

Solution: We have
$$\left(\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right)^{10}$$
[using $a^3+b^3=(a+b)(a^2-ab+b^2)$ and $a^2-b^2=(a-b)(a+b)$]
$$=\left(\frac{(x^{1/3}+1)\cancel{(x^{2/3}-x^{1/3}+1)}}{\cancel{(x^{2/3}-x^{1/3}+1)}}-\frac{1}{\sqrt{x}}\cdot\frac{(\sqrt{x}+1)\cancel{(\sqrt{x}-1)}}{\cancel{(\sqrt{x}-1)}}\right)^{10}$$
$$=\left(x^{1/3}+1-\frac{x^{1/2}+1}{x^{1/2}}\right)=(x^{1/3}+1-1-x^{-1/2})=(x^{1/3}-x^{-1/2})^{10}$$
Therefore,
$$T_{r+1}={}^{10}C_r(x^{1/3})^{10-r}(-x^{-1/2})^r\Rightarrow(-1)^r\,{}^{10}C_r x^{(20-5r)/6}$$
For T_{r+1} to be independent of x,
$$20-5r=0\Rightarrow r=4$$
Therefore, $T_5=T_{4+1}=(-1)^4\,{}^{10}C_4=210.$

Hence, the correct answer is option (B).

9. If $X=\{4^n-3n-1: n\in N\}$ and $Y=\{9(n-1): n\in N\}$, where N is the set of natural numbers, then $X\cup Y$ is equal to
(A) X **(B)** Y **(C)** N **(D)** $Y-X$

[JEE MAIN 2014 (OFFLINE)]

Solution:
$$X=\{4^n-3n-1: n\in N\}$$
$$4^n-3n-1=(1+3)^n-3n-1$$
$$=\cancel{{}^nC_0 3^0}+\cancel{{}^nC_1 3^1}+{}^nC_2 3^2+\cdots+{}^nC_n 3^n-\cancel{3n}-\cancel{1}$$
$=9({}^nC_2+\cdots+{}^nC_n 3^{n-2})=9k$, where k is an integer, when $n\geq 2$

When $n=1$,
$$4^n-3n-1=4-3-1=0=9k$$
Therefore, X contains multiples of 9.
Now,
$$Y=\{9(n-1): n\in N\}=\{0, 9, 18, \cdots\}=\text{all multiples of 9}$$
Thus, $X\cup Y=Y$.

Hence, the correct answer is option (B).

10. If the coefficients of x^3 and x^4 in the expansion of $(1+ax+bx^2)(1-2x)^{18}$ in powers of x are both zero, then (a, b) is equal to
(A) $\left(14, \frac{272}{3}\right)$ **(B)** $\left(16, \frac{272}{3}\right)$
(C) $\left(16, \frac{251}{3}\right)$ **(D)** $\left(14, \frac{251}{3}\right)$

[JEE MAIN 2014 (OFFLINE)]

Solution:
$(1+ax+bx^2)(1-2x)^{18}$
$$=(1+ax+bx^2)\{{}^{18}C_0(-2x)^0+{}^{18}C_1(-2x)^1+{}^{18}C_2(-2x)^2+{}^{18}C_3(-2x)^3+{}^{18}C_4(-2x)^4+\cdots\}$$
Coefficient of $x^3={}^{18}C_3(-2)^3+a\times{}^{18}C_2(-2)^2+b\times{}^{18}C_1(-2)$
$$=-\frac{8\times\cancel{18}^{3}\times17\times16}{\cancel{3}\times2\times1}+a\times\cancel{4}^{2}\times\frac{18\times17}{\cancel{2}\times1}-2b\times18$$
$$=-4\{2\times3\times17\times16-9\times17a+9b\}=0$$
$\Rightarrow 153a-9b=1632\Rightarrow 51a-3b=544$

Only $\left(16, \frac{272}{3}\right)$ satisfies the equation.

Hence, the correct answer is option (B).

11. The number of terms in the expansion of $(1+x)^{101}(1+x^2-x)^{100}$ in powers of x is
(A) 302 **(B)** 301 **(C)** 202 **(D)** 101

[JEE MAIN 2014 (ONLINE SET 1)]

Solution:
$(1+x)^{101}(1+x^2-x)^{100}=(1+x)^{100}(1-x+x^2)^{100}(1+x)^1$
$$=(1^3+x^3)^{100}(1+x)$$
$$=\left\{{}^{100}C_0(x^3)^0+{}^{100}C_1(x^3)^1+{}^{100}C_2(x^3)^2+\cdots+{}^{100}C_{100}(x^3)^{100}\right\}(1+x)$$
$$=(1+{}^{100}C_1x^3+{}^{100}C_2x^6+\cdots+x^{300})(1+x)$$
$$=\underbrace{1+{}^{100}C_1x^3+\ldots+x^{300}}_{101\text{ term}}+\underbrace{x+{}^{100}C_1x^4+\cdots+x^{301}}_{101\text{ term}}$$
Therefore, total terms = 202 terms (including terms with x).

Hence, the correct answer is option (C).

12. The coefficient of x^{50} in the binomial expansion of $(1+x)^{1000}+x(1+x)^{999}+x^2(1+x)^{998}+\cdots+x^{1000}$ is
(A) $\frac{(1000)!}{(50)!(950)!}$ **(B)** $\frac{(1000)!}{(49)!(951)!}$
(C) $\frac{(1001)!}{(51)!(950)!}$ **(D)** $\frac{(1001)!}{(50)!(951)!}$

[JEE MAIN 2014 (ONLINE SET 2)]

Solution: Coefficient of x^{50} is

$$^{1000}C_{50} + {}^{999}C_{49} + \cdots + {}^{952}C_2 + {}^{951}C_1 + {}^{950}C_0$$

$${}^{951}C_1 + {}^{950}C_0 = {}^{952}C_1$$

$${}^{952}C_2 + {}^{952}C_1 = {}^{953}C_2$$

$$\ldots = {}^{101}C_{50}$$

Therefore, $\dfrac{1001!}{50!\,951!}$.

Hence, the correct answer is option (D).

13. If $\left(2+\dfrac{x}{3}\right)^{55}$ is expanded in the ascending powers of x and the coefficients of powers of x in two consecutive terms of the expansion are equal, then these terms are:

(A) 7th and 8th **(B)** 8th and 9th
(C) 28th and 2th **(D)** 27th and 28th

[JEE MAIN 2014 (ONLINE SET 3)]

Solution: General term is

$$T_{r+1} = {}^{55}C_r 2^{55-r}\left(\frac{x}{3}\right)^r$$

Let the two consecutive term be T_{r+1} and T_{r+2}.

Therefore, according to question

$${}^{55}C_r 2^{55-r}\frac{1}{3^r} = {}^{55}C_{r+1}2^{55-r-1}\frac{1}{3^{r+1}}$$

$$\Rightarrow \frac{55!}{r!(55-r)!}\times 2^{55-r-55+r+1} = \frac{55!}{r+1!(55-r-1)!}\frac{1}{3^{r+1-r}}$$

$$\Rightarrow 2\times 3\times(r+1)r!\times(54-r)! = r!(55-r)(54-r)!$$

$$\Rightarrow 6\,(r+1) = 55 - r \Rightarrow 6\,r + r = 55 - 6$$

$$\Rightarrow 7\,r = 49 \Rightarrow r = 7$$

Therefore, terms are T_{7+1} & T_{7+2}, that is, T_8 and T_9.

Hence, the correct answer is option (B).

14. If $1+x^4+x^5 = \sum_{i=0}^{5} a_i\,(1+x)^i$ for all x in R, then a_2 is:

(A) –4 **(B)** 6 **(C)** –8 **(D)** 10

[JEE MAIN 2014 (ONLINE SET 3)]

Solution:

$1 + x^4 + x^5$

$$= a_0 + a_1\,(1+x)^1 + a_2\,(1+x)^2 + a_3\,(1+x)^3 + a_4\,(1+x)^4 + a_5\,(1+x)^5$$

Comparing coefficients of x^5

$$1 = a_5 \qquad (1)$$

Comparing coefficients of x^4

$$1 = {}^4C_4 a_4 + {}^5C_4 a_5 \text{ or } 1 = a_4 + 5\,a_5 \Rightarrow 1 = a_4 + 5$$
$$\Rightarrow a_4 = -4 \qquad (2)$$

Comparing coefficients of x^3

$$0 = {}^3C_3 a_3 + {}^4C_3 a_4 + {}^5C_3 a_5 \Rightarrow 0 = a_3 + 4\,a_4 + 10\,a_5$$
$$\Rightarrow 0 = a_3 + 4\,(-4) + 10\,(1) \Rightarrow a_3 = 6 \qquad (3)$$

Comparing coefficients of x^2

$$0 = a_2({}^2C_2) + a_3({}^3C_2) + a_4({}^4C_2) + a_5({}^5C_2)$$
$$\Rightarrow 0 = a_2 + 6\,(3) + (-4)\,6 + 1\,(10)$$
$$\Rightarrow 0 = a_2 + 18 - 24 + 10 \Rightarrow a_2 = -4$$

Hence, the correct answer is option (A).

15. The coefficient of x^{1012} in the expansion of $(1 + x^n + x^{253})^{10}$ (where $n \le 22$ is any positive integer) is

(A) 1 **(B)** $^{10}C_4$ **(C)** $4n$ **(D)** $^{253}C_4$

[JEE MAIN 2014 (ONLINE SET 4)]

Solution:

$$\{1+(x^n+x^{253})\}^{10} = {}^{10}C_0(x^n+x^{253})^0 + \cdots + \underbrace{{}^{10}C_4(x^n+x^{253})^4} + \cdots + {}^{10}C_{10}(x^n+x^{253})^{10}$$

$$\downarrow$$

$${}^{10}C_4\left\{{}^4C_0(x^n)^{4-0}(x^{253})^0 + \cdots + {}^4C_4(x^n)^{n-4}(x^{253})^4\right\}$$

Therefore, required coefficient $= {}^{10}C_4 \times {}^4C_4 = {}^{10}C_4$.

Hence, the correct answer is option (B).

16. If the coefficients of the three successive terms in the binomial expansion of $(1 + x)^n$ are in the ratio 1:7:42, then the first of these terms in the expansion is:

(A) 6th **(B)** 7th **(C)** 8th **(D)** 9th

[JEE MAIN 2015 (ONLINE SET 1)]

Solution: Let T_{r-1}, T_r, T_{r+1} be the three successive terms of $(1+x)^n$. Then

$${}^nC_{r-2} : {}^nC_{r-1} : {}^nC_r = 1:7:42$$

$$\Rightarrow \frac{{}^nC_{r-1}}{{}^nC_{r-2}} = 7;\ \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{42}{7} = 6$$

$$\Rightarrow \frac{n-r+2}{r-1} = 7;\ \frac{n-r+1}{r} = 6$$

$$\Rightarrow r = 8, n = 55 \Rightarrow T_{r-1} = T_7$$

Hence, first of the three given terms will be 7th term.

Hence, the correct answer is option (B).

17. The term independent of x in the binomial expansion of $\left(1-\dfrac{1}{x}+3x^5\right)\left(2x^2-\dfrac{1}{x}\right)^8$ is

(A) 400 **(B)** 496 **(C)** –400 **(D)** –496

[JEE MAIN 2015 (ONLINE SET 2)]

Solution: Given expression is

$$\left(1-\frac{1}{x}+3x^5\right)\left(2x^2-\frac{1}{x}\right)^8 = \left(1-\frac{1}{x}+3x^5\right).\sum_{r=0}^{8}{}^8C_r(2x^2)^{8-r}\left(\frac{-1}{x}\right)^r$$

$$= \left(1-\frac{1}{x}+3x^5\right)\cdot\sum_{r=0}^{8}{}^8C_r(-1)^r(2)^{8-r}\cdot x^{16-3r}$$

Therefore, term independent of x is

$$= \underbrace{-{}^8C_5(-1)^5(2)^3}_{\text{for } r=5} + \underbrace{3.{}^8C_7(-1)^7(2)^1}_{\text{for } r=7}$$

$= 56 \times 8 - 3(8)(2) = 400$

Hence, the correct answer is option (A).

18. If the number of terms in the expansion of $\left(1-\frac{2}{x}+\frac{4}{x^2}\right)^n$, $x \neq 0$, is 28, then the sum of the coefficients of all terms in this expansion is

(A) 729 **(B)** 64 **(C)** 2187 **(D)** 243

[JEE MAIN 2016 (OFFLINE)]

Solution: We have

$$\left(1-\frac{2}{x}+\frac{4}{x^2}\right)^n$$

The number of terms in the expansion is ${}^{n+2}C_2 = 28$. So,

$$\frac{(n+2)(n+1)}{2} = 28$$
$$\Rightarrow (n+2)\,(n+1) = 56$$
$$\Rightarrow n^2 + 3n - 54 = 0$$
$$\Rightarrow n = 6$$

The sum of coefficient is

$$(1 - 2 + 4)^6 = 3^6 = 729$$

This is possible only when we are not considering the number of dissimilar term.

Note: If we consider the dissimilar term, then number of terms is $2n + 1$ and hence,

$$2n + 1 = 28 \Rightarrow n = \frac{27}{2}$$

which is not possible (and hence, the question may be considered wrong).

Hence, the correct answer is option (A).

19. For $x \in R$, $x \neq -1$, if

$(1+x)^{2016} + x(1+x)^{2015} + x^2(1+x)^{2014} + \cdots + x^{2016} = \sum_{i=0}^{2016} a_i x^i$,

then a_{17} is equal to

(A) $\frac{2017!}{17!\ 2000!}$ **(B)** $\frac{2016!}{17!\ 1999!}$

(C) $\frac{2016!}{16!}$ **(D)** $\frac{2017!}{2000!}$

[JEE MAIN 2016 (ONLINE SET 1)]

Solution: We have

$$(1+x)^{2016}\frac{\left(\frac{x}{1+x}\right)^{2017}-1}{\frac{1}{1+x}-1} = \sum_{i=0}^{2016} a_i x^i$$

$$(1+x)^{2017}\frac{(1+x)^{2017}-x^{2017}}{(1+x)^{2017}} = \sum_{i=0}^{2016} a_i x^i$$

$$(1+x)^{2017} - x^{2017} = \sum_{i=0}^{2016} a_i x^i$$

Therefore,

$$a_{17} = \text{Coefficient of } x^{17} \text{ in } (x+1)^{2017} = {}^{2017}C_{17} = \frac{2017!}{17!2000!}$$

Hence, the correct answer is option (A).

20. If the coefficients of x^{-2} and x^{-4} in the expansion of $\left(x^{1/3}+\frac{1}{2x^{1/3}}\right)^{18}$, $(x > 0)$, are m and n, respectively, then $\frac{m}{n}$ is equal to

(A) 27 **(B)** 182 **(C)** $\frac{5}{4}$ **(D)** $\frac{4}{5}$

[JEE MAIN 2016 (ONLINE SET 2)]

Solution: General term in $\left(x^{1/3}+\frac{1}{2x^{1/3}}\right)^{18}$ is

$$T_{r+1} = {}^{18}C_r x^{r/3}\left(\frac{1}{2x^{1/3}}\right)^{18-r}$$
$$= {}^{18}C_r \cdot \frac{1}{2^{18-r}}\frac{x^{r/3}}{x^{(18-r)/3}}$$
$$= {}^{18}C_r \frac{1}{2^{18-r}} x^{(2r-18)/3}$$

Now,

$$\frac{2r}{3} - 6 = -2 \;\Rightarrow\; \frac{2r}{3} = 4 \;\Rightarrow\; r = 6$$

and

$$\frac{2r}{3} - 6 = -4 \;\Rightarrow\; \frac{2r}{3} = 2 \;\Rightarrow\; r = 3$$

Therefore, the coefficient of x^{-2} is

$${}^{18}C_6\left(\frac{1}{2^{12}}\right) = m$$

and the coefficient of x^{-4} is

$${}^{18}C_3\left(\frac{1}{2^{15}}\right) = n$$

Hence,

$$\frac{m}{n} = \frac{{}^{18}C_6}{{}^{18}C_3} \times \frac{2^{15}}{2^{12}} = 2^3\left(\frac{18!\times 3!\times 15!}{6!\times 12!\times 18!}\right) = 2^3\left(\frac{3!\times 15!}{6!\times 12!}\right)$$
$$= \frac{8\times 6\times 15\times 14\times 13}{6\times 5\times 4\times 3\times 2} = 14\times 13 = 182$$

Hence, the correct answer is option (B).

Previous Years' Solved JEE Advanced/ IIT-JEE Questions

1. For $r = 0, 1, \cdots, 10$, let A_r, B_r and C_r denote the coefficients of x^r in the expansions of $(1+x)^{10}$, $(1+x)^{20}$ and $(1+x)^{30}$, respectively. Then $\sum_{r=1}^{10} A_r(B_{10}B_r - C_{10}A_r)$ is equal to

(A) $B_{10} - C_{10}$ **(B)** $A_{10}(B_{10}^2 - C_{10}A_{10})$

(C) 0 **(D)** $C_{10} - B_{10}$

[IIT-JEE 2010]

Solution: Let

$$y = \sum_{r=1}^{10} A_r\left(B_{10}B_r - C_{10}A_r\right) = B_{10}\left(\sum_{r=1}^{10} A_rB_r - C_{10}\sum_{r=1}^{10} A_r^2\right)$$

A_r = coefficient of x^r in $(1+x)^{10} = {}^{10}C_r$

$$B_r = \text{coefficient of } x^r \text{ in } (1+x)^{20} = {}^{20}C_r$$
$$C_r = \text{coefficient of } x^r \text{ in } (1+x)^{30} = {}^{30}C_r$$

$$\sum_{r=1}^{10} A_rB_r = A_1B_1 + A_2B_2 + A_3B_3 + \dots + A_{10}B_{10} + A_0B_0 - A_0B_0$$
$$= {}^{10}C_0 \cdot {}^{20}C_0 + {}^{10}C_1 \cdot {}^{20}C_1 + \dots + {}^{10}C_{10} \cdot {}^{20}C_{10} - {}^{10}C_0 \cdot {}^{20}C_0$$
$$\Rightarrow \text{Coefficient of } x^{10} \text{ in } (1+x)^{30} - 1 = {}^{30}C_{10} - 1 = C_{10} - 1$$
$$\sum_{r=1}^{10} (A_r)^2 = \left({}^{10}C_1\right)^2 + \left({}^{10}C_2\right)^2 + \dots + \left({}^{10}C_{10}\right)^2 + \left({}^{10}C_0\right)^2 - \left({}^{10}C_0\right)^2$$
$$\Rightarrow y = C_{10} - 1 - (B_{10} - 1) = C_{10} - B_{10}$$

Hence, the correct answer is option (D).

2. The coefficients of three consecutive terms of $(1+x)^{n+5}$ are in the ratio 5:10:14. Then $n =$ ______.

[JEE ADVANCED 2013]

Solution: Let us consider that the consecutive terms be T_{r+2}, T_{r+1} and T_r.

Using
$$\frac{T_{r+1}}{T_n} = \frac{n-r+1}{r}$$

Therefore
$$\frac{T_{r+1}}{T_r} = \frac{10}{5} \Rightarrow n - 3r - 6 = 0 \quad (1)$$

,

Also
$$\frac{T_{r+2}}{T_{r+1}} = \frac{14}{10}$$
$$\Rightarrow 5n - 14r + 30 = 0 \quad (2)$$

Solving Eqs. (1) and (2), we get $n = 6$.

Hence, the correct answer is (6).

3. Coefficient of x^{11} in the expansion of $(1+x^2)^4(1+x^3)^7(1+x^4)^{12}$ is

(A) 1051 **(B)** 1106 **(C)** 1113 **(D)** 1120

[JEE ADVANCED 2014]

Solution: Expanding using binomial theorem

$$\{{}^4C_0(x^2)^0 + {}^4C_1(x^2)^1 + {}^4C_2(x^2)^2 + {}^4C_3(x^2)^3 + ({}^4C_4)(x^2)^4\}$$
$$\times \{{}^7C_0(x^3)^0 + {}^7C_1(x^3)^1 + ({}^7C_2)(x^3)^2 + ({}^7C_3)(x^3)^3 + ({}^7C_4)(x^3)^4 + \cdots\}$$
$$\times \{{}^{12}C_0(x^4)^0 + {}^{12}C_1(x^4)^1 + {}^{12}C_2(x^4)^2 + \cdots\}$$

[Neglecting higher power of x]

$$= \{1 + 4x^2 + 6x^4 + 4x^6 + x^8\}\{1 + 7x^3 + 21x^6 + 35x^9\}\{1 + 12x^4 + 64x^8\}$$

Coefficient of x^{11} in

$$\{1 + 7x^3 + 21x^6 + 35x^9 + 4x^2 + 28x^5 + 84r^8 + 140x^{11} + 6x^4 + 42x^7$$
$$+ 126x^{10} + 4x^6 + 28x^9 + x^8 + 7x^{11}\} \times \{1 + 12x^4 + 66x^8\}$$
$$= 462 + 140 + 504 + 7$$
$$= 1113$$

Hence, the correct answer is option (C).

4. The coefficient of x^9 in the expansion of $(1+x)(1+x^2)(1+x^3)\cdots(1+x^{100})$ is ______.

[JEE ADVANCED 2015]

Solution: Given expression is $(1+x)(1+x^2)(1+x^3)\cdots(1+x^{100})$.
Coefficient of x^9 in $(1+x)(1+x^2)(1+x^3)\cdots(1+x^9)$, that is,
Terms containing x^9

$$= (1 \cdot x^9 + x^1 \cdot x^8 + x^2 \cdot x^7 + x^3 \cdot x^6 + x^4 \cdot x^5 + x^1 \cdot x^2 \cdot x^6 + x^1 \cdot x^3 \cdot x^5 + x^1 \cdot x^4 \cdot x^5)$$

$\Rightarrow$ Term containing x^9 is $8x^9$ E

Therefore, coefficient of $x^9 = 8$.

Hence, the correct answer is (8).

5. Let m be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1+x)^2 + (1+x)^3 + \cdots + (1+x)^{49} + (1+mx)^{50}$ is $(3n+1)\,{}^{51}C_3$ for some positive integer n. Then the value of n is ________.

[JEE ADVANCED 2016]

Solution: It is given that the coefficient of x^2 in $(1+x)^2 + (1+x)^3 + \cdots + (1+x)^{49} + (1+mx)^{50}$

$$(3n+1)\,{}^{51}C_3$$

Now,

$${}^2C_2 + {}^3C_2 + {}^4C_2 + \cdots + {}^{49}C_2 + m^2\,{}^{50}C_2 = (3n+1)\,{}^{51}C_3$$
$$\Rightarrow {}^3C_3 + {}^3C_2 + {}^4C_2 + \cdots + {}^{49}C_2 + m^2\,{}^{50}C_2 = (3n+1)\,{}^{51}C_3$$
(since, ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$)
$$\Rightarrow {}^4C_3 + {}^4C_2 + \cdots + {}^{49}C_2 + m^2\,{}^{50}C_2 = (3n+1)\,{}^{51}C_3$$
$$\Rightarrow {}^{49}C_3 + {}^{49}C_2 + m^2\,{}^{50}C_2 = (3n+1)\,{}^{51}C_3$$
$$\Rightarrow {}^{50}C_3 + m^2\,{}^{50}C_2 = (3n+1)\,{}^{51}C_3$$
$$\Rightarrow {}^{50}C_3 + {}^{50}C_2 + m^2\,{}^{50}C_2 = (3n+1)\,{}^{51}C_3 + {}^{50}C_2$$
$$\Rightarrow {}^{51}C_3 + m^2\,{}^{50}C_2 = 3n\,{}^{51}C_3 + {}^{51}C_3 + {}^{50}C_2$$
$$\Rightarrow -{}^{50}C_2 + m^2\,{}^{50}C_2 = 3n\,{}^{51}C_3$$
$$\Rightarrow {}^{50}C_2\,(m^2-1) = 3n \cdot \frac{51}{3}\,{}^{50}C_2 \quad \left({}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1}\right)$$
$$\Rightarrow (m^2-1) = 51n$$

Therefore,

$$m^2 = 51n + 1$$
$$\Rightarrow n = \frac{m^2-1}{51}$$
$$\Rightarrow m = 16 \Rightarrow n = 5 \qquad (m, n \in I^+)$$

Hence, the correct answer is (5).

Practice Exercise 1

1. $2^{3n} - 7n - 1$ is divisible by

(A) 64 **(B)** 36 **(C)** 49 **(D)** 25

2. For each $n \in N$, $2^{3n} - 1$ is divisible by

(A) 8 **(B)** 16 **(C)** 32 **(D)** None of these

3. If the ratio of the 7th term from the beginning to the 7th term from the end in the expansion of $\left(\sqrt[3]{2} + \frac{1}{\sqrt[3]{3}}\right)^x$ is $\frac{1}{6}$, then x is

(A) 9 **(B)** 6 **(C)** 12 **(D)** None of these

4. The term independent of x in the expansion of $\left(2x + \frac{1}{3x}\right)^6$ is

(A) $\frac{160}{9}$ **(B)** $\frac{80}{9}$ **(C)** $\frac{160}{27}$ **(D)** $\frac{80}{3}$

5. The coefficient of x^n in the expansion of $\left(\frac{1}{1-x}\right)\left(\frac{1}{3-x}\right)$ is

(A) $\frac{3^{n+1}-1}{2\cdot 3^{n+1}}$ **(B)** $\frac{3^{n+1}-1}{3^{n+1}}$

(C) $2\left(\frac{3^{n+1}-1}{3^{n+1}}\right)$ **(D)** None of these

6. If the binomial expansion of $(a+bx)^{-2}$ is $\frac{1}{4}-3x+\cdots$, where $a > 0$, then (a, b) is

(A) (2, 12) **(B)** (2, 8)

(C) (–2, 12) **(D)** None of these

7. $(4-5x^2)^{-1/2}$ can be expanded as a power series of x if

(A) $|x|<\sqrt{5/2}$ **(B)** $|x|<2/\sqrt{5}$

(C) $-1<x<1$ **(D)** None of these

8. If the coefficient of m^{th}, $(m+1)^{th}$ and $(m+2)^{th}$ terms in the expansion $(1+x)^n$ are in AP, then

(A) $n^2+4(4m+1)+4m^2-2=0$

(B) $n^2+n(4m+1)+4m^2+2=0$

(C) $(n-2m)^2=n+2$

(D) $(n+2m)^2=n+2$

9. If $(1-x+x^2)^n = a_0+a_1x+a_2x^2+\cdots+a_{2n}x^{2n}$, then $a_0+a_2+a_4+\cdots+a_{2n}$ is equal to

(A) $\frac{3^n+1}{2}$ **(B)** $\frac{3^n-1}{2}$

(C) $3^n-\frac{1}{2}$ **(D)** $3^n+\frac{1}{2}$

10. The positive value of a, so that the coefficients of x^5 and x^{15} are equal in the expansion of $\left(x^2+\frac{a}{x^3}\right)^{10}$ is

(A) $\frac{1}{2\sqrt{3}}$ **(B)** $\frac{1}{\sqrt{3}}$ **(C)** 1 **(D)** $2\sqrt{3}$

11. The term independent of x in the expansion of $(1+x)^n(1+1/x)^n$ is

(A) $C_0^2+2C_1^2+3C_2^2+\cdots+(n+1)C_n^2$

(B) $(C_0+C_1+\cdots+C_n)^2$

(C) $C_0^2+C_1^2+\cdots+C_n^2$

(D) None of these

12. The coefficient of x^3 in $\left(\sqrt{x^5}+\frac{3}{\sqrt{x^3}}\right)^6$ is

(A) 0 **(B)** 120

(C) 420 **(D)** 540

13. The coefficient of y in the expansion of $(y^2+c/y)^5$ is

(A) $10c^3$ **(B)** $20c^2$

(C) $10c$ **(D)** $20c$

14. If the coefficients of x^2 and x^3 in the expansion of $(3+kx)^9$ are equal, then the value of k is

(A) $-\frac{9}{7}$ **(B)** $\frac{9}{7}$

(C) $\frac{7}{9}$ **(D)** None of these

15. The coefficient of x^n in $\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots+\frac{(-1)^n x^n}{n!}\right)^2$ is

(A) $\frac{(-n)^n}{n!}$ **(B)** $\frac{(-2)^n}{n!}$

(C) $\frac{1}{(n!)^2}$ **(D)** $-\frac{1}{(n!)^2}$

16. If $C_0, C_1, C_2, \cdots, C_n$ are binomial coefficients, then

$$\lim_{n\to\infty}\left[C_n-\left(\frac{2}{3}\right)C_{n-1}+\left(\frac{2}{3}\right)^2C_{n-2}+\cdots+(-1)^n\left(\frac{2}{3}\right)^nC_0\right]$$ is

(A) 0 **(B)** 1 **(C)** –1 **(D)** 2

17. Let n be an odd natural number and $A=\sum_{r=1}^{\frac{n-1}{2}}\frac{1}{{}^nC_r}$, then value of $\sum_{r=1}^{n}\frac{r}{{}^nC_r}$ is equal to

(A) $n(A-1)$ **(B)** $n(A+1)$

(C) $\frac{nA}{2}$ **(D)** nA

18. The sum of coefficients of even powers of x in the expansion of $\left(x+\frac{1}{x}\right)^{11}$ is

(A) $11\times {}^{11}C_5$ **(B)** $\frac{11}{2}\times {}^{11}C_6$

(C) $11({}^{11}C_5+{}^{11}C_6)$ **(D)** 0

19. If in the expansion of $\left(2^x+\frac{1}{4^x}\right)^n$, $T_3/T_2=7$ and the sum of the coefficient of 2nd and 3rd terms is 36, then the value of x is

(A) –1/3 **(B)** –1/2 **(C)** 1/3 **(D)** 1/2

20. The coefficient of middle term in the expansion of $(1+x)^{2n}$ is

(A) $2\,{}^nC_n$ **(B)** $\frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}2^n$

(C) $2\cdot 6\cdots(4n-2)$ **(D)** None of these

21. If ${}^{13}C_r$ is denoted by C_r, then the value of $C_1+C_5+C_7+C_9+C_{11}$ is equal to

(A) $2^{12}-287$ **(B)** $2^{12}-165$

(C) $2^{12}-C_2-C_{13}$ **(D)** None of these

22. The greatest positive integer, which divides $(n+16)(n+17)(n+18)(n+19)$, for all $n\in N$, is

(A) 2 **(B)** 4 **(C)** 24 **(D)** 120

23. If n is a positive integer which of the following will always be integers?

I. $(\sqrt{2}+1)^{2n}+(\sqrt{2}-1)^{2n}$

II. $(\sqrt{2}+1)^{2n}-(\sqrt{2}-1)^{2n}$

III. $(\sqrt{2}+1)^{2n+1}+(\sqrt{2}-1)^{2n+1}$

IV. $(\sqrt{2}+1)^{2n+1}-(\sqrt{2}-1)^{2n+1}$

(A) Only I and III **(B)** Only I and II
(C) Only I and IV **(D)** Only II and III

24. Coefficient of x^5 in the expansion of $(1+x^2)^5(1+x)^4$ is
(A) 61 **(B)** 59 **(C)** 0 **(D)** 60

25. The term independent of x in $\left[\sqrt{\frac{x}{3}}+\sqrt{\left(\frac{3}{2x^2}\right)}\right]^{10}$ is
(A) 1 **(B)** 5/12 **(C)** ${}^{10}C_1$ **(D)** None of these

26. If $(1+x+x^2)^n = a_0+a_1x+a_2x^2+\cdots+a_{2n}x^{2n}$, then the value of $a_0+a_3+a_6+\cdots$ is
(A) $a_1+a_4+a_7+$ **(B)** $a_1+a_2+a_3+$
(C) 2^{n+1} **(D)** None of these

27. The value of ${}^{2n}C_n - {}^nC_1 \cdot {}^{2n-2}C_n + {}^nC_2 \cdot {}^{2n-4}C_n - \cdots$ is equal to
(A) 3^n **(B)** 4^n
(C) 5^n **(D)** None of these

28. If $|x|<1$, then the coefficient of x^n in the expansion of $(1+x+x^2+x^3+\cdots)^2$ is
(A) n **(B)** $n-1$
(C) $n+2$ **(D)** $n+1$

29. If $(1+ax)^n = 1+8x+24x^2+\cdots$ then
(A) $a=3$ **(B)** $n=5$
(C) $a=2$ **(D)** None of these

30. The two successive terms in the expansion of $(1+x)^{24}$ whose coefficients are in the ratio 4:1 are
(A) 3rd and 4th **(B)** 4th and 5th
(C) 5th and 6th **(D)** 6th and 7th

31. The coefficient of x^k $(0\le k\le n)$ in the expansion of $E=1+(1+x)+(1+x)^2+\cdots+(1+x)^n$ is
(A) ${}^{n+1}C_{k+1}$ **(B)** ${}^nC_k$
(C) ${}^{n+1}C_{n-k-1}$ **(D)** None of these

32. The coefficient of x^n in the expansion of $(1-x)^{-2}$ is
(A) $(-1)^n(n+1)$ **(B)** $(n+1)$
(C) $(-1)^n n$ **(D)** None of these

33. If roots of the equation
$$({}^mC_0+{}^mC_1\cdots{}^mC_m)x^2+({}^nC_0+{}^nC_2+{}^nC_4\cdots)x+({}^nC_1+{}^nC_3+{}^nC_5\cdots)=0$$
are real, then find the minimum value of $n-m$.
(A) 1 **(B)** 2
(C) 3 **(D)** −1

34. A number is said to be a nice number if it has exactly 4 factors (including one and number itself). Let $n = 2^3\times3^2\times5^3\times7\times11^2$, then the number of factors, which are nice numbers, is
(A) 36 **(B)** 12
(C) 10 **(D)** 147

35. $\sum_{r=1}^{n}(-1)^{r+1}\cdot\frac{{}^nC_r}{r+1}$ is equal to
(A) $-\frac{1}{n+1}$ **(B)** $-\frac{1}{n}$ **(C)** $\frac{1}{n+1}$ **(D)** $\frac{n}{n+1}$

36. $\sum_{r=0}^{300}a_rx^r=(1+x+x^2+x^3)^{100}$. If $a=\sum_{r=0}^{300}a_r$, then $\sum_{r=0}^{300}r\cdot a_r$ is equal to
(A) $300a$ **(B)** $100a$ **(C)** $150a$ **(D)** $75a$

37. The number of terms in the expansion of $(1+x)(1+x^3)(1+x^6)(1+x^{12})(1+x^{24})\cdots(1+x^{3\times2^n})$ is
(A) 2^{n+3} **(B)** 2^{n+4} **(C)** 2^{n+5} **(D)** None of these

38. The number of terms in $(1+x)^{101}(1+x^2-x)^{100}$ is
(A) 302 **(B)** 301 **(C)** 202 **(D)** 101

39. If coefficient of $x^2y^3z^4$ in $(x+y+z)^n$ is A, then coefficient of x^4y^4z is
(A) $2A$ **(B)** $\frac{nA}{2}$ **(C)** $\frac{A}{2}$ **(D)** None of these

40. Let r^{th} term of a series be given by $T_r=\frac{r}{1-3r^2+r^4}$. Then $\lim_{n\to\infty}\sum_{r=1}^{n}T_r$ is
(A) 3/2 **(B)** 1/2 **(C)** −1/2 **(D)** −3/2

41. The coefficient of a^4b^5 in the expansion of $(a+b)^9$ is
(A) $\frac{9!}{4!5!}$ **(B)** $\frac{9!}{6!3!}$
(C) $\frac{4!5!}{9!}$ **(D)** None of these

42. If the coefficient in the third term of the expansion of $\left(x^2+\frac{1}{4}\right)^n$ when expanded in decreasing powers of x is 31, then n is equal to
(A) 16 **(B)** 20 **(C)** 30 **(D)** 32

43. The sum of coefficients in the expansion of $(1+x-3y^2)^{2163}$ is
(A) 1 **(B)** −1 **(C)** 2^{2163} **(D)** None of these

44. The sum of the rational terms in the expansion of $(\sqrt{2}+3^{1/5})^{10}$ is
(A) 20 **(B)** 21 **(C)** 40 **(D)** 41

45. In the expansion of $(1+x)^{50}$, let S be the sum of coefficients of odd power of x, then S is
(A) 0 **(B)** 2^{49} **(C)** 2^{50} **(D)** 2^{51}

46. The coefficient of x^{53} in $\sum_{r=0}^{100}{}^{100}C_r(x-3)^{100-r}2^r$ is
(A) ${}^{100}C_{51}$ **(B)** ${}^{100}C_{52}$ **(C)** ${}^{-100}C_{53}$ **(D)** ${}^{100}C_{54}$

47. The coefficient of x^m in $(1+x)^r+(1+x)^{r+1}+(1+x)^{r+2}+\cdots+(1+x)^n$, $r\le m\le n$ is
(A) ${}^{n+1}C_{m+1}$ **(B)** ${}^{n-1}C_{m-1}$ **(C)** ${}^nC_m$ **(D)** ${}^nC_{m+1}$

48. If $p(x) = x^n$, then the value of $p(1) + \frac{p'(1)}{1!} + \frac{p''(1)}{2!} + \cdots + \frac{p^{(n)}(1)}{n!}$, where $p^{(r)}(x)$ stands for the r^{th} derivative of $p(x)$ with respect to x, is
(A) 2^n **(B)** n **(C)** 2^{n-1} **(D)** None of these

49. If $\frac{1}{\sqrt{2x+1}} \times \left\{\left(\frac{1+\sqrt{2x+1}}{2}\right)^n - \left(\frac{1-\sqrt{2x+1}}{2}\right)^n\right\}$ is a polynomial of degree 5, then n is equal to
(A) 9 **(B)** 10 **(C)** 11 **(D)** None of these

50. If n is even, then the coefficient of x in the expansion of $(1+x)^n\left(1-\frac{1}{x}\right)^n$ is
(A) ${}^nC_2$ **(B)** ${}^{2n}C_n$ **(C)** 0 **(D)** 1

51. The sum of ${}^{21}C_{10} + {}^{21}C_9 + \cdots + {}^{21}C_0$ is equal to
(A) 2^{20} **(B)** 2^{21} **(C)** 2^{19} **(D)** None of these

52. In the expansion of $\left(x^3 - \frac{1}{x^2}\right)^{15}$, the constant term is
(A) ${}^{15}C_6$ **(B)** $-{}^{15}C_6$ **(C)** ${}^{15}C_4$ **(D)** $-{}^{15}C_4$

53. 3^{51} when divided by 8 leaves the remainder,
(A) 1 **(B)** 6 **(C)** 5 **(D)** 3

54. The greatest positive integer which divides $n(n+1)(n+2)(n+3)$, for all $n \in N$, is
(A) 2 **(B)** 6 **(C)** 24 **(D)** 120

55. If $\frac{T_2}{T_3}$ in the expansion of $(a+b)^n$ and $\frac{T_3}{T_4}$ in the expansion of $(a+b)^{n+3}$ are equal, then n is equal to
(A) 3 **(B)** 4 **(C)** 5 **(D)** 6

56. Show that the sum of the product of the C_i's taken two at a time and represented by $\sum_{1 \le i < j \le n} \sum C_i C_j$ is equal to $2^{2n-1} - \frac{(2n)!}{2(n!)^2}$.

57. Show that $C_1^2 - 2C_2^2 + 3 \times C_3^2 - \cdots - (2n)C_{2n}^2 = (-1)^{n-1} n \times C_n$ where $C_r = {}^{2n}C_r$.

58. If $\sum_{r=0}^{2n} a_r(x-2)^r = \sum_{r=0}^{2n} b_r(x-3)^r$ and $a_k = 1$ for all $k \ge n$, then show that $b_n = {}^{(2n+1)}C_{n+1}$.

59. Prove that the coefficient of x^r in the expansion
$(x+3)^{n-1} + (x+3)^{n-2}(x+2) + (x+3)^{n-3}(x+2)^2 + \cdots + (x+2)^{n-1}$ is $(3^{n-r} - 2^{n-r})^n - C_r$.

Practice Exercise 2

Single/Multiple Correct Choice Type Questions

1. Let $T_1 = 7$, $T_2 = 7^7$, $T_3 = 7^{7^7}$ and so on. The digit at the tens places of number T_{1000} is
(A) 8 **(B)** 0 **(C)** 6 **(D)** 4

2. If the polynomial $f(x) = 1 - x + x^2 - x^3 + \cdots + x^{19} + x^{20}$ is expressed as $g(y) = a_0 + a_1y + a_2y^2 + \cdots + a_{20}y^{20}$, where $y = x - 4$, then the value of $a_0 + a_1 + a_2 + \cdots + a_{20}$ is
(A) $\frac{5^{21}-1}{6}$ **(B)** $\frac{5^{20}}{6}$
(C) $\frac{1+5^{20}}{6}$ **(D)** $\frac{1+5^{21}}{6}$

3. ${}^{2n+3}C_1 + {}^{2n+3}C_2 + \cdots + {}^{2n+3}C_n - {}^{2n+3}C_{2n+3} - {}^{2n+3}C_{2n+2} - \cdots - {}^{2n+3}C_{n+3}$ is equal to
(A) a **(B)** ${}^{2n+3}C_{n+1}$ **(C)** -1 **(D)** 0

4. If $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \cdots + {}^nC_nx^n$ where ${}^nC_0, {}^nC_1, {}^nC_2, \cdots$ are binomial coefficients. Then $2(C_0 + C_3 + C_6 + \cdots) + (C_1 + C_4 + C_7 + \cdots)(1+\omega) + (C_2 + C_5 + C_8 + \cdots)(1+\omega^2)$, where ω is the cube root of unity and n is a multiple of 3, is equal to
(A) $2^n + 1$ **(B)** $2^{n-1} + 1$
(C) $2^{n+1} - 1$ **(D)** $2^n - 1$

5. If $b_1 = 2$ and $b_n = n(1 + b_{n-1})\ \forall\ n \ge 2$, then $\lim_{n\to\infty} \frac{b_{n+2}}{(n+2)!}$ is equal to
(A) e **(B)** $2e$ **(C)** $e-1$ **(D)** $e+1$

6. If α and β are the roots of equation $x^2 + 4x + p = 0$, where $p = \sum_{r=0}^{n} {}^nC_r \frac{1+rx}{(1+nx)^r}(-1)^r$, then the value of $|\alpha - \beta|$ is
(A) 2 **(B)** 6 **(C)** 4 **(D)** None of these

7. If $(x+1)(x+2)(x+3)\cdots(x+n) = A_0 + A_1x + A_2x^2 + \cdots + A_nx^n$, then $A_1 + 2A_2 + \cdots + nA_n$ is equal to
(A) $(n-1)!\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}\right)$ **(B)** $\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}\right)$
(C) $(n+1)!\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}\right)$ **(D)** None of these

8. $\sum_{r=0}^{10} 2^{10}\, {}^{20}C_r\, {}^{20-r}C_{10-r}$ is equal to
(A) ${}^{20}C_{10}$ **(B)** ${}^{20}C_{10}\left(\frac{3}{2}\right)^{10}$
(C) ${}^{20}C_{10}\,3^{10}$ **(D)** ${}^{20}C_{10}\,2^{10}$

9. If $\frac{2}{1!\,13!} + \frac{2}{3!\,11!} + \frac{2}{5!\,9!} + \frac{1}{7!\,7!} = \frac{2^m}{n!}$, then
(A) $m + n = 27$ **(B)** $m = 1 + n$
(C) $m^2 + n^2 = 2$ **(D)** $n = 1 + m$

10. In the expansion of $(1 + 2x + 3x^2)^{10}$
(A) sum of coefficients is equal to 6^{10}.
(B) number of total terms is 21.
(C) number of total terms is ${}^{12}C_2$.
(D) coefficients of x^{20} is 3^{10}.

Comprehension Type Questions

Paragraph for Questions 11–13: If $(1 + px + x^2)^n = 1 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n}$

11. Which of the following is true for $1 < r < 2n$?

(A) $(np+pr)a_r = (r+1)a_{r+1} + (r-1)a_{r-1}$

(B) $(np-pr)a_r = (r+1)a_{r+1} + (r-1-2n)a_{r-1}$

(C) $(np-pr)a_r = (r+1)a_{r+1} + (r-1-n)a_{r-1}$

(D) $(2np+pr)a_r = (r+1+n)a_{r+1} + (r+1-n)a_{r-1}$

12. The remainder obtained when $a_1 + 5a_2 + 9a_3 + 13a_4 + \cdots + (8n-3)a_{2n}$ is divided by $(p+2)$ is

(A) 1 **(B)** 2 **(C)** 3 **(D)** 0

13. The value of $a_1 + 3a_2 + 5a_3 + 7a_4 + \cdots + (4n-1)a_{2n}$, when $p = -3$ and $n \in$ even is

(A) n **(B)** $2n-1$ **(C)** $2n-2$ **(D)** $2n$

Paragraph for Questions 14 –16: The quantities $(1+x)$, $(1+x+x^2)$, $(1+x+x^2+x^3)$, $\cdots$, $(1+x+x^2+\cdots+x^n)$ are multiplied together and terms of the product are arranged in the increasing powers of x in the form $a_0 + a_1x + a_2x^2 + \cdots$, then

14. The number of terms in the product is

(A) n^2 **(B)** $n(n+1)$

(C) $\frac{n(n+1)}{2}$ **(D)** $\frac{n^2+n+2}{2}$

15. The coefficients of the equidistant term from the beginning and end are

(A) always equal. **(B)** sometimes equal.

(C) never equal. **(D)** cannot be discussed.

16. The sum of odd coefficients = sum of even coefficients is equal to

(A) $n!$ **(B)** $(n+1)!$

(C) $\frac{(n+1)!}{2}$ **(D)** None of these

Matrix Match Type Questions

17. Match the following:

Column I	Column II
(A) If the binomial coefficients of the r^{th}, $(r+1)^{th}$ and $(r+2)^{th}$ term in the expansion, $(1+x)^{14}$ are in AP, then r is equal to	**(p)** 5
(B) The sum of coefficients in the polynomial expansion of $\{(1+x+x^2+\ldots+x^{n-1})(1-x)\}^m$ is $(m, n \in N)$	**(q)** 3
(C) Sum of the series $\sum_{r=1}^{n}(-1)^r\, 3\, {}^nC_r(1+i^r+i^{2r}+\cdots+i^{(n-1)r})$; where $n=4k$, $k \in I$ and $i=\sqrt{-1}$, is	**(r)** 0
(D) If $3^{37} = 80\lambda + k$, where $\lambda \in N$, then $3k$ is equal to	**(s)** 9

18. If $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \cdots + {}^nC_nx^n$, then the value of

Column I	Column II
(A) $\left(1+\frac{{}^nC_1}{{}^nC_0}\right)\left(1+\frac{{}^nC_2}{{}^nC_1}\right)\cdots\left(1+\frac{{}^nC_n}{{}^nC_{n-1}}\right)$ is	**(p)** $\frac{n(n+1)}{2}$
(B) $\frac{{}^nC_1}{{}^nC_0}+\frac{2\,{}^nC_2}{{}^nC_1}+\frac{3\,{}^nC_3}{{}^nC_2}+\cdots+\frac{n\,{}^nC_n}{{}^nC_{n-1}}$ is	**(q)** $\frac{1}{(n+1)(n+2)}$
(C) $\frac{{}^nC_0}{2}-\frac{{}^nC_1}{3}+\frac{{}^nC_2}{4}-\cdots+(-1)^n\frac{{}^nC_n}{n+2}$ is	**(r)** $\frac{(n+1)^n}{n!}$
(D) $\frac{{}^nC_1}{2}+\frac{{}^nC_3}{4}+\frac{{}^nC_5}{6}+\cdots+\frac{{}^nC_n}{n+1}$ is	**(s)** $\frac{2^n-1}{n+1}$

Integer Type Question

19. If $R = (15+\sqrt{220})^{19} + (15+\sqrt{220})^{82}$, then the digit at the unit place of $[R]-1$ is (where [.] denotes the greatest integer function).

Answer Key

Practice Exercise 1

1. (C)	**2.** (D)	**3.** (A)	**4.** (C)	**5.** (A)	**6.** (A)
7. (B)	**8.** (C)	**9.** (A)	**10.** (A)	**11.** (C)	**12.** (D)
13. (A)	**14.** (B)	**15.** (B)	**16.** (A)	**17.** (B)	**18.** (D)
19. (A)	**20.** (B)	**21.** (A)	**22.** (C)	**23.** (C)	**24.** (D)
25. (D)	**26.** (A)	**27.** (D)	**28.** (D)	**29.** (C)	**30.** (C)
31. (A)	**32.** (B)	**33.** (C)	**34.** (B)	**35.** (D)	**36.** (C)
37. (D)	**38.** (C)	**39.** (C)	**40.** (C)	**41.** (A)	**42.** (D)
43. (B)	**44.** (D)	**45.** (B)	**46.** (C)	**47.** (A)	**48.** (A)
49. (C)	**50.** (C)	**51.** (A)	**52.** (B)	**53.** (D)	**54.** (C)
55. (C)					

Practice Exercise 2

1. (D)	**2.** (D)	**3.** (C)	**4.** (D)	**5.** (D)	**6.** (C)	**7.** (C)
8. (C)	**9.** (A), (D)	**10.** (A), (B), (D)	**11.** (B)	**12.** (C)	**13.** (D)	**14.** (D)

15. (A) **16.** (C) **17.** (A) → (p), (s); (B) → (r); (C) → (r); (D) → (s) **18.** (A) → (r); (B) → (p); (C) → (q); (D) → (s) **19.** 9

Solutions

Practice Exercise 1

1. For $n = 1$, $2^{3n} - 7n - 1$ has value $2^3 - 7 - 1 = 0$

for $n = 2$, $2^{3n} - 7n - 1$ has value $64 - 14 - 1 = 49$

which is divisible by 49 and not by 36 or 64.

2. For $n \geq 1$,

$$2^{3n} - 1 = (2^3)^n - 1 = 8^n - 1$$
$$= (8 - 1)\,[8^{n-1} + 8^{n-2} + \cdots + 1] \text{ (By GP)}$$
$$= 7 \times \text{positive integer}$$

3. T_7 in $\left(\sqrt[3]{2} + \dfrac{1}{\sqrt[3]{3}}\right)^x = {}^xC_6(2^{1/3})^{x-6}\left(\dfrac{1}{3^{1/3}}\right)^6$

7^{th} term from the end in $\left(\sqrt[3]{2}+\dfrac{1}{\sqrt[3]{3}}\right)^x = T_7$ in $\left(\dfrac{1}{\sqrt[3]{3}}+\sqrt[3]{2}\right)^x$

$$= {}^xC_6\left(\frac{1}{3^{1/3}}\right)^{x-6}(2^{1/3})^6$$

Therefore,

$$\frac{{}^xC_6(2^{1/2})^{x-6}\left(\dfrac{1}{3^{1/3}}\right)^6}{{}^xC_6\left(\dfrac{1}{3^{1/3}}\right)^{x-6}(2^{1/3})^6} = \frac{1}{6}$$

$$\Rightarrow \frac{(2^{1/3})^{x-12}}{\left(\dfrac{1}{3^{1/3}}\right)^{x-12}} = \frac{1}{6} \Rightarrow 2^{\frac{x-12}{3}} \times 3^{\frac{x-12}{3}} = \frac{1}{6}$$

$$\Rightarrow 6^{\frac{x-12}{3}} = 6^{-1} \Rightarrow \frac{x-12}{3} = -1 \Rightarrow x = 9$$

4. $T_{r+1} = {}^6C_r(2x)^{6-r}\left(\dfrac{1}{3x}\right)^r = {}^6C_r\dfrac{2^{6-r}}{3^r}x^{6-2r}$

Let T_{r+1} be independent of x. Then $6 - 2r = 0$ or $r = 3$. Therefore,

$$T_{r+1} = T_{3+1} = {}^6C_3\frac{2^{6-3}}{3^3}x^{6-2(3)}$$
$$= \frac{20 \times 8}{27} = \frac{160}{27}$$

5. $\dfrac{1}{(1-x)(3-x)} = (1-x)^{-1}(3-x)^{-1}$

$$= 3^{-1}(1-x)^{-1}\left(1-\frac{x}{3}\right)^{-1}$$

$$= \frac{1}{3}[1+x+x^2+\ldots+x^n]+\left[1+\frac{x}{3}+\frac{x^2}{3^2}+\ldots+\frac{x^{n-1}}{3^{n-1}}+\frac{x^n}{3^n}\right]$$

Coefficient of $x^n = \dfrac{1}{3^{n+1}} + \dfrac{1}{3^n} + \dfrac{1}{3^{n-1}} + \ldots (n+1)$ terms

$$= \frac{1}{3^{n+1}}\frac{[3^{n+1}-1]}{3-1} = \frac{3^{n+1}-1}{2\cdot 3^{n+1}}$$

6. $(a+bx)^{-2} = a^{-2}\left(1+\dfrac{b}{a}x\right)^{-2}$

$$= \frac{1}{a^2}\left[1+(-2)\left(\frac{b}{a}x\right)+\cdots\right] = \frac{1}{a^2} - \frac{2b}{a^3}x + \cdots$$

Also,

$$(a+bx)^{-2} = \frac{1}{4} - 3x + \cdots$$

Therefore

$$\frac{1}{a^2} = \frac{1}{4} \quad (1)$$

and

$$-\frac{2b}{a^3} = -3 \quad (2)$$

Eq. (1) $\Rightarrow a^2 = 4 \Rightarrow a = 2$ and from Eq. (2) $b = 12$

7. $(4-5x^2)^{-1/2} = 4^{-1/2}\left(1-\dfrac{5}{4}x^2\right)^{-1/2}$

$$= \frac{1}{2}\left(1+\left(-\frac{5}{4}x^2\right)\right)^{-1/2}$$

Therefore, $\left|-\dfrac{5}{4}x^2\right| < 1$ or $\left|-\dfrac{5}{4}\right||x^2| < 1$ or $\dfrac{5}{4}x^2 < 1$

or $x^2 < \dfrac{4}{5}$ or $|x| < \dfrac{2}{\sqrt{5}}$

8. We have ${}^nC_{m-1}, {}^nC_m, {}^nC_{m+1}$ in AP. So,

$$2\,{}^nC_m = {}^nC_{m-1} + {}^nC_{m+1}$$

$$\Rightarrow \frac{2(n!)}{m!(n-m)!} = \frac{n!}{(m-1)!(n-m+1)!} + \frac{n!}{(m+1)!(n-m-1)!}$$

$$\Rightarrow \frac{2}{m(n-m)} = \frac{1}{(n-m+1)(n-m)} + \frac{1}{m(m+1)}$$

$$\Rightarrow 2(m+1)(n-m+1) = m(m+1) + (n-m+1)(n-m)$$

On simplification, we get

$$n^2-4mn+4m^2-n-2=0 \Rightarrow (n-2m)^2=n+2$$

9. We have

$$(1-x+x^2)^n=a_0+a_1x+a_2x^2+\cdots+a_{2n}x^{2n}$$

Putting $x=1$ and -1, we get

$$1=a_0+a_1+a_2+a_3+\cdots+a_{2n}$$

and $$3^n=a_0-a_1+a_2-a_3+\cdots+a_{2n}$$

Adding, we get

$$1+3^n=2(a_0+a_2+a_4+\cdots+a_{2n})$$

$$a_0+a_2+a_4+\cdots+a_{2n}=\frac{3^n+1}{2}$$

10. $$T_{r+1}={}^{10}C_r(x^2)^{10-r}\left(\frac{a}{x^3}\right)^r={}^{10}C_ra^rx^{20-5r}$$

$$20-5r=5 \Rightarrow r=3$$

Therefore,

$$T_{r+1}=T_{3+1}={}^{10}C_3a^3x^{20-5(3)}=120a^3x^5$$

So, coefficient of $x^5=120\,a^3$.

Also,

$$20-5r=15 \Rightarrow r=1$$

Thus,

$$T_{r+1}=T_{1+1}={}^{10}C_1a^1x^{20-5(1)}=10ax^{15}$$

Therefore, coefficient of $x^{15}=10a$.

Hence, $120a^3=10a$ or $a=\frac{1}{2\sqrt{3}}$.

11. We have

$$(1+x)^n\left(1+\frac{1}{x}\right)^n=(C_0+C_1x+C_2x^2+\cdots+C_nx^n)$$
$$\times\left(C_0+\frac{C_1}{x}+\frac{C_2}{x^2}+\cdots+\frac{C_n}{x^n}\right)$$

Term independent of x on the RHS is

$$C_0^{\,2}+C_1^{\,2}+C_2^{\,2}+\cdots+C_n^{\,2}$$

12. $$T_{r+1}={}^6C_r(x^{5/2})^{6-r}\left(\frac{3}{x^{3/2}}\right)^r$$

$$={}^6C_r3^rx^{15-\frac{5r}{2}-\frac{3r}{2}}={}^6C_r3^rx^{15-4r}$$

Let T_{r+1} contains x^3. Then, $15-4r=3$ or $r=3$.

Thus,

$$T_{r+1}=T_{3+1}={}^6C_3(3)^3x^{15-4(3)}$$
$$=20\times27\times x^3=540x^3$$

Therefore, coefficient of $x^3=540$.

13. $$(r+1)^{\text{th}}\text{ terms}={}^5C_ry^{10-2r}\cdot r\cdot y^{-r}$$

Power of $y=1$

$$\Rightarrow 10-3r=1 \Rightarrow r=3$$

Required coefficient $={}^5C_2\cdot c^3=10\,c^3$

14. T_{r+1} in $(3+kx)^9={}^9C_r3^{9-r}(kx)^r$

$$={}^9C_r3^{9-r}k^rx^r$$

Therefore, coefficient of $x^r={}^9C_r3^{9-r}k^r$.

Now,

coefficient of x^2 = coefficient of x^3

$$\Rightarrow {}^9C_23^{9-2}k^2={}^9C_33^{9-3}k^3$$
$$\Rightarrow 36\times3^7k^2=84\times3^6k^3$$
$$\Rightarrow 36=28k \Rightarrow k=\frac{9}{7}$$

15. Coefficient of x^n in

$$\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots+\frac{(-1)^nx^n}{n!}\right)^2$$

= Coefficient of x^n in

$$\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots\right)^2$$

Coefficient of x^n in $(e^{-x})^2$ = Coefficient of x^n in $e^{-2x}=\frac{(-2)^n}{n!}$

16. Take $x=\frac{2}{3}$

$$\lim_{n\to\infty}[C_n-C_{n-1}x+C_{n-2}+\cdots+(-1)C_0x^n]$$
$$=\lim_{n\to\infty}[C_0-C_1x+C_2x^2+\cdots+(-1)^nC_nx^n]=\lim_{n\to\infty}[1-x]^n$$
$$=\lim_{n\to\infty}\left(1-\frac{2}{3}\right)^n=\lim_{n\to\infty}\frac{1}{3^n}=0$$

17. $$\sum_{r=1}^{n}\frac{r}{{}^nC_r}=\sum_{r=0}^{n-1}\frac{(n-r)}{{}^nC_r}$$
$$=\sum_{r=0}^{n-1}\frac{n}{{}^nC_r}-\sum_{r=0}^{n-1}\frac{r}{{}^nC_r}$$
$$2\sum_{r=1}^{n}\frac{r}{{}^nC_r}=n\sum_{r=0}^{n-1}\frac{1}{{}^nC_r}+n+n=n2A+2n$$
$$\Rightarrow \sum_{r=1}^{n}\frac{r}{{}^nC_r}=n(A+1)$$

18. $(r+1)^{\text{th}}$ term $={}^{11}C_r(x)^{11-r}\cdot x^{-r}$
$$={}^{11}C_rx^{11-2r}$$

Even power of x exists only if $11-2r$ = an even number which is not possible.

Sum of coefficient = 0

19. Given that

$${}^nC_1+{}^nC_2=36$$
$$\Rightarrow n=8, n\neq-9$$

Also,

$$\frac{{}^nC_2(2^x)^{n-2}\cdot\left(\frac{1}{4x}\right)^2}{{}^nC_1(2^x)^{n-1}\cdot\left(\frac{1}{4x}\right)^1}=7$$

$$\frac{28(2^x)^6 \cdot \left(\frac{1}{4x}\right)^2}{8(2^x)^7 \cdot \left(\frac{1}{4x}\right)^1} = 7 \Rightarrow x = \frac{-1}{3}$$

20. Coefficient of the middle term $= {}^{2n}C_n$

$$= \frac{(1\cdot 2\cdot 3\cdots 2n)}{n!n!} = \frac{2^n(1\cdot 3\cdot 5\cdots 2n-1)}{n!}$$

21. $(1 + x)^{13} = C_0 + C_1 x + C_2 x^2 + \cdots + C_{13} x^{13}$

$(1 - x)^{13} = C_0 - C_1 x + C_2 x^2 - \cdots - C_{13} x^{13}$

Put $x = 1$

$2^{13} = C_0 + C_1 + C_2 + \cdots + C_{13}$

$0 = (C_0 + C_2 + C_4 + C_6 + \cdots) - (C_1 + C_3 + \cdots)$

$2^{13} = 2\,(C_0 + C_2 + C_4 + \cdots C_{12})$

$2^{12} = C_0 + C_2 + C_4 + \cdots C_{12}$

Now,

$= C_1 + C_5 + C_7 + C_9 + C_{11} = C_{12} + C_2 + C_4 + C_6 + C_8$

[using, ${}^nC_n = {}^nC_{n-r}$]

$= 2^{12} - 1 - {}^{13}C_{10}$

$= 2^{12} - 287$

22. Since the product of any r consecutive integers is divisible by $r!$ and not by $(r + 1)!$

Therefore, the given product is divisible by $4! = 24$.

23. In I and IV only even powers of $\sqrt{2}$ occurs whereas in II and III only odd powers of $\sqrt{2}$ occurs.

24. $(1 + x^2)^5(1 + x)^4 = (1 + 5x^2 + 10x^4 + \cdots)(1 + x)^4$

$\Rightarrow$ Coefficient of $x^5 = 5 \times {}^4C_3 + 10 \times {}^4C_1 = 20 + 40 = 60$

25. General term in the expansion of

$${}^{10}C_r\left(\frac{x}{3}\right)^{\frac{r}{2}}\left(\frac{3}{2x^2}\right)^{\frac{10-r}{2}} = {}^{10}C_r x^{\frac{3r}{2}-10} \cdot \frac{3^{5-r}}{2^{\frac{10-r}{2}}}$$

For constant term,

$$\frac{3r}{2} = 10 \Rightarrow r = \frac{20}{3}$$

which is not an integer.

Therefore, there will be no constant term.

26. $(1 + x + x^2)^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

Put $x = w$ and w^2, we get

$0 = (a_0 + a_3 + a_6 + \cdots) + w\,(a_1 + a_4 + a_7 + \cdots) + w^2\,(a_2 + a_5 + a_8 + \cdots)$ (1)

$0 = (a_0 + a_3 + a_6 + \cdots) + w^2\,(a_1 + a_4 + a_7 + \cdots) + w\,(a_2 + a_5 + a_8 + \cdots)$ (2)

From Eqs. (1) and (2) we get,

$a_0 + a_3 + a_6 \cdots = a_1 + a_4 + a_7 + \cdots$

27. ${}^{2n}C_n - {}^nC_1\,{}^{2n-2}C_n + {}^nC_2\,{}^{2n-4}C_n - \cdots$

$=$ coefficient of x^n in $[{}^nC_0\,(1 + x)^{2n} - {}^nC_1\,(1 + x)^{2n-2} + {}^nC_2\,(1 + x)^{2n-4} - \cdots]$

$=$ coefficient of x^n in $[1 - (1 + x)^2]^n = 2^n\,(-1)^n$

28. $(1 + x + x^2 + x^3 + \cdots)^2 = \left(\frac{1}{1-x}\right)^2 = (1 - x)^{-2}$

$= 1 + 2x + 3x^2 + 4x^3 + \cdots$

Coefficient of $x^n = (n + 1)$

29. $(1 + a\,x)^n = 1 + n\,ax + \frac{n(n-1)}{2}\,a^2x^2 + \cdots = 1 + 8x + 24x^2 + \cdots$

$\Rightarrow n\,a = 8$

$n\,(n - 1)\,a^2 = 48 \Rightarrow n = 4, a = 2$

30. Let the coefficient of successive terms be ${}^{24}C_r$ and ${}^{24}C_{r+1}$, then

$$\frac{{}^{24}C_r}{{}^{24}C_{r+1}} = 4 \Rightarrow \frac{r+1}{(24-r)} = 4 \Rightarrow r = 19$$

${}^{24}C_{19}, {}^{24}C_{20} \Rightarrow {}^{24}C_5, {}^{24}C_4 \Rightarrow 6^{th}$ and 5^{th} terms

31. $E = \frac{(1+x)^{n+1} - 1}{(1+x) - 1} = \frac{{}^{n+1}C_0 + {}^{n+1}C_1x + {}^{n+1}C_2x^2 + \cdots - 1}{x}$

$= {}^{n+1}C_1 + {}^{n+1}C_2x + {}^{n+1}C_3x^2 + \cdots$

Coefficient of $x^k = {}^{n+1}C_{k+1}$

32. Since

$$(1 - x)^{-2} = 1 + 2x + 3x^2 + \cdots + (n + 1)x^n + \cdots$$

Hence, (B) is the correct answer.

33. Roots are real if

$(2^{n-1})^2 - 4(2^m)2^{n-1} \ge 0$

$2^{2n-2} - 2^{m+n+1} \ge 0$

$2n - 2 \ge m + n + 1$

$n - m \ge 3$

Minimum value of $n - m = 3$.

34. Any number having exactly 4 factors is of the form $m = p^3$ (p prime) or $m = p \cdot q$ (where p and q are distinct primes).

So, we have ${}^5C_2 + 2 = 12$ such factors.

35. Given

$$\sum_{r=1}^{n}(-1)^{r+1}\cdot\frac{{}^nC_r}{r+1} = \frac{1}{n+1}\sum_{r=1}^{n}(-1)^{r+1}\cdot{}^{n+1}C_{r+1}$$

$$= \frac{1}{n+1}(0 - 1 + (n+1)) = \frac{n}{n+1}$$

36. $$\sum_{r=0}^{300} a_r x^r = (1 + x + x^2 + x^3)^{100}$$

Clearly, a_r is the coefficient of x^r in the expansion of $(1 + x + x^2 + x^3)^{100}$.

Replacing x by $\frac{1}{x}$ in the given equation, we get

$$\sum_{r=0}^{300} a_r\left(\frac{1}{x}\right)^r = \frac{1}{x^{300}}(x^3 + x^2 + x + 1)^{100}$$

$$\Rightarrow \sum_{r=0}^{300} a_r x^{300-r} = (1 + x + x^2 + x^3)^{100}$$

Here, a_r represents the coefficient of x^{300-r} in $(1 + x + x^2 + x^3)^{100}$

Thus, $a_r = a_{300-r}$.

Let,

$$I=\sum_{r=0}^{300} r\cdot a_r=\sum_{r=0}^{300}(300-r)a_{300-r}$$
$$=\sum_{r=0}^{300}(300-r)a_r=\sum_{r=0}^{300}a_r-\sum_{r=0}^{300}r\cdot a_r$$
$$\Rightarrow 2I=300a$$
$$\Rightarrow I=150a$$

37. After expansion, no two terms will have the same powers of x or the terms are non-overlapping.
Therefore, the total number of terms = $2\times2\times2\times\cdots$ $(n+2)$ times $=2^{n+2}$ as a particular power of x can be chosen from each bracket in 2 ways.

38. $(1+x)^{101}(1+x^2-x)^{100}=(1+x)(1+x^3)^{100}$
$=(1+x)[C_0+C_1x^3+C_2x^6+\cdots+C_{100}x^{300}]$
$=C_0+C_0x+C_1x^3+C_1x^4+C_2x^6+C_2x^7+\cdots+C_{100}x^{300}+C_{100}x^{301}$
Therefore, the total number of terms = 101 + 101
= 202

39. Since $x^2y^3z^4$ is occurring in the expansion of $(x+y+z)^n$, so n should be 9 only. Now,

$$A=\frac{9!}{2!\times3!\times4!}=1260$$

Coefficient of x^4y^4z is

$$\frac{9!}{4!\times4!}=630=A/2$$

40. T_r can be written as

$$T_r=\frac{r}{(r^2-1)^2-r^2}=\frac{1}{2}\left(\frac{1}{r^2-1-r}-\frac{1}{r^2-1+r}\right)$$
$$\sum_{r=1}^{\infty}T_r=\frac{1}{2}\sum_{r=1}^{\infty}\left(\frac{1}{r^2-1-r}-\frac{1}{r^2-1+r}\right)$$
$$\frac{1}{2}\left[(-1-1)+\left(1-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{11}\right)+\cdots-0\right]$$
$$\left(\text{as }\lim_{r\to\infty}\frac{1}{r^2-1+r}=0\right)$$
$$=-\frac{1}{2}$$

41. Coefficient of a^4b^5 will be $\frac{9!}{4!\times5!}$.

42. The third term will be

$$^nC_2\left(\frac{1}{4}\right)^2=31\Rightarrow\frac{n(n-1)}{2\times16}=31$$
$$\Rightarrow n(n-1)=32\cdot31\Rightarrow n=32$$

43. For sum of coefficient put $x=1$ and $y=1$.
Hence, (B) is the correct answer.

44. There will be only two rational terms, the first term and the second term

$$2^5+3^2=41$$

45. $(1+x)^{50}=1+{}^{50}C_1x+{}^{50}C_2x^2+{}^{50}C_3x^3+\cdots+{}^{50}C_{49}x^{49}+{}^{50}C_{50}x^{50}$
Therefore, sum of coefficients of odd powers of x

$$^{50}C_1+{}^{50}C_3+\cdots+{}^{50}C_{49}=2^{50-1}=2^{49}$$

46. $\sum_{r=0}^{100}{}^{100}C_r(x-3)^{100-r}2^r=((x-3)+2)^{100}=(x-1)^{100}=(1-x)^{100}$

$$\sum_{r=0}^{100}{}^{100}C_r(-x)^r=\sum_{r=0}^{100}(-1)^r\,{}^{100}C_rx^r$$

Therefore, coefficient of $x^{53}=(-1)^{53}\,{}^{100}C_{53}=-{}^{100}C_{53}$.

47. Required coefficient = coefficient of x^m in

$$\frac{(1+x)^r\{(1+x)^{n-r+1}-1\}}{1+x-1}$$

(therefore, given series is a GP of $n-(r-1)$ terms with common ratio $1+x$)
= coefficient of x^{m+1} in $(1+x)^{n+1}-(1+x)^r$
$={}^{n+1}C_{m+1}$ (note then $m+1>r$)

48. Here, $p(x)=x^n$, so
$p'(x)=nx^{n-1}$,
$p''(x)=n(n-1)x^{n-2}$,
$p'''(x)=n(n-1)(n-2)x^{n-3},\cdots,$
$p^{(r)}(x)=n(n-1)(n-2)\ldots(n-r+1)x^{n-r}$

$$=\frac{n!}{(n-r)!}=x^{n-r}=r!\,{}^nC_rx^{n-r}$$

Therefore, $p(1)+\frac{p'(1)}{1!}+\frac{p''(1)}{2!}+\cdots+\frac{p^n(1)}{1!}$

$$=1+{}^nC_1+{}^nC_2+\cdots+{}^nC_n=2^n$$

49. Now

$$\frac{1}{\sqrt{2x+1}}\times\left\{\left(\frac{1+\sqrt{2x+1}}{2}\right)^n-\left(\frac{1-\sqrt{2x+1}}{2}\right)^n\right\}$$
$$\frac{1}{2^n}\frac{2}{\sqrt{2x+1}}\times\left\{{}^nC_1\sqrt{2x+1}+{}^nC_3(\sqrt{2x+1})^3+{}^nC_5(\sqrt{2x+1})^5+\cdots\right\}$$
$$=\frac{1}{2^{n-1}}\left\{\begin{matrix}{}^nC_1+{}^nC_3(2x+1)+{}^nC_5(2x+1)^2+{}^nC_7(2x+1)^3\\+{}^nC_9(2x+1)^4+{}^nC_{11}(2x+1)^5+\cdots\end{matrix}\right\}$$

Since, this polynomial is given to be of degree 5, therefore, n can be 11 or 12.

50. $(1+x)^n\left(1-\frac{1}{x}\right)^n=\frac{(1+x)^n(1-x)^n}{x^n}x^{-n}(1-x^2)^n$

Since n is even then only even power of x will occur in the expansion. Hence, coefficient of x is equal to zero.

51. $(1+x)^{21}={}^{21}C_0+{}^{21}C_1x+{}^{21}C_2x^2+\cdots+{}^{21}C_{10}x^{10}+\cdots+{}^{21}C_{21}x^{21}$
Put $x=1$, we get
$\Rightarrow({}^{21}C_0+{}^{21}C_1+{}^{21}C_2+\cdots+{}^{21}C_{10})+({}^{21}C_{11}+\cdots+{}^{21}C_{21})=2^{21}$
$\Rightarrow 2({}^{21}C_0+{}^{21}C_1+\cdots+{}^{21}C_{10})=2^{21}$
$\Rightarrow {}^{21}C_0+{}^{21}C_1+\cdots+{}^{21}C_{10}=2^{20}$

52. $T_{r+1}=(-1)^r\,{}^{15}C_r(x^3)^{15-r}\left(\frac{1}{x^2}\right)^r=(-1)^r\,{}^{15}C_rx^{45-3r-2r}$

For term independent of x

$$45 - 5r = 0 \Rightarrow r = 9$$

Therefore, term independent of x will be $= -{}^{15}C_9 = -{}^{15}C_6$.

53. $3^{51} = 3 \cdot 3^{50} = 3(8+1)^{25}$
$= 3({}^{25}C_0 8^{25} + \cdots + {}^{25}C_{21} 8) + 3$

Hence, 3 will be the reminder.

54. Since product of r consecutive integer is divisible by $r!$.
Hence (C) is the correct answer.

55. $$\frac{{}^nC_1 a^{n-1}b}{{}^nC_2 a^{n-2}b^2} = \frac{{}^{n+3}C_2 a^{n+1}b^2}{{}^{n+3}C_3 a^n b^3} \Rightarrow \frac{2}{\frac{n(n-1)}{2}} = \frac{\frac{(n+3)(n+2)}{2}}{\frac{(n+3)(n+2)(n+1)}{6}}$$

$$\Rightarrow \frac{2}{n-1} = \frac{3}{n+1} \Rightarrow 2n+2 = 3n-3 \Rightarrow n = 5$$

56. We know that $C_0 + C_1 + C_2 + \ldots + C_n = 2^n$ squaring,

$$(C_0 + C_1 + C_2 + \ldots + C_n)^2 = 2^{2n}$$

That is,

$$\left(C_0^2 + C_1^2 + \ldots + C_n^2\right) + 2\sum\sum C_i C_j = 2^{2n}$$

Therefore,

$$2\sum\sum C_i C_j = 2^{2n} - \left\{C_0^2 + C_1^2 + \ldots + C_n^2\right\}$$

Hence,

$$2\sum\sum C_i C_j = 2^{2n-1} - \frac{(2n)!}{2(n!)(n!)} = 2^{2n-1} - \frac{(2n)!}{2(n!)^2}$$

57. Let $S = C_1^2 - 2\cdot C_2^2 + 3\cdot C_3^2 - \cdots - (2n)C_{2n}^2$

$$= -(2n)C_0^2 + (2n-1)C_1^2 - (2n-2)C_2^2 - \ldots + C_1^2$$

$$\left(\text{since, } C_r = {}^{2n}C_r = {}^{2n}C_{2n-r} = C_{2n-r}\right)$$

Adding,

$$2S = (-2n)\left\{C_0^2 - C_2^2 + \ldots + C_{2n}^2\right\}$$
$$= (-2n)(-1)^n\ {}^{2n}C_n$$

Therefore,

$$S = (-1)^{n+1} \cdot n \cdot {}^{(2n)}C_n = (-1)^{n-1} \cdot nC_n$$

58. $\sum_{r=0}^{2n} a_r (x-2)^r$ is a polynomial in x of degree $2n$ expressed in powers of $x-2$, whereas $\sum_{r=0}^{2n} b_r (x-3)^r$ is an equivalent polynomial in x of the same degree expressed in powers of $(x-3)$ in which the coefficient $(x-3)^n$ is b_n. Compare the coefficient of $(x-3)^n$.

Now,

$$\text{LHS } = \sum_{r=0}^{2n} a_r (x-2)^r$$
$$= \sum_{r=0}^{2n} a_r \{(x-3)+1\}^r$$

Expanding this summation fully, we get

$$a_0 + a_1(x-3+1)^1 + a_2\{(x-3)+1\}^2 + a_3 \{(x-3)+1\}^3 + \cdots + a_{n-1}\{(x-3)+1\}^{n-1}$$
$$+ a_n\{(x-3)+1\}^n + a_{n+1}\{(x-3)+1\}^{n+1} + \cdots + a_{2n}\{(x-3)+1\}^{2n}$$

In this, collecting the coefficient of $(x-3)^n$ and remembering $a_k = 1$ for $k \geq n$,

The coefficient of $(x-3)^n = a_n + a_{n+1}\,{}^{(n+1)}C_1 + a_{n+2}\,{}^{(n+2)}C_2 + \ldots + a_{2n}\,{}^{2n}C_n$

$$= \underbrace{1 + {}^{(n+1)}C_1}_{} + {}^{(n+2)}C_2 + \ldots + {}^{2n}C_n$$

$$= {}^{(n+2)}C_1 + {}^{(n+2)}C_2 + {}^{(n+3)}C_3 + \ldots {}^{2n}C_n$$

$$= {}^{(n+3)}C_2 + {}^{(n+3)}C_3 + \ldots {}^{2n}C_n$$

....

....

$$= {}^{2n}C_{n-1} + {}^{2n}C_n$$

$= {}^{(2n+1)}C_n$, which is also equal to ${}^{(2n+1)}C_{n+1}$

59. The expression $= (x+3)^{n-1}\{1 + r + r^2 + \ldots + r^{n-1}\}$

where $r = \dfrac{x+2}{x+3}$

$$= (x+3)^{n-1}\left(\frac{1-r^n}{1-r}\right) \text{ being the sum of a GP}$$

$$= (x+3)^{n-1}\left(\frac{1-\left(\frac{x+2}{x+3}\right)^n}{\left(1-\frac{x+2}{x+3}\right)}\right) = (x+3)^n - (x+2)^n$$

$$= (3+x)^n - (2+x)^n$$

Therefore, coefficient of x^r is

$${}^nC_r\, 3^{n-r} - {}^nC_r\, 2^{n-r} = {}^nC_r\,(3^{n-r} - 2^{n-r})$$

Practice Exercise 2

1. 7^4 ends with 01, so 7^{4k+r} ends with same two digit number as does 7^r.

Therefore, the given number ends with the same two digits as 7^3.

2. $$f(x) = \frac{1+x^{21}}{1+x} \Rightarrow g(y) = \frac{1+(4+y)^{21}}{5+y}$$

$$\Rightarrow a_0 + a_1 y + \cdots + a_{20}y^{20} = \frac{1+(4+y)^{21}}{5+y}$$

Therefore,

$$a_0 + a_1 + \cdots + a_{20} = \frac{1+5^{21}}{6}$$

3. ${}^{2n+3}C_1 + {}^{2n+3}C_2 + \cdots + {}^{2n+3}C_n - {}^{2n+3}C_0 - {}^{2n+3}C_1 - \cdots - {}^{2n+3}C_n$

[using ${}^nC_r = {}^nC_{n-r}$]

$= -\,{}^{2n+3}C_0 = -1$

4. $(1+w)^n = C_0 + C_1 w + C_2 w + \cdots + C_n w^n$

$(1+1)^n = C_0 + C_1 + C_2 + \cdots + C_n$

$(1+w)^n + (1+1)^n = 2C_0 + C_1(1+w) + C_2(1+w^2) + C_3(1+w^3) + C_4(1+w) + C_5(1+w^2) + C_6(1+w^3) + \cdots + C_n(1+w^n)$

$2(C_0+C_3+C_6+\cdots)+(C_1+C_4+C_7+\cdots)(1+w)+(C_2+C_5+C_6+\cdots)(1+w^2)=-w^n+2^n$

$\Rightarrow 2^n-1$ (since, n in a multiple of 3, $w^n=1$)

5. $\lim\limits_{n\to\infty}\dfrac{b_{n+2}}{(n+2)!}=\dfrac{(n+2)(1+b_{n+1})}{(n+2)!}=\lim\limits_{n\to\infty}\left(\lim\limits_{n\to\infty}\dfrac{1}{(n+1)!}+\dfrac{b_{n+1}}{(n+1)!}\right)$

$$=\lim_{n\to\infty}\left(\frac{1}{(n+1)!}+\frac{1}{n!}+\cdots+\frac{1}{1!}+2\right)=1+e$$

6. $\sum\limits_{r=0}^{n}{}^nC_r\dfrac{1+rx}{(1+nx)^r}(-1)^r=\sum\limits_{r=0}^{n}{}^nC_r\left(\dfrac{(-1)}{1+nx}\right)^r+\sum\limits_{r=0}^{n}{}^nC_r\dfrac{rx}{(1+nx)^r}(-1)^r$

$$=\left(1-\frac{1}{1+nx}\right)^n+n\sum_{r=1}^{n}{}^{n-1}C_{r-1}\frac{x(-1)^r}{(1+nx)^r}$$

$$=\left(\frac{nx}{1+nx}\right)^n-\left(\frac{nx}{1+nx}\right)^n=0$$

$\Rightarrow P=0$

$\Rightarrow x^2+4x=0$

$\Rightarrow x=-4,0$

$\Rightarrow |\alpha-\beta|=4$

7. Put $x=1$ in the given equation, we get

$$(n+1)!=A_0+A_1+\cdots+A_n$$

Taking log on both sides, we get

$\log(x+1)+\log(x+2)+\cdots+\log(x+n)=\log(A_0+A_1x+A_2x^2+\cdots+A_nx^n)$

Differentiating, we get

$$\frac{1}{x+1}+\frac{1}{x+2}+\cdots+\frac{1}{x+n}=\frac{A_1+2A_2x+\cdots+nA_nx^{n-1}}{A_0+A_1x+\cdots+A_nx^n}$$

Putting $x=1$, we get

$$\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}=\frac{A_1+2A_2+\cdots+nA_n}{A_0+A_1+\cdots+A_n}$$

$$A_1+2A_2+\cdots+nA_n=(A_0+A_1+\cdots+A_n)\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}\right)$$

Put $x=1$ in $(x+1)(x+2)\cdots(x+n)=A_0+A_1x+\cdots+A_nx^n$

$$(n+1)!=A_0+A_1+A_2+\cdots+A_n$$

Hence,

$$A_1+2A_2+\cdots+nA_n=(n+1)!\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}\right)$$

8. $\sum\limits_{r=0}^{10}2^{10}\,{}^{20}C_r\,{}^{20-r}C_{10-r}=\sum\limits_{r=0}^{10}2^{10}\dfrac{20!}{r!(20-r)!}\times\dfrac{(20-r)!}{10!(10-r)!}$

$$=\sum_{r=0}^{10}2^{10}\frac{20!}{10!\,10!}\times\frac{10!}{r!(10-r)!}$$

$$=\sum_{r=0}^{10}2^{10}\,{}^{20}C_{10}\,{}^{10}C_r={}^{20}C_{10}\,3^{10}$$

9. $\dfrac{2}{1!\,13!}+\dfrac{2}{3!\,11!}+\dfrac{2}{5!\,9!}+\dfrac{1}{7!\,7!}=\dfrac{1}{14!}(2\,{}^{14}C_1+2\,{}^{14}C_3+2\,{}^{14}C_5+{}^{14}C_7)$

$$=\frac{1}{14!}2^{14-1}=\frac{2^{13}}{14!}$$

$\Rightarrow m=13$ and $n=14$

10. $(1+2x+3x^2)^{10}=a_0+a_1x+\cdots+a_{20}x^{20}$

So, there are 21 terms.

Also,

$$6^{10}=a_0+a_1+\cdots+a_{20}$$

Using multinomial theorem

$$(1+2x+3x^2)^{10}=\sum\frac{10!}{r_1!r_2!r_3!}(1)^{r_1}(2x)^{r_2}(3x^2)^{r_3}$$

$$\left.\begin{aligned}r_1+r_2+r_3&=10\\ r_2+2r_3&=20\end{aligned}\right\}\text{for coefficient of }x^{20}$$

$r_2+2r_3=20$ and $r_1+r_2+r_3=10$

$r_2=0,\ r_3=10\Rightarrow r_1=0$

$r_2=2,\ r_3=9\Rightarrow r_1=-1$ (Not Possible)

Only value of r_1, r_2, r_3 exists (0, 0, 10).

Hence, coefficient of x^{20} is 3^{10}.

Hence, (A), (B) and (D) are the correct answers.

11. Differentiating the expansion, we have

$n(p+2x)(1+px+x^2)^{n-1}=a_1+2a_2x+3a_3x^2+\cdots+2na_{2n}x^{2n-1}$

Multiplying by $(1+px+x^2)$, we get

$n(p+2x)(1+a_1x+a_2x^2+\cdots)=(1+px+x^2)(a_1+2a_2x+3a_3x^2+\cdots+2na_{2n}x^{2n-1})$

Comparing coefficient of x^r on both sides, we get

$$n[pa_r+2a_{r-1}]=(r+1)a_{r+1}+pra_r+(r-1)a_{r-1}$$

Therefore,

$$(np-pr)a_r=(r+1)a_{r+1}+(r-1-2n)a_{r-1}$$

12. $a_1+5a_2+9a_3+\cdots+(8n-3)a_{2n}=\sum\limits_{r=1}^{2n}(4r-3)a_r$

$$=4\sum_{r=1}^{2n}ra_r-3\sum_{r=1}^{2n}a_r$$

$$(1+px+x^2)^n=1+a_1x+a_2x^2+\cdots+a_{2n}x^{2n}$$

So,

$$\sum_{r=1}^{2n}a_r=(p+2)^n-1$$

Differentiating the expansion and substituting $x=1$, we get

$$\sum_{r=1}^{2n}ra_r=n(p+2)^n$$

Therefore,

$$\sum_{r=1}^{2n}(4r-3)a_r = 4n(p+2)^n - 3((p+2)^n - 1)$$
$$= (4n-3)(p+2)^n + 3$$

13. $$a_1 + 3a_2 + 5a_3 + \cdots + (4n-1)a_{2n} = \sum_{r=1}^{2n}(2r-1)a_r$$
$$= 2\sum_{r=1}^{2n} ra_r - \sum_{r=1}^{2n} a_r$$
$$\Rightarrow 2n(p+2)^n - ((p+2)^n - 1)$$
$$\Rightarrow (2n-1)(p+2)^n + 1$$

Now, $p = -3$ and $n \in$ even

$$\Rightarrow (2n-1) + 1 = 2n$$

14. Let $P = (1+x)(1+x+x^2)\cdots(1+x+x^2+\cdots+x^n)$

Clearly, the term containing the highest power of x in

$$P = x \cdot x^2 \cdot x^3 \cdots x^n = x^{n(n+1)/2}$$

So, the total number of terms in the product $= \dfrac{n(n+1)}{2} + 1$

$$= \frac{n^2+n+2}{2}$$

15. Let

$$m = \frac{n(n+1)}{2}$$

Then,

$$P = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m$$
$$\Rightarrow (1+x)(1+x+x^2)\cdots(1+x+x^2+\cdots+x^n)$$
$$= a_0 + a_1x + a_2x^2 + \cdots + a_mx^m \quad (1)$$

Replacing x by $\dfrac{1}{x}$, we get

$$\frac{(1+x)(1+x+x^2)\cdots(1+x+x^2+\cdots+x^n)}{x^{1+2+3+\cdots+n}} = \frac{a_0x^m + a_1x^{m-1} + \cdots + a_m}{x^m}$$

$$\Rightarrow \frac{a_0 + a_1x + \cdots + a_mx^m}{x^m} = \frac{a_0x^m + a_1x^{m-1} + \cdots + a_m}{x^m}$$

Equating coefficients of equal power, we get

$$a_0 = a_m, a_1 = a_{m-1}, \cdots, a_m = a_0$$

So, coefficient of the equidistant term from the beginning and end are always equal.

16. Putting $x = 1$ and $x = -1$, respectively, we obtain

$$a_0 + a_1 + a_2 + \cdots + a_n = (n+1)!$$
$$a_0 - a_1 + a_2 - \cdots + (-1)^m a_n = 0$$

Adding and subtracting, we get

$$a_0 + a_2 + a_4 + \cdots = a_1 + a_3 + a_5 + \cdots = \frac{(n+1)!}{2}$$

17. (A) $2({}^{14}C_r) = {}^{14}C_{r-1} + {}^{14}C_{r+1}$

$$\Rightarrow 2 = \frac{{}^{14}C_{r-1}}{{}^{14}C_r} + \frac{{}^{14}C_{r+1}}{{}^{14}C_r} = \frac{r}{14-(r-1)} + \frac{14-r}{r+1} \Rightarrow r^2 - 14r + 45 = 0$$

Therefore, $r = 5, 9$.

(B) $$\left\{\left(\frac{1-x^n}{1-x}\right)(1-x)\right\}^m = (1-x^n)^m;$$

Therefore, sum of coefficients = 0.

(C) $$\sum_{r=1}^{n}(-1)^r\, 3\, {}^nC_r\left(\frac{1-i^{nr}}{1-i^r}\right) = 0 \quad [\text{Since, } n = 4k]$$

(D) $3^{37} = 3 \cdot 3^{4 \cdot 9} = 3(81)^9 = 3(80+1)^9$

$$= 3\left({}^9C_0(80)^9 + {}^9C_1(80)^8 + \cdots + {}^9C_9\right)$$
$$= 80\left[3\left({}^9C_0(80)^8 + {}^9C_1(80)^7 + \cdots\right)\right] + 3$$
$$= 80\lambda + 3$$
$$\Rightarrow k = 3$$

Therefore, $3k = 9$.

18. (A) We have

$$\prod_{r=1}^{n}\left(1 + \frac{{}^nC_r}{{}^nC_{r-1}}\right) = \prod_{r=1}^{n}\left(1 + \frac{n-r+1}{r}\right)$$
$$= \prod_{r=1}^{n}\left(\frac{n+1}{r}\right) = \frac{(n+1)^n}{n!}$$

(B) Here,

$$\sum_{r=1}^{n}\frac{r\, {}^nC_r}{{}^nC_{r-1}} = \sum_{r=1}^{n}(n-r+1) = n(n+1) - \frac{n(n+1)}{2} = \frac{n(n+1)}{2}$$

(C) Multiply both sides of the given expression by x, we get

$$(1+x)^n \cdot x = {}^nC_0x + {}^nC_1x^2 + {}^nC_2x^3 + \cdots + {}^nC_nx^{n-1}$$

Integrating both sides w.r.t. x from 0 to -1, we get

$$\frac{1}{(n+1)(n+2)} = \frac{{}^nC_0}{2} - \frac{{}^nC_1}{3} + \frac{{}^nC_2}{4} - \frac{{}^nC_3}{5}\cdots$$

(D) Here,

$$\frac{{}^nC_r}{r+1} = \frac{1}{n+1}\, {}^{n+1}C_{r+1}$$

and hence the given expression is

$$= \frac{1}{n+1}\left[{}^{n+1}C_0 + {}^{n+1}C_2 + {}^{n+1}C_4 + \cdots + {}^{n+1}C_{n+1} - {}^{n+1}C_0\right]$$
$$= \frac{2^{n+1-1} - 1}{n+1} = \frac{2^n - 1}{n+1}$$

19. If $\alpha = 15 + \sqrt{220}$ and $\beta = 15 - \sqrt{220}$, then

$$\alpha^{19} + \beta^{19} = 10k_1$$

and

$$\alpha^{82} + \beta^{82} = 10k_2$$

where $\beta^{19} + \beta^{82} < 1$

$$\Rightarrow \alpha^{19} + \alpha^{82} = 10(k_1 + k_2) - (\beta^{19} + \beta^{82})$$

Therefore, $[E] = 10(k_1 + k_2) - 1$.

So, digit at unit place = 9.

Solved JEE 2017 Questions

JEE Main 2017

1. If $(27)^{999}$ is divided by 7, then the remainder is

(A) 6 **(B)** 1

(C) 1 **(D)** 3

(ONLINE)

Solution: We can rewrite $(27)^{999}$ as

$$(27)^{999} = (28-1)^{999}$$

That is, now, we have $\frac{(28-1)^{999}}{7}$.

Here, every term after the binomial expansion is divisible by 7 except the last term; therefore, we can consider the term $\frac{28n-1}{7}$ and adding and subtracting 7 from this term we get

$$\frac{28n+7-7-1}{7} = \frac{(28n-7)+(7-1)}{7} = \frac{7(4n-1)+6}{7}$$

Thus, the remainder is 6.

Hence, the correct answer is option (A).

2. The coefficient of x^{-5} in the binomial expansion of $\left(\frac{x+1}{x^{\frac{2}{3}}-x^{\frac{1}{3}}+1}-\frac{x-1}{x-x^{\frac{1}{2}}}\right)^{10}$, where $x \neq 0, 1$, is

(A) 1 **(B)** −4

(C) −1 **(D)** 4

(ONLINE)

Solution: The given binomial expansion is

$$\left(\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right)^{10}$$

$$\frac{x+1}{x^{2/3}-x^{1/3}+1} \Rightarrow \frac{(x^{1/3})^3+1}{x^{2/3}-x^{1/3}+1} \Rightarrow \frac{(x^{1/3}+1)(x^{2/3}-x^{1/3}+1)}{x^{2/3}-x^{1/3}-1} \Rightarrow x^{1/3}+1$$

$$\frac{x-1}{x-x^{1/2}} \Rightarrow \frac{(x^{1/2}-1)(x^{1/2}+1)}{x^{1/2}(x^{1/2}-1)} \Rightarrow \frac{x^{1/2}+1}{x^{1/2}} = 1+x^{-1/2}$$

Therefore, the general term is

$$^{10}C_r(x^{1/3})^r(-1)^{10-r}(x^{-1/2})^{10-r}$$

For the coefficient x^{-5}, we have

$$\frac{r}{3}-\frac{1}{2}(10-r) = -5$$

$$\Rightarrow \frac{r}{3}-5+\frac{r}{2} = -5$$

$$\Rightarrow \frac{r}{3}+\frac{r}{2} = 0 \Rightarrow r = 0$$

Therefore,

$$10C_r(-1)^{10-r}]_{r=0} \Rightarrow 10C_0(-1)^{10} = 1$$

Hence, the correct answer is option (A).

9 Sequence and Series

9.1 Sequence

A sequence is a function of natural numbers with co-domain that is the set of real numbers and its terms are in a definite order.

$$f: N \to R \quad \text{defined as } t_n = f(n), \quad n \in N$$

is called a sequence and denoted by

$$\{t_1, t_2, t_3, \ldots\} = \{f(1), f(2), f(3), \ldots\}$$

Some more examples of sequences:

(a) 2, 4, 6, 8, …
(b) 5, 3, 1, −1, …
(c) 1, 3, 9, 27, …
(d) 32, 16, 8, 4, …

A sequence is said to be finite or infinite accordingly as it has the finite or infinite number of terms.

Illustration 9.1 If $f: N \to R$ where $f(n) = \dfrac{n}{(2n+1)^2}$, find the sequence in an ordered pair form.

Solution:

$$t_n = \frac{n}{(2n+1)^2}$$

Putting $n = 1, 2, 3, 4, \ldots$ successively, we get

$$t_1 = \frac{1}{(2.1+1)^2} = \frac{1}{3^2} = \frac{1}{9}$$

$$t_2 = \frac{2}{(2.2+1)^2} = \frac{2}{5^2} = \frac{2}{25}$$

$$t_3 = \frac{3}{(2.3+1)^2} = \frac{3}{7^2} = \frac{3}{49}$$

$$t_4 = \frac{4}{(2.4+1)^2} = \frac{4}{9^2} = \frac{4}{81}$$

Hence, it sequences in an ordered pair form

$$\left\{\left(1, \frac{1}{9}\right), \left(2, \frac{2}{25}\right), \left(3, \frac{3}{49}\right), \left(4, \frac{4}{81}\right), \ldots\right\}$$

Illustration 9.2 Write down the sequence whose n^{th} terms are

(A) $\left(\dfrac{2n+2}{4}\right)$ **(B)** $(-1)^n\left(\dfrac{3n+2}{5}\right)$

(C) $\dfrac{1}{n^2}\sin\left(\dfrac{n\pi}{3}\right)$

Solution:

(A) $t_n = \left(\dfrac{2n+2}{4}\right)$

Putting $n = 1, 2, 3, 4, \ldots$ successively, we get

$$1, \frac{3}{2}, 2, \frac{5}{2}, 3, \ldots$$

which is the required sequence.

(B) $t_n = (-1)^n\left(\dfrac{3n+2}{5}\right)$

Putting $n = 1, 2, 3, 4, \ldots$ successively, we get

$$t_1 = -1, \frac{8}{5}, -\frac{11}{5}, \frac{14}{5} \ldots$$

which is the required sequence.

(C) $t_n = \dfrac{1}{n^2}\sin\left(\dfrac{n\pi}{3}\right)$

Putting $n = 1, 2, 3, 4, \ldots$ successively, we get

$$t_1 = \frac{1}{1^2}\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$t_2 = \frac{1}{2^2}\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{8}$$

$$t_3 = \frac{1}{3^2}\sin\left(\frac{3\pi}{3}\right) = 0$$

$$t_4 = \frac{1}{4^2}\sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{32}$$

Hence, the required sequence is

$$\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{8}, 0, -\frac{\sqrt{3}}{32}$$

Illustration 9.3 A sequence of numbers a_1, a_2, a_3 satisfies the relation $a_{n+1} = a_n + a_{n-1}$ for $n \geq 2$. Find a_4 if $a_1 = a_2 = 1$.

Solution: Put $n = 2$. Then

$$a_3 = a_2 + a_1 = 1 + 1 = 2$$

Again using $n = 3$, we get

$$a_4 = a_3 + a_2 = 2 + 1 = 3$$

Illustration 9.4 If a sequence of numbers $a_1, a_2, \ldots, a_n$ satisfies the relation $a_{n+1}^2 = a_n \cdot a_{n+2} + (-1)^n$ then find a_3, if $a_1 = 2$ and $a_2 = 5$.

Solution: Put $n = 1$ in the given relation. We get

$$a_2^2 = a_1a_3 + (-1)^1 \Rightarrow 5^2 = 2a_3 - 1 \Rightarrow 2a_3 = 26 \Rightarrow a_3 = 13$$

Illustration 9.5 A sequence of numbers u_0, u_1, u_2, u_3 satisfies the relation $u_{n+1} = 3u_n - 2u_{n-1}$. Find u_2 if $u_0 = 2$ and $u_1 = 3$.

Solution: Put $n = 1$ in the given relation. We get

$$u_2 = 3u_1 - 2u_0 = 3.3 - 2.2 = 9 - 4 = 5$$

9.2 Progression

When terms of a sequence are written under specific conditions, then the sequence is called a **progression**.

A progression is represented as $t_1, t_2, \ldots, t_n$ or $a_1, a_2, \ldots, a_n$ where t_1 or a_1 means the first term, and t_n or a_n means the n^{th} term. t_k or a_k is called the general term of the progression and its n^{th} term is always expressible in terms of n. The number of terms of progression can be finite or infinite.

9.2.1 Arithmetic Progression (AP)

A sequence is called an arithmetic progression (AP) if its terms continually increase or decrease by the same number. The fixed number by which terms increase or decrease is called the common difference.

OR

A sequence of numbers $\{a_n\}$ is called an AP if there is a number d, such that $d = a_n - a_{n-1}$ for all n. d is called the common difference (CD) of the AP.

Example: Common difference of

(a) 2, 6, 10, 14 is 4
(b) 10, 5, 0, –5, –10 is –5
(c) $a, a + d, a + 2d, a + 3d$ is d.

9.2.1.1 General Term of an AP

Let a be the first term and d the difference on an AP. Let $T_1, T_2, T_3, \ldots, T_n$ denote 1st, 2nd, 3rd, …., n^{th} terms, respectively. Then we have

$$T_2 - T_1 = d$$
$$T_3 - T_2 = d$$
…………….
…………….
$$T_n - T_{n-1} = d$$

Upon adding these, we get

$$T_n - T_1 = (n-1)d \Rightarrow T_n = T_1 + (n-1)d$$

But $T_1 = a$. Therefore, general term $= T_n = a + (n-1)d$.

Thus, if a is the first term and d is the common difference of an AP, then the AP is $a, a+d, a+2d, \ldots, a+(n-1)d$ or $a, a+d, a+2d, \ldots$, accordingly as it is finite or infinite.

If the number of terms of an AP is n and the value of the last term is l, then

$$l = T_n = a + (n-1)d$$

If a is the first term and d is the common difference, then AP can be written as

$$a, a+d, a+2d, \ldots, a+(n-1)d$$

The n^{th} term of AP is

$$T_n = a + (n-1)\,d$$

where $d = T_n - T_{n-1}$.

The n^{th} term of this AP from the last, if last term l is given is

$$T_n' = l - (n-1)d$$

If the n^{th} term of AP from stating is T_n and from last is T_n', then

$$T_n + T_n' = a + l$$

Illustration 9.6 The n^{th} term of an AP is $4n - 1$. Write down the first 4 terms and the 18th term of the AP.

Solution: Given $T_n = 4n - 1$. Putting $n = 1, 2, 3, 4, \ldots, 18$, we get

$$T_1 = 3, T_2 = 7, T_3 = 11, T_4 = 15 \text{ and } T_{18} = 71$$

Illustration 9.7 The 8th term of a series in the AP is 23 and the 102th term is 305 in the series. Find the series.

Solution: Given

$$T_8 = a + 7d = 23$$
$$T_{102} = a + 101d = 305$$

Solving the two equations, we get

$$a = 2, d = 3$$

Now the series is 2, 5, 8, 11, …

Illustration 9.8 If p times the p^{th} term of an AP is equal to q times the q^{th} term, show that the $(p+q)^{\text{th}}$ term is zero.

Solution: Given that $p \cdot t_p = q \cdot t_q$.

If a is the first term and d is the common difference then

$$p[a + (p-1)\,d] = q[a + (q-1)d]$$
$$\Rightarrow pa + p(p-q)d = qa + q(q-1)d$$
$$\Rightarrow (p-q)a = q^2d - qd - p^2d + pd$$
$$\Rightarrow (p-q)a = d(q^2 - p^2) - d\,(q-p)$$
$$\Rightarrow (p-q)a = d(q+p)\,(q-p) - d\,(q-p)$$
$$\Rightarrow -a = d(q+p-1)$$
$$\Rightarrow a + [(q+p) - 1]d = 0$$
$$\Rightarrow t_{p+q} = 0$$

Illustration 9.9 If a, b and c are the x^{th}, y^{th} and z^{th} terms of an AP, show that

(A) $a(y-z) + b(z-x) + c(x-y) = 0$
(B) $x(b-c) + y\,(c-a) + z(a-b) = 0$

Solution: Let A be the first term and D be the common difference. The x^{th}, y^{th}, z^{th} terms are given by

$$T_x = A + (x-1)D = a \quad (1)$$
$$T_y = A + (y-1)D = b \quad (2)$$
$$T_z = A + (z-1)D = c \quad (3)$$

Equation (2) – Eq. (3), Eq. (3) – Eq. (1) and Eq. (1) – Eq. (2), respectively, give

$$(b-c)=(y-z)D \Rightarrow (y-z)=\frac{b-c}{D},$$
$$(c-a)=(z-x)D \Rightarrow (z-x)=\frac{c-a}{D},$$
$$(a-b)=(x-y)D \Rightarrow (x-y)=\frac{a-b}{D}$$

(A) Now substituting the values of $(y-z)$, $(z-x)$ and $(x-y)$ in LHS of the expression (A), we get

$$\text{LHS} = \frac{a(b-c)}{D}+\frac{b(c-a)}{D}+\frac{c(a-b)}{D}$$
$$= \frac{ab-ac+bc-ab+ca-cb}{D} = 0 = \text{RHS}$$

(B) Now substituting the values of $(b-c)$, $(c-a)$ and $(a-b)$ in LHS of the expression (B), we get
$\text{LHS} = x(y-z)D + y(z-x)D + z(x-y)D$
$= \{xy - xz + yz - xz + zx - zy)\}D = 0 = \text{RHS}$

Your Turn 1

1. A sequence $\{a_n\}$ is given by the formula $a_n = 10 - 3n$. Prove that it is an AP.
2. Which term of the sequence –3, –7, –11, –15, … is –403? Also find which term, if any, of the given sequence is –500.
Ans. 101^{st} term, not any term
3. Find an AP of 8 terms whose first term is $\frac{1}{2}$ and last term is $\frac{17}{6}$.
Ans. $\frac{1}{2}, \frac{5}{6}, \frac{7}{6}, \ldots$
4. The fourth term of an AP is equal to 3 times of the first term and the 7^{th} term is more than twice the third term by 1. Find the first term and the common difference.
Ans. First term = 3, common difference = 2
5. If the p^{th} term of an AP is c and the q^{th} term is d, find the r^{th} term.
Ans. $\frac{c(r-q)-d(r-p)}{p-q}$
6. A man saves Rs 320 in the month of January, Rs 360 in the month of February, Rs 400 in the month of March. If he continues his savings in the same way, find his savings in the month of November in the same year. **Ans.** 720
7. Show that the sum of $(m+n)^{th}$ and $(m-n)^{th}$ terms of an AP is equal to twice the m^{th} term.
8. Show that there is no AP which consists only of distinct prime numbers.
9. If the m^{th} term of an AP is $\frac{1}{n}$ and the n^{th} term is $\frac{1}{m}$, then shown that its $(mn)^{th}$ term is 1.
10. If the p^{th} term of an AP is q and the q^{th} term is p, prove that its n^{th} term is $(p+q-n)$.

9.2.1.2 Sum of *n* Terms of an AP

Let a be the first term, d the common difference and l the last term of the given AP. If S_n is the sum of n terms, then

$$S_n = a + (a+d) + (a+2d) + \cdots + (l-2d) + (l-d) + l$$

Re-writing S_n in the reverse order, we get

$$S_n = l + (l-d) + (l-2d) + \cdots + (a+2d) + (a+d) + a$$

Adding columnwise, we get

$$2S_n = (a+l) + (a+l) + (a+l) + \cdots + n \text{ times} = n(a+l)$$

$$\Rightarrow S_n = \frac{n}{2}(a+l) = \frac{n}{2}[a+a+(n-1)d] \ [\text{since } l = a+(n-1)d]$$

Therefore,

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

If S_n is the sum of n terms of an AP whose first term is a and the last term is l, then

$$S_n = \frac{n}{2}(a+l)$$

If S_n is the sum of first n terms of an AP whose last term is l and the common difference is d, then

$$S_n = \frac{n}{2}\{a+l\} = \frac{n}{2}\{l-(n-1)d+l\} = \frac{n}{2}\{2l-(n-1)d\}$$

> If a is the first term and d is the common difference of the AP, then the n^{th} term a_n is given by
>
> $$a_n = a + (n-1)d$$
>
> The sum S_n of the first n terms of such an AP is given by
>
> $$S_n = \frac{n}{2}(2a+(n-1)d) = \frac{n}{2}\ (a+l)$$
>
> where l is the last term.

Illustration 9.10 Find the sum of n terms of a series whose 7^{th} term is 30 and 13^{th} term is 54. Hence, or otherwise, find the sum of r terms and 50 terms of the series. Assume the series is in AP.

Solution: Let the first term of AP be a and the common difference be d. The n^{th} term is

$$T_n = a + (n-1)d$$

So the 7^{th} and 13^{th} terms are

$$T_7 = a + 6d = 30 \quad (1)$$
$$T_{13} = a + 12d = 54 \quad (2)$$

Solving Eqs. (1) and (2), we get $a = 6$ and $d = 4$.

Now using

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

we get
$$S_n = \frac{n}{2}[2\times 6+(n-1)\times 4] = 2n(n+2)$$

Now find the sum of r terms, S_r. Using $S_n = 2n(n+2)$, we replace n by r to get S_r.

$$S_r = 2r(r+2)$$

To find the sum of 50 terms, we can use

$$S_n = 2n(n+2)$$
$$\Rightarrow S_{50} = 2 \times 50(50+2) = 5200$$

Illustration 9.11 The sum of n terms of two series in AP is in the ratio $5n+4:9n+6$. Find the ratio of their 13th term.

Solution: Let a_1 and a_2 be the first terms of two APs and d_1 and d_2 be their respective common differences. Then

$$\frac{\frac{n}{2}[2a_1+(n-1)d_1]}{\frac{n}{2}[2a_2+(n-1)d_2]} = \frac{5n+4}{9n+6}$$

$$\Rightarrow \frac{a_1+\frac{(n-1)}{2}d_1}{a_2+\frac{(n-1)}{2}d_2} = \frac{5n+4}{9n+6} \quad (1)$$

Now the ratio of 13th terms $= \dfrac{a_1+12d_2}{a_2+12d_2}$

Comparing this with LHS of Eq. (1) we get

$$\frac{n-1}{2} = 12 \Rightarrow n = 25$$

Hence, we have

$$\Rightarrow \frac{a_1+12d_1}{a_2+12d_2} = \frac{5(25)+4}{9(25)+6} = \frac{129}{231}$$

Illustration 9.12 The sum of first p, q and r terms of an AP is a, b and c, respectively. Show that

$$\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)=0$$

Solution: Let A be the first term and D be the common difference of the AP. Then

$$\Rightarrow a = \frac{p}{2}[2A+(p-1)D]$$

We can write

$$\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q) = \sum\frac{a}{p}(q-r)$$

Now

$$\text{LHS} = \sum\frac{a}{p}(q-r)$$
$$= \sum\frac{1}{2}(q-r)[2A+(p-1)D]$$
$$= \frac{1}{2}\sum 2A(q-r)+\frac{1}{2}\sum(q-r)D(p-1)$$
$$= A\sum(q-r)+\frac{D}{2}\sum[p(q-r)]-\frac{D}{2}\sum(q-r)$$
$$= 0+0-0 = 0 = \text{RHS}$$

Illustration 9.13 The sum of n, $2n$ and $3n$ terms of an AP is S_1, S_2 and S_3, respectively. Prove that $S_3 = 3(S_2 - S_1)$.

Solution: The S_1, S_2, S_3 terms can be written as

$$S_1 = \frac{n}{2}[2a(n-1)d]$$
$$S_2 = \left(\frac{2n}{2}[2a+(2n-1)d]\right)$$
$$S_3 = \frac{3n}{2}[2a+(3n-1)d]$$

Now

$$S_2 - S_1 = \frac{2n}{2}[2a+(2n-1)d]-\frac{n}{2}[2a+(n-1)d]$$
$$= \frac{n}{2}[2a+(3n-1)d]$$
$$3(S_2-S_1) = \frac{3n}{2}[2a+(3n-1)d] = S_3$$

9.2.1.3 Assuming Quantities in AP

If terms are given in AP and their sum is known, then the terms must be picked up in following way:

- For three terms $(a-d)$, a, $(a+d)$
- For four terms $(a-3d)$, $(a-d)$, $(a+d)$, $(a+3d)$
- For five terms $(a-2d)$, $(a-d)$, a, $(a+d)$, $(a+2d)$

Note: In general, if we take $(2r+1)$ terms in AP, we take them as

$$a-rd, a-(r-1)d, \ldots, a-d, a, a+d, \ldots, a+rd$$

And if we take $2r$ terms in AP, we take them as

$$(a-(2r-1)d), (a-(2r-3)d), \ldots, (a+(2r-3)d), (a+(2r-1)d)$$

Illustration 9.14 Sum of three numbers in AP is -3 and their product is 8. Find the numbers.

Solution: Let the three numbers be $(a-d)$, a, $(a+d)$. Given

$$a-d+a+a+d = 3 \Rightarrow 3a = -3 \Rightarrow a = -1$$

Given their product is

$$(a-d)a(a+d) = 8 \Rightarrow a(a^2-d^2) = 8 \Rightarrow (-1)(1-d^2) = 8$$
$$\Rightarrow -1+d^2 = 8 \Rightarrow d^2 = 9 \Rightarrow d = \pm 3$$

Therefore, the numbers are $-4, -1, 2$ or $2, -1, -4$.

Illustration 9.15 Find four numbers in AP whose sum is 20 and the sum of whose square is 120.

Solution: Let the four numbers be given by $a-3d$, $a-d$, $a+d$, $a+3d$. As per the given condition,

$$20 = (a-3d)+(a-d)+(a+d)+(a+3d)$$
$$\Rightarrow 4a = 20 \Rightarrow a = 5$$

Also given is the sum of square = 120. So

$$(a-3d)^2+(a-d)^2+(a+d)^2+(a+3d)^2 = 120$$
$$\Rightarrow 4a^2+20d^2 = 120 \Rightarrow a^2+5d^2 = 30$$
$$\Rightarrow 25+5d^2 = 30 \Rightarrow 5d^2 = 5 \Rightarrow d = \pm 1$$

Therefore, the numbers are 2, 4, 6, 8 or 8, 6, 4, 2.

Illustration 9.16 Divide 32 into four parts which are in AP such that the product of first and last terms and the product of middle terms is to the ratio 7:15.

Solution: Let the four parts be $(a-3d)$, $(a-d)$, $(a+d)$, $(a+3d)$. Now

$$a-3d+a-d+a+d+a+3d=32 \Rightarrow a=8$$

Also

$$\frac{(a-3d)(a+3d)}{(a-d)(a+d)}=\frac{7}{15} \Rightarrow \frac{a^2-9d^2}{a^2-d^2}=\frac{7}{15}$$

$$\Rightarrow a^2=16d^2 \Rightarrow a=\pm 4\,d \Rightarrow d=\pm\frac{1}{2}$$

Hence, required parts are 13/2, 15/2, 17/2, 19/2.

Properties of AP

- If a fixed number is added (subtracted) to each term of a given AP, then the resulting sequence is also an AP with the same common difference as that of the given AP.
- If each term of an AP is multiplied by a fixed number (say k) (or divided by a non-zero fixed number), the resulting sequence is also an AP with the common difference multiplied by k.
- If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are two APs with common differences d and d', respectively, then a_1+b_1, a_2+b_2, $a_3+b_3, \ldots$ is also an AP with the common difference $d+d'$.
- If $a_1, a_2, a_3, \ldots, a_n$ are in AP, then $a_1+a_n=a_2+a_{n-1}=a_3+a_{n-2}=\ldots$ and so on.
- If the n^{th} term of any sequence is a linear expression in n, then the sequence is an AP whose common difference is the coefficient of n.
- If the sum of n terms of any sequence is quadratic in n, then the sequence is an AP, whose common difference is twice the coefficient of n^2.
- If three terms are in AP, then the middle term is called the arithmetic mean (AM) between the other two, i.e. if a, b, c are in AP then $b=\frac{a+c}{2}$ is the AM of a and c.
- If $a_1, a_2, \ldots, a_n$ are n numbers, then the arithmetic mean (A) of these numbers is $A=\frac{1}{n}(a_1+a_2+a_3+\cdots+a_n)$.
- If n arithmetic means $a_1, a_2, \ldots, a_n$ are inserted between the numbers a and b then $a_1+a_2+a_3+\cdots+a_n=n\frac{(a+b)}{2}$.

Illustration 9.17 If a^2+2bc, b^2+2ac, c^2+2ab are in AP, show that $\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$ are in AP.

Solution: Given that a^2+2bc, b^2+2ac, c^2+2ab are in AP. Then $(a^2+2bc)-(ab+bc+ca)$, $(b^2+2ab)-(ab+ab+ca)$, $(c^2+2ab)-(ab+bc+ca)$ are in AP.

So $(a^2+bc-ab-ca)$, $(b^2+ca-ab-ab)$, $(c^2+ab-bc-ca)$ are in AP.

$\Rightarrow (a-b)(a-c), (b-c)(b-a), (c-a)(c-b)$ are in AP.

$$\Rightarrow \frac{-1}{b-c}, \frac{-1}{c-a}, \frac{-1}{a-b} \text{ are in AP.}$$

$$\Rightarrow \frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b} \text{ are in AP.}$$

Illustration 9.18 If $a_1, a_2, \ldots, a_n$ are in AP ($a_i>0$ for all i), show that

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\cdots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}=\frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$$

Solution:

$$\text{LHS}=\frac{\sqrt{a_1}-\sqrt{a_2}}{a_1-a_2}+\frac{\sqrt{a_2}-\sqrt{a_3}}{a_2-a_3}+\cdots+\frac{\sqrt{a_{n-1}}-\sqrt{a_n}}{a_{n-1}-a_n}$$

If d is the common difference, then

$$\text{LHS}=\frac{\sqrt{a_1}-\sqrt{a_2}}{-d}+\frac{\sqrt{a_2}-\sqrt{a_3}}{-d}+\cdots+\frac{\sqrt{a_{n-1}}-\sqrt{a_n}}{-d}$$

$$=-\frac{1}{d}\left[\sqrt{a_1}-\sqrt{a_2}+\sqrt{a_2}-\sqrt{a_3}+\cdots+\sqrt{a_{n-1}}-\sqrt{a_n}\right]$$

$$=-\frac{1}{d}\frac{(a_1-a_n)}{\sqrt{a_1}+\sqrt{a_n}}=\frac{(a_n-a_1)}{d}\cdot\frac{1}{\sqrt{a_1}+\sqrt{a_n}}$$

$$=\frac{a_1+(n-1)d-a_1}{d}\cdot\frac{1}{\sqrt{a_1}+\sqrt{a_n}}$$

$$=\frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}=\text{RHS}$$

Illustration 9.19 If $a_1, a_2, a_3, \ldots, a_n$ be an AP of non-zero terms, then prove that

$$\frac{1}{a_1a_2}+\frac{1}{a_2a_3}+\cdots+\frac{1}{a_{n-1}a_n}=\frac{n-1}{a_1a_n}$$

Solution: Let d be the common difference of the given AP. Then

$$a_2-a_1=a_3-a_2=a_n-a_{n-1}=d \text{ (say)}$$

Now,

$$\frac{1}{a_1a_2}+\frac{1}{a_2a_3}+\frac{1}{a_3a_4}+\cdots+\frac{1}{a_{n-1}a_n}$$

$$=\frac{1}{d}\left[\frac{d}{a_1a_2}+\frac{d}{a_2a_3}+\frac{d}{a_3a_3}+\cdots+\frac{d}{a_{n-1}a_n}\right]$$

$$=\frac{1}{d}\left[\frac{a_2-a_1}{a_1a_2}+\frac{a_3-a_2}{a_2a_3}+\cdots+\frac{a_n-a_{n-1}}{a_{n-1}a_n}\right]$$

$$=\frac{1}{d}\left[\frac{1}{a_1}-\frac{1}{a_2}+\frac{1}{a_2}-\frac{1}{a_3}+\cdots+\frac{1}{a_{n-1}}-\frac{1}{a_n}\right]$$

$$=\frac{1}{d}\left[\frac{1}{a_1}-\frac{1}{a_n}\right]$$

$$=\frac{1}{d}\left[\frac{a_n-a_1}{a_1a_n}\right]=\frac{1}{d}\left[\frac{a_1+(n-1)d-a}{a_1a_n}\right]$$

$$=\frac{n-1}{a_1a_n}=\text{RHS}$$

Illustration 9.20 Find the sum of first 24 terms of the AP $a_1, a_2, a_3, \ldots$, if it is known that $a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$.

Solution: As we know in an AP, the sum of the terms equidistant from the beginning and the end is always same and is equal to the sum of first and last terms. Therefore

$$a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \ldots$$

So

$$a_1 + a_{24} = a_5 + a_{20} = a_{10} + a_{15}$$

$$\Rightarrow 3(a_1 + a_{24}) = 225 \Rightarrow a_1 + a_{24} = 75$$

So

$$S_{24} = 24/2\,(a_1 + a_{24}) = 12 \times 75 = 780$$

Illustration 9.21 If for a sequence (T_n), $S_n = 2n^2 + 3n + 1$ find T_n, and T_1 and T_2.

Solution: Given

$$S_n = 2n^2 + 3n + 1$$
$$S_{(n-1)} = 2(n-1)^2 + 3(n-1) + 1$$
$$= 2[n^2 - 2n + 1] + 3n - 2$$
$$= 2n^2 - n$$
$$T_n = S_n - S_{n-1} = 2n^2 + 3n + 1 - 2n^2 + n = 4n + 1$$

Hence, $T_1 = 6$ and $T_2 = 9$.

Illustration 9.22 If for a sequence (a_n), $S_n = 3\cdot(2^n - 1)$ find its first term.

Solution: Given

$$S_n = 3(2^n - 1)$$
$$S_{n-1} = 3(2^{n-1} - 1)$$

So

$$a_n = S_n - S_{n-1} = 3(2^n - 1) - 3(2^{n-1} - 1)$$
$$= 3(2^n - 2^{n-1}) = 3\cdot 2^{n-1}$$

Therefore $a_1 = 3$.

Your Turn 2

1. If the angles of a triangle are in AP and tangent of the smallest angle is 1 then find all the angles of the triangle.

   Ans. 45°, 60°, 75°

2. If $a_1, a_2, a_3, a_4, a_5, a_6$ are in AP, then prove that the system of equations $a_1x + a_2y = a_3$, $a_4x + a_5y = a_6$ is consistent.

3. Let S_n denote the sum up to n terms of an AP, if $S_n = n^2P$ and $S_m = m^2P$, where m, n and p are positive integers and $m \neq n$, then find S_p.

   Ans. p^3

4. Let $a_1, a_2, a_3, \ldots$ be an AP. Prove that

$$\sum_{n=1}^{2m} (-1)^{n-1} a_n^2 = \frac{m}{2m-1}\left(a_1^2 - a_{2m}^2\right)$$

9.2.2 Geometric Progression (GP)

A sequence of non-zero numbers is called a Geometrical Progression (GP) if the ratio of a term and the term preceding it is always a constant quantity. This constant ratio is called the common ratio of the GP.

Thus, if $t_1, t_2, t_3, \ldots$ are in the GP then the common ratio is

$$r = \frac{t_n}{t_{n-1}},\ n \in N$$

Therefore, $t_n = r \times t_{n-1}$.

It follows that, in a GP, $t_n = t_{n-1}\, r$, i.e. any term (except the first) is obtained by multiplying its preceding term by a fixed (non-zero) number r.

9.2.2.1 General Term of a GP

Let a be the first term and r ($\neq 0$) be the common ratio of a GP. Let $T_1, T_2, T_3, \ldots, T_n$ denote 1st, 2nd, 3rd, …, nth terms, respectively, then we have

$$T_2 = T_1 \cdot r$$
$$T_3 = T_2 \cdot r$$
$$T_4 = T_3 \cdot r$$
$$\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots$$
$$T_n = T_{n-1} \cdot r$$

On multiplying these, we get

$$T_2 \cdot T_3 \cdot T_4 \cdots T_n = T_1 \cdot T_2 \cdot T_3 \cdots T_{n-1} \cdot r^{n-1}$$

$$\Rightarrow\ T_n = T_1 \cdot r^{n-1} \text{ but } T_1 = a$$

Therefore, the general term $= T_n = a\,r^{n-1}$
If the nth term of this GP of last term is l, then

$$T_n = l = ar^{n-1}$$

If the nth term of this GP is from the end and if l is the last term, then

$$T_n' = \frac{l}{r^{n-1}} = l \times \left(\frac{1}{r}\right)^{n-1}$$

If T_n is the nth term from beginning and T_n' is the nth term from end then

$$T_n \times T_n' = a \times l$$

where l is the last term.

Illustration 9.23 a, b, c are three consecutive terms of an AP and x, y, z as three consecutive terms of a GP, then prove that $x^{b-c} \cdot y^{c-a} \cdot z^{a-b} = 1$.

Solution: Let r is the common ratio of a GP, then

$$y = x \times r,\ z = x \times r^2$$
$$x^{b-c} \times y^{c-a} \times z^{a-b} = x^{b-c}\,(x \times r)^{c-a}\,(x \times r^2)^{a-b}$$
$$= x^{b-c+c-a+a-b} \times r^{c-a+2a-2b} \quad [2b = a + c,\ a, b, c \text{ are in AP}]$$
$$= x^0 \times r^{c+a-2b} = x^0 \times r^{2b-2b} = 1$$

Illustration 9.24 The fourth, seventh and last terms of a GP are 10, 80 and 2560. Find a, r and n.

Solution: The last term of a GP is

$$T_n = ar^{n-1}$$

So

$$T_4 = 10 = a\,r^3 \quad (1)$$

$$T_7 = 80 = a\,r^6 \quad (2)$$
$$T_n = 2560 = a\,r^{n-1} \quad (3)$$

Divide Eq. (2) by Eq. (1). We get

$$\frac{80}{10} = \frac{ar^6}{ar^3} \Rightarrow r^3 = 8 \Rightarrow r = 2$$

Putting the value of r in Eq. (1), we get

$$a = \frac{10}{8}$$

Putting the value of a and r in Eq. (3), we get

$$2560 = \frac{10}{8} \times 2^{n-1} \Rightarrow n = 12$$

9.2.2.2 Sum of *n* Terms of a GP

Let a be the first term and r ($\neq 1$) be the common ratio of the given GP. If S_n denotes the sum of n terms, then

$$S_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} \quad (1)$$

Multiplying both sides of Eq. (1) by r, we get

$$r\,S_n = ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} + ar^n \quad (2)$$

Subtracting Eq. (2) from Eq. (1), we get

$$S_n(1 - r) = a - ar^n$$
$$\Rightarrow (1 - r)S_n = a(1 - r^n)$$
$$\Rightarrow S_n = a\left(\frac{1-r^n}{1-r}\right), r \neq 1 \quad (3)$$

This result can also be written as

$$S_n = a\left(\frac{r^n - 1}{r-1}\right) \quad (4)$$

In case if $r = 1$, then $S_n = a + a + a + \cdots$ to n terms $= na$

If $r < 1$

$$S_n = a\left(\frac{1-r^n}{1-r}\right)$$

If $r > 1$

$$S_n = a\left(\frac{r^n-1}{r-1}\right)$$

If l is the last term of the GP then $l = ar^{n-1}$

$$S_n = a\left(\frac{1-r^n}{1-r}\right) = \frac{a - ar^n}{1-r} = \frac{a-(ar^{n-1}r)}{(1-r)} \quad S_n = \frac{a-lr}{1-r}$$

Sum of infinite terms (S_∞) is

$$S_\infty = \frac{a}{1-r} \text{ (for } |r| < 1)$$

Note: When $|r| > 1$, the series is divergent and so its sum is not possible.

> If 'a' is the first term and 'r' is the common ratio of a GP it can be written as $a, ar, ar^2, \ldots$. The n^{th} term 'a_n' is given by $a_n = ar^{n-1}$. The sum S_n of the first n terms is
>
> $$S_n = \begin{cases} \dfrac{a(r^n - 1)}{r-1}, & r \neq 1 \\ na, & r = 1 \end{cases}$$
>
> If $-1 < r < 1$, then the sum of the infinite GP is
>
> $$a + ar + ar^2 + \cdots = \frac{a}{1-r}$$

Illustration 9.25 Find the sum of the given series 2, 6, 18, … up to 7 terms.

Solution: Given $a = 2$, $r = 3$. The sum of the given series is

$$S_7 = \frac{a(r^7-1)}{(r-1)} = \frac{2(3^7-1)}{(3-7)}$$
$$= \frac{2(3^7-1)}{-4} = -\frac{1}{2}(3^7-1)$$

Illustration 9.26 Find the sum of the series $(a^2 - b^2), (a-b), \left(\frac{a-b}{a+b}\right) \ldots$ upto n.

Solution: The first term is $A = a^2 - b^2$, and the common ratio is $r = \frac{1}{a+b}$. The sum of the series is given by

$$S_n = \frac{(a^2-b^2)\left[\left(\frac{1}{a+b}\right)^n - 1\right]}{\left[\frac{1}{a+b} - 1\right]}$$
$$= \frac{(a-b)(a+b)[(a+b)^{-n}-1]}{\frac{(1-a-b)}{(a+b)}}$$
$$= \frac{(a-b)(a+b)^2[(a+b)^{-n}-1]}{(1-a-b)}$$
$$= \frac{(a-b)[(a+b)^n - 1]}{(a+b)^{n-2}[(a+b)-1]}$$

Illustration 9.27 Find the sum of 10 terms of the GP

1, 1/2, 1, 1/4, 1/8, …

Solution: Given $a = 1$, $r = 1/4$. So, $S_{10} = \frac{1[1-(1/2)^{10}]}{1-1/2} = \frac{2[2^{10}-1]}{2^{10}}$ $= 2^{-9}[2^{10}-1] = \frac{1023}{512}$.

Illustration 9.28 Find the sum up to 7 terms of the sequence

$$\left(\frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3}\right), \left(\frac{1}{5^4} + \frac{2}{5^5} + \frac{3}{5^6}\right), \left(\frac{1}{5^7} + \frac{2}{5^8} + \frac{3}{5^9}\right), \ldots$$

Solution: The given sequence can be written as

$$\left(\frac{1}{5}+\frac{2}{5^2}+\frac{3}{5^3}\right),\ \frac{1}{5^3}\left(\frac{1}{5}+\frac{2}{5^2}+\frac{3}{5^3}\right),\ \frac{1}{5^3}\left(\frac{1}{5}+\frac{2}{5^2}+\frac{3}{6^3}\right)$$

This is a GP with

$$a=\frac{1}{5}+\frac{2}{5^2}+\frac{3}{5^3}=\frac{38}{125}$$

$$r=\frac{1}{5^3}$$

Hence the sum upto 7 terms is

$$S_7=a\left(\frac{1-r^n}{1-r}\right)=\frac{38}{125}\left(\frac{1-(1/5^3)^n}{1-(1/5^3)}\right)=\frac{19}{62}\left(1-\frac{1}{5^{21}}\right)$$

Illustration 9.29 Find the sum of the following infinite series:

(A) $1-\frac{1}{3}+\frac{1}{3^2}-\frac{1}{3^3}+\frac{1}{3^4}+\cdots\infty$ **(B)** $8+4\sqrt{2}+4+\cdots\infty$

Solution:

(A) $a=1$ and $r=-\frac{1}{3}$

$$S_\infty=\frac{1}{1+1/3}=3/4$$

(B) $a=8$ and $r=\frac{1}{\sqrt{2}}$

$$S_\infty=\frac{8}{1-\frac{1}{\sqrt{2}}}=\frac{8\sqrt{2}}{\sqrt{2}-1}=\frac{8\sqrt{2}(\sqrt{2}+1)}{1}=16+8\sqrt{2}$$

Illustration 9.30 The sum of first two terms of an infinite GP is 5 and each term is three times the sum of the succeeding terms. Find the GP.

Solution: Let the GP be $a, ar, ar^2, \ldots, \infty$. Given

$$a+ar=5 \quad (1)$$

As given in question $a_n=3[a_{n+1}+a_{n+2}+\cdots+\infty]$. So

$$ar^{n-1}=3\,[ar^n+ar^{n+1}+\cdots+\infty]$$
$$=3ar^n[1+r+r^2+\cdots+\infty]$$
$$\Rightarrow 1=3r\,\frac{[1]}{1-r}$$
$$\Rightarrow 1-r=3r$$
$$\Rightarrow 1=4r$$
$$\Rightarrow r=\frac{1}{4}$$

Now, putting in Eq. (1)

$$a+a/4=5\Rightarrow a=4\Rightarrow r=\frac{1}{4}, a=4$$

Illustration 9.31 The fifth term of a GP is $\frac{1}{3}$ and the ninth term is $\frac{16}{243}$. Find the fourth term. Also, find the sum of first 10 terms of the GP.

Solution: Let b is the first term and r be the common ratio. Then

$$T_5=br^4=\frac{1}{3} \quad (1)$$

$$T_9=br^8=\frac{16}{243} \quad (2)$$

Dividing Eq. (2) by Eq. (1) we get $r=\frac{2}{3}$.

Substitute for r in Eq. (1) to get $b=\frac{27}{16}$. So we have

$$T_4=br^3=\frac{27}{16}\left(\frac{2}{3}\right)^3=\frac{1}{2}$$

Now

$$S_n=\frac{b(1-r^n)}{1-r}$$

Hence

$$S_{10}=\frac{\frac{27}{16}\left[1-\left(\frac{2}{3}\right)^{10}\right]}{1-\frac{2}{3}}=\frac{81}{16}\left(\frac{3^{10}-2^{10}}{3^{10}}\right)$$

Illustration 9.32 How many terms of the series $\sqrt{3}, 3, 3\sqrt{3}, \ldots$ amount to $39+13\sqrt{3}$?

Solution: Here the first term (b) is $\sqrt{3}$ and the common ratio (r) is $\sqrt{3}$. So

$$S_n=\frac{b(1-r^n)}{1-r}$$
$$\Rightarrow 39+13\sqrt{3}=\frac{\sqrt{3}\left[1-(\sqrt{3})^n\right]}{1-\sqrt{3}}$$
$$\Rightarrow \frac{(39+13\sqrt{3})(1-\sqrt{3})}{\sqrt{3}}=1-(\sqrt{3})^n$$
$$\Rightarrow (\sqrt{3})^n-1=26$$
$$\Rightarrow n=6$$

Illustration 9.33 The sum of infinite numbers of terms of a GP is 15 and the sum of their squares is 45. Find the series.

Solution: Let the first term of infinite series be b and the common ratio be r. Now for the series with squares of each term, the first term will be b^2 and the common ratio will be r^2.

$$S_\infty=\frac{b}{1-r}$$

$$\Rightarrow \frac{b}{1-r}=15 \quad (1)$$

and

$$\frac{b^2}{1-r^2}=45 \quad (2)$$

On dividing Eq. (2) by Eq. (1), we get

$$\frac{b}{1+r}=3 \quad (3)$$

From Eqs. (1) and (3), we get

$$\frac{1+r}{1-r}=5 \Rightarrow r=\frac{2}{3}$$

and so $b=5$

Hence, the series is $5, \frac{10}{3}, \frac{20}{9}, \ldots$.

Your Turn 3

1. Find the sum of p terms of a GP, whose pth term is 2^p.

Ans. $(2^{p+1}-2)$

2. If f is a function satisfying $f(x+y)=f(x)\,f(y)$ for all $x, y \in N$ such that $f(1)=3$ and $\sum_{x=1}^{n} f(x)=120$, find the value of n.

Ans. 4

3. Find the sum of geometric series $(x+y)+(x^2+xy+y^2)+(x^3+x^2y+xy^2+y^2)+\ldots$ to n terms.

Ans. $\frac{1}{x-y}\left[x^2\left(\frac{x^n-1}{x-1}\right)-y^2\left(\frac{y^n-1}{y-1}\right)\right]$

4. Find the sum of an infinitely decreasing GP whose first term is equal to $b+2$ and the common ratio is equal to $2/c$, where b is the least value of the product of the roots of the equation $(m^2+1)x^2-3x\,(m^2+1)^2=0$ and c is the greatest value of the sum of its roots.

Ans. 9

5. If $b=a+a^2+a^3+\ldots\infty$, prove that $a=\frac{b}{1+b}$.

6. If $x=\sum_{n=0}^{\infty}\cos^{2n}\theta$, $y=\sum_{n=0}^{\infty}\sin^{2n}\phi$, $z=\sum_{n=0}^{\infty}\cos^{2n}\phi$, where $0<\theta$, $\phi<\frac{\pi}{2}$ then prove that $xz+yz-z=xy$.

7. If $|x|<1$ and $|y|<1$, find the sum to infinity of the series $(x+y)+(x^2+xy+y^2)+(x^3+x^2y+xy^2+y^3)+\cdots$

Ans. $\frac{x+y-xy}{(1-x)(1-y)}$

8. If each term of an infinite GP is twice the sum of the terms following it then find the common ratio of the GP.

Ans. $\frac{1}{3}$

9.2.2.3 Assuming Quantity of GP

1. For three terms $\frac{a}{r}, a, ar$

2. For four terms $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$

3. For five terms $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$

Note: In general, if we take $(2k+1)$ terms in GP, we take them as

$$\frac{a}{r^k}, \frac{a}{r^{k-1}}, \ldots, \frac{a}{r}, a, ar, \ldots, ar^k$$

And if we have to take $2k$ terms in GP, we take them as

$$\frac{a}{r^{2k-1}}, \frac{a}{r^{2k-1}}, \ldots, ar^{2k-3}, ar^{2k-1}$$

Illustration 9.34 If a, b, c, d are in GP, show that

(A) $(a-d)^2=(b-c)^2+(c-a)^2+(d-b)^2$
(B) a^2-b^2, b^2-c^2 and c^2-d^2 are also in GP

Solution:

(A) a, b, c, d are in GP. Then

$$b^2=ac, c^2=bd, bc=ad \quad (1)$$

Now expanding the RHS, we get

$$\begin{aligned}\text{RHS} &= (b^2+c^2-2bc)+(c^2+a^2-2ac)+(d^2+b^2-2bd)\\ &= 2(b^2-ac)+2(c^2-bd)+a^2+d^2-2bc\\ &= 2(0)+2(0)+a^2+d^2-2ad \quad \text{[from Eq. (1)]}\\ &= (a-d)^2=\text{LHS}\end{aligned}$$

(B) Now we have to prove that a^2-b^2, b^2-c^2 and c^2-d^2 are in GP, that is, we have to show

$$(b^2-c^2)^2=(a^2-b^2)\,(c^2-d^2)$$

Consider the RHS:

$$\begin{aligned}(a^2-b^2)\,(c^2-d^2) &= a^2c^2-b^2c^2-a^2d^2+b^2d^2\\ &= b^4-b^2c^2-b^2c^2+c^4\\ &= (b^2-c^2)^2=\text{LHS}\end{aligned}$$

Properties of GP

- If each term of a GP is multiplied (divided) by a fixed non-zero constant, then the resulting sequence is also a GP with the same common ratio as that of the given GP.
- If each term of a GP (with common ratio r) is raised to the power k, then the resulting sequence is also a GP with common ratio r^k.
- If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are two GPs with common ratios r and r', respectively, then the sequence $a_1b_1, a_2b_2, a_3b_3, \ldots$ is also a GP with common ratios r and r'.
- If $a_1, a_2, \ldots, a_n$ are in GP, then $a_1\,a_n=a_2\,a_{n-1}=a_3\,a_{n-2}=\ldots$
- If three terms are in GP, then the middle term is called the geometric mean (GM) between the two. So, if a, b, c are in GP then $b=\sqrt{ac}$ is the geometric mean of a and c.
- If $a_1, a_2, \ldots, a_n$ are non-zero positive numbers then their GM (G) is given by $G=(a_1, a_2, a_3, \ldots, a_n)^{1/n}$.
- If $G_1, G_2, \ldots, G_n$ are n geometric means between a and b, then $G_1\,G_2, \ldots, G_n=(\sqrt{ab})^n$.

Illustration 9.35 Find three numbers in GP whose sum is 65 and whose product is 3375.

Solution: Let the numbers be *a/r, a, ar*. Then

$$\frac{a}{r}+a+ar=65 \quad (1)$$

and

$$\frac{a}{r}\times a\times ar=3375 \quad (2)$$

$$\Rightarrow a^3=3375$$

$$\Rightarrow a=15$$

Now from Eq. (1)

$$a\left[\frac{1}{r}+1+r\right]=65$$

$$\Rightarrow 15\left[\frac{1}{r}+1+r\right]=65$$

$$\Rightarrow 3\,[1+r+r^2]=13r$$

$$\Rightarrow 3+3r+3r^2-13r=0$$

$$\Rightarrow 3r^2-10r+3=0$$

$$\Rightarrow r=3, \frac{1}{3}$$

Illustration 9.36 The product of three numbers in GP is 216. If 2, 8, 6 are added then the result converts to AP. Find the numbers.

Solution: Let the numbers *a/r, a, ar*. Then

$$\frac{a}{r}\times a\times ar=216\Rightarrow a^3=216\Rightarrow a=6$$

So the numbers become

$$\frac{6}{r}, 6, 6r$$

According to the given condition the numbers become

$$\frac{6}{r}+2, 6+8, 6r+6$$

Since they are in AP, we have

$$\Rightarrow 14=\frac{\frac{6}{r}+2+6r+6}{2}$$

$$\Rightarrow 14=\frac{6\left(\frac{1}{r}+r\right)+8}{2}$$

$$\Rightarrow 14=3\left(\frac{1}{r}+r\right)+4$$

$$\Rightarrow 14=\frac{3(1+r^2)+4r}{r}$$

$$\Rightarrow 14r=3+3r^2+4r$$

$$\Rightarrow 3r^2-10r+3=0$$

$$\Rightarrow r=3, 1/3$$

Your Turn 4

1. If the p^{th}, q^{th}, r^{th} terms of an AP are in GP, then find the common ratio of the GP.

   Ans. $\frac{r-q}{q-p}$

2. A GP consists of $2n$ terms. If the sum of the terms occupying the odd places is S_1 and that of the terms in the even places is S_2, then find the common ratio of the progression.

   Ans. S_2/S_1

3. If *a*, *b*, *c* and *d* are in GP, then show that ax^2+c divides ax^3+bx^2+cx+d.

4. If G_1 and G_2 are two geometric means, and A_1 is the arithmetic mean between two positive numbers then show that

$$\frac{G_1^2}{G_2}+\frac{G_2^2}{G_1}=2A_1$$

5. Show that $\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix}=0$, if *a*, *b*, *c* are in GP.

9.2.3 Harmonic Progression (HP)

A sequence is said to be in harmonic progression if and only if the reciprocal of its terms form an arithmetic progression.

For example, $\frac{1}{2},\frac{1}{4},\frac{1}{6},\ldots$ forms an HP because 2, 4, 6, … are in AP.

If *a*, *b*, *c* are in an HP, then $\frac{1}{a},\frac{1}{b},\frac{1}{c}$ forms an AP.

> The n^{th} term a_n of the HP is $a_n=\frac{1}{a+(n-1)d}$, where $a=\frac{1}{a_1}$ and $d=\frac{1}{a_2}-\frac{1}{a_1}$.

9.2.3.1 General Term of HP

If *a* and *b* are first two terms of an HP then

$$t_n=\frac{1}{\frac{1}{a}+(n-1)\left(\frac{1}{b}-\frac{1}{a}\right)}$$

Illustration 9.37 In an HP the p^{th} term is qr and the q^{th} term is rp. Show that the r^{th} term is pq.

Solution: Let *A* and *D* be the first term and the common difference of the AP formed by the reciprocals of given HP.

The p^{th} term of AP is $\frac{1}{qr}$ and the q^{th} term of AP is $\frac{1}{rp}$. So

$$\frac{1}{qr}=A+(p-1)D \quad (1)$$

and

$$\frac{1}{rp}=A+(q-1)D \quad (2)$$

We will solve these two equations to get A and D.

Subtracting Eq. (2) from Eq. (1), we get

$$\frac{p-q}{pqr} = (p-q)D \Rightarrow D = \frac{1}{pqr}$$

Hence,

$$\frac{1}{qr} = A + \frac{p-1}{pqr} \Rightarrow A = \frac{1}{pqr}$$

Now the r^{th} term of AP $= T_r = A + (r-1)D$. So

$$T_r = \frac{1}{pqr} + \frac{r-1}{pqr} = \frac{1}{pq}$$

Hence, the r^{th} term of the given HP is pq.

Properties of HP

- If a and b are two non-zero numbers, then the harmonic mean of a and b is a number H such that the numbers a, H, b are in HP. We have $H = \frac{2ab}{a+b}$.
- If $a_1, a_2, \ldots, a_n$ are 'n' non-zero numbers, then the harmonic mean H of these numbers is given by

$$\frac{1}{H} = \frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)$$

Illustration 9.38 The sum of three numbers in HP is 26 and the sum of their reciprocals is $\frac{3}{8}$. Find the numbers.

Solution: Three numbers in HP are taken as

$$\frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}$$

By the given condition, we have

$$\frac{1}{a-d} + \frac{1}{a} + \frac{1}{a+d} = 26 \quad (1)$$

$$(a-d) + a + (a+d) = \frac{3}{8} \quad (2)$$

From Eqs. (1) and (2)

$$a = \frac{1}{8} \text{ and } d = \pm\frac{1}{24}$$

Hence, the numbers are 12, 8, 6 or 6, 8, 12.

Your Turn 5

1. If H is the harmonic mean of a and b then find the value of $\frac{H}{a} + \frac{H}{b} - 2$. **Ans.** 0
2. If $\frac{bc}{ad} = \frac{b+c}{a+d} = \frac{3(b-c)}{(a-d)}$, then show that a, b, c and d are in HP.
3. Show that if $a(b-c)x^2 + b(c-a)xy + c(a-b)y^2$ is a perfect square, the quantities a, b, c are in harmonical progression.
4. If a, b, c are in AP and a^2, b^2, c^2 are in HP, then prove that $-a/2, b, c$ are in GP or else $a = b = c$.

9.3 Different Means of Two Numbers

1. Arithmetic mean (AM) of any two numbers a and b:

$$A = \frac{a+b}{2}$$

Arithmetic mean of n numbers

$$a_1, a_2, \ldots, a_n \text{ is } A = \frac{1}{n}(a_1 + a_2 + \cdots + a_n)$$

2. Geometric mean (GM):
   - **(a)** $G = \sqrt{ab}$ is the geometric mean of two positive numbers a and b.
   - **(b)** $G = (a_1, a_2, \ldots, a_n)^{1/n}$ is the geometric mean of n positive numbers $a_1, a_2, a_3, \ldots, a_n$.
3. Harmonic mean (H) of any two non-zero numbers a and b:

$$H = \frac{2}{\frac{1}{a} + \frac{1}{b}} = \frac{2ab}{a+b}$$

Harmonic mean of n non-zero numbers $a_1, a_2, a_3, \ldots, a_n$.

$$H = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} = \frac{n}{\sum_{j=1}^{n} \frac{1}{a_j}}$$

9.4 Relation between AM, GM and HM

If A, G and H are AM, GM and HM of positive numbers $a_1, a_2, \ldots, a_n$ ($a_1 \le a_2 \le \ldots \le a_n$) then

$$a_1 \le H \le G \le A \le a_n \quad (1)$$

1. The equality at any place in Eq. (1) holds if and only if the numbers $a_1, a_2, \ldots, a_n$ are equal.
2. Step (1) is true for weighted means also.
3. $G^2 = AH$, if $n = 2$ only.

Illustration 9.39 If A and G are arithmetic mean (AM) and geometric mean (GM), respectively, between two numbers a and b, find the roots of the equation: $x^2 - 2Ax + G^2 = 0$.

Solution: Let α and β be the roots of the given equation. Then

$$\alpha + \beta = 2A \text{ and } \alpha\beta = G^2$$

Also, A is the AM between a and b and G is GM between a and b. So

$$A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}$$

$$\alpha + \beta = a + b \text{ and } \alpha\beta = ab$$

So, the roots are a and b.

Illustration 9.40 If a, b, c are in AP, x is the GM of a, b and y is GM of b, c, show that b^2 is the AM of x^2 and y^2.

Solution:

$$a, b, c \text{ are in AP} \Rightarrow 2b = a + c \quad (1)$$

$$x \text{ is GM of } a, b \Rightarrow x = \sqrt{ab} \quad (2)$$

$$y \text{ is GM of } b, c \Rightarrow y = \sqrt{bc} \quad (3)$$

Squaring Eqs. (2) and (3) and adding, we get

$$x^2 + y^2 = ab + bc = b(a + c)$$

From Eq. (1), we get $a + c = 2b$. So

$$x^2 + y^2 = 2b^2 \Rightarrow b^2 = \frac{x^2 + y^2}{2}$$

Hence, b^2 is arithmetic mean (AM) of x^2 and y^2.

Illustration 9.41 If one GM, G, and two AMs, p and q, be inserted between two quantities, show that $G^2 = (2p - q)(2q - p)$.

Solution: Let a and b be two quantities. Then $G^2 = ab$ and a, p, q, b are in AP. Hence

$$p = a + \frac{(b-a)}{3} = \frac{b+2a}{3}$$
$$q = a + 2\frac{b-a}{3} = \frac{2b+a}{3}$$

Now

$$\text{RHS} = (2p - q)(2q - p)$$
$$= \left(\frac{2}{3}(b+2a) - \frac{2ab+a}{3}\right)\left(\frac{2(2b+a)}{3} - \frac{b+2a}{3}\right)$$
$$= \frac{1}{9}(2b+4a-2b-a)(4b+2a-b-2a)$$
$$= \frac{1}{9}(3a)(3b) = ab = G^2 = \text{LHS}$$

Your Turn 6

1. The p^{th}, q^{th} and r^{th} terms of an HP are x, y and z, respectively. Show that $\frac{q-r}{x} + \frac{r-p}{y} + \frac{p-q}{z} = 0$.
2. Show that product of n GMs between two numbers a and b is $(ab)^{n/2}$.
3. The HM between two numbers is 4, their AM A and GM G satisfy the relation $2A + G^2 = 27$. Find the two numbers.

   Ans. 3 and 6 or 6 and 3
4. If a, b, c are in HP, show that $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are also in AP.

9.5 Insertion of Means between Two Numbers

Let a and b be two given numbers.

1. **Arithmetic means:** If $a, A_1, A_2, \ldots, A_n, b$ are in AP, then $A_1, A_2, \ldots, A_n$ are n AMs between a and b. If d is the common difference, then

$$b = a + (n + 2 - 1)d \Rightarrow d = \frac{b-a}{n+1}$$

$$A_i = a + id = a + i\cdot\frac{b-a}{n+1} = \frac{a(n-i+1)+ib}{n+1}, i = 1, 2, 3, \ldots, n$$

Note: The sum of n AMs,

$$A_1 + A_2 + \cdots + A_n = \frac{n}{2}(a+b)$$

2. **Geometric means:** If $a, G_1, G_2, \ldots, G_n, b$ are in GP, then $G_1, G_2, \ldots, G_n$ are n GMs between a and b. If r is the common ratio, then

$$b = ar^{n+1} \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{(n+1)}}$$

$$G_i = ar^i = a\left(\frac{b}{a}\right)^{\frac{i}{n+1}} = a^{\frac{n+1-i}{n+1}} \cdot b^{\frac{i}{n+1}}, i = 1, 2, \ldots, n$$

Note: The product of n GMs $G_1 G_2 \cdots G_n = (\sqrt{ab})^n$.

3. **Harmonic means:** If $a, H_1, H_2, \cdots, H_n, b$ are in HP, then $H_1, H_2, \cdots, H_n$ are the n HMs between a and b. If d is the common difference of the corresponding AP then

$$\frac{1}{b} = \frac{1}{a} + (n + 2 - 1)\, d \Rightarrow d = \frac{a-b}{ab(n+1)}$$

$$\frac{1}{H_i} = \frac{1}{a} + id = \frac{1}{a} + i\frac{a-b}{ab(n+1)} = \frac{ab(n+1)}{b(n-i+1)+ia}, i = 1, 2, 3, \ldots, n$$

9.6 Weighted Means of Numbers

Let $a_1, a_2, \ldots, a_n$ be n given numbers. If weights of $a_1, a_2, \ldots, a_n$ are $w_1, w_2, \ldots, w_n$, respectively, then their weighted arithmetic mean, weighted geometric mean, and weighted harmonic mean are, respectively, defined by

$$\frac{a_1w_1 + a_2w_2 + \cdots + a_nw_n}{w_1 + w_2 + \cdots + w_n},\ \left(a_1^{w_1} \cdot a_2^{w_2} \cdots a_n^{w_n}\right)^{\frac{1}{w_1+w_2+\cdots+w_n}}$$

and

$$\frac{w_1 + w_2 + \cdots + w_n}{\frac{w_1}{a_1} + \frac{w_2}{a_2} + \cdots + \frac{w_n}{a_n}}$$

9.7 Arithmetico-Geometric Series

The series whose each term is formed by multiplying the corresponding terms of an AP and a GP is called an Arithmetico-geometric series.

For example:

- $1 + 2x + 4x^2 + 6x^3 + \cdots$
- $a + (a + d)r + (a + 2d)r^2 + \cdots$

1. **Summation of n terms of an arithmetico-geometric series:**

   Let $S_n = a + (a + d)r + (a + 2d)r^2 + \cdots + [a + (n - 1)d]\, r^{n-1}, d \neq 0, r \neq 1$.

   Multiply by 'r' and rewrite the series in the following way:

   $rS_n = ar + (a + d)r^2 + (a + 2d)r^3 + \cdots + [a + (n - 2)d]r^{n-1} + [a + (n - 1)d]r^n$ on subtraction,

$$S_n(1 - r) = a + d(r + r^2 + \cdots + r^{n-1}) - [a + (n - 1)d]r^n$$

or, $$S_n(1 - r) = a + \frac{dr(1-r^{n-1})}{1-r} - [a+(n-1)d]\cdot r^n$$

or, $$S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]}{1-r}\cdot r^n$$

2. Summation of infinite terms series:
If $|r| < 1$, then $(n-1)r^n, r^{n-1} \to 0$, as $n \to \infty$. Thus

$$S_\infty = S = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$$

9.8 Sum of Miscellaneous Series

1. **Difference method:** Let $T_1, T_2, T_3, \ldots, T_n$ be the terms of a sequence and let
$(T_2 - T_1) = T_1'; (T_3 - T_2) = T_2', \ldots, (T_n - T_{n-1}) = T_{n-1}'$
   - If $T_1', T_2', \ldots, T_{n-1}'$ are in AP, then T_n will be of the form $an^2 + bn + c$; $a, b, c \in R$.
   - Again if $T_1', T_2', T_3', \ldots$ are not in AP, but $T_1' - T_2', T_2' - T_3', \ldots$ are in AP, then T_n is of the form $an^3 + bn^2 + cn + d$; $a, b, c, d \in R$.
   - If $T_1', T_2', \ldots, T_{n-1}'$ are in GP, then $T_n = ar^n + b$, r is the CR of the GP $T_1', T_2', T_3' \ldots$ and $a, b \in R$.
   - Again if $T_1', T_2', T_3', \ldots$ are not in GP but $T_2' - T_1', T_3' - T_2', \ldots, T_{n-1}' - T_{n-2}'$ are in GP, then T_n is of the form $ar^n + bn + c$; r is the CR of the GP $T_2' - T_1', T_3' - T_2', T_4' - T_3', \ldots$ and $a, b, c \in R$.

2. **$V_n - V_{n-1}$ method:** Let $T_1, T_2, T_3, \ldots$ be the terms of a sequence and $T_k = V_k - V_{k-1}$, for some positive integer k. Then

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (V_k - V_{k-1}) = V_n - V_0$$

Illustration 9.42 Find the n^{th} term and the sum to n terms of the series $1 + 6 + 23 + 58 + 117 + 206 + \cdots$.

Solution:

The terms of the series	1	6	23	58	117	206	…
	u_1	u_2	u_3	u_4	u_5	u_6	…

The 1st order of differences	$(u_2 - u_1)$	$(u_3 - u_2)$	$(u_4 - u_3)$	$(u_5 - u_4)$	$(u_6 - u_5)$	…
	5	17	35	59	89	…
	v_2	v_2	v_3	v_4	v_5	

The 2nd order of differences	$(v_2 - v_1)$	$(v_3 - v_2)$	$(v_4 - v_3)$	$(v_5 - v_4)$
	12	18	24	30

The 3rd order of differences	12	18	24	30
	6	6	6	

It may be noted that the terms of the successive order of difference series are obtained from the immediately preceding series by taking the difference of two consecutive terms.

If at any stage, in finding the successive order of difference series terms, all the terms reduce to the same number (as in this problem, this happens at the 3rd order of difference), then the n^{th} term u_n of the given series is a polynomial in n of degree equal to that order of difference series whose terms are all the same. Thus, in this problem, u_n is of degree 3 and hence u_n can be taken as either $an^3 + bn + cn + d$ or conveniently as $u_n = An(n-1)(n-2) + Bn(n-1) + Cn + D$.

The constants A, B, C and D are determined as follows:

Put $n = 1$, $u_1 = 1$ and $u_1 = C + D$	Therefore, $C + D = 1$
Put $n = 2$, $u_2 = 6$ and $u_2 = 2B + 2C + D$	Therefore, $2B + 2C + D = 6$
Put $n = 3$, $u_3 = 23$ and $u_3 = 6A + 6B + 3C + D$	Therefore, $6A + 6B + 3C + D = 23$
Put $n = 4$, $u_4 = 58$ and $u_4 = 24A + 12B + 4C + D$	Therefore, $24A + 12B + 4C + D = 58$

Solving for A, B, C and D we get $A = 1, B = 3, C = -1$ and $D = 2$. Hence,

$$u_n = n(n-1)(n-2) + 3n(n-1) - n + 2 = n^3 - 2n + 2$$

Therefore,

$$\sum_{n=1}^{n} u_n = \sum n^3 - 2\sum n + 2\sum 1 = \frac{n^2(n+1)^2}{4} - \frac{2n(n+1)}{2} + 2n$$

$$= \frac{n}{4}\{n^3 + 2n^2 - 3n + 4\}$$

Illustration 9.43 Find the sum of n terms of the series $1 + 4 + 11 + 26 + 57 + 120 + \cdots$.

Solution:

	1	4	11	26	57	120
First order of differences	3	7	15	31	63	
Second order of differences	4	8	16	32		

The 2nd order of difference is a GP of common ratio 2. In this case, $u_n = A$ (common ratio)^n + (first degree polynomial in n since the 2nd order difference is a GP)

$$u_n = A \cdot 2^{n-1} + Bn + C$$

Obtain A, B and C using u_1, u_2 and u_3. Then

$$S_n = \sum_{n=1}^{n} u_n = A(20 + 21 + 22 + \cdots + 2^{n-1}) + B\Sigma n + C\Sigma(1)$$

$$= A(2^n - 1) + \frac{Bn(n+1)}{2} + Cn$$

Illustration 9.44 Find the sum of the n terms of the series: $3 + 7 + 13 + 21 + 31 + \cdots$.

Solution: The given series is neither an AP nor a GP, but the difference of the successive terms are in AP

Series: 3 7 13 21 31 …
Difference: 4 6 8 10 …

In such cases, we find the n^{th} term as follows: Let S be the sum of the first n terms. Then

$$S = 3 + 7 + 13 + 21 + 31 + \cdots + T_n$$

or

$$S = 3 + 7 + 13 + 21 + 31 + \cdots + T_{n-1} + T_n$$

On subtracting, we get

$$0 = 3 + (4 + 6 + 8 + 10 + \cdots\} - T_n$$

$$\Rightarrow T_n = 3 + \{4 + 6 + 8 + 10 + \cdots + (n-1) \text{ terms}\}$$

$$\Rightarrow T_n = 3 + \frac{n-1}{2}[2(4) + (n-2)2]$$

$$\Rightarrow T_n = n^2 + n + 1$$

Now

$$S = \sum_{k=1}^{n} T_k = \sum k^2 + \sum k + \sum 1$$

$$\Rightarrow S=\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}+n=\frac{n}{3}(n^2+3n+5)$$

Illustration 9.45 Find the sum of the series: $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\dots$ to n terms.

Solution: Let

$$S=\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\dots+\frac{1}{(3n-2)(3n+1)}$$

$$\Rightarrow 3S=\frac{3}{1\cdot 4}+\frac{3}{4\cdot 7}+\frac{3}{7\cdot 10}+\dots+\frac{3}{(3n-2)(3n+1)}$$

$$\Rightarrow 3S=\frac{4-1}{1\cdot 4}+\frac{7-4}{4\cdot 7}+\frac{10-7}{7\cdot 10}+\dots+\frac{(3n+1)-(3n-2)}{(3n-2)\times(3n+1)}$$

$$\Rightarrow 3S=\left(\frac{1}{1}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{10}\right)+\dots+\left(\frac{1}{(3n-2)}-\frac{1}{3n+1}\right)$$

$$\Rightarrow S=\frac{1}{1}-\frac{1}{3n+1}\quad=\frac{n}{3n+1}$$

Your Turn 7

1. Find the sum of n terms of the series $\frac{1}{3}+\frac{3}{3\cdot 7}+\frac{5}{3\cdot 7\cdot 11}+\frac{7}{3\cdot 7\cdot 11\cdot 15}+\cdots$

   Ans. $\frac{1}{2}-\frac{1}{2\cdot 3\cdot 7\cdot 11\cdots(4n-1)}$

2. Show that $\sum_{r=0}^{n}\frac{2^r}{x^{2^r}+1}=\frac{1}{x-1}-\frac{2^{n+1}}{x^{2^{n+1}}-1}$.

3. Show that the value of $1+(1+2)x+(1+2+3)x^2+\dots$ to n terms is $\frac{1-x^n}{(1-x)^3}=\frac{n(n+3)}{2(1-x)^2}x^n+\frac{n(n+1)}{2(1-x)^2}x^{n+1}$.

Illustration 9.46 Sum the series to n terms: $4+44+444+4444+\cdots$.

Solution: Let

$$S_n=4+44+444+4444+\cdots \text{ to } n \text{ terms}$$
$$=4(1+11+111+1111+\cdots n \text{ terms})$$
$$=4/9\,\{10-1)+100-1)\,(1000-1)+\cdots n \text{ terms}\}$$
$$=4/9\,\{(10+10^2+10^3+\cdots n \text{ terms})-(1+1+1+\cdots n \text{ terms})\}$$
$$=\frac{4}{9}\left(\frac{10(10^n-1)}{10-1}-n\right)$$
$$=\frac{4}{81}[10(10^n-1)-9n]$$

Illustration 9.47 The series of natural numbers is divided into groups: (1); (2, 3, 4); (5, 6, 7, 8, 9) and so on. Show that the sum of the numbers in the n^{th} group is $(n-1)^3+n^3$.

Solution: Note that the last term of each group is the square of a natural number. Hence, the first term in the n^{th} group is $=(n-1)^2+1=n^2-2n+2$.

There is 1 term in 1st group, 3 in 2nd, 5 in 3rd, 7 in 4th, $\cdots$.

The number of terms in the n^{th} group $=n^{th}$ term of $(1,3,5,7,\dots)=2n-1$

Common difference in the n^{th} group $=1$

$$\text{Sum}=\frac{2n-1}{2}[2(n^2-2n+2)+(2n-2)1]$$
$$=\frac{2n-1}{2}[2n^2-2n+2]=(2n-1)(n^2-n+1)$$
$$=2n^3-3n^2+3n-1=n^3+(n-1)^3$$

Your Turn 8

1. Show that $\frac{a_1}{a_2}+\frac{a_2}{a_3}+\dots+\frac{a_{n-1}}{a_n}+\frac{a_n}{a1}>n$, where $a_1, a_2, \dots, a_n$ are different positive integers.
2. Show that $b^2c^2+c^2a^2+a^2b^2>abc\,(a+b+c)$, where a, b, c are different positive integers.
3. If a, b, c are unequal and positive, show that $\frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b}<\frac{1}{2}(a+b+c)$.
4. Find the sum of the series $0.9+0.99+0.999+0.9999+\dots$ up to n terms.

   Ans. $n-\frac{1}{9}\left(1-\frac{1}{10^n}\right)$

Sum of n terms $S_n=ab+(a+d)br+(a+2d)br^2+\cdots+(a+(n-2)d)\,br^{n-2}+(a+(n-1)d)br^{n-1}$ is

$$S_n=\frac{ab}{1-r}+\frac{dbr(1-r^{n-1})}{(1-r)^2}-\frac{(a+(n-1)d)br^n}{1-r}$$

and $$\lim_{n\to\infty}S_n=S=\frac{ab}{1-r}+\frac{dbr}{(1-r)^2}$$

9.9 Sum of First *n* Natural Numbers

Sum of first n natural numbers: $1+2+3+\cdots+n=\frac{n(n+1)}{2}$

Sum of the squares of first n natural numbers: $1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$

Sum of the cubes of first n natural numbers: $1^3+2^3+3^3+\cdots+n^3=\left\{\frac{n(n+1)}{2}\right\}^2$

Illustration 9.48 Find the sum of n terms of the series $1^2+3^2+5^2+\dots$ to n terms.

Solution: Let T_n be the n^{th} term of this series. Then

$$T_n=[1+(n-1)\times 2]^2=(2N-1)^2=4n^2-4n+1$$

Therefore,

$$1^2+3^2+5^2+\cdots \text{ to } n \text{ terms }=\sum_{k=1}^{n}T_k$$

$$= \sum_{k=1}^{n} (4k^2 - 4k + 1)$$

$$= 4\left(\sum_{k=1}^{n} k^2\right) - 4\left(\sum_{k=1}^{n} k\right) + \sum_{k=1}^{n} 1$$

$$= 4\frac{n(n+1)(2n+1)}{6} - 4\left\{\frac{n(n+1)}{2}\right\} + n$$

$$= \frac{n}{3}\{2(n+1)(2n+1) - 6(n+1) + 3\}$$

$$= \frac{n}{3}\{4n^2 + 6n + 2 - 6n - 6 + 3\} = \frac{n}{3}(4n^2 - 1)$$

Illustration 9.49 Find the sum of the series $2^2 + 4^2 + 6^2 + \cdots + (2n)^2$.

Solution: Let T_n be the n^{th} term of this series. Then

$$T_n = (2n)^2 = 4n^2$$

Therefore,

$$2^2 + 4^2 + 6^2 + \cdots + (2n)^2 = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} 4k^2$$

$$= 4\sum_{k=1}^{n} k^2 = 4\left\{\frac{n(n+1)(2n+1)}{6}\right\} = \frac{2}{3}n(n+1)(2n+1)$$

Illustration 9.50 Find the sum of n terms of the series $1\cdot 2^2 + 2\cdot 3^2 + 3\cdot 4^2 + \cdots$.

Solution: Let T_n be the n^{th} term of the given series. Then

T_n = (n^{th} term of the sequence formed by first digits in each term) × (n^{th} term of the sequence formed by second digits in each term)

$\Rightarrow T_n$ = (n^{th} term of 1, 2, 3, …) × (n^{th} term of $2^2, 3^2, 4^2, \ldots$)

$\Rightarrow T_n = n(n+1)^2 = n^3 + 2n^2 + n$

Let S_n denote the sum to n terms of the given series. Then,

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (k^3 + 2k^2 + k)$$

$$= \sum_{k=1}^{n} k^3 + 2\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k$$

$$= \left\{\frac{n(n+1)}{2}\right\}^2 + 2\left\{\frac{n(n+1)(2n+1)}{6}\right\} + \left\{\frac{n(n+1)}{2}\right\}$$

$$= \frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} + \frac{2}{3}(2n+1) + 1\right\}$$

$$= \frac{n(n+1)}{2}\frac{\{3n^2 + 3n + 8n + 4 + 6\}}{6}$$

$$= \frac{n(n+1)}{2}\left\{\frac{3n^2 + 11n + 10}{6}\right\} = \frac{n(n+1)(n+2)(3n+5)}{12}$$

Illustration 9.51 Sum the series $3\cdot 8 + 6\cdot 11 + 9\cdot 14 + \cdots$ to n terms.

Solution: Let T_n be the n^{th} term of the given series. Then

T_n = (n^{th} term of 3, 6, 9, …) × (n^{th} term of 8, 11, 14, …)
$= [3 + (n-1) \times 3] \times [8 + (n-1) \times 3] = 3n(3n+5) = 9n^2 + 15n$

Let S_n denote the sum to n terms of the given series. Then

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (9k^2 + 15k) = 9\sum_{k=1}^{n} k^2 + 15\sum_{k=1}^{n} k$$

$$= 9\left\{\frac{n(n+1)(2n+1)}{6}\right\} + 15\left\{\frac{n(n+1)}{2}\right\}$$

$$= \frac{3}{2}n(n+1)[2n+1+5]$$

$$= 3n(n+1)(n+3)$$

Illustration 9.52 Find the sum of the following series to n terms:

$$\frac{1^3}{1} + \frac{1^3 + 2^3}{1+3} + \frac{1^3 + 2^3 + 3^3}{1+3+5} + \ldots$$

Solution: Let T_n be the n^{th} term of the given series. Then

$$T_n = \frac{1^3 + 2^3 + 3^3 + \cdots + n^3}{1 + 3 + 5 + \cdots + (2n-1)} = \frac{\left\{\frac{n(n+1)}{2}\right\}^2}{\frac{n}{2}(1 + (2n-1))} = \frac{(n+1)^2}{4} = \frac{1}{4}(n^2 + 2n + 1)$$

Let S_n denote the sum of n terms of the given series. Then

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (k^2 + 2k + 1) = \frac{1}{4}\left[\sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1\right]$$

$$= \frac{1}{4}\left[\frac{n(n+1)(2n+1)}{6} + 2\left(\frac{n(n+1)}{2}\right) + n\right] = \frac{n}{24}(2n^2 + 9n + 13)$$

Illustration 9.53 Find the sum to n terms of the series

$$1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \cdots$$

Solution: Let T_n be the n^{th} term of the given series. Then

$$T_n = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} = \frac{1}{6}(2n^3 + 3n^2 + n)$$

Let S_n be the sum of n terms of the given series. Then

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} \frac{1}{6}(2k^3 + 3k^2 + k)$$

$$= \frac{2}{6}\left(\sum_{k=1}^{n} k^3\right) + \frac{3}{6}\sum_{k=1}^{n} k^2 + \frac{1}{6}\sum_{k=1}^{n} k$$

$$= \frac{1}{3}\left(\sum_{k=1}^{n} k^3\right) + \frac{1}{2}\left(\sum_{k=1}^{n} k^2\right) + \frac{1}{6}\left(\sum_{k=1}^{n} k\right)$$

$$= \frac{1}{3}\left(\frac{n(n+1)^2}{2}\right)^2 + \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{1}{6}\left(\frac{n(n+1)}{2}\right)$$

$$= \frac{n(n+1)}{12}[n(n+1) + (2n+1) + 1] = \frac{n(n+1)}{12}(n^2 + 3n + 2)$$

$$= \frac{n}{12}(n+1)^2(n+2)$$

Illustration 9.54 Find sum of n terms of the series $3\cdot 7 + 5\cdot 10 + 7\cdot 13 + \cdots$.

Solution: As usual, find the n^{th} term. Note that 3, 5, 7, …, i.e. the first number of each term is in AP and 7, 10, 13, …, i.e. the second number of each term is also in AP. Now

$$T_n = (2n+1)(3n+4) \quad \text{[product to } n \text{ term of two AP(s)]}$$
$$= 6n^2 + 11n + 4$$

Taking summation of both sides

$$S_n = \sum T_n = 6\sum n^2 + 11\sum n + 4\sum 1$$
$$= 6\frac{n(n+1)(2n+1)}{6} + 11\frac{n(n+1)}{2} + 4n$$
$$= \frac{n}{2}(4n^2 + 17n + 21)$$

Illustration 9.55 Sum the series: $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \ldots$ to n terms.

Solution: First determine the n^{th} term

$$T_n = (1^2 + 2^2 + 3^3 + \cdots + n^2)$$
$$= \sum n^2 = \frac{n(n+1)(2n+1)}{6}$$
$$= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

Now,

$$S_n = \sum T_n = \frac{1}{3}\sum n^3 + \frac{1}{2}\sum n^2 + \frac{1}{6}\sum n$$
$$= \frac{1}{3}\frac{n^2(n+1)^2}{4} + \frac{1}{2}\frac{n(n+1)(2n+1)}{6} + \frac{1}{6}\frac{n(n+1)}{2}$$

Simplifying this we get, $S_n = \dfrac{n(n+1)^2(n+2)}{12}$

Your Turn 9

Find the sum of the following series to n terms (1 – 4)

1. $2^2 + 4^2 + 6^2 + 8^2 + \cdots$ **Ans.** $\frac{2n}{3}(n+1)(2n+1)$
2. $1^3 + 3^3 + 5^3 + 7^3 + \cdots$ **Ans.** $n^2(2n^2 - 1)$
3. $1\cdot2\cdot5 + 2\cdot3\cdot6 + 3\cdot4\cdot7 + \cdots$ **Ans.** $\frac{n}{12}(n+1)(3n^2 + 23n + 34)$
4. $1 + (1+2)\,(1+2+3) + (1+2+3+4) + \cdots$ **Ans.** $\frac{n}{6}(n+1)(n+2)$
5. Find the sum of the series whose n^{th} term is

(A) $2n^3 + 3n^2 - 1$ **Ans.** $\frac{n}{2}(n^3 + 4n^2 + 4n - 1)$

(B) $(2n-1)^2$ **Ans.** $\frac{n}{3}(2n+1)(2n-1)$

(C) $1\cdot3 + 3\cdot5 + 5\cdot7 + 7\cdot9 + \cdots$ **Ans.** $\frac{2n(n+1)(2n+1)}{3} - n$

(D) $2\cdot3\cdot1 + 3\cdot4\cdot4 + 4\cdot5\cdot7 + \cdots$

Ans. $\left(3\left(\frac{n(n+1)}{2}\right)^2 + 5\frac{n(n+1)(2n+1)}{3} + 9\frac{n(n+1)}{2} + 2n\right)$

9.10 Inequalities

AM ≥ GM ≥ HM: Let $a_1, a_2, \ldots, a_n$ be n positive real numbers. Then we define their arithmetic mean (A), geometric mean (G) and harmonic mean (H) as

$$A = \frac{a_1 + a_2 + \cdots + a_n}{n}$$
$$G = (a_1 a_2, \ldots, a_n)^{1/n}$$

and
$$H = \frac{n}{\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots \frac{1}{a_n}\right)}$$

It can be shown that $A \ge G \ge H$. Moreover, equality holds at either place if and only if $a_1 = a_2 = \cdots = a_n$.

Illustration 9.56 Show that if $s = a_1 + a_2 + \cdots + a_n$,

$$\frac{s}{s-a_1} + \frac{s}{s-a_2} + \cdots + \frac{s}{s-a_n} > \frac{n^2}{(n-1)} \quad \text{unless } a_1 = a_2 = \cdots = a_n$$

Solution: We have

$$\frac{\frac{s}{s-a_1} + \frac{s}{s-a_2} + \cdots + \frac{s}{s-a_n}}{n} > \left(\frac{s}{s-a_1}\cdot\frac{s}{s-a_2}\cdots\frac{s}{s-a_n}\right)^{1/n} \quad (1)$$

unless $\frac{s}{s-a_1} = \frac{s}{s-a_2} = \cdots = \frac{s}{s-a_n}$

That is, unless $a_1 = a_2 = \cdots = a_n$

Also,

$$\frac{\frac{s-a_1}{s} + \frac{s-a_2}{s} + \cdots + \frac{s-a_n}{s}}{n} > \left(\frac{s-a_1}{s}\cdot\frac{s-a_2}{s}\cdots\frac{s-a_n}{s}\right)^{1/n} \quad (2)$$

Multiplying Eqs. (1) and (2) we get

$$\left(\frac{\frac{s}{s-a_1} + \frac{s}{s-a_2} + \cdots + \frac{s}{s-a_n}}{n}\right) \times \left(\frac{ns - (a_1 + a_2 + \cdots + a_n)}{ns}\right) > 1$$

$$\Rightarrow \frac{\frac{s}{s-a_1} + \frac{s}{s-a_2} + \cdots + \frac{s}{s-a_n}}{n} \times \frac{(n-1)}{n} > 1$$

$$\Rightarrow \frac{s}{s-a_1} + \frac{s}{s-a_2} + \cdots + \frac{s}{s-a_n} > \frac{n^2}{(n-1)}$$

> ***AM ≥ GM ≥ HM:*** Let $a_1, a_2, \ldots, a_n$ be n positive real numbers, then their arithmetic mean (A), geometric mean (G) and harmonic mean (H) as $A = \frac{a_1 + a_2 + \cdots + a_n}{n}$, $G = (a_1 a_2, \ldots, a_n)^{1/n}$ and $H = \frac{n}{\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots \frac{1}{a_n}\right)}$.
>
> It can be shown that $A \ge G \ge H$. Equality holds at either place if and only if $a_1 = a_2 = \cdots = a_n$.

Weighted means: Let $a_1, a_2, \ldots, a_n$ be n positive real numbers and $m_1, m_2, \ldots, m_n$ be n positive rational numbers. Then we define weighted arithmetic mean (A^*), weighted geometric mean (G^*) and weighted harmonic mean (H^*) as

$$A^* = \frac{m_1a_1 + \cdots + m_na_n}{m_1 + m_2 + \cdots m_n},$$

$$G^* = \left(a_1^{m_1} \cdot a_2^{m_2} \cdots\cdots a_n^{m_n}\right)^{\frac{1}{(m_1+m_2+\cdots+m_n)}}$$

and $$H^* = \frac{m_1 + m_2 + \cdots + m_n}{\left(\frac{m_1}{a_1} + \frac{m_2}{a_2} + \cdots + \frac{m_n}{a_n}\right)}.$$

It can be shown that $A^* \geq G^* \geq H^*$. Moreover, equality holds at either place if and only if $a_1 = a_2 = \cdots = a_n$.

Illustration 9.57 If a, b, c are positive real numbers, then prove that $[(1 + a)(1 + b)(1 + c)]^7 > 7^7 a^4 b^4 c^4$.

Solution:

$$(1 + a)(1 + b)(1 + c) = 1 + ab + a + b + c + abc + ac + bc$$

$$\Rightarrow \frac{(1+a)(1+b)(1+c) - 1}{7} \geq (ab \cdot a \cdot b \cdot c \cdot abc \cdot ac \cdot bc)^{1/7}$$

(using $AM \geq GM$)

$$\Rightarrow (1 + a)(1 + b)(1 + c) - 1 > 7\,(a^4 \cdot b^4 \cdot c^4)^{1/7}$$

$$\Rightarrow (1 + a)(1 + b)(1 + c) > 7\,(a^4 \cdot b^4 \cdot c^4)^{1/7}$$

$$\Rightarrow (1 + a)^7 (1 + b)^7 (1 + c)^7 > 7^7 (a^4 \cdot b^4 \cdot c^4)$$

9.10.1 Proving Inequalities

1. Any inequality has to be solved using a clever manipulation of the previous results.
2. Any inequality involving the sides of a triangle can be reduced to an inequality involving only positive real numbers, which is generally easier to prove.

   For the triangle, we have the constraints $a + b > c$, $b + c > a$, $a + c > b$

   Do the following: Put $x = s - a$, $y = s - b$, $z = s - c$. Then

   $$x + y + z = 3s - 2s = s$$

   and $$a = y + z, b = x + z, c = x + y$$

   Substitute $a = y + z, b = x + z, c = x + y$ in the inequality involving a, b, c to get an inequality involving x, y, z

   Also, note that the condition $a + b > c$ is equivalent to

   $$a + b + c > 2c, \text{ i.e., } 2s > 2c, \text{ or } s - c > 0, \text{ i.e., } z > 0$$

   Similarly,

   $$b + c > a \equiv x > 0, a + c > b \equiv y > 0$$

   Therefore, the inequality obtained after the substitution is easier to prove (involving only positive real numbers without any other constraints).

Illustration 9.58 If a, b, c are the sides of a triangle and $s = \frac{a+b+c}{2}$, prove that $8(s - a)(s - b)(s - c) \leq abc$.

Solution: Let $x = s - a$, $y = s - b$, $z = s - c$. Then $a = y + z$, $b = x + z$, $c = x + y$. The inequality reduces to

$$8xyz \leq (x + y)(y + z)(x + z)\ x, y, z \geq 0$$

which follows easily from AM $\geq$ GM inequality

$$x + y \geq 2\sqrt{xy},\ y + z \geq 2\sqrt{yz},\ x + z \geq 2\sqrt{xz}$$

Therefore, $(x + y)(x + z)(x + z) \geq 8xyz$. Substituting the values of x, y, z we get the required result.

9.10.2 Arithmetic Mean of m^{th} Power

Let $a_1, a_2, \ldots, a_n$ be n positive real numbers (not all equal) and let m be a real number. Then

$$\frac{a_1^m + a_2^m + \cdots + a_n^m}{n} > \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^m \text{ if } m \in R - [0, 1]$$

However, if $m \in (0, 1)$, then

$$\frac{a_1^m + a_2^m + \cdots + a_n^m}{n} < \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^m$$

Obviously if $m \in \{0, 1\}$, then

$$\frac{a_1^m + a_2^m + \cdots + a_n^m}{n} = \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^m$$

Illustration 9.59 Prove that $a^4 + b^4 + c^4 > abc(a + b + c)$ where a, b, c are distinct positive real numbers.

Solution: Using m^{th} power inequality, we get

$$\frac{a^4 + b^4 + c^4}{3} > \left(\frac{a+b+c}{3}\right)^4$$

$$\Rightarrow \frac{a^4 + b^4 + c^4}{3} > \left(\frac{a+b+c}{3}\right)\left(\frac{a+b+c}{3}\right)^3$$

$$\Rightarrow \frac{a^4 + b^4 + c^4}{3} > \left(\frac{a+b+c}{3}\right)[(abc)^{1/3}]^3, \quad \text{since (AM > GM)}$$

$$\Rightarrow \frac{a^4 + b^4 + c^4}{3} > \left(\frac{a+b+c}{3}\right) abc$$

$$\Rightarrow a^4 + b^4 + c^4 > abc(a + b + c)$$

Illustration 9.60 Show that the greatest value of $xyz\,(d - ax - by - cz)$ is $\frac{d^4}{4^4 abc}$ (given $a, b, c, x, y, z > 0$, $ax + by + cz < d$).

Solution: Consider 4 factors, $ax, by, cz, d - ax - by - cz$. Now,

$$\frac{ax + by + cz + d - ax - by - cz}{4} \geq \sqrt[4]{(ax)(by)(cz)(d - ax - by - cz)}$$

$$\Rightarrow \frac{d^4}{4^4} \geq abc\, xyz\,(d - ax - by - cz)$$

$$\Rightarrow xyz\,(d - ax - by - cz) \leq \frac{d^4}{4^4 abc}$$

Illustration 9.61 If s is the sum of the n^{th} powers, p is the sum of the products of m together of the n quantities $a_1, a_2, \ldots, a_n$ each of which is greater than 1, show that

$$(n-1)!\, s > (n-m)!.\, m! \cdot p$$

Solution: p contains ${}^nC_m$ terms, and each term of $p < a_1 a_2, \ldots, a_n$. So

$$p < {}^nC_m\,(a_1 a_2, \ldots, a_n)$$

$$\Rightarrow a_1 a_2, \ldots, a_n > \frac{p}{{}^nC_m} \quad (1)$$

Now,

$$\frac{a_1^n + a_2^n + \cdots + a_n^n}{n} > \left(a_1^n a_2^n \cdots a_n^n\right)^{1/n}$$

$$\Rightarrow \frac{s}{n} > a_1 a_2 \cdots a_n > \frac{p}{{}^nC_m} \quad \text{[from Eq. (1)]}$$

$$\Rightarrow {}^nC_m \frac{s}{n} > p \Rightarrow (n-1)!\, s > (n-m)!\, m! \cdot p$$

Illustration 9.62 Prove that $\left(\frac{a+b}{2}\right)^{a+b} \geq a^b \cdot b^a$ where $a, b \in N$.

Solution: Let us consider b quantities each equal to a and a quantities each equal to b. Then since AM > GM

$$\frac{(a+a+a+\cdots b \text{ times}) + (b+b+b+\cdots a \text{ times})}{a+b}$$

$$\geq a \cdot a \cdot a \cdots b \text{ times}\,(b \cdot b \cdot b \cdots a \text{ times})]^{1/(a+b)}$$

$$\Rightarrow \frac{ab+ab}{a+b} \geq (a^b b^a)^{1/(a+b)} \Rightarrow \frac{2ab}{a+b} \geq (a^b b^a)^{1/(a+b)}$$

But $\frac{a+b}{2}$ being AM of a and b is greater than $\frac{2ab}{a+b}$ their HM. Therefore,

$$\frac{a+b}{2} \geq \frac{2ab}{a+b} \geq (a^b b^a)^{1/(a+b)} \quad \Rightarrow \left(\frac{a+b}{2}\right)^{a+b} \geq a^b \cdot b^a$$

Illustration 9.63 If $x_1 x_2 x_3 \cdots x_n = y^n$, then show that

$$(1+x_1)(1+x_2)\cdots(1+x_n) \geq (1+y)^n$$

Solution:

$$(1+x_1)(1+x_2)\cdots(1+x_n) = 1 + (x_1 + x_2 + \cdots + x_n) + \sum x_1 x_2 + \sum x_1 x_2 x_3 + \cdots + x_1 x_2 x_3 \cdots x_n$$

Now,

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq (x_1 x_2 \cdots x_n)^{1/n}$$

$$\Rightarrow x_1 + x_2 + \cdots + x_n \geq ny$$

and

$$\frac{\sum x_1 x_2}{{}^nC_2} \geq \left(x_1^{n-1} x_2^{n-1} \cdots x_n^{n-1}\right)^{1/{}^nC_2} = (x_1 x_2 \cdots x_n)^{\frac{n-1}{{}^nC_2}} = y^2$$

$$\Rightarrow \sum x_1 x_2 \geq {}^nC_2 y^2$$

Only $\sum x_1 x_2 x_3 \geq {}^nC_3 y^3$. Therefore, from Eq. (1)

$$(1+x_1)(1+x_2)\cdots(1+x_n) \geq 1 + ny + {}^nC_2 y^2 + {}^nC_3 y^3 + \cdots + y^n = (1+y)^n$$

Your Turn 10

1. If $0 < \theta < \pi/2$, then find the least value of $\tan\theta + \cot\theta$.

Ans. 2

2. If x and y are positive quantities whose sum is 4, show that

$$\left(x + \frac{1}{x}\right)^2 + \left(y + \frac{1}{y}\right)^2 \geq 12\frac{1}{2}$$

3. If $a, b, c > 0$ show that $\frac{bc}{b+c} + \frac{ca}{c+a} + \frac{ab}{a+b} \leq \frac{a+b+c}{2}$.

4. Show that $a^2(1+b^2) + b^2(1+c^2) + c^2(1+a^2) \geq 6abc$.

5. If m and n are positive quantities, prove that $\left(\frac{mn+1}{m+1}\right)^{m+1} \geq n^m$.

6. Prove that $\left(\frac{bc+ac+ab}{a+b+c}\right)^{a+b+c} \geq \sqrt{(bc)^a (ac)^b (ab)^c}$ [where $a, b, c > 0$].

9.11 Exponential

1. **The number *e*:** The sum of the series $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \infty$ is denoted by the number e, that is,

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

2. The number e lies between 2 and 3. Approximate value of $e = 2.718281828$.
3. e is an irrational number.

9.11.1 Exponential Function

The function f defined as $f(x) = e^x$, $x \in R$ is called the exponential function. The graph of exponential function is given in Fig. 9.1.

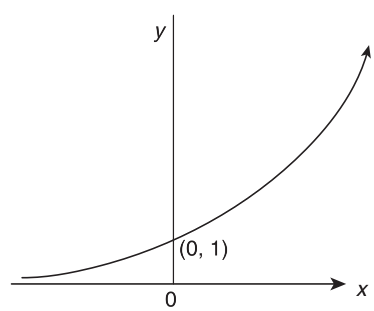


Figure 9.1

9.11.2 Exponential Series

The exponential series is given by

$$e^x = \sum_{n=0}^{\infty}\left(\frac{x^n}{n!}\right) = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\infty \qquad (1)$$

Putting $x = 1$, we get

$$e = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\infty$$

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

The number 'e' lies between 2 and 3 and it is an irrational number.

Putting $x = -x$ in Eq. (1), we get

$$e^{-x} = \sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^n}{n!} = 1-\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots+\infty \qquad (2)$$

From Eqs. (1) and (2), we have

$$\frac{e^x+e^{-x}}{2} = \sum_{n=0}^{\infty}\frac{x^{2n}}{2n!} = 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots+\infty$$

$$\frac{e^x-e^{-x}}{2} = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!} = x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots+\infty$$

Also

$$e^{ax} = 1+\frac{(ax)}{1!}+\frac{(ax)^2}{2!}+\frac{(ax)^3}{3!}+\cdots+\frac{(ax)^n}{n!}+\cdots+\infty$$

General term is $T_{r+1} = \frac{(ax)^r}{r!}$

Coefficient of x^n in $e^{ax} = \frac{a^n}{n!}$

$a^x = e^{x\ln a} = 1+(\log a)x+\frac{(\log a)^2}{2!}x^2+\frac{(\log a)^3}{3!}x^3+\cdots\infty$ for $a > 0, x \in R$.

9.11.2.1 Some Standard Deductions from Exponential Series

1. $e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\infty$
2. $e^{-x} = 1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots+\frac{(-1)^n}{n!}x^n+\cdots+\infty$ (Replace x by $-x$)
3. $e = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\infty$ {Putting $x = 1$ in Eq. (1)}
4. $e^{-1} = 1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots+\infty$ {Putting $x = -1$ in Eq. (2)}
5. $\frac{e^x+e^x}{2} = 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots+\infty$
6. $\frac{e^x-e^x}{2} = x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots+\infty$
7. $ax = 1+x(\log_e a)+\frac{x^2}{2!}(\log_e a)^2+\frac{x^3}{3!}(\log_e a)3+\cdots(a>0)$

Illustration 9.64 Find the coefficient of x^{10} in the series of e^{2x}.

Solution: The exponential series is given by

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

Replacing x by $2x$, we get

$$e^{2x} = 1+\frac{(2x)}{1!}+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+\cdots = \sum_{n=0}^{\infty}\frac{(2x)^n}{n!}$$

Clearly, $T_{n+1} = (n+1)^{\text{th}}$ term $= \frac{(2x)^n}{n!} = \frac{2^n}{n!}x^n$

If $(n+1)^{\text{th}}$ term contains x^{10}, then $n = 10$. Therefore

$$\text{Coefficient of } x^{10} = \frac{2^{10}}{10!} = \frac{4}{14175}$$

Illustration 9.65 If $y = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\infty$, then $x =$

(A) $\log_e y$ **(B)** $\log_e \frac{1}{y}$

(C) e^y **(D)** $e^{-y}!$

Solution: The given series on the RHS is an exponential series. Hence

$$y = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots = e^x \Rightarrow x = \log_e y$$

Illustration 9.66 $1+\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+\cdots+\infty =$

(A) e^{-1} **(B)** e

(C) $\frac{e+e^{-1}}{2}$ **(D)** $\frac{e-e^{-1}}{2}$

Solution:

$$1+\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+\cdots+\infty = \frac{e-e^{-1}}{2}$$

9.12 Logarithm

9.12.1 Definition

Consider the equation $ax = N$. Here, x is called the logarithm of N to the base a. It is also designated as $\log_a N$. Therefore

$$\log_a N = x \text{ or } ax = N, a > 0, a \neq 1 \text{ and } N = 0$$

Note:

1. The logarithm of a number is unique, i.e. no number can have two different logs to a given base.

2. From the definition of the logarithm of the number to a given base 'a':

$$a^{\log_a N} = N, a > 0, a \neq 1, N > 0$$

is known as the fundamental logarithmic identity.

3. $\log_e a = \log 10^a \cdot \log_e 10$ or $\log_{10} a = \dfrac{\log_e a}{\log_e 10} = 0.434 \ \log_e a$

9.12.2 Properties of Logarithms

Let M and N are arbitrary positive numbers such that $a > 0, a \neq 1, b > 0, b \neq 1$. Then

1. $\log_a MN = \log_a M + \log_a N$
2. $\log_a \dfrac{M}{N} = \log_a M - \log_a N$
3. $\log_a N^\alpha = \alpha \log_a N$
4. $\log_{a^\beta} N^\alpha = \dfrac{\alpha}{\beta} \log_a N \qquad (a \neq 0, b \neq 0)$
5. $\log_a N = \dfrac{\log_b N}{\log_b a}$
6. $\log_b a \cdot \log_a b = 1 \Rightarrow \log_b a = \dfrac{1}{\log_a b}$

9.12.3 Logarithmic Inequality

Let a be a real number, such that

1. For $a > 1$ the inequality $\log_a x > \log_a y$ and $x > y$ are equivalent
2. If $a > 1$ then $\log_a x < \alpha \Rightarrow 0 < x < a^\alpha$
3. If $a > 1$ then $\log_a x < \alpha \Rightarrow x > a^\alpha$
4. For $0 < a < 1$ the inequalities $0 < x < y$ and $\log_a x > \log_a y$ are equivalent
5. If $0 < a < 1$ then $\log_a x < a \Rightarrow x > a^\alpha$

9.12.4 Important Discussion

1. Given a number N, logarithm can be expressed as

$\log_{10} N$ = Integer + fraction (+ve)
↓ ↓
Characteristics Mantissa

(a) The mantissa part of log of a number is always kept positive.
(b) If the characteristics of $\log_{10} N$ is n then the number of digits in N is $(n + 1)$.
(c) If the characteristics of $\log_{10} N$ is $(-n)$ then there exists $(n - 1)$ number of zeros after decimal point of N.

2. If the number and the base are on the same side of unity, then the logarithm is positive; and if the number and the base are on different side of unity, then the logarithm is negative.

9.12.5 Logarithmic Series

If $-1 < x \leq 1$:

1. $\log_e(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots \infty$
2. $\log_e(1-x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots \infty$
3. $\log(1+x) - \log(1-x) = \log\left(\dfrac{1+x}{1-x}\right) = 2\left(x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \cdots\right)$
4. $\log(1+x) + \log(1-x) = \log(1-x^2) = -2\left(\dfrac{x^2}{2} + \dfrac{x^4}{4} + \dfrac{x^6}{6} + \cdots\right)$

Logarithmic series:
If $|x| < 1$, then

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \infty \qquad (1)$$

or

$$\log_e(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

Putting $x = -x$ in Eq. (1), we get

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \infty \qquad (2)$$

Subtracting Eq. (2) from Eq. (1) we get

$$\log(1+x) - \log(1-x) = \log\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots \infty\right)$$

Illustration 9.67 Using the series for log 2, prove that the value of log 2 lies between 0.61 and 0.76.

Solution: We have

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots \infty$$

Putting $x = 1$ in this series, we get

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots$$

$$= \frac{1}{2} + \frac{1}{12} + \frac{1}{30} + \cdots \geq \frac{37}{60} > 0.616$$

Also,

$$\log 2 = 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \left(\frac{1}{6} - \frac{1}{7}\right) - \cdots$$

$$= 1 - \frac{1}{6} - \frac{1}{20} - \frac{1}{42} - \cdots \leq 1 - \frac{1}{6} - \frac{1}{20} - \frac{1}{42} = \frac{319}{420} < 0.76$$

Hence, $0.61 < \log 2 < 0.76$.

Illustration 9.68 $\dfrac{2}{1} \cdot \dfrac{1}{3} + \dfrac{3}{2} \cdot \dfrac{1}{9} + \dfrac{4}{3} \cdot \dfrac{1}{27} + \dfrac{5}{4} \cdot \dfrac{1}{81} + \cdots \infty =$

(A) $\dfrac{1}{2} - \log_e \dfrac{2}{3}$ **(B)** $-\log_e \dfrac{2}{3}$

(C) $\dfrac{1}{2} + \log_e\left(\dfrac{2}{3}\right)$ **(D)** None of these

Solution:

$$S = \frac{2}{1} \cdot \frac{1}{3} + \frac{3}{2} \cdot \frac{1}{9} + \frac{4}{3} \cdot \frac{1}{27} + \cdots + \frac{n+1}{n} \cdot \frac{1}{3^n} + \cdots \infty$$

where $T_n = \frac{n+1}{n}\cdot\frac{1}{3^n} = \left(1+\frac{1}{n}\right)\frac{1}{3^n} = \frac{1}{3^n} + \frac{1}{n\cdot 3^n}$

Now

$$S = \sum T_n = \sum\frac{1}{3^n} + \sum\frac{1}{n\cdot 3}n = \frac{\frac{1}{3}}{1-\frac{1}{3}} + \left\{-\log_e\left(1-\frac{1}{3}\right)\right\} = \frac{1}{2} - \log_e\left(\frac{2}{3}\right)$$

Trick: As the sum of the series up to 3 or 4 terms is approximately 0.9, obviously, (A) gives the value nearer to 0.9.

Illustration 9.69 $\left(\frac{a-b}{a}\right)+\frac{1}{2}\left(\frac{a-b}{a}\right)^2+\frac{1}{3}\left(\frac{a-b}{a}\right)^3+\cdots=$

(A) $\log_e(a-b)$ **(B)** $\log_e\left(\frac{a}{b}\right)$

(C) $\log_e\left(\frac{b}{a}\right)$ **(D)** $e^{\left(\frac{a-b}{a}\right)}$

Solution:

$$\left(\frac{a-b}{a}\right)+\frac{1}{2}\left(\frac{a-b}{a}\right)^2+\frac{1}{3}\left(\frac{a-b}{a}\right)^3+\cdots=\log_e\left(1-\frac{a-b}{a}\right)$$

$$=-\log_e\left(\frac{b}{a}\right)=\log_e\left(\frac{a}{b}\right)$$

9.13 Difference between the Exponential and Logarithmic Series

1. In the exponential series $e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\infty$ all the terms carry positive signs, whereas in the logarithmic series $\log_e(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\infty$, the terms are alternatively positive and negative.
2. In the exponential series, the denominator of the terms involves factorial of natural numbers. But in the logarithmic series the terms do not contain factorials.
3. The exponential series is valid for all the values of x. The logarithmic series is valid when $|x|<1$.

- $\sum_{n=0}^{\infty}\frac{1}{(2n)!} = 1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots\infty = \frac{e+e^{-1}}{2} = \sum_{n=1}^{\infty}\frac{1}{(2n-2)!}$
- $\sum_{n=1}^{\infty}\frac{1}{(2n-1)!} = \frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\cdots\infty = \frac{e-e^{-1}}{2} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}$
- $\sum_{n=0}^{\infty}\frac{n}{n!} = e = \sum_{n=1}^{\infty}\frac{n}{n!}$
- $\sum_{n=0}^{\infty}\frac{n^2}{n!} = 2e = \sum_{n=1}^{\infty}\frac{n^2}{n!}$
- $\sum_{n=0}^{\infty}\frac{n^3}{n!} = 5e = \sum_{n=1}^{\infty}\frac{n^3}{n!}$
- $\sum_{n=0}^{\infty}\frac{n^4}{n!} = 15e = \sum_{n=1}^{\infty}\frac{n^4}{n!}$

Additional Solved Examples

1. If $\frac{a+bx}{a-bx} = \frac{b+cx}{b-cx} = \frac{c+dx}{c-dx}$ where $x \neq 0$, then show that a, b, c and d are in GP.

Solution: Consider

$$\frac{a+bx}{a-bx} = \frac{b+cx}{b-cx}$$

$\Rightarrow ab + b^2x - acx - bcx^2 = ab + acx - b^2x - bcx^2$

$\Rightarrow 2b^2x = 2acx \Rightarrow b^2 = ac$ (as $x \neq 0$) (1)

Similarly,

$\frac{b+cx}{b-cx} = \frac{c+dx}{c-dx}$ gives $c^2 = bd$ (2)

From Eqs. (1) and (2)

$\frac{b}{a} = \frac{c}{b} = \frac{d}{c} \Rightarrow a, b, c, d$ are in GP

Alternative method:

$$\frac{a+bx}{a-bx} = \frac{b+cx}{b-cx}$$

$$\Rightarrow \frac{(a+bx)-(a-bx)}{(a+bx)+(a-bx)} = \frac{(b+cx)-(b-cx)}{(b+cx)+(b-cx)}$$

(Applying componendo and dividendo)

$\Rightarrow \frac{2bx}{2a} = \frac{2cx}{2b} \Rightarrow \frac{b}{a} = \frac{c}{b}$ (1)

Similarly,

$\frac{b+cx}{b-cx} = \frac{c+dx}{c-ax} \Rightarrow \frac{c}{b} = \frac{d}{c}$ (2)

From Eqs. (1) and (2) we get

$$\frac{b}{a} = \frac{c}{b} = \frac{d}{c}$$

Therefore, a, b, c, d are in GP.

2. If the n^{th} term of a certain series, whose first 3 terms are 2, 1 and –3, is of the form $a + bn + c\cdot 2^n$, where a, b, c are constants, then show that a, c, b are in AP. Also find the sum of first n terms of the series.

Solution: We have

$t_n = a + bn + c\cdot 2^n$

$t_1 = a + b + 2c = 2$ (1)

$t_2 = a + 2b + 4c = -1$ (2)

$t_3 = a + 3b + 8c = -3$ (3)

Solving Eqs. (1), (2) and (3), we get

$$a = 5, b = -4, c = 1/2$$

Clearly, $a + b = 2c$

Therefore, a, c, b are in AP

Now $t_n = 5 - 4n + 2^{n-1}$. Therefore,

$$S_n = \Sigma t_n = 5\Sigma 1 - 4\Sigma n + \Sigma 2^{n-1} = 5n - 4\frac{n(n+1)}{2} + 2^n - 1$$

$$= 2^n - 2n^2 + 3n - 1$$

3. Find the sum S_n of the cubes of the first n terms of an AP whose terms are integers. Also, show that the sum of the first n terms of the AP is a factor of S_n.

Solution: Let the first term of the AP be a and common difference be d. Then

$$a, (a+d), (a+2d), \ldots, [a+\overline{(n-1)}d] \text{ are the terms.}$$

Now, $S_n = a^3 + (a+d)^3 + (a+2d)^3 + \cdots + [a+(n-1)d]^3$

$$= na^3 + 3a^2d[1+2+3+\cdots+(n-1)] + 3ad^2[1^2+2^2+3^2 + \cdots + (n-1)^2] + d^3[1^3+2^3+3^3+\cdots+(n-1)^3]$$

$$= na^2 + 3a^2\cdot d\frac{(n-1)n}{2} + 3ad^2\frac{(n-1)\cdot n(2n-1)}{6} + \frac{(n-1)^2\cdot n^2}{4}\cdot d^3$$

$$= \frac{n}{2}\left[2a^2 + 3\cdot(n-1)a^2d + (n-1)(2n-1)ad^2 + \frac{n(n-1)^2}{2}d^3\right]$$

$$= \frac{n}{2}\left[a^2\{2a+(n-1)d\} + (n-1)\ ad\{2a+(n-1)d\} + \frac{n(n-1)}{2}d^2 \{2a+(n-1)d\}\right]$$

$$= \frac{n}{2}\{2a+(n-1)d\}\left[a^2 + (n-1)ad + \frac{n(n-1)d^2}{2}\right]$$

$$= S\left[a^2 + (n-1)ad + \frac{n(n-1)d^2}{2}\right]$$

where S is the sum of first n terms of the AP. Hence, S is a factor of S_n.

4. Show that the sum of the n terms of the series

$$\frac{3}{1^2} + \frac{5}{1^2+2^2} + \frac{7}{1^2+2^2+3^2} + \frac{9}{1^2+2^2+3^2+4^2} + \cdots \text{ is } \frac{6n}{n+1}$$

Solution: Let t_n be the n^{th} term of the given series. Then obviously

$$t_n = \frac{3+(n-1)2}{1^2+2^2+3^2+\cdots+n^2} = \frac{2n+1}{\sum n^2}$$

$$\Rightarrow t_n = \frac{6\cdot(2n+1)}{n(n+1)(2n+1)} = \frac{6}{n(n+1)}$$

$$\Rightarrow t_n = \frac{6}{n} - \frac{6}{n+1} \qquad (V_n - V_{n+1} \text{ form})$$

If S_n is the required sum, then

$$S_n = t_1 + t_2 + t_3 + t_4 + \cdots t_{n-1} + t_n$$

$$= 6\left[\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\right]$$

$$= \frac{6}{1} - \frac{6}{n+1} = \frac{6n}{n+1}$$

5. Find the sum

$$\sum_{k=1}^{n}\tan^{-1}\left[\frac{2k}{2+k^2+k^4}\right]$$

Hence, show that

$$\frac{\pi}{4} = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{2}{11}\right) + \tan^{-1}\left(\frac{3}{46}\right) + \tan^{-1}\left(\frac{4}{137}\right) + \ldots$$

upto infinity.

Solution: We first try to express $\tan^{-1}\left[\frac{2k}{2+k^2+k^4}\right]$ in the form $\tan^{-1}\left[\frac{(x-y)}{(1+xy)}\right]$ as follows: Let $x - y = 2k$ and $xy = 1 + k^2 + k^4$. Then

$$\Rightarrow x(x-2k) = 1 + k^2 + k^4$$
$$\Rightarrow x^2 - 2kx + k^2 = 1 + 2k^2 + k^4$$
$$\Rightarrow (x-k)^2 = (k^2+1)^2$$
$$\Rightarrow x - k = k^2 + 1$$
$$\Rightarrow x = k^2 + k + 1$$
$$\Rightarrow y = x - 2k = k^2 - k + 1$$

Hence,

$$\sum_{k=1}^{n}\tan^{-1}\left[\frac{2k}{2+k^2+k^4}\right] = \sum_{k=1}^{n}\tan^{-1}\left[\frac{(k^2+k+1)-(k^2-k+1)}{1+(k^2+k+1)(k^2-k+1)}\right]$$

$$= \sum_{k=1}^{n}[\tan^{-1}(k^2+k+1) - \tan^{-1}(k^2-k+1)] \qquad (V_k - V_{k-1} \text{ form})$$

Since

$$(k^2+k+1)\ (k^2-k+1) = k^4+k^2+1 > -1$$
$$= (\tan^{-1}3 - \tan^{-1}1) + (\tan^{-1}7 - \tan^{-1}3) + (\tan^{-1}13 - \tan^{-1}7) + \cdots + [\tan^{-1}(n^2+n+1) - \tan^{-1}(n^2-n+1)]$$

{By substituting $k = 1, 2, \ldots, n$}

$$= \tan^{-1}(n^2+n+1) - \tan^{-1}1 = \tan^{-1}(n^2+n+1) - \frac{\pi}{4}$$

Since

$$\sum_{k=1}^{n}\tan^{-1}\left(\frac{2k}{2+k^2+k^4}\right) = \tan^{-1}(n^2+n+1) - \frac{\pi}{4}, \quad \text{for all } n \in N$$

taking limit as n tends to infinity we get

$$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{2}{11}\right) + \tan^{-1}\left(\frac{3}{46}\right) + \cdots \text{ to infinity}$$

$$= \lim_{n\to\infty}\left(\tan^{-1}(n^2+n+1) - \frac{\pi}{4}\right)$$

$$= \tan^{-1}\infty - \frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

6. If a, b and c are in arithmetic progression and a^2, b^2 and c^2 are in harmonic progression, then prove that either $a = b = c$ or $2b^2 = -ac$.

Solution: Given that

$$2b = a + c \qquad (1)$$

and

$$b^2 = \frac{2a^2c^2}{a^2+c^2} \qquad (2)$$

Squaring Eq. (1) and using the result in Eq. (2) we get

$$b^2 = \frac{2a^2c^2}{4b^2 - 2ac}$$

$$\Rightarrow (ac - b^2)\ (ac + 2b^2) = 0$$
$$\Rightarrow b^2 = ac \text{ or } 2b^2 = -\,ac$$

If $b^2 = ac$ then $\left(\frac{a+c}{2}\right)^2 = ac$, using Eq. (1) $\Rightarrow a = c$

So $a = b = c$, as a, b, c are in AP.

7. If the $(m+1)^{th}$, $(n+1)^{th}$ and $(r+1)^{th}$ terms of an AP are in GP and m, n, r are in HP, show that the ratios of the common difference to the first term in the AP is $(-2/n)$.

Solution: Let a be the first term and d be the common difference of the AP. Let x, y, z be the $(m+1)^{th}$, $(n+1)^{th}$ and $(r+1)^{th}$ terms of the AP. Then $x = a + md$, $y = a + nd$ and $z = a + rd$, since x, y, z are in GP. Therefore,

$$\Rightarrow y^2 = xz$$
$$\Rightarrow (a+nd)^2 = (a+rd)(a+md)$$
$$\Rightarrow \frac{d}{a} = \frac{r+m-2n}{n^2-rm}$$

Now m, n, r are in HP. So

$$\frac{2}{n} = \frac{1}{m} + \frac{1}{r} \Rightarrow \frac{2}{n} = \frac{m+r}{mr}$$

Hence

$$\frac{d}{a} = \frac{2\left(\frac{r+m}{2} - n\right)}{n\left(n - \frac{rm}{n}\right)} = \frac{2\left(\frac{mr}{n} - n\right)}{n\left(n - \frac{rm}{n}\right)} = \frac{-2}{n}$$

8. If the m^{th}, n^{th} and p^{th} terms of an AP and GP be equal and be, respectively, x, y, z. Then prove that

$$x^{y-z} \cdot y^{z-x} \cdot z^{x-y} = 1 \text{ or } x^y y^z z^x = x^z \cdot y^x \cdot z^y$$

Solution: Let a be the first term and d be the common difference of the AP. Then

$$x = a + (m-1)d, y = a + (n-1)d, z = a + (p-1)d$$

Let A be the first term and R be the common ratio of the GP. Then

$$x = AR^{m-1}, y = AR^{n-1}, z = AR^{p-1}$$
$$\Rightarrow x^{y-z} \cdot y^{z-x} \cdot z^{x-y} \cdot (AR^{n-1})^{z-x} \cdot (AR^{p-1})^{x-y}$$
$$= A^0 \cdot (R^{m-1})^{(n-p)d} \cdot (R^{n-1})^{(p-m)d} \cdot (R^{p-1})^{(m-n)d}$$
$$= R^0 = 1$$
$$\Rightarrow x^y y^z z^x = x^z \cdot y^x \cdot z^y$$

9. Does there exist a GP containing 27, 8 and 12 as three of its terms? If it exists, how many such progressions are possible.

Solution: Let 8 be the m^{th}, 12 be the n^{th} and 27 be the t^{th} terms of a GP whose first term is A and the common ratio is R. Then

$$8 = AR^{m-1}, 12 = AR^{n-1}, 27 = AR^{t-1}$$
$$\Rightarrow \frac{8}{12} = R^{m-n} = \frac{2}{3}, \frac{12}{27} = R^{n-t} = \left(\frac{2}{3}\right)^2, \frac{8}{27} = R^{m-t} = \left(\frac{2}{3}\right)^3$$
$$\Rightarrow 2m - 2n = n - t \text{ and } 3m - 3n = m - t$$
$$\Rightarrow 2m + t = 3n \text{ and } 2m + t = 3n$$
$$\Rightarrow \frac{2m+t}{3} = n$$

There are infinity of sets of values of m, n, t which satisfy this relation. For example, take $m = 1$. Then

$$\frac{2+t}{3} = n = k \Rightarrow n = k, t = 3k - 2$$

By giving different values to k we get integral values of n and t. Hence, there are an infinite number of GPs whose terms are 27, 8, 12 (may not be consecutive).

10. Find the natural number a for which $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$, where the function f satisfies $f(x+y) = f(x)f(y)$, for all natural numbers x, y and further $f(1) = 2$.

Solution: It is given that $f(x+y) = f(x) f(y)$ and $f(1) = 2$. Now

$$f(2) = f(1+1) = f(1) f(1) = 2.2 = 2^2$$
$$f(3) = f(1) f(2) = 2.2^2 = 2^3$$
$$f(k) = 2^k \text{ and } f(a) = 2^a$$

Hence,

$$\sum_{k=1}^{n} f(a+k) = \sum_{k=1}^{n} f(a)f(k) = f(a)\sum_{k=1}^{n} f(k) = f(a)\sum_{k=1}^{n} 2^k$$
$$= f(a)[2 + 2^2 + 2^3 + \cdots + 2^n]$$

From this we have

$$16(2^n - 1) = f(a) \cdot \frac{2(2^n-1)}{2-1}$$
$$\Rightarrow 16(2^n - 1) = 2^a \cdot 2(2^n - 1)$$
$$\Rightarrow 2^4 = 2^{a+1} \Rightarrow a + 1 = 4 \Rightarrow a = 3$$

11. If $H_1, H_2, H_3, \ldots, H_n$ are n-harmonic means lying between a and b, then show that

$$\frac{H_1+a}{H_1-a} + \frac{H_n+b}{H_n-b} = 2n.$$

Solution: Given that $a, H_1, H_2, H_3, \ldots, H_n, b$ are in HP. Then $\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \ldots, \frac{1}{H_n}, \frac{1}{b}$ are in AP. Let d be its common difference. So

$$\frac{1}{b} = \frac{1}{a} + (n+2-1)d \qquad \left\{\text{since } \frac{1}{b} \text{ is } (n+2)^{th} \text{ term}\right\}$$

Thus,

$$d = \frac{a-b}{ab(n+1)}$$

Now

$$\frac{1}{H_1} = \frac{1}{a} + d = \frac{1}{a} + \frac{a-b}{ab(n+1)}$$
$$\Rightarrow \frac{1}{H_1} = \frac{bn+a}{ab(n+1)} \Rightarrow \frac{a}{H_1} = \frac{bn+a}{b(n+1)}$$

Using the componendo and dividendo rule, we get

$$\frac{H_1+a}{H_1-a} = \frac{2bn+(a+b)}{b-a} \qquad (1)$$

Again,

$$\frac{1}{H_n} = \frac{1}{b} - d = \frac{1}{b} - \frac{a-b}{ab(n+1)}$$
$$\Rightarrow \frac{1}{H_n} = \frac{an+b}{ab(n+1)} \Rightarrow \frac{b}{H_n} = \frac{an+b}{a(n+1)}$$
$$\Rightarrow \frac{H_n+b}{H_n-b} = \frac{2an+(a+b)}{a-b} \qquad (2)$$

From Eqs. (1) and (2), we get

$$\frac{H_1+a}{H_1-a}+\frac{H_n+b}{H_n-b}=\frac{2(b-a)\cdot n}{(b-a)}=2n$$

12. First two terms of each of an AP, a GP and a HP are x and y. If their $(n+2)^{\text{th}}$ terms are in GP, then show that

$$\frac{y^{2n+2}-x^{2n+2}}{xy(y^{2n}-x^{2n})}=\frac{n+1}{n}$$

Solution: For AP

$$d=y-x$$

$$t_{n+2}=x+(n+2-1)(y-x)=x+(n+1)(y-x) \quad (1)$$

For GP

$$r=\frac{y}{x}$$

$$u_{n+2}=x\cdot\left(\frac{y}{x}\right)^{n+2-1}=x\left(\frac{y}{x}\right)^{n+1} \quad (2)$$

For HP

$$d=\frac{1}{y}-\frac{1}{x}$$

$$\frac{1}{V_{n+2}}=\frac{1}{x}+(n+2-1)\cdot\left(\frac{1}{y}-\frac{1}{x}\right)=\frac{1}{x}+\frac{(n+1)(x-y)}{xy}=\frac{y+(n+1)(x-y)}{xy}$$

Therefore,

$$v_{n+2}=\frac{xy}{y+(n+1)(x-y)}$$

Now, it is given that $t_{n+2}, u_{n+2}, v_{n+2}$ are in GP. Therefore

$$u_{n+2}^2=v_{n+2}\cdot t_{n+2}$$

$$\Rightarrow x^2\left(\frac{y}{x}\right)^{2n+2}=\left(\frac{xy}{y+(n+1)(x-y)}\right)\cdot[x+(n+1)(y-x)]$$

$$\Rightarrow\left(\frac{y}{x}\right)^{2n+2}=\frac{[yx+(n+1)(y^2-xy)]}{[xy+(n+1)(x^2-xy)]}$$

$$\Rightarrow\left(\frac{y}{x}\right)^{2n+2}=\frac{(n+1)y^2-nxy}{(n+1)x^2-nxy}$$

$$\Rightarrow y^{2n+2}[(n+1)x^2-nxy]=x^{2n+2}[(n+1)y^2-nxy]$$

$$\Rightarrow\left(\frac{n+1}{n}\right)x^2y^{2n+2}-xy^{2n+3}=\left(\frac{n+1}{n}\right)x^{2n+2}y^2-x^{2n+3}y$$

$$\Rightarrow\left(\frac{n+1}{n}\right)(x^2\cdot y^{2n+2}-x^{2n+2}y^2)=xy\cdot(y^{2n+2}-x^{2n+2})$$

$$\Rightarrow\frac{n+1}{n}=\frac{xy(y^{2n+2}-x^{2n+2})}{x^2y^2(y^{2n}-x^{2n})}$$

$$\Rightarrow\frac{n+1}{n}=\frac{y^{2n+2}-x^{2n+2}}{xy(y^{2n}-x^{2n})}$$

13. If $a_1, a_2, \ldots, a_n$ are n positive real numbers, then show that

$$\frac{\left(1+a_1+a_1^2\right)\left(1+a_1+a_2^2\right)\cdots\left(1+a_n+a_n^2\right)}{a_1a_2a_3\cdots a_n}\geq 3^n$$

Solution: Since

$$a_1+\frac{1}{a_1}\geq 2$$

Therefore,

$$1+a_1+\frac{1}{a_1}\geq 3$$

Similarly,

$$1+a_2+\frac{1}{a_2}\geq 3$$

$$1+a_3+\frac{1}{a_3}\geq 3$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$1+a_n+\frac{1}{a_n}\geq 3$$

Thus,

$$\left(1+a_1+\frac{1}{a_1}\right)\left(1+a_2+\frac{1}{a_2}\right)\left(1+a_3+\frac{1}{a_3}\right)\cdots\left(1+a_n+\frac{1}{a_n}\right)\geq 3^n$$

Therefore,

$$\frac{\left(1+a_1+a_1^2\right)\left(1+a_2+a_2^2\right)\cdots\left(1+a_n+a_n^2\right)}{a_1a_2a_3\cdots a_n}\geq 3^n$$

14. A cricketer plays n matches ($n\geq 1$). The total number of runs scored by him is $\left(\frac{n+1}{4}\right)\left(2^{n+1}-n-2\right)$. If he scores $k\cdot 2^{n-k+1}$ runs in k^{th} match ($1\leq k\leq n$), find the value of n.

Solution: Total number of runs scored is given by

$$\sum_{k=1}^{n}k\cdot 2^{n-k+1}=2^{n+1}\sum_{k=1}^{n}\frac{k}{2^k}$$

$$=2^{n+1}\left[1\left(\frac{1}{2}\right)+2\left(\frac{1}{2}\right)^2+3\left(\frac{1}{2}\right)^3+\cdots+n\left(\frac{1}{2^n}\right)\right]$$

$$=2[2^{n+1}-n-2] \quad \text{(sum of AGP)}$$

According to question

$$\left(\frac{n+1}{4}\right)(2^{n+1}-n-2)=2[2^{n+1}-n-2]$$

$$\Rightarrow\left(\frac{n+1}{4}\right)=2$$

$$\Rightarrow n=7$$

Previous Years' Solved JEE Main/AIEEE Questions

1. In a geometric progression consisting of positive terms, each term equals the sum of the next two terms. Then the common ratio of this progression equals

(A) $\frac{1}{2}(1-\sqrt{5})$ **(B)** $\frac{1}{2}\sqrt{5}$

(C) $\sqrt{5}$ **(D)** $\frac{1}{2}(\sqrt{5}-1)$

[AIEEE 2007]

Solution: Given that $ar^{n-1}=ar^n+ar^{n+1}$, which implies that

$$1 = r + r^2 \Rightarrow r^2 + r - 1 = 0 \Rightarrow r = \frac{\sqrt{5}-1}{2} \qquad \left(r \neq \frac{-\sqrt{5}-1}{2}\right)$$

Hence, the correct answer is option (D).

2. If p and q are positive real numbers such that $p^2 + q^2 = 1$, then the maximum value of $(p + q)$

(A) 2 **(B)** 1/2

(C) $\frac{1}{\sqrt{2}}$ **(D)** $\sqrt{2}$ **[AIEEE 2007]**

Solution: Using AM ≥ GM, we have,

$$\frac{p^2+q^2}{2} \geq \sqrt{p^2q^2} \Rightarrow \frac{1}{2} \geq pq \Rightarrow pq \leq \frac{1}{2}$$

Now,

$$(p+q)^2 = p^2 + q^2 + 2pq \Rightarrow (p+q)^2 \leq 1 + 2 \times \frac{1}{2}$$

$$\Rightarrow (p+q)^2 \leq 2 \Rightarrow p + q \leq \sqrt{2}$$

Hence, the correct answer is option (D).

3. The first two terms of a geometric progression add up to 12. The sum of the third and fourth terms is 48. If the terms of the geometric progression are alternately positive and negative, then the first term is

(A) –4 **(B)** –12

(C) 12 **(D)** 4 **[AIEEE 2008]**

Solution: Let $a, ar, ar^2, \ldots$ be the terms of a GP. Then

$$a + ar = 12 \qquad (1)$$

$$ar^2 + ar^3 = 48 \qquad (2)$$

Dividing Eq. (2) by Eq. (1), we have

$$\frac{ar^2(1+r)}{a(r+1)} = 4 \Rightarrow r^2 = 4 \quad \text{if } r \neq -1$$

Since the terms are alternately positive and negative, therefore $r = -2$. Also, we find $a = -12$ by using Eq. (1).

Hence, the correct answer is option (B).

4. The sum to the infinity of the series $1 + \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \frac{14}{3^4} + \cdots$ is

(A) 2 **(B)** 3

(C) 4 **(D)** 6 **[AIEEE 2009]**

Solution: Let

$$S = 1 + \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \frac{14}{3^4} + \cdots \qquad (1)$$

Therefore,

$$\frac{1}{3}S = \frac{1}{3} + \frac{2}{3^2} + \frac{6}{3^3} + \frac{10}{3^4} + \cdots \qquad (2)$$

Subtracting Eq. (2) from Eq. (1), we get

$$S\left(1 - \frac{1}{3}\right) = 1 + \frac{1}{3} + \frac{4}{3^2} + \frac{4}{3^3} + \frac{4}{3^4} + \cdots \infty = \frac{4}{3} + \frac{4}{3^2}\left(\frac{1}{1-(1/3)}\right)$$

$$= \frac{4}{3} + \frac{4}{3^2}\left(\frac{3}{2}\right) \Rightarrow \frac{2}{3}S = \frac{4}{3} + \frac{2}{3} = 2 \Rightarrow S = 3$$

Hence, the correct answer is option (B).

5. A person is to count 4500 currency notes. Let a_n denote the number of notes he counts in the n^{th} minute. If $a_1 = a_2 = \cdots = a_{10} = 1500$ and $a_{10}, a_{11}, \ldots$ are in AP with common difference –2, then the time taken by him to count all notes is

(A) 34 min **(B)** 125 min

(C) 135 min **(D)** 24 min

[AIEEE 2010]

Solution: Till 10th minute the number of notes counted = 1500.

Now suppose he takes n minutes in addition to 10 minutes to count all the notes. Therefore,

$$3000 = \frac{n}{2}[2 \times 148 + (n-1)(-2)] = n[148 - n + 1]$$

$$n^2 - 149n + 3000 = 0 \Rightarrow \quad n = 125, 24$$

where $n = 125$ is not possible. Therefore, the total time = 24 + 10 = 34 min.

Hence, the correct answer is option (A).

6. A man saves Rs 200 in each of the first three months of his service. In each of the subsequent months his saving increases by Rs 40 more than the saving of immediately previous month. His total saving from the start of service will be Rs 11040 after

(A) 19 months **(B)** 20 months

(C) 21 months **(D)** 18 months

[AIEEE 2011]

Solution: We have

1	2	3	4	5	6	…
200	200	200	240	280	…	…

Sum = 11040

First 3 months' saving = 600

Now,

$$600 + \frac{n}{2}[2 \times 240 + (n-1)40] = 11040$$

$$\Rightarrow 600 + 240n + n(n-1)20 = 11040 \Rightarrow n(n-1) + 12n + 30 = 552$$

$$\Rightarrow n^2 + 11n - 552 = 0$$

$$\Rightarrow n = \frac{-11 \pm \sqrt{121 + 2088}}{2} = \frac{-11 \pm \sqrt{2209}}{2} = \frac{-11 \pm 47}{2} = 18, -\frac{58}{2}$$

Therefore, the total months = $3 + 18 = 21$

Hence, the correct answer is option (C).

7. **Statement-1:** The sum of the series 1 + (1 + 2 + 4) + (4 + 6 + 9) + (9 + 12 + 16) + ⋯ + (361 + 380 + 400) is 8000.

Statement-2: $\sum_{k=1}^{n}(k^3 - (k-1)^3) = n^3$ for any natural number n.

(A) Statement 1 is false, Statement 2 is true

(B) Statement 1 is true, Statement 2 is true; Statement 2 is a correct explanation for Statement 1

(C) Statement 1 is true, Statement 2 is true; Statement 2 is not a correct explanation for Statement 1

(D) Statement 1 is true, Statement 2 is false

[AIEEE 2012]

Solution: Statement 1 has 20 terms whose sum is 8000 and statement 2 is true and supporting statement 1, since, k^{th} bracket is

$$(k-1)^2 + k(k-1) + k^2 = 3k^2 - 3k + 1$$

Hence, the correct answer is option (B).

8. If 100 times the 100th term of an AP with non-zero common difference equals the 50 times its 50th term, then the 150th term of this AP is

(A) –150 **(B)** 150 times its 50th term
(C) 150 **(D)** zero

[AIEEE 2012]

Solution:

$$100(T_{100}) = 50(T_{50}) \Rightarrow 2[a+99d] = a+4d \Rightarrow a+149d = 0 \Rightarrow T_{150} = 0$$

Hence, the correct answer is option (D).

9. If x, y, z are in AP and $\tan^{-1}x$, $\tan^{-1}y$ and $\tan^{-1}z$ are also in AP, then

(A) $2x = 3y = 6z$ **(B)** $6x = 3y = 2z$
(C) $6x = 4y = 3z$ **(D)** $x = y = z$

[JEE MAIN 2013]

Solution: If x, y, z are in AP, we have,

$$2y = x + z \quad (1)$$

If $\tan^{-1}x$, $\tan^{-1}y$ and $\tan^{-1}z$ are also in AP, we have

$$2\tan^{-1} y = \tan^{-1} x + \tan^{-1} z$$

Therefore,

$$\tan^{-1}\left(\frac{2y}{1-y^2}\right) = \tan^{-1}\left(\frac{x+z}{1-xz}\right) \quad (2)$$

Using Eq. (1) in Eq. (2), we get

$$\tan^{-1}\left(\frac{x+z}{1-y^2}\right) = \tan^{-1}\left(\frac{x+z}{1-xz}\right)$$

$$\Rightarrow y^2 = xz \quad \text{or}$$

$$x + z = 0$$

$$\Rightarrow x = y = z$$

Hence, the correct answer is option (D).

10. The sum of first 20 terms of the sequence 0.7, 0.77, 0.777, …, is

(A) $\frac{7}{9}(99-10^{-20})$ **(B)** $\frac{7}{81}(179+10^{-20})$

(C) $\frac{7}{9}(99+10^{-20})$ **(D)** $\frac{7}{81}(179-10^{-20})$

[JEE MAIN 2013]

Solution: We have $t_r = 0.777, \ldots, r$, which is expressed as

$$7(10^{-1}+10^{-2}+10^{-3}+\cdots+10^{-r} = 7\times\frac{10^{-1}(1-(10^{-1})^r)}{(1-10^{-1})}$$

$$= 7\times\frac{10^{-1}\times 10(1-10^r)}{9} = \frac{7}{9}(1-10^r)$$

Therefore,

$$S_{20} = \sum_{r=1}^{20} t_r = \frac{7}{9}\left(20 - \sum_{r=1}^{20} 10^{-r}\right)$$

$$= \frac{7}{9}\left(20 - \frac{1}{9}(1-10^{-20})\right) = \frac{7}{81}(179+10^{-20})$$

Hence, the correct answer is option (B).

11. If $(10)^9 + 2(11)^1(10)^8 + 3(11)^2(10)^7 + \cdots + 10(11)^9 = k(10)^9$, then k is equal to

(A) 100 **(B)** 110

(C) $\frac{121}{10}$ **(D)** $\frac{441}{100}$

[JEE MAIN 2014 (OFFLINE)]

Solution: Let

$$S = 1\times 10^9 + 2\times 11^1\times 10^8 + 3\times 11^2\times 10^7 + \cdots + 9\times 10\times 11^8 + 10\times 11^9 \quad (1)$$

Therefore,

$$\frac{11}{10}S = 11\times 10^8 + 2\times 11^2\times 10^7 + 3\times 11^3\times 10^6 + \cdots + 9\times 11^9 \quad (2)$$

On subtracting Eq. (2) from Eq. (1), we get

$$\frac{1}{10}S = -11\times 10^8 - 11^2\times 10^7 - \cdots 11^9 + 11^{10} - 10^9$$

$$= -10^9 - 11\times 10^8 - 11^2\times 10^7 - \cdots 11^9 + 11^{10}$$

$$= -10^9\frac{\{(11/10)^{10}-1\}}{(11/10)-1} + 11^{10} = \frac{-10^9\{(11/10)^{10}-1\}}{1/10} + 11^{10}$$

$$\Rightarrow \frac{1}{10}S = -10^{10}\left\{\frac{11^{10}}{10^{10}} - 1\right\} + 11^{10} = -11^{10} + 10^{10} + 11^{10}$$

$$\Rightarrow S = 10^{11} = 100\times 10^9$$

Therefore, $k = 100$.

Hence, the correct answer is option (A).

12. Three positive numbers form an increasing GP. If the middle term in this GP is doubled, the new numbers are in AP. Then the common ratio of the GP is

(A) $2-\sqrt{3}$ **(B)** $2+\sqrt{3}$

(C) $\sqrt{2}+\sqrt{3}$ **(D)** $3+\sqrt{2}$

[JEE MAIN 2014 (OFFLINE)]

Solution: Let the numbers be a, ar, ar^2. Now

$$2[2ar] = a + ar^2 \Rightarrow 4ar = a + ar^2 \Rightarrow r^2 - 4r + 1 = 0$$

$$\Rightarrow r = \frac{4\pm\sqrt{16-4}}{2} = 2\pm\sqrt{3}$$

Since, the numbers form an increasing GP so $r > 1$. Therefore, $r = 2+\sqrt{3}$.

Hence, the correct answer is option (B).

13. Given an AP whose terms are all positive integers. The sum of its first 9 terms is greater than 200 and less than 220. If the second term in it is 12, then its 4th term is

(A) 8 **(B)** 16
(C) 20 **(D)** 24

[JEE MAIN 2014 (ONLINE SET-1)]

Solution: Let

$$200 < a_1 + a_2 + a_3 + \cdots + a_9 < 220 \text{ or } 200 < \frac{9}{2}[2a_1 + 8d] < 220$$

Therefore,

$$200 < 9a_1 + 36d < 220 \quad (1)$$

Since, $a_2 = 12 \Rightarrow a_1 + d = 12$. Therefore, $d = 12 - a_1$. Now from Eq. (1)

$$200 < 9\,a_1 + 36\,(12 - a_1) < 220$$
$$\Rightarrow 200 < 9\,a_1 + 432 - 36\,a_1 < 220$$
$$\Rightarrow 200 < 432 - 27\,a_1 < 220$$
$$\Rightarrow -232 > 27\,a_1 > 212$$

Therefore, $a_1 = 8 \Rightarrow d = 12 - 8 = 4$, $a_3 = 16$ and $a_4 = 20$

Hence, the correct answer is option (C).

14. If the sum

$$\frac{3}{1^2} + \frac{5}{1^2 + 2^2} + \frac{7}{1^2 + 2^2 + 3^2} + \cdots + \text{ up to 20 terms}$$

is equal to $\frac{k}{21}$, then k is equal to

(A) 120 **(B)** 180
(C) 240 **(D)** 60

[JEE MAIN 2014 (ONLINE SET-1)]

Solution:

$$T_n = \frac{2n+1}{\frac{n(n+1)(2n+1)}{6}} = \frac{6}{n(n+1)}$$

$$\Rightarrow T_n = 6\left\{\frac{n+1-n}{n(n+1)}\right\} = +\left\{\frac{1}{n} - \frac{1}{n+1}\right\}$$

Therefore,

$$T_1 = 6\left\{\frac{1}{1} - \frac{1}{2}\right\},\quad T_2 = 6\left\{\frac{1}{2} - \frac{1}{3}\right\},\quad T_3 = 6\left\{\frac{1}{3} - \frac{1}{4}\right\}$$

$$S_2 = 6\left\{1 - \frac{1}{21}\right\} = 6 \times \frac{20}{21} = \frac{40}{7} \Rightarrow \frac{40}{7} = \frac{k}{21} \Rightarrow k = \frac{40 \times \cancel{21}^{3}}{\cancel{7}} = 120$$

Hence, the correct answer is option (A).

15. In a geometric progression, if the ratio of the sum of first 5 terms to the sum of their reciprocals is 49, and the sum of the first and third terms is 35, then the first term of this geometric progression is

(A) 7 **(B)** 21
(C) 28 **(D)** 42

[JEE MAIN 2014 (ONLINE SET-2)]

Solution: Let the sum of first 5 terms of the GP be

$$\frac{a}{r^2} + \frac{a}{r} + a + ar + ar^2$$

Now according to question

$$\frac{\frac{a}{r^2} + \frac{a}{r} + a + ar + ar^2}{\frac{r^2}{a} + \frac{r}{a} + \frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2}} = 49 \Rightarrow a^2 = 49 \Rightarrow a = \pm 7$$

Let $a = 7$. Now

$$\frac{7}{r^2} + 7 = 35 \Rightarrow \frac{7}{r^2} = 28 \Rightarrow r^2 = \frac{1}{4} \Rightarrow r = \pm\frac{1}{2}$$

Thus, first term is

$$\frac{7}{\left(\frac{1}{2}\right)^2} = 7 \times 4 = 28$$

Hence, the correct answer is option (C).

16. The sum of the first 20 terms common between the series $3 + 7 + 11 + 15 + \cdots$ and $1 + 6 + 11 + 16 + \cdots$ is

(A) 4000 **(B)** 4020
(C) 4200 **(D)** 4220

[JEE MAIN 2014 (ONLINE SET-2)]

Solution:

$$3 + 7 + 11 + 15 + 19 + 23 + 27 + 31 + 35 + \cdots$$
$$\updownarrow \qquad\qquad \nearrow\!\!\!\swarrow$$
$$1 + 6 + 11 + 16 + 21 + 26 + 31 + \cdots$$

Therefore, the common term series is 11 + 31 + 51 + … 20 terms. Therefore,

$$S = \frac{20}{2}[2 \times 11 + (20-1)20] = 10[22 + 380] = 10 \times 402 = 4020$$

Hence, the correct answer is option (B).

17. Let G be the geometric mean of two positive numbers a and b, and M be the arithmetic mean of $\frac{1}{a}$ and $\frac{1}{b}$. If $\frac{1}{M}$:G is 4:5, then a:b can be

(A) 1:4 **(B)** 1:2
(C) 2:3 **(D)** 3:4

[JEE MAIN 2014 (ONLINE SET-3)]

Solution: A, G, b be in $GP \Rightarrow G = \sqrt{ab}$

$$\frac{1}{a},\ M,\ \frac{1}{b} \text{ be in AP} \Rightarrow \frac{\frac{1}{a} + \frac{1}{b}}{2} = \frac{a+b}{2ab} \Rightarrow \frac{1}{M} = \frac{2ab}{a+b}$$

Therefore, according to the question

$$\frac{\frac{2ab}{a+b}}{\sqrt{ab}} = \frac{4}{5} \Rightarrow \frac{2ab}{a+b} \times \frac{1}{\sqrt{ab}} = \frac{4}{5} \Rightarrow \frac{\sqrt{2ab}}{a+b} = \frac{4}{5} \Rightarrow \frac{a+b}{2\sqrt{ab}} = \frac{5}{4}$$

Applying componendo and dividendo, we get

$$\frac{a+b+2\sqrt{a}\sqrt{b}}{a+b-2a\sqrt{a}\sqrt{b}} = \frac{9}{1} \Rightarrow \frac{(\sqrt{a}+\sqrt{b})^2}{(\sqrt{a}-\sqrt{b})^2} = \frac{9}{1} \Rightarrow \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{3}{1}$$

Again applying componendo and dividendo, we get

$$\frac{2\sqrt{a}}{2\sqrt{b}} = \frac{3+1}{3-1} \Rightarrow \sqrt{\frac{a}{b}} = \frac{\cancel{4}^{2}}{\cancel{2}} \Rightarrow \frac{a}{b} = \frac{4}{1}$$

We can start as $b\ G\ a$ in GP and $\frac{1}{b} M \frac{1}{a}$ in AP. Then

$$\frac{b}{a} = \frac{4}{1} \Rightarrow \frac{a}{b} = \frac{1}{4}$$

Hence, the correct answer is option (A).

18. The least positive integer n such that $1-\frac{2}{3}-\frac{2}{3^2}-\cdots-\frac{2}{3^{n-1}}<\frac{1}{100}$, is

(A) 4 **(B)** 5

(C) 6 **(D)** 7

[JEE MAIN 2014 (ONLINE SET-3)]

Solution:

$$1-\frac{2}{3}\left[1+\frac{1}{3}+\cdots+\frac{1}{3^{n-2}}\right]=1-\frac{2}{3}\left[\frac{1\left[1-\left(\frac{1}{3}\right)\right]^{n-1}}{1-\frac{1}{3}}\right]$$

Now, $\left(\frac{1}{3}\right)^{n-1}<\frac{1}{100}$

When $n=7$, $\left(\frac{1}{3}\right)^{6}<\frac{1}{100}$ true

When $n=6$, $\left(\frac{1}{3}\right)^{5}<\frac{1}{100}$ true

When $n=5$, $\left(\frac{1}{3}\right)^{4}>\frac{1}{100}$

Therefore, least $n=6$.

Hence, the correct answer is option (C).

19. The number of terms in an AP is even; the sum of the odd terms in it is 24 and that the even terms is 30. If the last term exceeds the first term by $10\frac{1}{2}$ then the number of terms in the AP is:

(A) 4 **(B)** 8

(C) 12 **(D)** 16

[JEE MAIN 2014 (ONLINE SET-4)]

Solution: Let number of terms $=2n$, $n\in N$. Given

$$a_1+a_3+\cdots+a_{n-1}=24 \quad (1)$$

$$a_2+a_4+\cdots+a_{2n}=30 \quad (2)$$

$$a_{2n}=a_1+10\frac{1}{2} \quad (3)$$

Therefore,

$$\frac{n}{2}[2a_1+(n-1)2d]=24 \quad (4)$$

and

$$\frac{n}{2}[2a_2+(n-1)2d]=30 \quad (5)$$

Now subtracting Eq. (5) from Eq. (4), we get

$$\frac{n}{2}[2a_2-2a_1]=6$$

Therefore,

$$\frac{n}{\not 2}\times\not 2\times d=6\Rightarrow \underbrace{nd=6\cdots(3)}_{\text{alt.}(1)-(2)\Rightarrow\ \underbrace{d+d+\cdots+d}_{n\text{ times}}=6}$$

Now from Eq. (3)

$$\not a_1+(2n-1)d=\not a_1+10\frac{1}{2} \text{ or } 2nd-d=10\frac{1}{2}\Rightarrow 2\times 6-d=10\frac{1}{2}$$

$$\Rightarrow d=12-10\frac{1}{2}=1\frac{1}{2}=\frac{3}{2}n=\frac{6}{3/2}=\frac{\not 6^2\times 2}{\not 3}=4$$

Thus, the number of terms $=2\times 4=8$

Hence, the correct answer is option (B).

20. Let $f(n)=\left[\frac{1}{3}+\frac{3n}{100}\right]n$, where $[n]$ denotes the greatest integer less than or equal to n. Then $\sum_{n=1}^{56}f(n)$ is equal to

(A) 56 **(B)** 689

(C) 1287 **(D)** 1399

[JEE MAIN 2014 (ONLINE SET-4)]

Solution:

$$f(1)=\left[\frac{1}{3}+\frac{3}{100}\right]1=0; \qquad f(2)=\left[\frac{1}{3}+\frac{6}{100}\right]2=0\cdots$$

$$f(22)=\left[\frac{1}{3}+\frac{66}{100}\right]22=\left[\frac{100+198}{300}\right]22=0$$

$$f(23)=\left[\frac{1}{3}+\frac{69}{100}\right]23=\left[\frac{100+207}{300}\right]23=23\cdots$$

$$f(55)=\left[\frac{1}{3}+\frac{165}{100}\right]55=\left[\frac{100+495}{300}\right]55=55$$

$$f(56)=\left[\frac{1}{3}+\frac{168}{100}\right]56=\left[\frac{100+504}{300}\right]56=2\times 56=112$$

Therefore,

$$\sum_{n=1}^{56}f(n)=0+(23+24+\cdots 55)+112=\frac{33}{2}[46+(33-1)1]+112$$

$$=\frac{33}{2}[46+33-1]+112$$

$$=\frac{33}{2}[78]+112=33\times 39\times 112=1399$$

Hence, the correct answer is option (D).

21. The sum of first 9 terms of the series

$$\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\cdots \text{ is}$$

(A) 96 **(B)** 142

(C) 192 **(D)** 71

[JEE MAIN 2015 (OFFLINE)]

Solution: The given series is

$$\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\cdots$$

Therefore,

$$t_n=\frac{1^3+2^3+3^3+\cdots n^3}{1+3+5+\cdots(2n-1)}=\frac{\left[\frac{n(n+1)}{2}\right]^2}{n^2}$$

So,

$$S_9 = \sum_{n=1}^{9} \frac{(n+1)^2}{4} = \frac{1}{4}\sum_{k=2}^{10} k^2 = \frac{1}{4}\left[\frac{(10)(11)(21)}{6} - 1\right] = 96$$

Hence, (A) is the correct answer.

22. If m is the AM of two distinct real numbers l and n ($l, n > 1$) and G_1, G_2 and G_3 are three geometric means between l and n, then $G_1^4 + 2G_2^4 + G_3^4$ equals

(A) 1 lm^2n **(B)** 4 lmn^2
(C) 4 $l^2m^2n^2$ **(D)** 4 l^2mn

[JEE MAIN 2015 (OFFLINE)]

Solution:

$$m = \frac{l+n}{2};\quad (1, n > 1) \qquad (1)$$

and $l, G_1, G_2, G_3 n$ are in GP. So

$$G_1 = l\left(\frac{n}{l}\right)^{1/4},\qquad G_2 = l\left(\frac{n}{l}\right)^{1/2},\qquad G_3 = l\left(\frac{n}{l}\right)^{3/4}$$

Therefore,

$$(G_1)^4 + 2(G_2)^4 + (G_3)^4 = l^2\left(\frac{n}{l} + \frac{2n^2}{l^2} + \frac{n^3}{l^3}\right) = \frac{n}{(n+l)^2} = 4m^2nl$$

[from Eq. (1)]

Hence, the correct answer is option (A).

23. Let the sum of the first three terms of an AP be 39 and the sum of its last four terms be 178. If the first term of this AP is 10, then the median of the AP is

(A) 26.5 **(B)** 28
(C) 29.5 **(D)** 31

[JEE MAIN 2015 (ONLINE SET-1)]

Solution: Let $a - d, a, a + d$ be three terms in AP. Then

$$(a - d) + a + (a + d) = 39$$

$\Rightarrow a = 13 \Rightarrow (a - d) = 1^{st}$ term = 10 (given) $\Rightarrow 13 - d = 10 \Rightarrow d = 3$

Given AP is 10, 13, 16, …. Let there be n terms in AP. Then
Sum of last four terms = 178 (given)

$$\Rightarrow \frac{4}{2}(2l + (4-1)(-3)) = 178$$

$$\Rightarrow 2(2l - 9) = 178$$

$$\Rightarrow 4l = 196 \Rightarrow 1 = 49 = t_n$$
$$\Rightarrow 10 + (n - 1)(3) = 49 \Rightarrow 3n = 42 \Rightarrow n = 14$$

Therefore, median of AP is

$$\frac{\frac{n}{2}\text{th term} + \left(\frac{n}{2}+1\right)\text{th term}}{2} = \frac{7\text{th term} + 8\text{th term}}{2}$$
$$= \frac{t_7 + t_8}{2} = \frac{t_1 + t_{14}}{2} = \frac{10 + 49}{2}$$
$$= \frac{59}{2} = 29.5$$

Hence, the correct answer is option (C).

24. The value of $\sum_{r=16}^{30}(r+2)(r-3)$ is equal to

(A) 7785 **(B)** 7780
(C) 7775 **(D)** 7770

[JEE MAIN 2015 (ONLINE SET-1)]

Solution:

$$\sum_{r=16}^{30}(r+2)(r-3) = \sum_{r=16}^{30}(r^2 - r - 6)$$
$$= \left(\sum_{r=1}^{30} r^2 - \sum_{r=1}^{15} r^2\right) - \left(\sum_{r=1}^{30} r - \sum_{r=1}^{15} r\right) - 6\sum_{r=1}^{30} 1$$
$$= \left[\frac{(30)(31)(61)}{6} - \frac{(15)(16)(31)}{6}\right] - \left[\frac{(30)(31)}{2} - \frac{(15)(16)}{2}\right] - 6(15)$$
$$= [(5)(31)(61) - 1240] = [465 - 120] - 90 = [7780]$$

Hence, the correct answer is option (B).

25. The sum of the 3rd and 4th terms of a GP is 60 and the product of its first three terms is 1000. If the first term of this GP is positive, then is the 7th term is

(A) 7290 **(B)** 320
(C) 640 **(D)** 2430

[JEE MAIN 2015 (ONLINE SET-2)]

Solution: We have

$$ar^2 + ar^3 = 60$$
$$a \cdot ar \cdot ar^2 = 1000,\ a > 0$$
$$\Rightarrow ar^2(1 + r) = 60;\ a^3r^3 = 1000,\ a > 0$$
$$\Rightarrow ar = 10 \Rightarrow \frac{ar^2(1+r)}{ar} = \frac{60}{10}$$
$$\Rightarrow r(r + 1) = 6 \Rightarrow r = 2 \Rightarrow a = 5$$
$$\Rightarrow T_7 = ar^6 = (5)(2)^6 = 320$$

Hence, the correct answer is option (B).

26. If $\sum_{n=1}^{5}\frac{1}{n(n+1)(n+2)(n+3)} = \frac{k}{3}$, then k is equal to

(A) $\frac{55}{336}$ **(B)** $\frac{17}{105}$
(C) $\frac{1}{6}$ **(D)** $\frac{19}{112}$

[JEE MAIN 2015 (ONLINE SET-2)]

Solution:

$$S = \sum_{n=1}^{5}\frac{1}{n(n+1)(n+2)(n+3)} = \frac{k}{3}$$

$$S = \frac{1}{3}\sum_{n=1}^{5}\frac{(n+3)-(n)}{n(x+1)(x+2)(x+3)}$$
$$= \frac{1}{3}\sum_{n=1}^{5}\left[\frac{1}{n(n+1)(n+2)} - \frac{1}{(n+1)(n+2)(n+3)}\right]$$
$$\Rightarrow S = \frac{1}{3}\left(\frac{1}{1\cdot 2\cdot 3} - \frac{1}{6\cdot 7\cdot 8}\right)$$

(Remaining terms cancel out)

$$\Rightarrow S = \frac{1}{3}\left(\frac{56-1}{6\cdot 7\cdot 8}\right) = \frac{1}{3}\left(\frac{55}{336}\right) \Rightarrow k = \frac{55}{336}$$

Hence, the correct answer is option (A).

27. If $\cos\alpha + \cos\beta = \frac{3}{2}$ and $\sin\alpha + \sin\beta = \frac{1}{2}$ and θ is the arithmetic mean of α and β, then $\sin 2\theta + \cos 2\theta$ is equal to

(A) $\frac{3}{5}$ **(B)** $\frac{4}{5}$

(C) $\frac{7}{5}$ **(D)** $\frac{8}{5}$

[JEE MAIN 2015 (ONLINE SET-2)]

Solution:

$$2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = \frac{3}{2} \text{ and } 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = \frac{1}{2}$$

$$\Rightarrow \tan\left(\frac{\alpha+\beta}{2}\right) = \frac{1}{3}$$

$$\Rightarrow \sin 2\theta + \cos 2\theta = \sin(\alpha+\beta) + \cos(\alpha+\beta)$$

$$= \frac{\frac{2}{3}}{1+\frac{1}{9}} + \frac{1-\frac{1}{9}}{1+\frac{1}{9}} = \frac{6}{10} + \frac{8}{10} = \frac{7}{5}$$

Hence, the correct answer is option (C).

28. If the 2nd, 5th and 9th terms of a non-constant AP are in GP, then the common ratio of this GP is

(A) $\frac{7}{4}$ **(B)** $\frac{8}{5}$

(C) $\frac{4}{3}$ **(D)** 1

[JEE MAIN 2016 (OFFLINE)]

Solution: Let a and b be the 1st term and the common difference of AP. Then, $t_2 = a + d$, $t_5 = a + 4d$ and $t_9 = a + 8d$ are in GP. That is, t_2, t_5 and t_9 in GP. So

$$\frac{t_5}{t_2} = \frac{t_9}{t_5} \Rightarrow t_5^2 = t_2 t_9$$

$$\Rightarrow (a+4d)^2 = (a+d)(a+8d)$$

$$\Rightarrow a^2 + 16d^2 + 8ad = a^2 + 8ad + ad + 8d^2$$

$$\Rightarrow 8d^2 - ad = 0$$

$$\Rightarrow d(8d - a) = 0$$

$$\Rightarrow d = 0$$

which is not possible and $a = 8d$.
Now, the common ratio of GP is

$$\frac{t_5}{t_2} = \frac{a+4d}{a+d} = \frac{12d}{9d} = \frac{4}{3}$$

Hence, the correct answer is option (C).

29. If the sum of the first 10 terms of the series

$$\left(1\frac{3}{5}\right)^2 + \left(2\frac{2}{5}\right)^2 + \left(3\frac{1}{5}\right)^2 + 4^2 + \left(4\frac{4}{5}\right)^2 + \cdots,$$

is $\frac{16}{5}m$, then m is equal to

(A) 99 **(B)** 102

(C) 101 **(D)** 100

[JEE MAIN 2016 (OFFLINE)]

Solution: We have

$$\left(\frac{8}{5}\right)^2 + \left(\frac{12}{5}\right)^2 + \left(\frac{16}{5}\right)^2 + \left(\frac{20}{5}\right)^2 + \cdots \text{ up to 10 terms} = \frac{16}{5}m$$

That is,

$$\left(\frac{4}{5}\right)^2 (2^2 + 3^2 + 4^2 + 5^2 + \cdots + 11^2) = \frac{16}{5}m$$

$$\Rightarrow 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + \ldots + 1^2 = 5m + 1^2$$

$$\Rightarrow \frac{11(11+1)[2(11)+1]}{6} = 5m + 1$$

$$\Rightarrow 22 \times 23 = 5m + 1$$

$$\Rightarrow 506 = 5m + 1$$

$$\Rightarrow 5m = 505$$

$$\Rightarrow m = 101$$

Hence, the correct answer is option (C).

30. The sum $\sum_{r=1}^{10}(r^2+1)\times(r!)$ is equal to

(A) $11 \times (11!)$ **(B)** $10 \times (11!)$

(C) $(11)!$ **(D)** $101 \times (10!)$

Solution: We have

$$T_r = (r^2+1)r! = (r+1)!r - [r \times (r!)] + r!$$

$$T_1 = [1 \times (2!)] - [1 \times (1!)] + 1!$$

$$T_2 = [2 \times (3!)] - [2 \times (2!)] + 2!$$

$$T_3 = [3 \times (4!)] - [3 \times (3!)] + 3!$$

$$T_{10} = [10 \times (11!)] - [10 \times (10!)] + 10!$$

Therefore,

$$T_1 + T_2 + \cdots + T_{10} = 10 \times (11!)$$

Hence, the correct answer is option (B).

31. Let $a_1, a_2, a_3, \ldots, a_n$ be in AP. If $a_3 + a_7 + a_{11} + a_{15} = 72$, then the sum of its first 17 terms is equal to

(A) 306 **(B)** 204

(C) 153 **(D)** 612

Solution: It is given that $a_1, a_2, \ldots, a_n$, is in AP. Let the common difference be d. Then

$$(a_1 + 2d) + (a_1 + 6d) + (a_1 + 10d) + (a_1 + 14d) = 72$$

$$\Rightarrow 4a_1 + 2d(1 + 3 + 5 + 7) = 72$$

$$\Rightarrow 4a_1 + 32d = 72$$

$$\Rightarrow 2a_1 + 16d = 36$$

Therefore, the sum of the first 17 terms is obtained as follows:

$$S_{17} = \frac{17}{2}(2a_1 + 16d) = \left(\frac{17}{2} \times 36\right) = 17 \times 18 = 306$$

Hence, the correct answer is option (A).

Previous Years' Solved JEE Advanced / IIT-JEE Questions

Paragraph for Questions 1 – 3: Let V_r denote the sum of the first r terms of an arithmetic progression (AP) whose first term is r and the common difference is $(2r-1)$. Let

$$T_r = V_{r+1} - V_r - 2 \text{ and } Q_r = T_{r+1} - T_r \text{ for } r = 1, 2, \ldots$$

[IIT-JEE 2007]

1. The sum $V_1 + V_2 + \cdots + V_n$ is

(A) $\frac{1}{12}n(n+1)(3n^2-n+1)$ **(B)** $\frac{1}{12}n(n+1)(3n^2+n+2)$

(C) $\frac{1}{12}n(2n^2-n+1)$ **(D)** $\frac{1}{3}(2n^2-2n+3)$

Solution: We have

$$v_r = \frac{r}{2}[2r+(r-1)(2r-1)] = \frac{1}{2}(2r^3-r^2+r)$$

Therefore,

$$\sum v_r = \sum \frac{1}{2}(2r^3-r^2+r)$$
$$= \sum r^3 - \frac{1}{2}\sum r^2 - \frac{1}{2}\sum r$$
$$= \frac{n^2(n+1)^2}{12} - \frac{n(n+1)(2n+1)}{12} + \frac{n(n+1)}{4}$$
$$= \frac{n(n+1)}{12}[3n(n+1)-(2n+1)+3]$$
$$= \frac{1}{12}n(n+1)(3n^2+n+2)$$

Hence, the correct answer is option (B).

2. T_r is always

(A) an odd number **(B)** an even number

(C) a prime number **(D)** a composite number

Solution: We have

$$v_r = \frac{1}{2}(2r^3-r^2+r)$$
$$v_{r+1} = \frac{1}{2}[2(r+1)^3-(r+1)^2+(r+1)]$$

Now

$$T_r = v_{r+1} - v_r = (r+1)^3 - \frac{1}{2}[(r+1)^2-r^2] + \frac{1}{2}(1)$$
$$= 3r^2+2r-1 = (r+1)(3r-1)$$

which is a composite number.

Hence, the correct answer is option (D).

3. Which one of the following is a correct statement?

(A) $Q_1, Q_2, Q_3, \ldots$ are in AP with common difference 5

(B) $Q_1, Q_2, Q_3, \ldots$ are in AP with common difference 6

(C) $Q_1, Q_2, Q_3, \ldots$ are in AP with common difference 11

(D) $Q_1 = Q_2 = Q_3 = \ldots$

Solution: We have

$$T_r = 3r^2+2r-1$$
$$T_{r+1} = 3(r+1)^2+2(r+1)-1$$

Now

$$Q_r = T_{r+1} - T_r = 3(r+1)+2(1) = 6r+5$$
$$Q_{r+1} = 6(r+1)+5$$
$$Q_{r+1} - Q_r = 6 = \text{Constant}$$

Therefore, Q_1, Q_2 and $Q_3, \ldots$ are in AP with common difference 6.

Hence, the correct answer is option (B).

Paragraph for Questions 4-6: Let A, G, H, denote the arithmetic, geometric and harmonic means, respectively, of two distinct positive numbers. For $n \geq 2$, let A_{n-1}, G_{n-1} and H_{n-1} have arithmetic, geometric and harmonic means as A_n, G_n, H_n, respectively.

4. Which one of the following statements is correct?

(A) $G_1 > G_2 > G_3 > \cdots$

(B) $G_1 < G_2 < G_3 < \cdots$

(C) $G_1 = G_2 = G_3 = \cdots$

(D) $G_1 < G_3 < G_5 < \cdots$ and $G_2 > G_1 > G_6 > \cdots$

[IIT-JEE 2007]

Solution: Let us consider

$$A_1 = \frac{a+b}{2};\ G_1 = \sqrt{ab};\ H_1 = \frac{2ab}{a+b}$$
$$A_n = \frac{A_{n-1}+H_{n-1}}{2};\ G_n = \sqrt{A_{n-1}\cdot H_{n-1}};\ H_n = \frac{2A_{n-1}\cdot H_{n-1}}{A_{n-1}+H_{n-1}}$$

Therefore,

$$G_2 = \sqrt{A_1 \cdot H_1} = \sqrt{ab}$$

Thus, it is clear that

$$G_1 = G_2 = G_3 = \cdots = \sqrt{ab}$$

and hence, we conclude that option (C) is the correct one.

Hence, the correct answer is option (C).

5. Which one of the following statements is correct?

(A) $A_1 > A_2 > A_3 > \cdots$

(B) $A_1 < A_2 < A_3 < \cdots$

(C) $A_1 > A_3 > A_5 > \cdots$ and $A_2 < A_4 < A_6 < \cdots$

(D) $A_1 < A_3 < A_5 < \cdots$ and $A_2 > A_4 > A_6 > \cdots$

[IIT-JEE 2007]

Solution: Since A_2 is the AM of A_1 and H_1 and $A_1 > H_1$, we get

$$A_1 > A_2 > H_1 \quad (1)$$

Also, A_3 is the AM of A_2 and H_2 and $A_2 > H_2$. Therefore,

$$A_2 > A_3 > H_2 \Rightarrow A_1 > A_2 > A_3 > \cdots$$

Hence, the correct answer is option (A).

6. Which one of the following statements is correct?

(A) $H_1 > H_2 > H_3 > \cdots$

(B) $H_1 < H_2 < H_3 < \cdots$

(C) $H_1 < H_3 > H_5 > \cdots$ and $H_2 < H_4 < H_6 < \cdots$

(D) $H_1 < H_3 < H_5 < \cdots$ and $H_2 > H_4 > H_6 > \cdots$

[IIT-JEE 2007]

Solution: As discussed in the solution of question 5, we have

$$A_1 > H_2 > H_1$$

and $$A_2 > H_3 > H_1$$

Therefore,

$$H_1 < H_2 < H_1 < \cdots$$

Hence, the correct answer is option (B).

7. Let $S_n = \sum_{k=1}^{n} \frac{n}{n^2+kn+k^2}$ and $T_n = \sum_{k=0}^{n-1} \frac{n}{n^2+kn+k^2}$ for $n = 1$, 2, 3, ... Then

(A) $S_n < \frac{\pi}{3\sqrt{3}}$ **(B)** $S_n > \frac{\pi}{3\sqrt{3}}$

(C) $T_n < \frac{\pi}{3\sqrt{3}}$ **(D)** $T_n > \frac{\pi}{3\sqrt{3}}$

[IIT-JEE 2008]

Solution:

$$S_n = \sum_{k=1}^{n} \frac{n}{n^2+kn+k^2}$$

$$\Rightarrow S_n < S_\infty$$

$$\Rightarrow S_n < \lim_{x\to\infty} \sum_{k=1}^{n} \frac{n}{n^2+nk+k^2} = \lim_{n\to\infty} \sum_{n\to\infty}^{n} \frac{\frac{1}{n}}{1+\frac{k}{n}+\left(\frac{k}{n}\right)^2}$$

$$= \int_0^1 \frac{dx}{1+x+x^2} = \frac{\pi}{3\sqrt{3}}$$

$$\Rightarrow S_n < \frac{\pi}{3\sqrt{3}}$$

Also

$$T_1 = 1 > \frac{\pi}{3\sqrt{3}} \Rightarrow T_n > \frac{\pi}{3\sqrt{3}}$$

Hence, the correct answers are options (A) and (D).

8. Suppose four distinct positive numbers a_1, a_2, a_3, a_4 are in GP. Let $b_1 = a_1$, $b_2 = b_1 + a_2$, $b_3 = b_2 + a_3$ and $b_4 = b_3 + a_4$.

Statement-1: The numbers b_1, b_2, b_3, b_4 are neither in AP nor in GP

Statement-2: The numbers b_1, b_2, b_3, b_4 are in HP

(A) Statement 1 is True, Statement 2 is True; Statement 2 is a correct explanation for Statement 1.

(B) Statement 1 is True, Statement 2 is True; Statement 2 is NOT a correct explanation for Statement 1.

(C) Statement 1 is True, Statement 2 is False.

(D) Statement 1 is False, Statement 2 is True.

[IIT-JEE 2008]

Solution: Since,

$$b_1 = a_1$$
$$b_2 = a_1 + a_2$$
$$b_3 = a_1 + a_2 + a_3$$
$$b_4 = a_1 + a_2 + a_3 + a_4$$

Hence, b_1, b_2, b_3, b_4 are neither in AP nor in GP and HP.

Hence, the correct answer is option (C).

9. Let $z = \cos\theta + i\sin\theta$. Then the value of $\sum_{m=1}^{15} \text{Im}(z^{2m-1})$ at $\theta = 2°$ is

(A) $\frac{1}{\sin 2°}$ **(B)** $\frac{1}{3\sin 2°}$

(C) $\frac{1}{2\sin 2°}$ **(D)** $\frac{1}{4\sin 2°}$

[IIT-JEE 2009]

Solution:

$$X = \sin\theta + \sin 3\theta + \cdots + \sin 29\theta$$

$$\Rightarrow 2(\sin\theta)X = 1 - \cos 2\theta + \cos 2\theta - \cos 4\theta + \cdots + \cos 28\theta - \cos 30\theta$$

$$\Rightarrow X = \frac{1-\cos 30\theta}{2\sin\theta} = \frac{1}{4\sin 2°}$$

Hence, the correct answer is option (D).

10. If the sum of first n terms of an AP is cn^2, then the sum of squares of these n terms is

(A) $\frac{n(4n^2-1)c^2}{6}$ **(B)** $\frac{n(4n^2+1)c^2}{3}$

(C) $\frac{n(4n^2-1)c^2}{3}$ **(D)** $\frac{n(4n^2+1)c^2}{6}$

[IIT-JEE 2009]

Solution: We have

$$t_n = c\{n^2 - (n-1)^2\} = c(2n-1)$$

Squaring both sides, we get

$$t_n^2 = c^2(4n^2 - 4n + 1)$$

$$\Rightarrow \sum_{n=1}^{n} t_n^2 = c^2\left\{\frac{4n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n\right\}$$

$$= \frac{c^2 n}{6}\{4(n+1)(2n+1) - 12(n+1) + 6\}$$

$$= \frac{c^2 n}{6}\{4n^2 + 6n + 2 - 6n - 6 + 3\} = \frac{c^2}{3} n(4n^2 - 1)$$

Hence, the correct answer is option (C).

11. Let S_k, $k = 1, 2, \ldots, 100$, denote the sum of the infinite geometric series whose first term is $\frac{k-1}{k!}$ and the common ratio is $\frac{1}{k}$.

Then the value of $\frac{100^2}{100!} + \sum_{k=1}^{100} \left|(k^2 - 3k + 1)S_k\right|$ is

[IIT-JEE 2010]

Solution:

$$S_k = \frac{\frac{k-1}{k!}}{1-\frac{1}{k}} = \frac{1}{(k-1)!}$$

Now

$$\sum_{k=2}^{100}\left|(k^2-3k+1)\frac{1}{(k-1)!}\right| = \sum_{k=2}^{100}\left|\frac{(k-1)^2-k}{(k-1)}\right|$$

$$=\sum\left|\frac{k-1}{(k-2)!}-\frac{k}{(k-1)!}\right|$$

$$=\left|\frac{2}{1!}-\frac{3}{2!}\right|+\left|\frac{3}{2!}-\frac{4}{3!}\right|+\cdots$$

$$=\frac{2}{1!}-\frac{1}{0!}+\frac{2}{1!}-\frac{3}{2!}+\frac{3}{2!}-\frac{4}{3!}+\cdots+\frac{99}{98!}-\frac{100}{99!}$$

$$=3-\frac{100}{99!}$$

Hence, the correct answer is (3).

12. Let a_1, a_2, a_3, …, a_{11} be real numbers satisfying a_1 = 15, 27 − 2a_2 > 0 and $a_k = 2a_{k-1} - a_{k-2}$ for k = 3, 4, …, 11. If $\frac{a_1^2+a_2^2+\cdots+a_{11}^2}{11}=90$, then the value of $\frac{a_1+a_2+\cdots+a_{11}}{11}$ is equal to

[IIT-JEE 2010]

Solution : $a_k = 2a_{k-1}-a_{k-2} \Rightarrow a_1, a_2, \ldots, a_{11}$ are in AP. Therefore,

$$\frac{a_1^2+a_2^2+\cdots+a_{11}^2}{11}=\frac{11a^2+35\times 11d^2 10ad}{11}=90$$

$$\Rightarrow 225+35d^2+150d=90$$

$$\Rightarrow 35d^2+150d+135=0 \Rightarrow d=-3,-9/7$$

Given $a_2 < \frac{27}{2}$. Therefore $d=-3$ and $d \neq -9/7$

This gives $\frac{a_1+a_2+\cdots+a_{11}}{11}=\frac{11}{2}[30-10\times 3]=0.$

Hence, the correct answer is (0).

13. Let a_1, a_2, a_3, …, a_{100} be an arithmetic progression with a_1 = 3 and $S_p=\sum_{i=1}^{p}a_i, 1\le p\le 100$. For any integer n with $1\le n\le 20$, let $m=5n$. If $\frac{S_m}{S_n}$ does not depend on n, then a_2 is _____.

[IIT-JEE 2011]

Solution: $a_1, a_2, a_3, \cdots, a_{100}$ are in AP. Then

$$a_1=3, S_p=\sum_{i=1}^{p}a_i, 1\le p\le 100$$

$$\frac{S_m}{S_n}=\frac{S_{5n}}{S_n}=\frac{\frac{5n}{2}(6+(5n-1)d}{\frac{n}{2}(6-d+nd)}$$

$\frac{S_m}{S_n}$ is independent of n of $6-d=0 \Rightarrow d=6$

$$a_2=a_1+d=3+6=9$$

Hence, the correct answer is (9).

14. Let a_1, a_2, a_3, … be in harmonic progression with a_1 = 5 and a_{20} = 25. The least positive integer n for which a_n < 0 is.

(A) 22 **(B)** 23

(C) 24 **(D)** 25 **[IIT-JEE 2012]**

Solution: a_1, a_2, a_3 are in HP. So

$$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \cdots \text{ are in AP}$$

$$\Rightarrow \frac{1}{a_n}=\frac{1}{a_1}+(n-1)d<0, \text{ where } \frac{\frac{1}{25}-\frac{5}{25}}{19}=d=\left(\frac{-4}{9\times 25}\right)$$

$$\Rightarrow \frac{1}{5}+(n-1)\left(\frac{-4}{19\times 25}\right)<0$$

$$\Rightarrow \frac{4(n-1)}{19\times 5}>1$$

$$\Rightarrow n-1>1\frac{19\times 5}{4}$$

$$\Rightarrow n>\frac{19\times 5}{4}+1 \Rightarrow n\ge 25$$

Hence, the correct answer is option (D).

Paragraph for Questions 15 and 16: Let a_n denote the number of all n-digit positive integers formed by the digits 0, 1 or both such that no consecutive digits in them are 0. Let b_n = the number of such n-digit integers ending with digit 1 and c_n = the number of such n-digit integers ending with digit 0. **[IIT-JEE 2012]**

15. The value of b_6 is

(A) 7 **(B)** 8

(C) 9 **(D)** 11

Solution: We have $a_n = b_n + c_n$.

Now $b_n = a_{n-1}\, c_n = a_{n-2}$. So

$$a_n = a_{n-1} + a_{n-2}$$

As $a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 5, a_5 = 8 \Rightarrow b_6 = 8$.

Hence, the correct answer is option (B).

16. Which of the following is correct?

(A) $a_{17}=a_{16}+a_{15}$ **(B)** $c_{17}\neq c_{16}+c_{15}$

(C) $b_{17}\neq b_{16}+c_{16}$ **(D)** $a_{17}=c_{17}+b_{16}$

Solution: As $a_n = a_{n-1} + a_{n-2}$, for n = 17 we have

$$a_{17}=a_{16}+a_{15}$$

Hence, the correct answer is option (A).

17. Let $S_n=\sum_{k=1}^{4n}(-1)^{\frac{k(k+1)}{2}}k^2$. Then S_n can take value(s)

(A) 1056 **(B)** 1088

(C) 1120 **(D)** 1332

[JEE ADVANCED 2013]

Solution: Given

$$S_n=\sum_{k=1}^{4n}(-1)^{\frac{k(k+1)}{2}}k^2$$

Therefore,

$$S_n = 1^2 - 2^2 + 3^2 + 4^2 + 5^2 - 6^2 + \cdots (4n-3)^2$$
$$-(4n-2)^2 + (4n-1)^2 + (4n)^2$$
$$S_n = (3^2 - 1^2) + (4^2 - 2^2) + (7^2 - 5^2) + (8^2 - 6^2)$$
$$+(11^2 - 9^2) + (12^2 - 10^2) + \cdots + (4n-3)^2 + (4n)^2 - (4n-2)^2$$
$$S_n = 2(1+3) + 2(4+2) + 2(7+5) + 2(8+6) + \cdots$$
$$+2(4n-1+4n-3) + 2(4n+4n-2)$$
$$S_n = 2[1+2+3+\cdots+4n] = \frac{2\cdot 4n(4n+1)}{2}$$

- From the value given in option (A), we get

$$4n(4n+1) = 1056$$
$$\Rightarrow 4n^2 + n - 264$$
$$\Rightarrow 4n^2 + n - 264 = 0$$
$$\Rightarrow n = 8$$

- From the value given in option (B), we get

$$4n(4n+1) = 1088$$

Solving this for n is not possible.

- From the value given in option (C), we get

$$4n(4n+1) = 1120$$

Solving this for n is also not possible.

- From the value given in option (D), we get

$$4n(4n+1) = 1332 \Rightarrow n = 9$$

Hence, the correct answers are options (A) and (D).

18. A pack contains n cards numbered from 1 to n. Two consecutive numbered cards are removed from the pack and the sum of the numbers on the remaining cards is 1224. If the smaller of the numbers on the removed cards is k, then $k - 20 =$ ______. **[JEE ADVANCED 2013]**

Solution: The smallest value of n for which

$$\frac{n(n+1)}{2} > 1224 \Rightarrow n(n+1) > 2448 \Rightarrow n > 49$$

For $n = 50$, we have

$$\frac{n(n+1)}{2} = 1275$$

Therefore,

$$k + (k+1) + 1275 - 1224 = 51$$

Therefore, $k = 25$ and thus

$$k - 20 = 5$$

Hence, the correct answer is (5).

19. Let a, b, c be positive integers such that $\frac{b}{a}$ is an integer. If a, b, c are in geometric progression and the arithmetic mean of a, b, c is $b + 2$, then the value of $\frac{a^2+a-14}{a+1}$ is ______.

[JEE ADVANCED 2014]

Solution: Let $a = a$, $b = ar$, $c = ar^2$, where r is integer. Since, $\frac{b}{a}$ is an integer, according to the question

$$\frac{a + ar + ar^2}{3} = ar + 2$$

Since (AM) = (b + 2), therefore

$$a + ar + ar^2 = 3ar + 6$$
$$\Rightarrow ar^2 - 2r + a = 6$$
$$\Rightarrow \underbrace{r^2 - 2r + 1}_{\text{integer}} = \frac{6}{\underset{\text{integer}}{\underbrace{a}}} \quad (1)$$
$$\Rightarrow (r-1)^2 = \frac{6}{a}$$

Now with $a = 1 - 5$, we do not get a perfect square and integer. Therefore, only possibility is that $a = r$. So

$$\frac{a^2 + a - 14}{a+1} = \frac{36 + 6 - 14}{6+1} = \frac{284}{7} = 4$$

Hence, the correct answer is (4).

20. Suppose that all the terms of an arithmetic progression (AP) are natural numbers. If the ratio of the sum of the first 7 terms to the sum of the first 11 terms is 6:11 and the seventh term lies in between 130 and 140, then the common difference of this AP is ______. **[JEE ADVANCED 2015]**

Solution:

$$\frac{S_7}{S_{11}} = \frac{6}{11} \quad (1)$$

From Eq. (1) we have

$$\Rightarrow \frac{\frac{7}{2}[2a+6d]}{\frac{11}{2}[2a+10d]} = \frac{6}{11}$$
$$\Rightarrow \frac{a+3d}{a+5d} = \frac{6}{7}$$
$$\Rightarrow \frac{t_4}{4_6} = \frac{6}{7}$$

Let $t_4 = 6k$, $t_6 = 7k$. So

$$2d = k \Rightarrow d = k/2 \text{ and } a + 3d = 6k$$
$$\Rightarrow a = 6k - 3k/2 = 9k/2$$

Therefore,

$$130 \le t_7 \le 140$$
$$\Rightarrow 130 \le \frac{9k}{2} + 3k \le 140$$
$$\Rightarrow 130 \le \frac{15k}{2} \le 140$$
$$\Rightarrow \frac{52}{3} \le k \in \frac{56}{3}$$

Since, $k \in \mathbb{N} \Rightarrow k = 18$. Hence

$$\Rightarrow d = \frac{k}{2} = \frac{18}{2} = 9$$

Hence, the correct answer is (9).

21. The value of $\sum_{k=1}^{13}\frac{1}{\sin\left(\frac{\pi}{4}+\frac{(k-1)\pi}{6}\right)\sin\left(\frac{\pi}{4}+\frac{k\pi}{6}\right)}$ is equal to

(A) $3-\sqrt{3}$ **(B)** $2(3-\sqrt{3})$

(C) $2(\sqrt{3}-1)$ **(D)** $2(2+\sqrt{3})$

[JEE ADVANCED 2016]

Solution: It is given that

$$\sum_{k=1}^{13}\frac{1}{\sin\left(\frac{\pi}{4}+\frac{(k-1)\pi}{6}\right)\sin\left(\frac{\pi}{4}+\frac{k\pi}{6}\right)}$$

Let $\alpha=\frac{\pi}{4}$ and $\beta=\frac{\pi}{6}$. Therefore,

$$\sum_{k=1}^{13}\frac{1}{\sin(\alpha+k\beta)\sin(\alpha+(k-1)\beta)}$$
$$=\frac{1}{\sin\beta}\sum_{k=1}^{13}\frac{\sin((\alpha+k+\beta)-(\alpha+(k-1)\beta))}{\sin(\alpha+k\beta)\sin(\alpha+(k-1)\beta)}$$
$$=\frac{1}{\sin\beta}\sum_{k=1}^{13}(\cot(\alpha+(k-1)\beta)-\cot(\alpha+k\beta))$$
$$=\frac{1}{\sin\beta}\{[\cot(\alpha)-\cot(\alpha+\beta)]+[\cot(\alpha+\beta)-\cot(\alpha+2\beta)]+\cdots$$
$$\cdots+[\cot(\alpha+12\beta)-\cot(\alpha+13\beta)]\}$$
$$=\frac{1}{\sin\beta}(\cot\alpha-\cot(\alpha+13\beta))$$
$$=\frac{1}{\sin(\pi/6)}\left(\cot\frac{\pi}{4}-\cot\left(\frac{\pi}{4}+\frac{13\pi}{6}\right)\right)$$
$$=2(1-2+\sqrt{3})$$
$$=2(\sqrt{3}-1)$$

Hence, the correct answer is option (C).

22. Let $b_i > 1$ for i = 1, 2, …, 101. Suppose $\log_e b_1$, $\log_e b_2$, …, $\log_e b_{101}$ are in arithmetic progression (AP) with the common difference $\log_e 2$. Suppose $a_1, a_2, \ldots, a_{101}$ are in AP such that $a_1 = b_1$ and $a_{51} = b_{51}$. If $t = b_1 + b_2 + \cdots + b_{51}$ and $s = a_1 + a_2 + \cdots + a_{51}$, then

(A) $s > t$ and $a_{101} + b_{101}$ **(B)** $s > t$ and $a_{101} + b_{101}$

(C) $s < t$ and $a_{101} + b_{101}$ **(D)** $s < t$ and $a_{101} + b_{101}$

[JEE ADVANCED 2016]

Solution: Let $b_i > 1$ and i = 1, 2, …, 101. Therefore,

$\log_e b_1, \log_e b_{21}, \ldots, \log_e b_{101}$ in AP (common difference $=\log_e 2$)

$b_1, b_{21}, \ldots, b_{101}$ in GP (common ratio = 2)

We see that $a_1, a_{21}, \ldots, a_{101}$ are in AP. Therefore,

$$a_1 = b_1 \text{ and } a_{51} = b_{51}$$

That is,

$$b_{51}=b_1(2^{50}) \Rightarrow \frac{b_{51}}{b_1}=2^{50}$$

Therefore,

$$a_{51}=a_1+50d$$

where d is the common difference of the second given AP.

So,

$$b_{51}=b_1+50d \Rightarrow 50d=b_{51}-b_1 \Rightarrow d=\frac{(b_{51}-b_1)}{50}$$

Now,

$$a_{101}=a_1+100d=a_1+100\frac{(b_{51}-b_1)}{50}$$
$$=(2b_{51}-b_1)-[2(2^{50})b_1-b_1]$$
$$=b_1(2^{51}-1)$$

and $b_{101}=b_1 2^{100}$

That is, $b_{101}>a_{101}$

Hence,

$$t=b_1+b_2+\cdots+b_{51}=\frac{b_1(2^{50}-1)}{(2-1)}=b_1(2^{50}-1)$$

and $s=\left(\frac{a_1+a_{51}}{2}\right)51=\frac{51}{2}(b_1+b_{51})=\frac{51}{2}(b_1+b_1 2^{50})=\frac{51b_1}{2}(2^{50}+1)$

Therefore, it is obvious that $s > t$.

Hence, the correct answer is option (B).

Practice Exercise 1

1. If the p^{th} term of an AP is q and the q^{th} term is p, then its r^{th} term will be

(A) $p+q+r$ **(B)** $p+q-r$

(C) $p+r-q$ **(D)** $p-q-r$

2. If $\tan n\theta=\tan m\theta$, then the different values of θ will be in

(A) AP **(B)** GP

(C) HP **(D)** None of these

3. n^{th} term of the series $3\cdot8+6\cdot11+9\cdot14+12\cdot17+\cdots$ will be

(A) $3n(3n+5)$ **(B)** $3n(n+5)$

(C) $n(3n+5)$ **(D)** $n(n+5)$

4. The sum of integers from 1 to 100 that are divisible by 2 or 5 is

(A) 3000 **(B)** 3050

(C) 4050 **(D)** None of these

5. If m^{th} terms of the series $63+65+67+69+\cdots$ and $3+10+17+24+\cdots$ are equal, then $m=$

(A) 11 **(B)** 12

(C) 13 **(D)** 15

6. If $2x, x+8, 3x+1$ are in AP, then the value of x will be

(A) 3 **(B)** 7

(C) 5 **(D)** –2

7. If the sum of n terms of an AP is $nA+n^2B$, where A, B are constants, then its common difference will be

(A) $A-B$ **(B)** $A+B$

(C) $2A$ **(D)** $2B$

8. If the 9th term of an AP is 35 and 19th term is 75, then its 20th term will be
(A) 78 **(B)** 79
(C) 80 **(D)** 81

9. The 9th term of the series $27+9+5\frac{2}{5}+3\frac{6}{7}+\cdots$ will be
(A) $1\frac{10}{17}$ **(B)** $\frac{10}{17}$
(C) $\frac{16}{27}$ **(D)** $\frac{17}{27}$

10. If a, b, c are in AP, then $\frac{(a-c)^2}{(b^2-ac)}=$
(A) 1 **(B)** 2
(C) 3 **(D)** 4

11. If $\log_3 2, \log_3(2^x-5)$ and $\log_3\left(2^x-\frac{7}{2}\right)$ are in AP, then x is equal to
(A) $1, \frac{1}{2}$ **(B)** $1, \frac{1}{3}$
(C) $1, \frac{3}{2}$ **(D)** None of these

12. Let T_r be the r^{th} term of an AP for $r=1,2,3,\cdots$. If for some positive integers m, n, we have $T_m=\frac{1}{n}$ and $T_n=\frac{1}{m}$, then T_{mn} equals
(A) $\frac{1}{mn}$ **(B)** $\frac{1}{m}+\frac{1}{n}$
(C) 1 **(D)** 0

13. If a,b,c,d,e are in AP then the value of $a+b+4c-4d+e$ in terms of a, if possible is
(A) $4a$ **(B)** $2a$
(C) 3 **(D)** None of these

14. If the ratio of the sum of n terms of two APs is $(7n+1):(4n+27)$, then the ratio of their 11th terms will be
(A) $2:3$ **(B)** $3:4$
(C) $4:3$ **(D)** $5:6$

15. The sum of the series $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}+\cdots$ to 9 terms is
(A) $-\frac{5}{6}$ **(B)** $-\frac{1}{2}$
(C) 1 **(D)** $-\frac{3}{2}$

16. The interior angles of a polygon are in AP. If the smallest angle is $120°$ and the common difference is $5°$, then the number of sides is
(A) 8 **(B)** 10
(C) 9 **(D)** 6

17. If $a_1, a_2, a_3, \ldots, a_n$ are in AP, where $a_i>0$ for all i, then the value of $\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\cdots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}=$
(A) $\frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$ **(B)** $\frac{n+1}{\sqrt{a_1}+\sqrt{a_n}}$
(C) $\frac{n-1}{\sqrt{a_1}-\sqrt{a_n}}$ **(D)** $\frac{n+1}{\sqrt{a_1}-\sqrt{a_n}}$

18. If $a_1, a_2, \ldots, a_n$ are in AP with common difference d, then the sum of the following series is
$\sin d(\operatorname{cosec} a_1 \cdot \operatorname{cosec} a_2+\operatorname{cosec} a_2 \cdot \operatorname{cosec} a_3+\cdots+\operatorname{cosec} a_{n-1}$ $\operatorname{cosec} a_n)$
(A) $\sec a_1-\sec a_n$ **(B)** $\cot a_1-\cot a_n$
(C) $\tan a_1-\tan a_n$ **(D)** $\operatorname{cosec} a_1-\operatorname{cosec} a_n$

19. If the sum of the series $2+5+8+11+\cdots$ is 60100, then the number of terms are
(A) 100 **(B)** 200
(C) 150 **(D)** 250

20. The sum of all natural numbers between 1 and 100 which are multiples of 3 is
(A) 1680 **(B)** 1683
(C) 1681 **(D)** 1682

21. The sum of $1+3+5+7+\cdots$ up to n terms is
(A) $(n+1)^2$ **(B)** $(2n)^2$
(C) n^2 **(D)** $(n-1)^2$

22. If the sum of the series $54+51+48+\cdots$ is 513, then the number of terms are
(A) 18 **(B)** 20
(C) 17 **(D)** None of these

23. The sum of the numbers between 100 and 1000 which is divisible by 9 is
(A) 55350 **(B)** 57228
(C) 97015 **(D)** 62140

24. The ratio of sum of m and n terms of an AP is $m^2:n^2$. Then the ratio of m^{th} and n^{th} terms will be
(A) $\frac{m-1}{n-1}$ **(B)** $\frac{n-1}{m-1}$
(C) $\frac{2m-1}{2n-1}$ **(D)** $\frac{2n-1}{2m-1}$

25. For a series whose n^{th} term is $\left(\frac{n}{x}\right)+y$, the sum of r terms will be
(A) $\left\{\frac{r(r+1)}{2x}\right\}+ry$ **(B)** $\left\{\frac{r(r-1)}{2x}\right\}$
(C) $\left\{\frac{r(r-1)}{2x}\right\}-ry$ **(D)** $\left\{\frac{r(r+1)}{2y}\right\}-rx$

26. The sum of the integers from 1 to 100 which are not divisible by 3 or 5 is
(A) 2489 **(B)** 4735
(C) 2317 **(D)** 2632

27. The sum of the first and third terms of an arithmetic progression is 12 and the product of first and second terms is 24. Then first term is
(A) 1 **(B)** 8
(C) 4 **(D)** 6

28. If the sum of the first $2n$ terms of $2, 5, 8, \ldots$ is equal to the sum of the first n terms of $57, 59, 61, \ldots$, then n is equal to
(A) 10 **(B)** 12
(C) 11 **(D)** 13

29. The sum of numbers from 250 to 1000 which are divisible by 3 is
(A) 135657 **(B)** 136557
(C) 161575 **(D)** 156375

30. 7^{th} term of an AP is 40. Then the sum of first 13 terms is
(A) 53 **(B)** 520
(C) 1040 **(D)** 2080

31. If $a_1, a_2, \ldots, a_{n+1}$ are in AP, then $\frac{1}{a_1a_2} + \frac{1}{a_2a_3} + \cdots + \frac{1}{a_na_{n+1}}$ is
(A) $\frac{n-1}{a_1a_{n+1}}$ **(B)** $\frac{1}{a_1a_{n+1}}$
(C) $\frac{n+1}{a_1a_{n+1}}$ **(D)** $\frac{n}{a_1a_{n+1}}$

32. If the sum of the first n terms of a series is $5n^2 + 2n$, then its second term is
(A) 7 **(B)** 17
(C) 24 **(D)** 42

33. The number of terms of the AP 3, 7, 11, 15, … to be taken so that the sum is 406 is
(A) 5 **(B)** 10
(C) 12 **(D)** 14

34. There are 15 terms in an arithmetic progression. Its first term is 5 and their sum is 390. The middle term is
(A) 23 **(B)** 26
(C) 29 **(D)** 32

35. If the sum of the 10 terms of an AP is 4 times to the sum of its 5 terms, then the ratio of first term and common difference is
(A) 1 : 2 **(B)** 2 : 1
(C) 2 : 3 **(D)** 3 : 2

36. Three numbers are in AP such that their sum is 18 and the sum of their squares is 158. The greatest number among them is
(A) 10 **(B)** 11
(C) 12 **(D)** None of these

37. If $\frac{3+5+7+\cdots \text{ to } n \text{ terms}}{5+8+11+\cdots \text{ to } 10 \text{ terms}} = 7$, then the value of n is
(A) 35 **(B)** 36
(C) 37 **(D)** 40

38. The arithmetic mean of first n natural number
(A) $\frac{n-1}{2}$ **(B)** $\frac{n+1}{2}$
(C) $\frac{n}{2}$ **(D)** n

39. The sum of n arithmetic means between a and b is
(A) $\frac{n(a+b)}{2}$ **(B)** $n(a+b)$
(C) $\frac{(n+1)(a+b)}{2}$ **(D)** $(n+1)(a+b)$

40. After inserting n AMs between 2 and 38, the sum of the resulting progression is 200. The value of n is
(A) 10 **(B)** 8
(C) 9 **(D)** None of these

41. The mean of the series $a, a+nd, a+2nd$ is
(A) $a+(n-1)d$ **(B)** $a+nd$
(C) $a+(n+1)d$ **(D)** None of these

42. If $f(x+y, x-y) = xy$, then the arithmetic mean of $f(x, y)$ and $f(y, x)$ is
(A) x **(B)** y
(C) 0 **(D)** 1

43. If $\log 2, \log(2^n - 1)$ and $\log(2^n + 3)$ are in AP, then $n =$
(A) 5/2 **(B)** $\log_2 5$
(C) $\log_3 5$ **(D)** 3/2

44. If the sum of two extreme numbers of an AP with four terms is 8 and the product of remaining two middle terms is 15, then the greatest number of the series will be
(A) 5 **(B)** 7
(C) 9 **(D)** 11

45. If the sides of a right-angled triangle are in AP, then the sides are proportional to
(A) 1:2:3 **(B)** 2:3:4
(C) 3:4:5 **(D)** 4:5:6

46. Three numbers are in AP whose sum is 33 and product is 792. Then the smallest number from these numbers is
(A) 4 **(B)** 8
(C) 11 **(D)** 14

47. If a, b, c, d, e, f are in AP, then the value of $e - c$ will be
(A) $2(c-a)$ **(B)** $2(f-d)$
(C) $2(d-c)$ **(D)** $d-c$

48. If the sum of three numbers of an arithmetic sequence is 15 and the sum of their squares is 83, then the numbers are
(A) 4, 5, 6 **(B)** 3, 5, 7
(C) 1, 5, 9 **(D)** 2, 5, 8

49. The four arithmetic means between 3 and 23 are
(A) 5, 9, 11, 13 **(B)** 7, 11, 15, 19
(C) 5, 11, 15, 22 **(D)** 7, 15, 19, 21

50. If the sum of three consecutive terms of an AP is 51 and the product of last and first terms is 273, then the numbers are
(A) 21, 17, 13 **(B)** 20, 16, 12
(C) 22, 18, 14 **(D)** 24, 20, 16

51. If $\frac{1}{p+q}, \frac{1}{r+p}, \frac{1}{q+r}$ are in AP, then
(A) p, q, r are in AP **(B)** p^2, q^2, r^2 are in AP
(C) $\frac{1}{p}, \frac{1}{q}, \frac{1}{r}$ are in AP **(D)** None of these

52. The difference between an integer and its cube is divisible by
(A) 4 **(B)** 6
(C) 9 **(D)** None of these

53. If a, b, c are in AP, then $(a+2b-c)(2b+c-a)(c+a-b)$ equals

(A) $\frac{1}{2}abc$ **(B)** abc

(C) $2\,abc$ **(D)** $4\,abc$

54. Four numbers are in arithmetic progression. The sum of first and last terms is 8 and the product of both middle terms is 15. The least number of the series is

(A) 4 **(B)** 3

(C) 2 **(D)** 1

55. If twice the 11^{th} term of an AP is equal to 7 times of its 21^{st} term, then its 25^{th} term is equal to

(A) 24 **(B)** 120

(C) 0 **(D)** None of these

56. If x, y, z are in AP and $\tan^{-1}x, \tan^{-1}y$ and $\tan^{-1}z$ are also in AP, then

(A) $x = y = z$ **(B)** $x = y = -z$

(C) $x = 1; y = 2; z = 3$ **(D)** $x = 2; y = 4; z = 6$

57. If x, y, z are in GP and $a^x = b^y = c^z$, then

(A) $\log_a c = \log_b a$ **(B)** $\log_b a = \log_c b$

(C) $\log_c b = \log_a c$ **(D)** None of these

58. If the p^{th}, q^{th} and r^{th} terms of a GP are a, b, c, respectively, then $a^{q-r} \cdot b^{r-p} \cdot c^{p-q}$ is equal to

(A) 0 **(B)** 1

(C) abc **(D)** pqr

59. If the third term of a GP is 4 then the product of its first 5 terms is

(A) 4^3 **(B)** 4^4

(C) 4^5 **(D)** None of these

60. The value of $0.2\dot{3}\dot{4}$ is

(A) $\frac{232}{990}$ **(B)** $\frac{232}{9990}$

(C) $\frac{232}{990}$ **(D)** $\frac{232}{9909}$

61. If the sum of three terms of GP is 19 and product is 216, then the common ratio of the series is

(A) $-\frac{3}{2}$ **(B)** $\frac{3}{2}$

(C) 2 **(D)** 3

62. The sum of the series $6 + 66 + 666 + \cdots$ up to n terms is

(A) $(10^{n-1} - 9n + 10)/81$ **(B)** $2(10^{n+1} - 9n - 10)/27$

(C) $2(10^n - 9n - 10)/27$ **(D)** None of these

63. If every term of a GP with positive terms is the sum of its two previous terms, then the common ratio of the series is

(A) 1 **(B)** $\frac{2}{\sqrt{5}}$

(C) $\frac{\sqrt{5}-1}{2}$ **(D)** $\frac{\sqrt{5}+1}{2}$

64. The sum of first two terms of a GP is 1 and every term of this series is twice of its previous term, then the first term will be

(A) 1/4 **(B)** 1/3

(C) 2/3 **(D)** 3/4

65. If the sum of n terms of a GP is 255 and n^{th} term is 128 and the common ratio is 2, then first term will be

(A) 1 **(B)** 3

(C) 7 **(D)** None of these

66. The sum of n terms of the series $1 + (1+x) + (1+x+x^2) + \cdots$ will be

(A) $\frac{1-x^n}{1-x}$ **(B)** $\frac{x(1-x^n)}{1-x}$

(C) $\frac{n(1-x) - x(1-x^n)}{(1-x)^2}$ **(D)** None of these

67. The two geometric means between the numbers 1 and 64 are

(A) 1 and 64 **(B)** 4 and 16

(C) 2 and 16 **(D)** 8 and 16

68. Let

$$x = 1 + a + a^2 + \cdots \infty (a < 1)$$
$$y = 1 + b + b^2 \cdots \infty (b < 1)$$

Then the value of $1 + ab + a^2b^2 + \cdots \infty$ is

(A) $\frac{xy}{x+y-1}$ **(B)** $\frac{xy}{x+y+1}$

(C) $\frac{xy}{x-y-1}$ **(D)** $\frac{xy}{x-y+1}$

69. The first term of a GP whose second term is 2 and sum to infinity is 8, will be

(A) 6 **(B)** 3

(C) 4 **(D)** 1

70. $0.4\dot{2}\dot{3} =$

(A) $\frac{419}{990}$ **(B)** $\frac{419}{999}$

(C) $\frac{417}{990}$ **(D)** $\frac{417}{999}$

71. The sum of infinite terms of a GP is x and on squaring the each term of it, the sum will be y. Then the common ratio of this series is

(A) $\frac{x^2-y^2}{x^2+y^2}$ **(B)** $\frac{x^2+y^2}{x^2-y^2}$

(C) $\frac{x^2-y}{x^2+y}$ **(D)** $\frac{x^2+y}{x^2-y}$

72. If the sum of an infinite GP and the sum of square of its terms is 3, then the common ratio of the first series is

(A) 1 **(B)** $\frac{1}{2}$

(C) $\frac{2}{3}$ **(D)** $\frac{3}{2}$

73. The value of $4^{1/3} \cdot 4^{1/9} \cdot 4^{1/27} \cdots \infty$ is
(A) 2 **(B)** 3
(C) 4 **(D)** 9

74. If $y = x + x^2 + x^3 + \cdots \infty$, then $x =$
(A) $\frac{y}{1+y}$ **(B)** $\frac{1-y}{y}$
(C) $\frac{y}{1-y}$ **(D)** None of these

75. If the sum of infinite terms of a GP is 3 and the sum of squares of its terms is 3, then its first term and common ratio are
(A) 3/2, 1/2 **(B)** 1, 1/2
(C) 3/2, 2 **(D)** None of these

76. The sum of infinite terms of the geometric progression $\frac{\sqrt{2}+1}{\sqrt{2}-1}, \frac{1}{2-\sqrt{2}}, \frac{1}{2} \cdots$ is
(A) $\sqrt{2}(\sqrt{2}+1)^2$ **(B)** $(\sqrt{2}+1)^2$
(C) $5\sqrt{2}$ **(D)** $3\sqrt{2}+\sqrt{5}$

77. If in an infinite GP the first term is equal to twice the sum of the remaining terms, then its common ratio is
(A) 1 **(B)** 2
(C) 1/3 **(D)** –1/3

78. If the sum of the series $1+\frac{2}{x}+\frac{4}{x^2}+\frac{8}{x^3}+\cdots\infty$ is a finite number, then
(A) $x > 2$ **(B)** $x > -2$
(C) $x > \frac{1}{2}$ **(D)** None of these

79. $0.5737373\cdots =$
(A) $\frac{284}{497}$ **(B)** $\frac{284}{495}$
(C) $\frac{568}{990}$ **(D)** $\frac{567}{990}$

80. The value of $\overline{0.037}$ where, $0.\overline{037}$ stands for the number $0.037037037 \cdots$ is
(A) $\frac{37}{1000}$ **(B)** $\frac{1}{27}$
(C) $\frac{1}{37}$ **(D)** $\frac{37}{999}$

81. If x is added to each of numbers 3, 9, 21 so that the resulting numbers may be in GP, then the value of x will be
(A) 3 **(B)** $\frac{1}{2}$
(C) 2 **(D)** $\frac{1}{3}$

82. If s is the sum of an infinite GP and the first term is a, then the common ratio r given by
(A) $\frac{a-s}{s}$ **(B)** $\frac{s-a}{s}$
(C) $\frac{a}{1-s}$ **(D)** $\frac{s-a}{a}$

83. The sum to infinity of the progression $9-3+1-\frac{1}{3}+\cdots$ is
(A) 9 **(B)** 9/2
(C) 27/4 **(D)** 15/2

84. If $(a+2b+2c)(a-2b+2c) = a^2+4c^2$, where a, b, c are non-zero numbers. Then a, b, c are in
(A) AP **(B)** GP
(C) HP **(D)** None of these

85. The product $(32)(32)^{1/6}(32)^{1/36} \cdots$ to ∞ is
(A) 16 **(B)** 32
(C) 64 **(D)** 0

86. If the m^{th} term of an HP is n and n^{th} is m, then the r^{th} term will be
(A) $\frac{r}{mn}$ **(B)** $\frac{mn}{r+1}$
(C) $\frac{mn}{r}$ **(D)** $\frac{mn}{r-1}$

87. Which number should be added to the numbers 13, 15, 19 so that the resulting numbers are consecutive terms of an HP?
(A) 7 **(B)** 6
(C) –6 **(D)** –7

88. The fifth term of the HP, $2, 2\frac{1}{2}, 3\frac{1}{3}, \ldots$ will be
(A) $5\frac{1}{5}$ **(B)** $3\frac{1}{5}$
(C) 1/10 **(D)** 10

89. If $a_1, a_2, a_3, \ldots, a_n$ are in HP, then $a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n$ will be equal to
(A) a_1a_n **(B)** na_1a_n
(C) $(n-1)a_1a_n$ **(D)** None of these

90. If x, y, z are in HP, then the value of expression $\log(x+z) + \log(x-2y+z)$ will be
(A) $\log(x-z)$ **(B)** $2\log(x-z)$
(C) $3\log(x-z)$ **(D)** $4\log(x-z)$

91. If 5^{th} term of an HP is $\frac{1}{45}$ and the 11^{th} term is $\frac{1}{69}$, then its 16^{th} term will be
(A) 1/89 (B) 1/85
(C) 1/80 (D) 1/79

92. The first term of a harmonic progression is 1/7 and the second term is 1/9. The 12^{th} term is
(A) 1/19 (B) 1/29
(C) 1/17 (D) 1/27

93. If a, b, c are three distinct positive real numbers which are in HP, then $\frac{3a+2b}{2a-b}+\frac{3c+2b}{2c-b}$ is

(A) greater than or equal to 10
(B) less than or equal to 10
(C) only equal to 10
(D) None of these

94. If a, b, c, d are in HP, then $ab+bc+cd$ is equal to

(A) $3ad$ **(B)** $(a+b)(c+d)$
(C) $3ac$ **(D)** None of these

95. If the 7$^{\text{th}}$ term of a harmonic progression is 8 and the 8$^{\text{th}}$ term is 7, then its 15$^{\text{th}}$ term is

(A) 16 **(B)** 14
(C) $\frac{27}{14}$ **(D)** $\frac{56}{15}$

96. If the 7$^{\text{th}}$ term of a HP is $\frac{1}{10}$ and the 12$^{\text{th}}$ term is $\frac{1}{25}$, then the 20$^{\text{th}}$ term is

(A) $\frac{1}{37}$ **(B)** $\frac{1}{41}$
(C) $\frac{1}{45}$ **(D)** $\frac{1}{49}$

97. If the 6$^{\text{th}}$ term of an HP is $\frac{1}{61}$ and its 10$^{\text{th}}$ term is $\frac{1}{105}$, then first term of that an HP is

(A) $\frac{1}{28}$ **(B)** $\frac{1}{39}$
(C) $\frac{1}{6}$ **(D)** $\frac{1}{17}$

98. In an HP, the p^{th} term is q and the q^{th} term is p. Then pq^{th} term is

(A) 0 **(B)** 1
(C) pq **(D)** $pq(p+q)$

99. The 4$^{\text{th}}$ term of an HP is $\frac{3}{5}$ and the 8$^{\text{th}}$ term is $\frac{1}{3}$. Then its 6$^{\text{th}}$ term is

(A) $\frac{1}{6}$ **(B)** $\frac{3}{7}$
(C) $\frac{1}{7}$ **(D)** $\frac{3}{5}$

100. If H is the harmonic mean between p and q, then the value of $\frac{H}{p}+\frac{H}{q}$ is

(A) 2 **(B)** $\frac{pq}{p+q}$
(C) $\frac{p+q}{pq}$ **(D)** None of these

101. If the harmonic mean between a and b is H, then the value of $\frac{1}{H-a}+\frac{1}{H-b}$ is

(A) $a+b$ **(B)** ab
(C) $\frac{1}{a}+\frac{1}{b}$ **(D)** $\frac{1}{a}-\frac{1}{b}$

102. HM between the roots of the equation $x^2-10x+11=0$ is

(A) $\frac{1}{5}$ **(B)** $\frac{5}{21}$
(C) $\frac{21}{20}$ **(D)** $\frac{11}{5}$

103. The harmonic mean of $\frac{a}{1-ab}$ and $\frac{a}{1+ab}$ is

(A) $\frac{a}{\sqrt{1-a^2b^2}}$ **(B)** $\frac{a}{1-a^2b^2}$
(C) a **(D)** $\frac{1}{1-a^2b^2}$

104. The sixth HM between 3 and $\frac{6}{13}$ is

(A) $\frac{63}{120}$ **(B)** $\frac{63}{12}$
(C) $\frac{126}{105}$ **(D)** $\frac{120}{63}$

105. If $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is the harmonic mean between a and b, then the value of n is

(A) 1 **(B)** -1
(C) 0 **(D)** 2

106. If the harmonic mean between a and b is H, then $\frac{H+a}{H-a}+\frac{H+b}{H-b}=$

(A) 4 **(B)** 2
(C) 1 **(D)** $a+b$

107. If a, b, c are in HP, then

(A) $a^2+c^2>b^2$ **(B)** $a^2+b^2>2c^2$
(C) $a^2+c^2>2b^2$ **(D)** $a^2+b^2>c^2$

108. If a, b, c, d are in HP, then

(A) $a+d>b+c$ **(B)** $ad>bc$
(C) Both (A) and (B) **(D)** None of these

109. If the arithmetic, geometric and harmonic means between two distinct positive real numbers are A, G and H, respectively, then the relation between them is

(A) $A>G>H$ **(B)** $A>G<H$
(C) $H>G>A$ **(D)** $G>A>H$

110. If $a^{1/x}=b^{1/y}=c^{1/z}$ and a, b, c are in GP, then x, y, z will be in

(A) AP **(B)** GP
(C) HP **(D)** None of these

111. If a, b, c are in GP and x, y are the arithmetic means between a, b and b, c, respectively, then $\frac{a}{x}+\frac{c}{y}$ is equal to

(A) 0 **(B)** 1
(C) 2 **(D)** $\frac{1}{2}$

112. If a^2, b^2, c^2 are in AP, then $(b+c)^{-1}, (c+a)^{-1}$ and $(a+b)^{-1}$ will be in

(A) HP **(B)** GP
(C) AP **(D)** None of these

113. If $x, 1, z$ are in AP and $x, 2, z$ are in GP, then $x, 4, z$ will be in

(A) AP **(B)** GP
(C) HP **(D)** None of these

114. If A_1, A_2 are the two AMs between two numbers a and b and G_1, G_2 are two GMs between same two numbers, then $\frac{A_1 + A_2}{G_1 \cdot G_2} =$

(A) $\frac{a+b}{ab}$ **(B)** $\frac{a+b}{2ab}$
(C) $\frac{2ab}{a+b}$ **(D)** $\frac{ab}{a+b}$

115. If the $(m+1)^{th}, (n+1)^{th}$ and $(r+1)^{th}$ terms of an AP are in GP and m, n, r are in HP, then the value of the ratio of the common difference to the first term of the AP is

(A) $-\frac{2}{n}$ **(B)** $\frac{2}{n}$
(C) $-\frac{n}{2}$ **(D)** $\frac{n}{2}$

116. If the AM is twice the GM of the numbers a and b, then $a:b$ will be

(A) $\frac{2-\sqrt{3}}{2+\sqrt{3}}$ **(B)** $\frac{2+\sqrt{3}}{2-\sqrt{3}}$
(C) $\frac{\sqrt{3}-2}{\sqrt{3}+2}$ **(D)** $\frac{\sqrt{3}+2}{\sqrt{3}-2}$

117. $x+y+z=15$ if 9, x, y, z, a, are in AP; while $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{5}{3}$ if 9, x, y, z, a are in HP. Then the value of a will be

(A) 1 **(B)** 2
(C) 3 **(D)** 9

118. In four numbers, first three are in GP and last three are in AP whose common difference is 6. If the first and last numbers are the same, then the first number will be

(A) 2 **(B)** 4
(C) 6 **(D)** 8

119. If a, b, c are in HP, then $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in

(A) AP **(B)** GP
(C) HP **(D)** None of these

120. If the first and $(2n-1)^{th}$ terms of an AP, GP and HP are equal and their n^{th} terms are, respectively, a, b and c, then

(A) $a \ge b \ge c$ **(B)** $a+c=b$
(C) $ac-b^2=0$ **(D)** Both (A) and (C)

121. The product of three terms of GP is 512. If 8 added to the first term and 6 added to the second term, so that the number may be in AP, then the numbers are

(A) 2, 4, 8 **(B)** 4, 8, 16
(C) 3, 6, 12 **(D)** None of these

122. If the ratio of HM and GM between two numbers a and b is $4:5$, then the ratio of the two numbers will be

(A) $1:2$ **(B)** $2:1$
(C) $4:1$ **(D)** $1:4$

123. $1+3+7+15+31+\cdots$ to n terms $=$

(A) $2^{n+1}-n$ **(B)** $2^{n+1}-n-2$
(C) 2^n-n-2 **(D)** None of these

124. $2+4+7+11+16+\cdots$ to n terms $=$

(A) $\frac{1}{6}(n^2+3n+8)$ **(B)** $\frac{n}{6}(n^2+3n+8)$
(C) $\frac{1}{6}(n^2-3n+8)$ **(D)** $\frac{n}{6}(n^2-3n+8)$

125. n^{th} term of the series $2+4+7+11+\cdots$ will be

(A) $\frac{n^2+n+1}{2}$ **(B)** n^2+n+2
(C) $\frac{n^2+n+2}{2}$ **(D)** $\frac{n^2+2n+2}{2}$

126. The sum of first n terms of the given series $1^2+2\cdot 2^2+3^2+2\cdot 4^2+5^2+2\cdot 6^2+\cdots$ is $\frac{n(n+1)^2}{2}$, when n is even. When n is odd, the sum will be

(A) $\frac{n(n+1)^2}{2}$ **(B)** $\frac{1}{2}n^2(n+1)$
(C) $n(n+1)^2$ **(D)** None of these

127. The sum of the series $1\cdot 3^2+2\cdot 5^2+3\cdot 7^2+\cdots$ up to 20 terms is

(A) 188090 **(B)** 189080
(C) 199080 **(D)** None of these

128. $\frac{2}{1!}+\frac{2+4}{2!}+\frac{2+4+6}{3!}+\cdots\infty=$

(A) e **(B)** $2e$
(C) $3e$ **(D)** None of these

129. $\frac{2}{3!}+\frac{4}{5!}+\frac{6}{7!}+\cdots\infty=$

(A) e **(B)** $2e$
(C) e^2 **(D)** $1/e$

130. $\frac{1^2\cdot 2}{1!}+\frac{2^2\cdot 3}{2!}+\frac{3^2\cdot 4}{3!}+\cdots\infty=$

(A) $6e$ **(B)** $7e$
(C) $8e$ **(D)** $9e$

131. $1+\frac{1+2}{2!}+\frac{1+2+3}{3!}+\frac{1+2+3+4}{4!}+\cdots\infty=$

(A) e **(B)** $3e$
(C) $e/2$ **(D)** $3e/2$

132. Sum of the infinite series $1+2+\frac{1}{2!}+\frac{2}{3!}+\frac{1}{4!}+\frac{2}{5!}+\cdots$ is

(A) e^2 **(B)** $e+e^{-1}$
(C) $\frac{e-e^{-1}}{2}$ **(D)** $\frac{3e-e^{-1}}{2}$

133. The sum of the series $\frac{1}{1\cdot 2}+\frac{1\cdot 3}{1\cdot 2\cdot 3\cdot 4}+\frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}+\cdots\infty$ is

(A) $15e$ **(B)** $e^{1/2}+e$
(C) $e^{1/2}-1$ **(D)** $e^{1/2}-e$

134. $1+\frac{\log_e x}{1!}+\frac{(\log_e x)^2}{2!}+\frac{(\log_e x)^3}{3!}+\cdots\infty=$

(A) $\log_e x$ **(B)** x
(C) x^{-1} **(D)** $-\log_e(1+x)$

135. $(1+3)\log_e 3+\frac{1+3^2}{2!}(\log_e 3)^2+\frac{1+3^3}{3!}(\log_e 3)^3+\cdots\infty=$

(A) 28 **(B)** 30
(C) 25 **(D)** 0

136. The coefficient of x^3 in the expansion of 3^x is

(A) $\frac{3^3}{6}$ **(B)** $\frac{(\log 3)^3}{3}$
(C) $\frac{\log(3^3)}{6}$ **(D)** $\frac{(\log 3)^3}{6}$

137. The value of $\sqrt{e}$ will be
(A) 1.648 **(B)** 1.547
(C) 1.447 **(D)** 1.348

138. The sum of the series $\frac{1}{2\cdot 3}+\frac{1}{4\cdot 5}+\frac{1}{6\cdot 7}+\cdots=$

(A) $\log(2/e)$ **(B)** $\log(e/2)$
(C) $2/e$ **(D)** $e/2$

139. The sum of $\frac{1}{2}+\frac{1}{3\cdot 2^3}+\frac{1}{5\cdot 2^5}+\cdots\infty$ is

(A) $\log_e\sqrt{\frac{3}{2}}$ **(B)** $\log_e\sqrt{3}$
(C) $\log_e\sqrt{\frac{1}{2}}$ **(D)** $\log_e 3$

140. $\log_a x$ is defined for $(a>0)$
(A) All real x **(B)** All negative real $x\neq 1$
(C) All positive real $x\neq 0$ **(D)** $a\geq e$

141. The sum of the series $\log_4 2-\log_8 2+\log_{16} 2\cdots$ is

(A) e^2 **(B)** $\log_e 2$
(C) $\log_e 3-2$ **(D)** $1-\log_e 2$

142. The coefficient of x^n in the expansion of $\log_e(1+3x+2x^2)$ is

(A) $(-1)^n\left[\frac{2^n+1}{n}\right]$ **(B)** $\frac{(-1)^{n+1}}{n}[2^n+1]$
(C) $\frac{2^n+1}{n}$ **(D)** None of these

143. If $n=(1999)!$ then $\sum_{x=1}^{1999}\log_n x$ is equal to
(A) 1 **(B)** 0
(C) $\sqrt[1999]{1999}$ **(D)** -1

144. $e^{\left(x-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\frac{1}{4}(x-1)^4+\cdots\right)}$ is equal to
(A) $\log x$ **(B)** $\log(x-1)$
(C) x **(D)** None of these

145. If x, $|x+1|$, $|x-1|$ are the three terms of an AP, its sum up to 20 terms is
(A) 90 or 175 **(B)** 180 or 350
(C) 360 or 700 **(D)** 720 or 1400

146. If a, b and c are positive real numbers then $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$ is greater than or equal to
(A) 3 **(B)** 6
(C) 27 **(D)** None of these

147. If a, b and c are positive real numbers, then the least value of $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ is
(A) 9 **(B)** 3
(C) 10/3 **(D)** None of these

148. If the sides of a right-angled triangle form an AP then the sine of the acute angle is
(A) $\frac{3}{5},\frac{4}{5}$ **(B)** $\frac{3}{4},\frac{3}{5}$
(C) $\frac{2}{5},\frac{3}{5}$ **(D)** None of these

149. If a, b, c, d and p are distinct real numbers such that $(a^2+b^2+c^2)p^2-2(ab+bc+cd)\,p+(b^2+c^2+d^2)\leq 0$, then a, b, c, d are in
(A) AP **(B)** GP
(C) HP **(D)** None of these

150. If S is the sum, p the product and R the sum of the reciprocals of n terms of a GP, then $(S/R)^n$ is equal to
(A) p^2 **(B)** p^3
(C) p **(D)** None of these

151. If $t_r=\frac{r+2}{r(r+1)}\times\frac{1}{2^{r+1}}$, then $\sum_{r=1}^{n}t_r$ is equal to

(A) $\frac{n2^n-1}{n+1}$ **(B)** $\frac{n+1}{2^{n+1}(n+2)}$

(C) $\frac{n}{2^n}-1$ **(D)** $\frac{(n+1)2^n-1}{2^{n+1}(n+1)}$

152. $\sum_{j=1}^{n}\sum_{i=1}^{n} i$ is equal to

(A) $\frac{n(n+1)}{2}$ **(B)** $\frac{n(n+1)^2}{2}$

(C) $\frac{n^2(n+1)}{2}$ **(D)** None of these

153. If $\frac{1}{a}+\frac{1}{a-2b}+\frac{1}{c}+\frac{1}{c-2b}=0$ and a, b, c are not in AP, then

(A) a, b, c are in GP **(B)** $a, \frac{b}{2}, c$ are in AP

(C) $a, \frac{b}{2}, c$ are in HP **(D)** $a, 2b, c$ are in HP

154. If $\frac{3+5+7+\cdots+n \text{ terms}}{5+8+11+\cdots+10 \text{ terms}}=7$, then the value of n is

(A) 35 **(B)** 36
(C) 37 **(D)** 40

155. If the sum of an infinite GP and the sum of the squares of its terms are both equal to 5, then the first term is

(A) $\frac{5}{3}$ **(B)** $\frac{2}{3}$

(C) $\frac{7}{3}$ **(D)** None of these

156. The sum of the infinite series $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\cdots$ is

(A) $\frac{1}{3}$ **(B)** $\frac{1}{5}$

(C) $\frac{2}{3}$ **(D)** None of these

157. If $\frac{a^n+b^n}{a^{n-1}+b^{n-1}}$ is the GM between distinct positive numbers a and b, then the value of n is

(A) 0 **(B)** 1
(C) 1/2 **(D)** None of these

158. If $\frac{a^n+b^n}{a^{n-1}+b^{n-1}}$ is the AM between distinct positive numbers a and b, then the value of n is

(A) 0 **(B)** 1
(C) –1 **(D)** None of these

159. If three positive real numbers a, b, c are in AP, with $abc = 4$, then the minimum value of b is

(A) $4^{1/3}$ **(B)** 3
(C) 2 **(D)** 1/2

160. If a, b and c are three positive real numbers, then the minimum value of the expression $\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}$ is

(A) 1 **(B)** 2
(C) 3 **(D)** 6

161. If a, b, c are three positive real numbers such that $b + c - a$, $c + a - b$ and $a + b - c$ are positive, the expression $(b + c - a)(c + a - b)(a + b - c) - abc$ is

(A) Positive **(B)** Negative
(C) Non-positive **(D)** Non-negative

162. $\frac{n}{(n+1)!}+\frac{(n+1)}{(n+2)!}+\cdots+\frac{(n+p)}{(n+p+1)!}$ is equal to

(A) $\frac{1}{n!}-\frac{1}{(n+p+1)!}$ **(B)** $\frac{1}{n!}-\frac{1}{(n-p-1)!}$

(C) $\frac{1}{n!}+\frac{1}{(n+p+1)!}$ **(D)** None of these

163. The sum of two numbers is $2\frac{1}{6}$ and even numbers of AMs are inserted between them. If the sum of these means exceeds their number by 1, then the number of means is

(A) 11 **(B)** 12
(C) 13 **(D)** 14

164. The first term of an infinite GP is 1 and any term is equal to the sum of all the succeeding terms. The common ratio of the GP is

(A) $\frac{1}{3}$ **(B)** $\frac{1}{2}$

(C) $\frac{1}{6}$ **(D)** $\frac{1}{4}$

165. Let $p = 3^{1/3}\, 3^{2/9}\, 3^{3/27} \ldots \infty$. Then $p^{1/3}$ is equal to

(A) $3^{2/3}$ **(B)** $\sqrt{3}$
(C) $3^{1/3}$ **(D)** $3^{1/4}$

166. If $x_i > 0$, $i = 1, 2, \ldots, 50$ and $x_1 + x_2 + \cdots + x_{50} = 50$, then the minimum value of $\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_{50}}$ equals

(A) 50 **(B)** $(50)^2$
(C) $(50)^3$ **(D)** $(50)^4$

167. If $H_1, H_2, \cdots, H_n$ are n harmonic means between two numbers a and b, then the value of $\frac{H_n+a}{H_1-a}+\frac{H_n+b}{H_n-b}$ is

(A) n **(B)** $2n$

(C) $\frac{1}{n}$ **(D)** $\frac{2}{n}$

168. The largest interval for which the series $1 + (x - 1) + (x - 1)^2 + \cdots \infty$ may be summed is

(A) $0 < x < 1$ **(B)** $0 < x < 2$
(C) $-1 < x < 1$ **(D)** $-2 < x < 2$

169. If $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$ up to $\infty = \frac{\pi^2}{6}$, then $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$ equals

(A) $\pi^2/6$ **(B)** $\pi^2/16$
(C) $\pi^2/8$ **(D)** None of these

170. Let T_r and S_r be the r^{th} term and sum up to r^{th} term of a series, respectively. If for odd number n, $S_n = n$ and $T_n = \frac{T_{n-1}}{n^2}$, then T_m (m being even) is

(A) $\frac{2}{1+m^2}$ **(B)** $\frac{2m^2}{1+m^2}$

(C) $\frac{(m+1)^2}{2+(m+1)^2}$ **(D)** $\frac{2(m+1)^2}{1+(m+1)^2}$

171. Let $a_1, a_2, a_3, \ldots$ be in AP and a_p, a_q, a_r be in GP. Then $a_q : a_p$ is equal to

(A) $\frac{r-p}{q-p}$ **(B)** $\frac{q-p}{r-q}$

(C) $\frac{r-q}{q-p}$ **(D)** None of these

172. If $H_1, H_2, H_3, \ldots, H_{2n+1}$ are in HP, then $\sum_{i=1}^{2n}(-1)^i\left(\frac{H_i+H_{i+1}}{H_i-H_{i+1}}\right)$ is equal to

(A) $2n-1$ **(B)** $2n+1$

(C) $2n$ **(D)** $2n+2$

173. Consider the sequence 1, 2, 2, 4, 4, 4, 4, 8, 8, 8, 8, 8, 8, 8, 8, … where n consecutive terms have value n. Then 1025^{th} term is

(A) 2^9 **(B)** 2^{10}

(C) 2^{11} **(D)** 2^8

174. If $S_n = \sum_{r=1}^{n} T_r = n(n+1)(n+2)(n+3)$ then $\sum_{r=1}^{10}\frac{1}{T_r}$ is equal to

(A) $\frac{55}{527}$ **(B)** $\frac{58}{528}$

(C) $\frac{59}{528}$ **(D)** None of these

175. a, b, c, d, e are five numbers in which the first three are in AP and the last three are in HP. If the three numbers in the middle are in GP then the numbers at the odd places are in

(A) AP **(B)** GP

(C) HP **(D)** None of these

176. The coefficient of x^{15} in the product of $(1-x)(1-2x)(1-2^2x)(1-2^3x)\cdots(1-2^{15}x)$ is equal to

(A) $2^{105}-2^{121}$ **(B)** $2^{121}-2^{105}$

(C) $2^{120}-2^{104}$ **(D)** None of these

177. Sum of n terms of the series $(2n-1) + 2(2n-3) + 3(2n-5) + \cdots$ is

(A) $\frac{n(n+1)(2n+1)}{6}$ **(B)** $\frac{n(n+1)(2n-1)}{6}$

(C) $\frac{n(n-1)(2n-1)}{6}$ **(D)** None of these

178. The sum of n terms of the series

$$\frac{5}{1\cdot 2}\cdot\frac{1}{3}+\frac{7}{2\cdot 3}\cdot\frac{1}{3^2}+\frac{9}{3\cdot 4}\cdot\frac{1}{3^3}+\frac{11}{4\cdot 5}\cdot\frac{1}{3^4}+\cdots \text{ is}$$

(A) $1-\frac{1}{n+1}\cdot\frac{1}{3^n}$ **(B)** $1+\frac{1}{n+1}\cdot\frac{1}{3^n}$

(C) $1-\frac{1}{n-1}\cdot\frac{1}{3^{n+1}}$ **(D)** None of these

179. Let $t_r = 2^{r/2}+2^{-r/2}$. Then $\sum_{r=1}^{10} t_r^2$ is equal to

(A) $\frac{2^{21}-1}{2^{10}}+20$ **(B)** $\frac{2^{21}-1}{2^{10}}+19$

(C) $\frac{2^{21}-1}{20}-1$ **(D)** None of these

180. In a sequence of $(4n + 1)$ terms the first $(2n + 1)$ terms are in AP whose common difference is 2 and the last $(2n + 1)$ terms are in GP whose common ratio is 0.5. If the middle terms of the AP and GP are equal then the middle term of the sequence is

(A) $\frac{n\cdot 2^{n+1}}{2^n-1}$ **(B)** $\frac{n\cdot 2^{n+1}}{2^{2n}-1}$

(C) $n\cdot 2^n$ **(D)** None of these

181. The sum to n terms of the series $6\cdot 9 + 12\cdot 21 + 20\cdot 37 + 30\cdot 57 + 42\cdot 81 + \cdots$ is

(A) $\frac{2}{5}n(n+1)(n+2)(n+3)(n+4)+\frac{1}{3}(n+1)(n+2)(n+3)$

(B) $\frac{2}{5}n(n+1)(n+2)(n+3)(n+4)-\frac{1}{3}(n+1)(n+2)(n+3)$

(C) $\frac{2}{5}n(n-1)(n-2)(n-3)(n-4)+\frac{1}{3}(n-1)(n-2)(n-3)$

(D) None of these

182. The n^{th} term of the series 10, 23, 60, 169, 494, ⋯ is

(A) $2\cdot 3^n+n+3$ **(B)** $2\cdot 3^n-n-3$

(C) $2\cdot 3^{n+1}+n+3$ **(D)** None of these

183. The value of $\sum_{n=0}^{m}\log\frac{a^{2n-1}}{b^{m-1}}$ $(a\neq 0, 1; b\neq 0, 1)$ is

(A) $m\log\frac{a^{2m}}{b^{m-1}}$ **(B)** $(m+1)\log\frac{a}{b^{m-1}}$

(C) $\frac{m}{2}\log\frac{a^{2m}}{b^{2m-2}}$ **(D)** $\frac{m}{2}\log\frac{a^{2m}}{b^{m+1}}$

184. A_r; $r = 1, 2, 3, \ldots, n$ are n points on the parabola $y^2 = 4x$ in the first quadrant. If $A_r \equiv (x_r, y_r)$ where $x_1, x_2, \ldots, x_n$ are in GP and $x_1 = 1, x_2 = 2$ then y_n is equal to

(A) $2^{\frac{n+1}{2}}$ **(B)** 2^{n+1}

(C) $(\sqrt{2})^{n+1}$ **(D)** $2^{n/2}$

185. Let $S_k = \lim_{n\to\infty}\sum_{i=0}^{n}\frac{1}{(k+1)^i}$. Then $\sum_{k=1}^{n} kS_k$ equals

(A) $\frac{n(n+1)}{2}$ **(B)** $\frac{n(n-1)}{2}$

(C) $\frac{n(n+2)}{2}$ **(D)** $\frac{n(n+3)}{2}$

186. The sum of the products of every pair of the first n natural numbers is

(A) $\frac{n(n+1)(3n^2-n-2)}{24}$ **(B)** $\frac{n(n+1)(3n^2+n+2)}{24}$

(C) $\frac{n(n-1)(3n^2-n-2)}{24}$ **(D)** None of these

187. In a given square, a diagonal is drawn and parallel line segments joining points on the adjacent sides are drawn on both sides of the diagonals. The length of the diagonal is $n\sqrt{2}$ cm. If the distance between consecutive line segments is $\frac{1}{\sqrt{2}}$ cm, then the sum of the lengths of all possible line segments and the diagonal is

(A) $n(n+1)\sqrt{2}$ cm **(B)** n^2 cm

(C) $n(n+2)$ cm **(D)** $n^2\sqrt{2}$ cm

188. The sum of the series $1{\cdot}2{\cdot}3 + 2{\cdot}3{\cdot}4 + 3{\cdot}4{\cdot}5 + \cdots$ to n terms is

(A) $\frac{n(n+1)(n+2)(n+3)}{5}$ **(B)** $\frac{n(n+1)(n+2)(n+3)}{4}$

(C) $\frac{n(n+1)(n+2)(3n+13)}{12}$ **(D)** None of these

189. If $a_1, a_2, a_3, \cdots$ are in HP and $f(k)=\sum_{r=1}^{n} a_r - a_k$, then $\frac{f(1)}{a_1}, \frac{f(2)}{a_2}, \frac{f(3)}{a_3}, \ldots, \frac{f(n)}{a_n}$ are in

(A) AP **(B)** GP

(C) HP **(D)** None of these

190. Let $a_1 = 0$ and $a_1, a_2, a_3, \cdots, a_n$ be real numbers such that $|a_i| = |a_{i-1}+1|$ for all $i = 0, 1, 2, \cdots, n$. If the AM of the numbers $a_1, a_2, a_3, \cdots, a_n$ has the value x, then

(A) $x < 1$ **(B)** $x < -\frac{1}{2}$

(C) $x \geq -\frac{1}{2}$ **(D)** $x \geq 1$

191. If a, b, c are in GP and $\log a - \log 2b, \log 2b - \log 3c$ and $\log 3c - \log a$ are AP, then a, b, c are the lengths of the sides of a triangle which is

(A) acute angled **(B)** obtuse angled

(C) right angled **(D)** no triangle will be formed

192. $\dfrac{\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots\infty}{1+\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+\cdots\infty}=$

(A) $\frac{e+1}{e-1}$ **(B)** $\frac{e-1}{e+1}$

(C) $\frac{e^2+1}{e^2-1}$ **(D)** $\frac{e^2-1}{e^2+1}$

193. $\frac{1}{1!}+\frac{4}{2!}+\frac{7}{3!}+\frac{10}{4!}+\cdots\infty=$

(A) $e+4$ **(B)** $2+e$

(C) $3+e$ **(D)** e

194. In the expansion of $\frac{1+x}{1!}+\frac{(1+x)^2}{2!}+\frac{(1+x)^3}{3!}+\cdots$, the coefficient of x^n will be

(A) $\frac{1}{n!}$ **(B)** $\frac{1}{n!}+\frac{1}{(n+1)!}$

(C) $\frac{e}{n!}$ **(D)** $e\left[\frac{1}{n!}+\frac{1}{(n+1)!}\right]$

195. If n is even, then in the expansion of $\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)^2$, the coefficient of x^n is

(A) $\frac{2^n}{n!}$ **(B)** $\frac{2^n-2}{n!}$

(C) $\frac{2^{n-1}-1}{n!}$ **(D)** $\frac{2^{n-1}}{n!}$

196. $1+\frac{1+2}{1!}+\frac{1+2+3}{2!}+\frac{1+2+3+4}{3!}+\cdots\infty=$

(A) 0 **(B)** 1

(C) $\frac{7e}{2}$ **(D)** $2e$

197. $1{\cdot}5+\frac{2\cdot 6}{1!}+\frac{3\cdot 7}{2!}+\frac{4\cdot 8}{3!}+\cdots$ is equal to

(A) $13e$ **(B)** $15e$

(C) $9e+1$ **(D)** $5e$

198. If $S=\sum_{n=0}^{\infty}\frac{(\log x)^{2n}}{(2n)!}$, then $S =$

(A) $x+x^{-1}$ **(B)** $x-x^{-1}$

(C) $\frac{1}{2}(x+x^{-1})$ **(D)** None of these

199. The sum of the series $\frac{4}{1!}+\frac{11}{2!}+\frac{22}{3!}+\frac{37}{4!}+\frac{56}{5!}+\cdots$ is

(A) $6e$ **(B)** $6e-1$

(C) $5e$ **(D)** $5e+1$

200. In the expansion of $\frac{a+bx+cx^2}{e^x}$, the coefficient of x^n will be

(A) $\frac{a(-1)^n}{n!}+\frac{b(-1)^{n-1}}{(n-1)!}+\frac{c(-1)^{n-2}}{(n-2)!}$

(B) $\frac{a}{n!}+\frac{b}{(n-1)!}+\frac{c}{(n-2)!}$

(C) $\frac{(-1)^n}{n!}+\frac{(-1)^{n-1}}{(n-1)!}+\frac{(-1)^{n-2}}{(n-2)!}$

(D) None of these

201. If m,n are the roots of the equation $x^2-x-1=0$, then the value of

$$\frac{\left(1+m\log_e 3+\frac{(m\log_e 3)^2}{2!}+\cdots\infty\right)\left(1+n\log_e 3+\frac{(n\log_e 3)^2}{2!}+\cdots\infty\right)}{\left(1+mn\log_e 3+\frac{(mn\log_e 3)^2}{2!}+\cdots\infty\right)}$$

(A) 9 **(B)** 3
(C) 0 **(D)** 1

202. $\frac{1}{3}+\frac{1}{2\cdot 3^2}+\frac{1}{3\cdot 3^3}+\frac{1}{4\cdot 3^4}+\cdots\infty=$

(A) $\log_e 2-\log_e 3$ **(B)** $\log_e 3-\log_e 2$
(C) $\log_e 6$ **(D)** None of these

203. If $|x|<1$, then the coefficient of x^5 in the expansion of $(1-x)\log_e(1-x)$ is

(A) 1/2 **(B)** 1/4
(C) 1/20 **(D)** 1/10

204. The sum of the series $1+\frac{3}{2!}+\frac{7}{3!}+\frac{15}{4!}+\cdots$ to ∞ is

(A) $e(e+1)$ **(B)** $e(e-1)$
(C) $3e-1$ **(D)** $3e$

205. $1+\frac{2}{3}-\frac{2}{4}+\frac{2}{5}-\cdots\infty=$

(A) $\log_e 3$ **(B)** $\log_e 4$
(C) $\log_e\left(\frac{e}{2}\right)$ **(D)** $\log_e\left(\frac{2}{3}\right)$

206. The value of $\log_e\left(1+ax^2+a^2+\frac{a}{x^2}\right)$ is

(A) $a\left(x^2-\frac{1}{x^2}\right)-\frac{a^2}{2}\left(x^4-\frac{1}{x^4}\right)+\frac{a^3}{3}\left(x^6-\frac{1}{x^6}\right)-\cdots$

(B) $a\left(x^2+\frac{1}{x^2}\right)-\frac{a^2}{2}\left(x^4+\frac{1}{x^4}\right)+\frac{a^3}{3}\left(x^6+\frac{1}{x^6}\right)-\cdots$

(C) $a\left(x^2+\frac{1}{x^2}\right)+\frac{a^2}{2}\left(x^4+\frac{1}{x^4}\right)+\frac{a^3}{3}\left(x^6+\frac{1}{x^6}\right)+\cdots$

(D) $a\left(x^2-\frac{1}{x^2}\right)+\frac{a^2}{2}\left(x^4-\frac{1}{x^4}\right)+\frac{a^3}{3}\left(x^6-\frac{1}{x^6}\right)+\cdots$

Practice Exercise 2

Single/Multiple Correct Choice Type Questions

1. If $S_{(n)}=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$, $(n\in N)$, then $S_{(1)}+S_{(2)}+\cdots+S_{(n-1)}$ is equal to

(A) $nS_{(n)}-n$ **(B)** $nS_{(n)}-1$
(C) $(n-1)S_{(n-1)}-n$ **(D)** $nS_{(n-1)}-n+1$

2. $T_r=\frac{1}{r\sqrt{r+1}+(r+1)\sqrt{r}}$, then (here $r\in N$)

(A) $T_r>T_{r+1}$ **(B)** $T_r<T_{r+1}$
(C) $\sum_{r=1}^{99}T_r=\frac{9}{10}$ **(D)** $\sum_{r=1}^{n}T_r<1$

3. Let $f(n)$ be the sum of first n terms of sequence 0, 1, 1, 2, 2, 3, 3, 4, 4, $\cdots$. Then

(A) $f(n)=\frac{n^2}{4}$, where n is an even number

(B) $f(n)=\frac{n^2-1}{4}$, where n is an odd number

(C) $f(n+m)-f(n-m)=nm$, where $n, m\in I^+ (n>m)$

(D) $f(n+m)-f(n-m)=\frac{4nm+1}{1}$, where $n, m\in I^+ (n>m)$

4. If $N=\underbrace{111\ldots1}_{n\text{ times}}$, then

(A) N is divisible by 71 whenever n is a multiple of 5
(B) N is divisible by 91 whenever n is a multiple of 6
(C) N is divisible by 41 whenever n is a multiple of 5
(D) N is divisible by 71 whenever n is a multiple of 6

5. If $x=\sum_{r=1}^{\infty}\frac{1}{(2r-1)2r}$, $y=\sum_{r=1}^{\infty}\frac{1}{2r(2r+1)}$, $z=\sum_{r=1}^{\infty}\frac{1}{(2r-1)2r(2r+1)}$, then

(A) $x+z=1$ **(B)** $x+y=1$
(C) $y+z=\frac{1}{2}$ **(D)** $x-z=\frac{1}{2}$

6. Given $a+b=50$, $a, b\in R^+$. If A, G and H are, respectively, the AM, GM and HM between the numbers a and b, such that the GM exceeds HM by 4, then (where $A>1$, $G>1$, $H>1$)

(A) $A+G=30H$ **(B)** $G+H=A+11$
(C) $4(G+H)=A-1$ **(D)** $A+G=3(H-1)$

7. For the series

$S=1+\frac{1}{(1+3)}(1+2)^2+\frac{1}{(1+3+5)}(1+2+3)^2+\frac{1}{(1+3+5+7)}(1+2+3+4)^2+\cdots$

(A) 7^{th} term is 16
(B) 7^{th} term is 18

(C) sum of first 10 terms is $\dfrac{505}{4}$

(D) sum of first 10 term is $\dfrac{405}{4}$

8. If $\sum_{r=1}^{n} r(r+1) = \dfrac{(n+a)(n+b)(n+c)}{3}$, where $a < b < c$, then

(A) $2b = c$ **(B)** $a^3 - 8b^3 + c^3 = 8abc$
(C) c is a prime number **(D)** $(a+b)^2 = 0$

9. In a GP the ratio of the sum of the first 11 terms to the sum of the last 11 terms is $\dfrac{1}{8}$ and the ratio of the sum of all the terms without the first nine to the sum of all the terms without the last nine is 2. Then the number of terms of the GP is less than

(A) 15 **(B)** 43
(C) 38 **(D)** 56

10. Let $a_n = \underset{n \text{ times}}{(111\ldots1)}$. Then

(A) a_{912} is not prime **(B)** a_{951} is not prime
(C) a_{480} is not prime **(D)** a_{91} is not prime

Comprehension Type Questions

Paragraph for Questions 11–15: Let $A_1, A_2, A_3, \cdots, A_m$ be the arithmetic means between –2 and 1027 and $G_1, G_2, G_3, \cdots, G_n$ be the geometric means between 1 and 1024. Product of the geometric means is 2^{45} and the sum of arithmetic means is 1025×171.

11. The value of n is

(A) 7 **(B)** 9
(C) 11 **(D)** None of these

12. The value of m is

(A) 340 **(B)** 342
(C) 344 **(D)** 346

13. The value of $G_1 + G_2 + G_3 + \cdots + G_n$ is

(A) 1022 **(B)** 2044
(C) 512 **(D)** None of these

14. The common difference of the progression $A_1, A_3, A_5, \cdots, A_{m-1}$ is

(A) 6 **(B)** 3
(C) 2 **(D)** 1

15. The numbers $2A_{171}, G_5^2 + 1, 2A_{172}$ are in

(A) AP **(B)** GP
(C) HP **(D)** AGP

Paragraph for Questions 16–18: There are two sets A and B each of which consists of three numbers in AP whose sum is 15 and where D and d are the common differences such that $D - d = 1$. If $\dfrac{p}{q} = \dfrac{7}{8}$ where p and q are the product of the numbers, respectively, and $d > 0$, in the two sets, then

16. Value of p is

(A) 100 **(B)** 120
(C) 105 **(D)** 110

17. Value of q is

(A) 100 **(B)** 120
(C) 105 **(D)** 110

18. Value of $D + d$ is

(A) 1 **(B)** 2
(C) 3 **(D)** 4

Paragraph for Questions 19–21: Four different integers form an increasing AP. One of these numbers is equal to the sum of the squares of the other three numbers. Then

19. The smallest number is

(A) –2 **(B)** 0
(C) –1 **(D)** 2

20. The common difference of the four numbers is

(A) 2 **(B)** 1
(C) 3 **(D)** 4

21. The sum of all the four numbers is

(A) 10 **(B)** 8
(C) 2 **(D)** 6

Matrix Match Type Questions

22. Match the following:

Column I	Column II
(A) Suppose that $F(n+1) = \dfrac{2F(n)+1}{2}$ for $n = 1, 2, 3, \cdots$ and $F(1) = 2$. Then $F(101)$ equals	**(p)** 42
(B) If $a_1, a_2, a_3, \cdots a_{21}$ are in AP and $a_3 + a_5 + a_{11} + a_{17} + a_{19} = 10$ then the value of $\sum_{i=1}^{21} a_i$ is	**(q)** 1620
(C) 10^{th} term of the sequence $S = 1 + 5 + 13 + 29 + \cdots$ is	**(r)** 52
(D) The sum of all two digit numbers which are not divisible by 2 or 3 is	**(s)** 2045
	(t) $2 + 4 + 6 + \cdots + 12$

23. Match the following:

Column I	Column II
(A) The arithmetic mean of the two positive numbers is 6 and their geometric mean G and harmonic mean H satisfy the relation $G^2 + 3H = 48$. Then the product of the two numbers is	**(p)** $\dfrac{2}{7}$
(B) The sum of the series $\dfrac{5}{1^2 \cdot 4^2} + \dfrac{11}{4^2 \cdot 7^2} + \dfrac{17}{7^2 \cdot 10^2} + \cdots$ is	**(q)** 32
(C) If the first two terms of a harmonic progression are $\dfrac{1}{2}$ and $\dfrac{1}{3}$, then the harmonic mean of the first four terms is	**(r)** $\dfrac{1}{3}$
(D) Geometric mean of –4 and –9	**(s)** 6
	(t) –6

Integer Type Questions

24. The integers from 1 to 1000 are written in order around a circle. Starting at 1, every fifteenth number is marked (that is 1, 16, 31, etc.). This process is continued until a number is reached which has already been marked. If the number of marked numbers is A, then find $\frac{A}{50}$.

25. A sequence is obtained by deleting all perfect squares from a set of natural numbers. Find the remainder when the 2003rd term of the new sequence is divided by 2048.

26. An eccentric person starts writing numbers from 1 to n in a row such that i^{th} number is written i^2 times. Then find the 500th digit from the starting.

27. Find the sum to infinity of a decreasing GP with the common ratio x such that $|x| < 1$; $x \neq 0$. The ratio of the fourth term to the second term is $\frac{1}{16}$ and the ratio of third term to the square of the second term is $\frac{1}{9}$.

28. If $\sum_{\alpha=4}^{n+3} 4(\alpha - 3) = An^2 + Bn + C$, then find the value of $A + B - C$.

29. If $(1 - P)(1 + 3x + 9x^2 + 27x^3 + 81x^4 + 243x^5) = 1 - P^6$, $P \neq 1$, then find the value of $\frac{P}{x}$.

30. If $(1^2 - a_1) + (2^2 - a_2) + (3^2 - a_3) + \cdots + (n^2 - a_n) = \frac{1}{3} n(n^2 - 1)$, then find the value of a_7.

31. The sum of the terms of an infinitely decreasing GP is equal to the greatest value of the function $f(x) = x^3 + 3x - 9$ on the interval $[-4, 3]$ and the difference between the first and second terms is $f'(0)$. Find the value of $27r$ where r is the common ratio.

Answer Key

Practice Exercise 1

1. (B)	**2.** (A)	**3.** (A)	**4.** (B)	**5.** (C)	**6.** (C)
7. (D)	**8.** (B)	**9.** (A)	**10.** (D)	**11.** (D)	**12.** (C)
13. (D)	**14.** (C)	**15.** (D)	**16.** (C)	**17.** (A)	**18.** (B)
19. (B)	**20.** (B)	**21.** (C)	**22.** (A)	**23.** (A)	**24.** (C)
25. (A)	**26.** (D)	**27.** (C)	**28.** (C)	**29.** (D)	**30.** (B)
31. (D)	**32.** (B)	**33.** (D)	**34.** (B)	**35.** (A)	**36.** (B)
37. (A)	**38.** (B)	**39.** (A)	**40.** (B)	**41.** (B)	**42.** (C)
43. (B)	**44.** (B)	**45.** (C)	**46.** (A)	**47.** (C)	**48.** (B)
49. (B)	**50.** (A)	**51.** (B)	**52.** (B)	**53.** (D)	**54.** (D)
55. (C)	**56.** (A)	**57.** (B)	**58.** (B)	**59.** (C)	**60.** (A)
61. (B)	**62.** (B)	**63.** (D)	**64.** (B)	**65.** (A)	**66.** (C)
67. (B)	**68.** (A)	**69.** (C)	**70.** (A)	**71.** (C)	**72.** (B)
73. (A)	**74.** (A)	**75.** (A)	**76.** (A)	**77.** (C)	**78.** (A)
79. (C)	**80.** (D)	**81.** (A)	**82.** (B)	**83.** (C)	**84.** (B)
85. (C)	**86.** (C)	**87.** (D)	**88.** (D)	**89.** (C)	**90.** (B)
91. (A)	**92.** (B)	**93.** (D)	**94.** (A)	**95.** (D)	**96.** (D)
97. (C)	**98.** (B)	**99.** (B)	**100.** (A)	**101.** (C)	**102.** (D)
103. (C)	**104.** (A)	**105.** (B)	**106.** (B)	**107.** (C)	**108.** (C)
109. (A)	**110.** (A)	**111.** (C)	**112.** (C)	**113.** (C)	**114.** (A)
115. (A)	**116.** (B)	**117.** (A)	**118.** (D)	**119.** (C)	**120.** (D)
121. (B)	**122.** (C), (D)	**123.** (B)	**124.** (B)	**125.** (C)	**126.** (B)
127. (A)	**128.** (C)	**129.** (D)	**130.** (B)	**131.** (D)	**132.** (D)
133. (C)	**134.** (B)	**135.** (A)	**136.** (D)	**137.** (A)	**138.** (B)
139. (B)	**140.** (C)	**141.** (D)	**142.** (B)	**143.** (A)	**144.** (D)
145. (A)	**146.** (A)	**147.** (A)	**148.** (A)	**149.** (B)	**150.** (A)
151. (D)	**152.** (C)	**153.** (D)	**154.** (A)	**155.** (A)	**156.** (A)
157. (C)	**158.** (B)	**159.** (A)	**160.** (D)	**161.** (C)	**162.** (A)
163. (B)	**164.** (B)	**165.** (D)	**166.** (A)	**167.** (B)	**168.** (B)
169. (C)	**170.** (D)	**171.** (C)	**172.** (C)	**173.** (B)	**174.** (D)
175. (B)	**176.** (A)	**177.** (A)	**178.** (A)	**179.** (B)	**180.** (A)
181. (A)	**182.** (A)	**183.** (B)	**184.** (A)	**185.** (D)	**186.** (A)
187. (D)	**188.** (B)	**189.** (A)	**190.** (C)	**191.** (B)	**192.** (B)
193. (B)	**194.** (C)	**195.** (D)	**196.** (C)	**197.** (A)	**198.** (C)
199. (B)	**200.** (A)	**201.** (A)	**202.** (B)	**203.** (C)	**204.** (B)
205. (B)	**206.** (B)				

Practice Exercise 2

1. (A), (D)   2. (A), (C), (D)   3. (A), (B), (C)   4. (B), (C)   5. (B), (C), (D)   6. (B), (D)
7. (A), (C)   8. (A), (B), (C)   9. (B), (D)   10. (A), (B), (C), (D)   11. (B)   12. (B)
13. (A)   14. (A)   15. (A)   16. (C)   17. (B)   18. (C)
19. (C)   20. (B)   21. (C)
22. (A) $\to$ (r), (B) $\to$ (p, t), (C) $\to$ (s), (D) $\to$ (q)   23. (A) $\to$ (q), (B) $\to$ (r), (C) $\to$ (p), (D) $\to$ (t)
24. 4   25. 0   26. 11   27. 12   28. 4   29. 3
30. 7   31. 18

Solutions

Practice Exercise 1

1. Given that

$$T_p = a+(p-1)d = q \quad (1)$$

and

$$T_q = a+(q-1)d = p \quad (2)$$

From Eqs. (1) and (2), we get

$$d = -\frac{(p-q)}{(p-q)} = -1$$

Putting value of d in Eq. (1), we get $a = p+q-1$

Now, r^{th} term is given by AP.

$$T_r = a+(r-1)d = (p+q-1)+(r-1)(-1) = p+q-r$$

Note: Students should remember this question as a formula.

2. We have

$$\tan n\theta = \tan m\theta \Rightarrow n\theta = N\pi + (m\theta) \Rightarrow \theta = \frac{N\pi}{n-m}$$

Putting $N = 1, 2, 3, \cdots$ we get

$$\frac{\pi}{n-m}, \frac{2\pi}{n-m}, \frac{3\pi}{n-m}, \cdots$$

which are obviously in AP.

Since,

$$\text{Common difference, } d = \frac{\pi}{n-m}$$

3. Given series $3\cdot 8 + 6\cdot 11 + 9\cdot 14 + 12\cdot 17 + \cdots$

First factors are 3, 6, 9, 12 whose n^{th} term is $3n$ and second factors are 8, 11, 14, 17.

$$t_n = [8+(n-1)3] = (3n+5)$$

Hence, n^{th} term of given series $= 3n(3n+5)$

4. The sum of integers from 1 to 100 that are divisible by 2 or 5 = sum of series divisible by 2 + sum of series divisible by 5 – sum of series divisible by 2 and 5

$$= (2+4+6+\cdots+100)+(5+10\cdots+100)$$
$$-(10+20+30+\cdots+100)$$
$$= \frac{50}{2}\{2\times 2+(50-1)2\} + \frac{20}{2}\{2\times 5+(20-1)5\}$$
$$-\frac{10}{2}\{10\times 2+(10-1)10\}$$
$$= 2550+1050-550 = 3050$$

5. Given series

$$63+65+67+69+\cdots \quad (1)$$

and

$$3+10+17+20+\cdots \quad (2)$$

Now from Eq. (1), m^{th} term $= (2m+61)$ and m^{th} term of Eq. (2) series $= (7m-4)$

Under given condition,

$$7m-4 = 2m+61 \Rightarrow 5m = 65 \Rightarrow m = 13$$

6. $2x, x+8, 3x+1$ are in AP. Therefore,

$$(x+8) = \frac{(2x)+(3x+1)}{2} = \frac{5x+1}{2}$$

$$\Rightarrow 2x+16 = 5x+1 \Rightarrow 3x = 15 \Rightarrow x = 5$$

7. Given that $S_n = nA + n^2B$

Putting $n = 1, 2, 3, \cdots$, we get

$$S_1 = A+B, S_2 = 2A+4B, S_3 = 3A+9B$$

....

....

Therefore,

$$T_1 = S_1 = A+B, T_2 = S_2 - S_1 = A+3B$$
$$T_3 = S_3 - S_2 = A+5B$$

....

....

Hence, the sequence is $(A+B), (A+3B), (A+5B), \ldots$

Here, $a = A+B$ and common difference $d = 2B$.

8. $T_9 = a+8d = 35$ and $T_{19} = a+18d = 75$

Solving the equations, we get $d = 4$ and $a = 3$

Hence, 20^{th} term of AP is

$$a+19d = 3+19\times 4 = 79$$

9. Given series

$$27+9+5\cdot\frac{2}{5}+3\cdot\frac{6}{7}+\cdots$$

$$= 27+\frac{27}{3}+\frac{27}{5}+\frac{27}{7}+\cdots+\frac{27}{2n-1}+\cdots$$

Hence, the n^{th} term of given series $T_n = \dfrac{27}{2n-1}$. So

$$T_9 = \frac{27}{2\times 9-1} = \frac{27}{17} = 1\frac{10}{17}$$

10. If a, b, c are in AP $\Rightarrow 2b = a+c$. So

$$\frac{(a-c)^2}{(b^2-ac)} = \frac{(a-c)^2}{\left\{\left(\frac{a+c}{2}\right)-ac\right\}}$$

$$= \frac{(a-c)^2 4}{[a^2+c^2+2ac-4ac]} = \frac{4(a-c)^2}{(a-c)^2} = 4$$

Trick: Put $a=1, b=2, c=3$. Then the required value is $\frac{4}{1} = 4$.

11. $\log_3 2, \log_3(2^x-5)$ and $\log_3\left(2^x - \frac{7}{2}\right)$ are in AP

$$\Rightarrow 2\log_3(2^x-5) = \log_3\left[(2)\left(2^x-\frac{7}{2}\right)\right]$$

$$\Rightarrow (2^x-5)^2 = 2^{x+1}-7 \Rightarrow 2^{2x} - 12\cdot 2^x - 32 = 0$$

$$\Rightarrow x = 2, 3$$

But $x = 2$ does not hold.

Hence, $x = 3$.

12. $T_m = a + (m-1)d\frac{1}{n}$ and $T_n = a+(n-1)d\frac{1}{m}$

On solving, $a = \frac{1}{mn}$ and $d = \frac{1}{mn}$

Therefore, $T_{mn} = a + (mn-1)d = \frac{1}{mn} + (mn-1)\frac{1}{mn} = 1$

13. It is not possible to express $a+b+4c-4d+e$ in terms of a. Hence, the correct answer is option (D).

14. Let S_n and S'_n be the sums of n terms of two AP's and T_{11} and T_{11} be the respective 11th terms. Then

$$\frac{S_n}{S'_n} = \frac{\frac{n}{2}[2a+(n-1)d]}{\frac{n}{2}[2a'+(n-1)d']} = \frac{7n+1}{4n+27}$$

$$\Rightarrow \frac{a\frac{(n-1)}{2}d}{a'+\frac{(n-1)}{2}d'} = \frac{7n+1}{4n+27}$$

Now put $n = 21$. We get

$$\frac{a+10d}{a'+10d'} = \frac{T_{11}}{T_{11}} = \frac{148}{111} = \frac{4}{3}$$

Note: If ratio of sum of n terms of two APs is given in terms of n and ratio of their p^{th} terms is to be found then put $n = 2p-1$. Here we put $n = 11\times 2 - 1 = 21$.

15. Given series $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}+\cdots$

Here $a = \frac{1}{2}$, common difference $d = -\frac{1}{6}$ and $n = 9$. So

$$S_9 = \frac{9}{2}\left[2\times\frac{1}{2} + (9-1)\left(-\frac{1}{6}\right)\right] = -\frac{3}{2}$$

16. Let the number of sides of the polygon be n.
Then the sum of interior angles of the polygon is

$$(2n-4)\frac{\pi}{2} = (n-2)\pi$$

Since the angles are in AP and $a = 120°, d = 5$, therefore

$$\frac{n}{2}[2\times 120 + (n-1)5] = (n-2)180$$

$$\Rightarrow n^2 - 25n + 144 = 0 \Rightarrow (n-9)(n-16) = 0 \Rightarrow n = 9, 16$$

But $n = 16$ gives $T_{16} = a+15d = 120° + 15\cdot 5° = 195°$, which is impossible as interior angle cannot be greater than 180°. Hence, $n = 9$.

17. Given

$$a_2 - a_1 = a_3 - a_2 = \cdots = a_n - a_{n-1} = d$$

where d is the common difference of the given AP.
Also, $a_n = a_1 + (n-1)d$. Then by rationalizing each term, we get

$$\frac{1}{\sqrt{a_2}+\sqrt{a_1}} + \frac{1}{\sqrt{a_3}+\sqrt{a_2}} + \cdots + \frac{1}{\sqrt{a_n}+\sqrt{a_{n-1}}}$$

$$= \frac{\sqrt{a_2}-\sqrt{a_1}}{a_2-a_1} + \frac{\sqrt{a_3}-\sqrt{a_2}}{a_3-a_2} + \cdots + \frac{\sqrt{a_n}-\sqrt{a_{n-1}}}{a_n-a_{n-1}}$$

$$= \frac{1}{d}\left\{\sqrt{a_2}-\sqrt{a_1}+\sqrt{a_3}-\sqrt{a_2}+\cdots\sqrt{a_n}-\sqrt{a_{n-1}}\right\}$$

$$= \frac{1}{d}\left\{\sqrt{a_n}-\sqrt{a_1}\right\} = \frac{1}{d}\left(\frac{a_n-a_1}{\sqrt{a_n}+\sqrt{a_1}}\right)$$

$$= \frac{1}{d}\left\{\left[\frac{(n-1)d}{\sqrt{a_n}+\sqrt{a_1}}\right]\right\} = \frac{n-1}{\sqrt{a_n}+\sqrt{a_1}}$$

18. Given $d = a_2 - a_1 = a_3 - a_2 = \cdots = a_n - a_{n-1}$
Therefore,

$$\sin d\{\operatorname{cosec} a_1 \operatorname{cosec} a_2 + \cdots + \operatorname{cosec} a_{n-1}\operatorname{cosec} a_n\}$$

$$= \frac{\sin(a_2-a_1)}{\sin a_1\cdot\sin a_2} + \cdots + \frac{\sin(a_n-a_{n-1})}{\sin a_{n-1}\sin a_n}$$

$$= (\cot a_1 - \cot a_2) + (\cot a_2 - \cot a_3) + \cdots + (\cot a_{n-1} - \cot a_n)$$

$$= \cot a_1 - \cot a_n$$

19. Given series $2+5+8+11+\cdots$. Here $a = 2, d = 3$. Let number of terms be n. Then

$$\text{Sum of AP} = \frac{n}{2}\{2a+(n-1)d\}$$

$$\Rightarrow 60100 = \frac{n}{2}\{2\times 2 + (n-1)3\} \Rightarrow 120200 = n(3n+1)$$

$$\Rightarrow 3n^2 + n - 120200 = 0 \Rightarrow (n-200)(3n+601) = 0$$

Hence, $n = 200$.

20. Given series 3, 6, 9, 12,…, 99. Here $n = \frac{99}{3} = 33, a = 3, d = 3$. Therefore,

$$S = \frac{33}{2}\{2\times 3 + (33-1)3\} = \frac{33}{2}\times 102 = 33\times 51 = 1683$$

21. $1+3+5+7+\cdots$ up to n terms

$$S_n = \frac{n}{2}\{2\times 1 + (n-1)2\} = n^2$$

22. According to condition

$$513 = \frac{n}{2}\{2\times 54 + (n-1)(-3)\}$$

$$\Rightarrow 1026 = n(111-3n) \Rightarrow 3n^2 - 111n + 1026 = 0$$

$$\Rightarrow (3n-57)(n-18) = 0$$

Hence, $n = 18$.

23. Series $108 + 117 + \cdots 999$ is an AP where $a = 108$, common difference $d = 9$,

$$n = \frac{999}{9} - \frac{99}{9} = 111 - 11 = 100$$

Hence, the required sum is

$$\frac{100}{2}(108+999) = 50\times 1107 = 55350$$

24. Given that

$$\frac{\frac{m}{2}[2a+(m-1)d]}{\frac{n}{2}[2a+(n-1)d]} = \frac{m^2}{n^2}$$

$$\Rightarrow \frac{2a+(m-1)d}{2a+(n-1)d} = \frac{m}{n} \Rightarrow \frac{a+\frac{1}{2}(m-1)d}{a+\frac{1}{2}(n-1)d} = \frac{m}{n}$$

$$\Rightarrow an\frac{1}{2}(m-1)nd = am + \frac{1}{2}(n-1)md$$

$$\Rightarrow a(n-m) + \frac{d}{2}[mn - n - mn + m] = 0$$

$$\Rightarrow a(n-m) + \frac{d}{2}(m-n) = 0 \Rightarrow a = \frac{d}{2} \text{ or } d = 2a$$

So, required ratio,

$$\frac{T_m}{T_n} = \frac{a+(m-1)d}{a+(n-1)d} = \frac{a+(m-1)2a}{a+(n-1)2a}$$

$$= \frac{1+2m-2}{1+2n-2} = \frac{2m-1}{2n-1}$$

Trick: Replace m by $2m-1$ and n by $2n-1$. Obviously, if S_m is of degree 2, then T_m is of degree 1, i.e. linear.

25. On putting $n = 1, 2, 3, \cdots$

First term of the series $a = \frac{1}{x} + y$ and the second term $= \frac{2}{x} + y$.

Therefore,

$$d = \left(\frac{2}{x} + y\right) - \left(\frac{1}{x} + y\right) = \frac{1}{x}$$

Sum of r terms of the series is

$$\frac{r}{2}\left[2\left(\frac{1}{x} + y\right) + (r-1)\frac{1}{x}\right] = \frac{r}{2}\left[\frac{2}{x} + 2y + \frac{r}{x} - \frac{1}{x}\right]$$

$$= \frac{r^2 - r + 2r}{2x} + ry = \left[\frac{r(r+1)}{2x} + ry\right]$$

26. Let

$$S = 1 + 2 + 3 + \cdots + 100$$
$$= \frac{100}{2}(1+100) = 50(101) - 5050$$

Let

$$S_1 = 3 + 6 + 9 + 12 + \cdots 99$$
$$= 3(1+2+3+4+\cdots+33)$$
$$= 3\cdot\frac{33}{2}(1+33) = 99\times 17 = 1683$$

Let

$$S_2 = 5 + 10 + 15 + \cdots + 100$$
$$= 5(1+2+3+\cdots+20)$$
$$= 5\cdot\frac{20}{2}(1+20) = 50\times 21 = 1050$$

Let

$$S_3 = 15 + 30 + 45 + \cdots + 90$$
$$= 15(1+2+3+\cdots+6)$$
$$= 15\cdot\frac{6}{2}(1+6) = 45\times 7 = 315$$

Therefore, required sum is

$$S = S - S_1 - S_2 + S_3$$
$$= 5050 - 1683 - 1050 + 315 = 2632$$

27. Let first 3 terms be $a-d$, a and $a+d$. Now

$$(a-d) + (a+d) = 12 \Rightarrow 2a = 12 \Rightarrow a = 6$$

and $\quad (a-d)a = 24 \Rightarrow 6(6-d) = 24 \Rightarrow d = 2$

Therefore, first term $= a - d = 6 - 2 = 4$.

28. Given,

$$\frac{2n}{2}\{2\cdot 2 + (2n-1)3\} = \frac{n}{2}\{2\cdot 57 + (n-1)2\}$$

$$\Rightarrow 2(6n+1) = 112 + 2n \Rightarrow 10n = 110$$

Therefores, $n = 11$.

29. The numbers divisible by 3 between 250 and 1000 are 252, 255, …, 999. Therefore,

$$T_n = 999 = 252 + (n-1)3 \Rightarrow 333 = 84 + n - 1 \Rightarrow n = 250$$

Therefore,

$$S=\frac{n}{2}[a+l]=\frac{250}{2}[252+999]=125\times1251=156375$$

30. 7^{th} term of an AP $=40$

$$a+6d=40$$

$$S_{13}=\frac{13}{2}[2a+(13-1)d]=\frac{13}{2}[2(a+6d)]=\frac{13}{2}\cdot2\cdot40=520$$

31. $a_1, a_2, a_3, \ldots, a_{n+1}$ are in AP and common difference $=d$. Let

$$S=\frac{1}{a_1a_2}+\frac{1}{a_2a_3}+\cdots+\frac{1}{a_na_{n+1}}$$

$$\Rightarrow S=\frac{1}{d}\left\{\frac{d}{a_1a_2}+\frac{d}{a_2a_3}+\cdots+\frac{d}{a_n\,a_{n+1}}\right\}$$

$$\Rightarrow S=\frac{1}{d}\left\{\frac{a_2-a_1}{a_1a_2}+\frac{a_3-a_2}{a_2a_3}+\cdots+\frac{a_{n+1}-a_n}{a_n\,a_{n+1}}\right\}$$

$$\Rightarrow S=\frac{1}{d}\left\{\frac{1}{a_1}-\frac{1}{a_2}+\frac{1}{a_2}-\frac{1}{a_3}+\cdots+\frac{1}{a_n}-\frac{1}{a_{n+1}}\right\}$$

$$\Rightarrow S=\frac{1}{d}\left\{\frac{1}{a_n}-\frac{1}{a_{n+1}}\right\}=\frac{1}{d}\left\{\frac{a_{n+1}-a_1}{a_1a_{n+1}}\right\}$$

$$\Rightarrow S=\frac{1}{d}\left(\frac{nd}{a_1a_{n+1}}\right)=\frac{n}{a_1a_{n+1}}$$

Trick: Check for $n=2$

32. $T_2=S_2-S_1$

$$=5(2)^2+2(2)-\{5(1)^2+2(1)\}=24-7=17$$

33. $S=\frac{n}{2}[2a+(n-1)d]$

$$\Rightarrow 406=\frac{n}{2}[6+(n-1)4]\Rightarrow 812=n[6+4n-4]$$

$$\Rightarrow 812=2n+4n^2\Rightarrow 406=2n^2+n$$

$$\Rightarrow 2n^2+n-406=0$$

$$\Rightarrow n=\frac{-1\pm\sqrt{1+4\cdot2\cdot406}}{2\cdot2}=\frac{-1\pm\sqrt{3249}}{4}=\frac{-1\pm57}{4}$$

Taking (+) sign, $n=\frac{-1+57}{4}=14$

34. According to the given condition

$$\frac{15}{2}[10+14\times d]=390\Rightarrow d=3$$

Hence, middle term, that is, 8^{th} term is given by

$$5+7\times3=26$$

35. Under conditions, we get

$$\frac{10}{2}\{2a+(10-1)d\}=4\left[\frac{5}{2}[2a+(5-1)d]\right]$$

$$\Rightarrow 2a+9d=4a+8d \text{ or } \frac{a}{d}=\frac{1}{2}$$

Hence, $a:d=1:2$

36. Let three numbers of AP be $a-d, a, a+d$.

$$\text{Sum}=a-d+a+a+d=18\Rightarrow a=6$$

and $$(a-d)^2+a^2+(a+d)^2=58$$

$$\Rightarrow(6-d)^2+36+(6+d)^2=158$$

$$=36+d^2+36+d^2=122=2d^2+72=122$$

$$=2d^2=50\Rightarrow d=5$$

Hence, numbers are 1, 6, 11, that is, the maximum number is 11.

37. We have $\frac{3+5+7+\cdots \text{ upto } n \text{ terms}}{5+8+11+\cdots \text{ upto 10 terms}}=7$

$$\Rightarrow\frac{\frac{n}{2}[6+(n-1)2]}{\frac{10}{2}[10+(10-1)3]}=7\Rightarrow\frac{n(2n+4)}{10\times37}=7$$

$$\Rightarrow n^2+2n-1295=0\Rightarrow(n+37)(n-35)=0$$

Hence, $n=35$

38. $\text{AM}=\frac{1+2+3+\cdots+n}{n}=\frac{\frac{1}{2}n(n+1)}{n}=\frac{n+1}{2}$

39. The sum of n arithmetic means between a and b $=\frac{n}{2}(a+b)$

Aliter: As we know $A_1+A_2+\cdots+A_n=nA$, where $A=\frac{a+b}{2}$

40. The resulting progression will have $n+2$ terms with 2 as the first term and 38 as the last term. Therefore, the sum of the progression

$$\frac{n+2}{2}(2+38)=20(n+2)$$

By hypothesis, $20(n+2)=200\Rightarrow n=8$

41. $\text{Mean}=\frac{a+(a+nd)+(a+2nd)}{3}=\frac{3a+3nd}{3}=a+nd$

42. Let $x+y=u$, $x-y=v$. Then

$$x=\frac{u+v}{2}, y=\frac{u-v}{2}$$

Therefore,

$$f(u,v)=\left(\frac{u+v}{2}\right)\cdot\left(\frac{u-v}{2}\right)$$

Now,

$$\frac{f(x,y)+f(y,x)}{2}=\frac{\left(\frac{x+y}{2}\cdot\frac{x-y}{2}\right)+\left(\frac{y+x}{2}\cdot\frac{y-x}{2}\right)}{2}=0$$

43. As $\log 2, \log(2^n-1)$ and $\log(2^n+3)$ are in AP, therefore

$$2\log(2^n-1)=\log2+\log(2^n+3)$$

$$\Rightarrow (2^n - 5)(2^n + 1) = 0$$

As 2^n cannot be negative, hence

$$2^n - 5 = 0 \Rightarrow 2^n = 5 \text{ or } n = \log_2 5$$

44. Let the four numbers be $a-3d, a-d, a+d, a+3d$. Now

$$(a - 3d) + (a + 3d) = 8 \Rightarrow a = 4$$

and

$$(a-d)(a+d) = 15 \Rightarrow a^2 - d^2 = 15 \Rightarrow d = 1$$

Thus, required numbers are 1, 3, 5, 7.
Hence, the greatest number is 7.

45. Let the sides of the triangle be $a-d, a, a+d$. Since hypotenuse is the greatest side, let it be given by, $a+d$. So

$$(a+d)^2 = a^2 + (a-d)^2$$

$$\Rightarrow a^2 + d^2 + 2ad = a^2 + a^2 - 2ad + d^2 \Rightarrow a = 4d$$

Therefore, ratio of the side $s = a-d : a : a+d$

$$= (4d-d) : 4d : (4d+d) = 3:4:5$$

46. Suppose that the three numbers are $a+d, a, a-d$. Therefore,

$$a+d+a+a-d = 33 \Rightarrow a = 11$$

$$a(a+d)(a-d) = 792 \Rightarrow 11(121-d^2) = 792 \Rightarrow d = 7$$

Then required numbers are 4, 11, 18. The smallest number is 4.

47. a, b, c, d, e, f are in AP. So

$$b-a = c-b = d-c = e-d = f-e = K$$

where K is the common difference. Now,

$$d-c = e-d \Rightarrow e+c = 2d$$

$$e-c+2c = 2d \Rightarrow e-c = 2(d-c)$$

Trick: Check by putting $a=1, b=2, c=3, d=4, e=5$ and $f=6$.

48. Let three numbers be $a-d, a, a+d$. We get

$$a-d+a+a+d = 15 \Rightarrow a = 5$$

and $$(a-d)^2 + a^2 + (a+d)^2 = 83$$

Now

$$a^2 + d^2 - 2ad + a^2 + a^2 + d^2 + 2ad = 83$$

$$\Rightarrow 2(a^2 + d^2) + a^2 = 83$$

Putting $a = 5$. We get

$$2(25 + d^2) + 25 = 83 \Rightarrow 2d^2 = 8 \Rightarrow d = 2$$

Thus, the numbers are 3, 5, 7.

Trick: Since $3+5+7=15$ and $3^2+5^2+7^2=83$.

49. Let four arithmetic means be A_1, A_2, A_3 and A_4. So $3, A_1, A_2, A_3, A_4, 23$.

$$T_6 = 23 = a + 5d \Rightarrow d = 4$$

Thus,

$$A_1 = 3+4 = 7, A_2 = 7+4 = 11$$

$$A_3 = 11+4 = 15, A_4 = 15+4 = 19$$

50. Let consecutive terms of an AP be $a-d, a, a+d$. Under given condition,

$$(a-d) + a + (a+d) = 51$$

$$\Rightarrow a = 17 \text{ and } (a-d)(a+d) = 273 \Rightarrow a^2 - d^2 = 273$$

$$\Rightarrow -d^2 = 273 - 289 \Rightarrow d = 4$$

Hence, consecutive terms are 13, 17, 21.

Trick: Both conditions are satisfied by (A), that is, 21, 17, 13.

51. Since $\frac{1}{p+q}, \frac{1}{r+p}$ and $\frac{1}{q+r}$ are in AP therefore,

$$\frac{1}{r+p} - \frac{1}{p+q} = \frac{1}{q+r} - \frac{1}{r+p}$$

$$\Rightarrow \frac{p+q-r-p}{(r+p)(p+q)} = \frac{r+p-q-r}{(q+r)(r+p)}$$

$$\Rightarrow \frac{q-r}{p+q} = \frac{p-q}{q+r} \text{ or } q^2 - r^2 = p^2 - q^2$$

$$\Rightarrow 2q^2 = r^2 + p^2$$

Therefore, p^2, q^2, r^2 are in AP.

52. It can easily be proved by putting $n = 2, 3, 4 \ldots$
The difference between an integer and its cube is divisible by 6.

53. $(a+2b-c)(2b+c-a)(c+a-b)$

$$= (a+a+c-c)(a+c+c-a)(2b-b) = 4abc$$

Since a, b, c are in AP, therefore, $2b = a+c$.

54. Let A_1, A_2, A_3 and A_4 be four numbers in AP.

$$A_1 + A_4 = 8 \quad (1)$$

and $$A_2 \cdot A_3 = 15 \quad (2)$$

The sum of terms equidistant from the beginning and end is constant and is equal to sum of first and last terms. Hence

$$A_2 + A_3 = A_1 + A_4 = 8 \quad (3)$$

From Eqs. (2) and (3),

$$A_2 + \frac{15}{A_2} = 8 \Rightarrow A_2^2 - 8A_2 + 15 = 0$$

$$A_2 = 3 \text{ or } 5 \text{ and } A_3 = 5 \text{ or } 3$$

As we know,

$$A_2 = \frac{A_1 + A_3}{2} \Rightarrow A_1 = 2A_2 - A_3$$

$$\Rightarrow A_1 = 2 \times 3 - 5 = 1 \text{ and } A_4 = 8 - A_1 = 7$$

Hence, the series is 1, 3, 5, 7.
So that the least number in the series is 1.

55. Let the first term of AP be a and common difference be d.
11^{th} term of AP $= a + 10d$
21^{st} term of AP $= a + 20d$

$$2(a+10d) = 7(a+20d) \Rightarrow 2a + 20d = 7a + 140d$$

$$5a+120d=0 \Rightarrow a+24d=0$$

Hence, the 25th term is 0.

56. $2\tan^{-1}y=\tan^{-1}x+\tan^{-1}z$

$$\Rightarrow \tan^{-1}\left(\frac{2y}{1-y^2}\right)=\tan^{-1}\left(\frac{x+z}{1-xz}\right)$$

$$\Rightarrow \frac{2y}{1-y^2}=\frac{x+z}{1-xz}$$

But $2y=x+z$. Therefore,

$$1-y^2=1-xz \Rightarrow y^2=xz$$

Since, x, y, z are both in GP and AP, therefore,

$$x=y=z$$

57. x, y, z are in GP. Then $y^2=x\cdot z$. Now

$$a^x=b^y=c^z=m$$

$$\Rightarrow x\log_e a=y\log_e b=z\log_e c=\log_e m$$

$$\Rightarrow x\log_a m,\ y\log_b m,\ z\log_c m$$

Again as x, y, z are in GP, so $\frac{y}{x}=\frac{z}{y}$

$$\Rightarrow \frac{\log_b m}{\log_a m}=\frac{\log_c m}{\log_b m} \Rightarrow \log_b a=\log_c b$$

58. Let

$$AR^{p-1}=a \quad (1)$$

$$AR^{q-1}=b \quad (2)$$

and $$AR^{r-1}=c \quad (3)$$

So

$$a^{q-r}b^{r-p}c^{p-q}=\{AR^{p-1}\}^{q-r}\{AR^{q-1}\}^{r-p}\{AR^{r-1}\}^{p-q}$$

$$=A^{(q-r+r-p+p-q)}R^{(pq-pr-q+r+qr-pq-r+p+pr-rq-p+q)}$$

$$=A^0R^0=1$$

Note: Such type of questions, i.e. containing terms of powers in cyclic order associated with negative sign, reduce to 1 mostly.

59. Given that $ar^2=4$. Then the product of first 5 terms is

$$a(ar)(ar^2)(ar^3)(ar^4)=a^5r^{10}=[ar^2]^5=4^5$$

60. $0\cdot 2\dot{3}\dot{4}=0\cdot 2343434\cdots$

$$=0.2+0.034+0.00034+0.0000034+\cdots$$

$$=0\cdot 2+\frac{34}{1000}+\frac{34}{100000}+\frac{34}{10000000}+\cdots\infty$$

$$=\frac{2}{10}+34\left[\frac{1}{10^3}+\frac{1}{10^5}+\frac{1}{10^7}+\cdots\infty\right]$$

$$=\frac{2}{10}+34\left[\frac{1/10^3}{1-1/1000}\right]=\frac{2}{10}+34\times\frac{1}{1000}\times\frac{100}{99}$$

$$=\frac{2}{10}+\frac{34}{990}=\frac{232}{990}$$

61. Let the three terms of GP be a, ar, ar^2. Then

$$a+ar+ar^2=19 \Rightarrow a[1+r+r^2]=19 \quad (1)$$

$$a\cdot ar\cdot ar^2=216 \Rightarrow a^3r^3=216 \Rightarrow ar=6 \quad (2)$$

Now dividing Eq. (2) by Eq. (1), we get

$$\frac{6}{r}+\frac{6}{r}r+\frac{6}{r}r^2=19 \Rightarrow \frac{6}{r}+6+6r=19$$

$$\Rightarrow r^2-\frac{13}{6}r+1=0$$

Hence, $r=\frac{3}{2}$

62. Given series $6+66+666+\cdots+$ up to n terms

$$=\frac{6}{9}(9+99+999+\cdots+\text{ up to } n \text{ terms})$$

$$=\frac{2}{3}(10+10^2+10^3+\cdots+\text{ up to } n \text{ terms } -n)$$

$$=\frac{2}{3}\left(\frac{10(10^n-1)}{10-1}-n\right)=\frac{1}{27}[20(10^n-1)-18n]$$

$$=\frac{2(10^{n+1}-9n-10)}{27}$$

63. Let first term and common ratio of GP be, respectively, a and r. Then under given condition

$$T_n=T_{n-1}+T_{n-2} \Rightarrow ar^{n-1}=ar^{n-2}+ar^{n-3}$$

$$\Rightarrow ar^{n-1}=ar^{n-1}r^{-1}+ar^{n-1}r^{-2}$$

$$\Rightarrow 1=\frac{1}{r}+\frac{1}{r^2} \Rightarrow r^2-r-1=0$$

$$\Rightarrow r=\frac{1\pm\sqrt{1+4}}{2}=\frac{1+\sqrt{5}}{2}$$

We take only (+) sign, since $r>1$.

64. Given $a+ar+1$ and $r=2$. Therefore

$$a+2a=1 \Rightarrow a=\frac{1}{3}$$

65. Given that

$$\frac{a(r^n-1)}{r-1}=255, \text{ since } r>1 \quad (1)$$

$$ar^{n-1}=128 \quad (2)$$

and Common ratio $r=2$ (3)

From Eqs. (1), (2) and (3), we get

$$a(2)^{n-1}=128 \quad (4)$$

and $$\frac{a(2^n-1)}{2-1}=255 \quad (5)$$

Now dividing Eq. (5) by Eq. (4), we get

$$\frac{2^n-1}{2^{n-1}}=\frac{255}{128} \Rightarrow 2-2^{-n+1}=\frac{255}{128}$$

$$\Rightarrow 2^{-n} = 2^{-8} \Rightarrow n = 8$$

Putting $n = 8$ in Eq. (4), we have

$$a \cdot 2^7 = 128 = 2^7 \text{ or } a = 1$$

66. $1 + (1+x) + (1+x+x^2) + \cdots$

$$+ (1 + x + x^2 + x^3 + \cdots + x^{n-1}) + \cdots$$

$$\text{Required sum} = \frac{1}{(1-x)}\{(1-x) + (1-x^2) + (1-x^3)$$

$$+ (1-x^4) + \cdots \text{ upto } n \text{ terms}\}$$

$$= \frac{1}{(1-x)}[n - \{x + x^2 + x^3 + \cdots \text{ upto } n \text{ terms}\}]$$

$$= \frac{1}{(1-x)}\left[n - \frac{x(1-x^n)}{1-x}\right] = \frac{n(1-x) - x(1-x^n)}{(1-x)^2}$$

67. Let 1, a, b, 64 $\Rightarrow a^2 = b$ and $b^2 = 64a$

$$\Rightarrow a = 4 \text{ and } b = 16$$

68. Since the series is GP, therefore,

$$x = \frac{1}{1-a} \Rightarrow a = \frac{x-1}{x}$$

and
$$y = \frac{1}{1-b} \Rightarrow b = \frac{y-1}{y}$$

Therefore,

$$1 + ab + a^2b^2 + \cdots \infty = \frac{1}{1-ab}$$

$$= \frac{1}{1 - \frac{x-1}{x} \cdot \frac{y-1}{y}} = \frac{xy}{x+y-1}$$

69. We have $ar = 2$ and $S_\infty = 8 = \dfrac{a}{1-r}$

$$\Rightarrow 8 = \frac{2}{r(1-r)} \text{ since } a = \frac{2}{r}$$

$$\Rightarrow 4r(1-r) = 1 \Rightarrow 4r - 4r^2 - 1 = 0$$

$$\Rightarrow 4r^2 - 4r + 1 = 0 \Rightarrow \left(r - \frac{1}{2}\right)(4r-2) = 0 \Rightarrow r = \frac{1}{2}$$

So, the first term is $a = 4$

70. We have $0.4\dot{2}\dot{3} = 0.4232323\ldots$

$$= 0.4 + 0.023 + 0.00023 + 0.0000023 + \cdots \infty$$

$$= \frac{4}{10} + \frac{23}{10^3} + \frac{23}{10^5} + \frac{23}{10^7} \cdots \infty$$

$$= \frac{4}{10} + \frac{23}{10^3}\left[1 + \frac{1}{10^2} + \frac{1}{10^4} + \cdots \infty\right]$$

$$= \frac{4}{10} + \frac{23}{1000}\left(\frac{1}{1 - \frac{1}{10^2}}\right) = \frac{4}{10} + \frac{23}{990} = \frac{419}{990}$$

71. We have

$$\frac{a}{1-r} = x \quad (1)$$

and
$$\frac{a^2}{1-r^2} = \frac{a}{1-r} \cdot \frac{a}{1+r} = y \quad (2)$$

$$\Rightarrow y = x \cdot \frac{a}{1+r} = x \cdot \frac{x(1-r)}{1+r} \Rightarrow \frac{y}{x^2} = \frac{1-r}{1+r}$$

$$\Rightarrow \frac{x^2}{y} = \frac{1+r}{1-r} \Rightarrow \frac{x^2}{y}(1-r) = 1 + r$$

$$\Rightarrow r\left[1 + \frac{x^2}{y}\right] = -1\frac{x^2}{y} \Rightarrow r = \frac{x^2 - y}{x^2 + y}$$

72. Let the first series be $a + ar + ar^2 + \cdots$. Then the second series is $a^2 + a^2r^2 + a^2r^4 + \cdots$. Their sum is given as 3. So, we have

$$\frac{a}{1-r} = 3 \text{ or } \Rightarrow a = 3(1-r)$$

and
$$\frac{a^2}{1-r^2} = 3 \Rightarrow a^2 = 3(1-r^2)$$

Eliminating a, we get

$$\{3(1-r)\}^2 = 3(1-r^2)$$

$$\Rightarrow 3(1-r) = (1+r), \text{ since, } \{r \neq 1\}$$

$$\Rightarrow 4r = 2 \text{ or } r = \frac{1}{2}$$

73. $4^{1/3} \cdot 4^{1/9} \cdot 4^{1/27} \cdots \infty$

Therefore, $S = 4^{1/3 + 1/9 + 1/27} \cdots \infty$

$$\Rightarrow S = 4^{\left(\frac{1/3}{1-1/3}\right)} = 4^{\frac{1/3}{2/3}} \Rightarrow S = 4^{1/2} \Rightarrow S = 2$$

74. $y = \dfrac{x}{1-x}$ (Infinite GP)

Therefore,

$$y - yx = x \text{ or } y = x(1+y), \text{ that is, } x = \frac{y}{1+y}$$

75.
$$\frac{a}{1-r} = 3 \quad (1)$$

and
$$\frac{a^2}{1-r^2} = 3 \quad (2)$$

From Eqs. (1) and (2),

$$\frac{a}{1+r} = 1 \Rightarrow a = 1 + r$$

From Eq. (1),

$$\frac{1+r}{1-r} = 3 \Rightarrow r = \frac{1}{2}$$

From Eq. (1), $a = 3/2$.

So, the first term = 3/2 and common ratio = 1/2.

76. $\frac{\sqrt{2}+1}{\sqrt{2}-1}, \frac{1}{\sqrt{2}(\sqrt{2}-1)}, \frac{1}{2}, \cdots$

Common ratio of the series $= \frac{1}{\sqrt{2}(\sqrt{2}+1)}$

Therefore, sum $= \frac{a}{1-r} = \left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) \Big/ \left(1 - \frac{1}{\sqrt{2}(\sqrt{2}+1)}\right)$

$$= \frac{(\sqrt{2}+1)}{(\sqrt{2}-1)} \cdot \frac{\sqrt{2}(\sqrt{2}+1)}{(1+\sqrt{2})} = \sqrt{2}(\sqrt{2}+1)^2$$

77. Given, $a = 2\left(\frac{ar}{1-r}\right) \Rightarrow 1-r = 2r \Rightarrow r = \frac{1}{3}$

78. Common ratio $(r) = \frac{2}{x}$

For sum to be finite $r < 1 \Rightarrow \frac{2}{x} < 1 \Rightarrow 2 < x \Rightarrow x > 2$.

79. Given series 0.5737373…

$$= 0.5 + 0.073 + 0.00073$$
$$= 0.5 + \frac{73}{1000} + \frac{73}{100000} + \cdots$$
$$= 0.5 + 73\left[\frac{1}{1000} + \frac{1}{100000} + \cdots\right]$$
$$= 0.5 + 73\left[\frac{1/1000}{1 - \frac{1}{100}}\right] = 0.5 + \frac{73}{1000} \cdot \frac{100}{99} = \frac{5}{10} + \frac{73}{990}$$
$$= \frac{495+73}{990} = \frac{568}{990}$$

80. Given series 0.037037037…

$$= 0.037 + 0.000037 + 0.0000000037 + \cdots$$
$$= \frac{37}{10^3} + \frac{37}{10^6} + \frac{37}{10^9} + \cdots$$
$$= 37\left[\frac{1}{10^3} + \frac{1}{10^6} + \frac{1}{10^9} + \cdots\right]$$
$$= 37\left[\frac{1/10^3}{1 - 1/10^3}\right] = 37\left[\frac{1}{10^3} \cdot \frac{10^3}{999}\right] = \frac{37}{999}$$

81. $3+x, 9+x, 21+x$ are in GP

$$\Rightarrow (9+x)^2 = (3+x)(21+x)$$
$$\Rightarrow 81 + x^2 + 18x = x^2 + 24x + 63$$
$$\Rightarrow 6x = 18 \Rightarrow x = 3$$

Trick: Check for (A), $3+3, 9+3, 21+3$ are in GP.

82. $s = \frac{a}{1-r} \Rightarrow s - sr = a \Rightarrow -sr = a - s \Rightarrow r = \frac{s-a}{s}$

83. Infinite series $9 - 3 + 1 - \frac{1}{3} \cdots \infty$ is a GP with $a = 9, r = \frac{-1}{3}$.

Therefore,

$$S_\infty = \frac{a}{1-r} = \frac{9}{1+\left(\frac{1}{3}\right)} = \frac{9 \times 3}{4} = \frac{27}{4}$$

84. $(a+2b+2c)(a-2b+2c) = a^2 + 4c^2$

$$\Rightarrow (a+2c)^2 - (2b)^2 = a^2 + 4c^2$$
$$\Rightarrow a^2 + 4ac + 4c^2 - 4b^2 = a^2 + 4c^2$$
$$\Rightarrow 4ac - 4b^2 = 0 \Rightarrow b^2 = ac$$

Hence, a, b, c are in GP.

85. $(32)(32)^{1/6}(32)^{1/36} \cdots \infty = (32)^{1+\frac{1}{6}+\frac{1}{36}+\cdots\infty}$

$$= (32)^{\frac{1}{1-(1/6)}} = (32)^{\frac{1}{5/6}} = (32)^{6/5} = 2^6 = 64$$

86. Given $T_m = n, T_n = m$ for HP. Therefore, for the corresponding AP m^{th} term $= \frac{1}{n}$, n^{th} term $= \frac{1}{m}$.

Let a and d be the first term and common difference of this AP. Then

$$a + (m-1)d = \frac{1}{n} \quad (1)$$

$$a + (n-1)d = \frac{1}{m} \quad (2)$$

Solving these, we get $a = \frac{1}{mn}$, $d = \frac{1}{mn}$

Now, r^{th} term of corresponding AP is

$$a + (r-1)d = \frac{1}{mn} + (r-1)\frac{1}{mn} = \frac{1+r-1}{mn} = \frac{r}{mn}$$

Therefore, r^{th} term of corresponding HP is $\frac{mn}{r}$.

Note: Students should remember this question as a fact.

87. Suppose that x is to be added. Then numbers 13, 15, 19 become new numbers $x+13, 15+x, 19+x$ which will be in HP. So

$$(15+x) = \frac{2(x+13)(19+x)}{x+13+x+19}$$
$$\Rightarrow x^2 + 31x + 240 = x^2 + 32x + 247 \Rightarrow x = -7$$

Trick: Such type of questions should be checked with the options.

88. Series $2, 2\frac{1}{2}, 3\frac{1}{3}, \cdots$ is in HP $\Rightarrow \frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \cdots$ will be in AP

Now the first term $a = \frac{1}{2}$ and common difference $d = -\frac{1}{10}$

So, 5^{th} term of the AP $= \frac{1}{2} + (5-1)\left(-\frac{1}{10}\right) = \frac{1}{10}$

Hence, 5^{th} term in HP is 10.

89. Since $a_1, a_2, a_3, \ldots, a_n$ are in HP, therefore $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3} \cdots \frac{1}{a_n}$ will be in AP which gives

$$\frac{1}{a_2} - \frac{1}{a_1} = \frac{1}{a_3} - \frac{1}{a_2} = \cdots = \frac{1}{a_n} - \frac{1}{a_{n-1}} = d$$

$$\frac{a_1-a_2}{a_1a_2}=\frac{a_3-a_2}{a_2a_3}=\cdots\frac{a_{n-1}-a_n}{a_{n-1}a_n}=d$$

$$\Rightarrow a_1-a_2=da_1a_2$$

$$a_2-a_3=da_2a_3$$

and $\quad a_{n-1}-a_n=da_na_{n-1}$

Adding these, we get

$$d(a_1a_2+a_2a_3+\cdots+a_na_{n-1})$$

$$=(a_1+a_2+\cdots+a_{n-1})-(a_2+a_3+\cdots+a_n)$$

$$=a_1-a_n \qquad (1)$$

Also, n^{th} term of this AP is given by

$$\frac{1}{a_n}=\frac{1}{a_1}+(n-1)d\Rightarrow d=\frac{a_1-a_n}{a_1a_n(n-1)}$$

Now substituting this value of d in Eq. (1). We get

$$(a_1-a_n)=\frac{a_1-a_n}{a_1a_n(n-1)}(a_1a_2+a_2a_3+\cdots+a_na_{n-1})$$

$$(a_1a_2+a_2a_3+\cdots+a_na_{n-1})=a_1a_n(n-1)$$

90. If x,y,z are in HP, then $y=\frac{2xz}{x+z}$. Now

$$\log_e(x+z)+\log_e(x-2y+z)$$

$$=\log_e\{(x+z)(x-2y+z)\}$$

$$=\log_e\left[(x+z)\left(x+z-\frac{4xz}{x+z}\right)\right]$$

$$=\log_e[(x+z)^2-4xz]=\log_e(x-z)^2=2\log_e(x-z)$$

91. Here 5^{th} term of the corresponding AP is

$$a+4d=45 \qquad (1)$$

and 11^{th} term of the corresponding AP is

$$a+10d=69 \qquad (2)$$

From Eqs. (1) and (2), we get $a=29, d=4$. Therefore, 6^{th} term of the corresponding AP is

$$a+15d=29+15\times 4=89$$

Hence, 16^{th} term of the HP is $\frac{1}{89}$.

92. Here the first term of AP is 7 and the second term is 9. Then 12^{th} term will be $7+11\times 2=29$

Hence, 12^{th} term of the HP is $\frac{1}{29}$.

93. We have $\frac{1}{a},\frac{1}{b},\frac{1}{c}$ are in AP. Let $\frac{1}{a}=p-q, \frac{1}{b}=p$ and $\frac{1}{c}=p+q$, where $p,q>0$ and $p>q$. Now, substitute these values in

$$\frac{3a+2b}{2a-b}+\frac{3c+2b}{2c-b}$$

Then it reduces to $10+\frac{14q^2}{p^2-q^2}$ which is obviously greater than 10 (as $p>q>0$).

Trick: Put $a=1, b=\frac{1}{2}, c=\frac{1}{3}$.

The expression has value $\frac{3+1}{2-\frac{1}{2}}+\frac{1+1}{\frac{2}{3}-\frac{1}{2}}=\frac{8}{3}+12>10$.

94. Since a,b,c,d are in HP, therefore, b is the HM of a and c that is, $b=\frac{2ac}{a+c}$ and c is the HM of b and d, that is, $c=\frac{2bd}{b+d}$. Therefore

$$(a+c)(b+d)=\frac{2ac}{b}\cdot\frac{2bd}{c}$$

$$\Rightarrow ab+ad+bc+cd=4ad\Rightarrow ab+bc+cd=3ad$$

Trick: Check for $a=1, b=\frac{1}{2}, c=\frac{1}{3}, d=\frac{1}{4}$.

95. Obviously, 7^{th} term of corresponding AP is $\frac{1}{8}$ and the 8^{th} term will be $\frac{1}{7}$. So

$$a+6d=\frac{1}{8} \text{ and } a+7d=\frac{1}{7}$$

Solving these, we get $d=\frac{1}{56}$ and $a=\frac{1}{56}$

Therefore, the 15^{th} term of this AP is

$$\frac{1}{56}+14\times\frac{1}{56}=\frac{15}{56}$$

Hence, the required 15^{th} term of the HP is $\frac{56}{15}$.

96. First term of an AP = 10 and the 12^{th} term = 25. Considering corresponding AP

$$a+6d=10 \text{ and } a+11d=25\ d=3, a=-8$$

$$\Rightarrow T_{20}=a+19d=8+57=49$$

Hence, the 20^{th} term of the corresponding HP is $\frac{1}{49}$.

97. T_6 of HP $=\frac{1}{61}\Rightarrow T_6$ of AP $=61$

and T_{10} of HP $=\frac{1}{105}\Rightarrow T_{10}$ of AP $=105$

So

$$a+5d=61 \qquad (1)$$

and

$$a+9d=105 \qquad (2)$$

From Eqs. (1) and (2), $a=6$

Therefore, the first term of HP $=\frac{1}{a}=\frac{1}{6}$

98. Let a be the first term and d be the common difference of the corresponding AP. Then

$$p^{th} \text{ term of AP } (T_p)=a+(p-1)d=\frac{1}{q} \qquad (1)$$

$$q^{\text{th}} \text{ term of AP } (T_q) = a+(q-1)d = \frac{1}{p} \quad (2)$$

From Eqs. (1) and (2),

$$(p-q)d = \frac{1}{q}-\frac{1}{p} = \frac{p-q}{pq} \Rightarrow d = \frac{1}{pq}$$

From Eq. (1),

$$a+(p-1)\frac{1}{pq} = \frac{1}{q} \Rightarrow a = \frac{1}{pq}$$

Therefore,

$$T_{pq} = a+(pq-1)d = \frac{1}{pq}+(pq-1)\frac{1}{pq} = 1$$

So, pq^{th} term is 1.

99. $a+3d = \frac{5}{3}$ and $a+7d = 3$

Solving we get, $a = \frac{2}{3}$, $d = \frac{1}{3}$

6^{th} term of AP $= a+5d = \frac{2}{3}+\frac{5}{3} = \frac{7}{3}$

6^{th} term of HP $= \frac{3}{7}$

100. As given $H = \frac{2pq}{p+q}$. Therefore

$$\frac{H}{p}+\frac{H}{q} = \frac{2q}{p+q}+\frac{2p}{p+q} = \frac{2(p+q)}{p+q} = 2$$

101. Putting $H = \frac{2ab}{a+b}$, we have

$$\frac{1}{H-a}+\frac{1}{H-b} = \frac{1}{\left(\frac{2ab}{a+b}-a\right)}+\frac{1}{\left(\frac{2ab}{a+b}-b\right)}$$

$$= \frac{a+b}{ab-a^2}+\frac{a+b}{ab-b^2} = \left(\frac{a+b}{b-a}\right)\left(\frac{1}{a}-\frac{1}{b}\right)$$

$$= \left(\frac{a+b}{b-a}\right)\left(\frac{b-a}{ab}\right) = \frac{a+b}{ab} = \frac{1}{a}+\frac{1}{b}$$

102. Let roots be α, β. Then HM $= \frac{2\alpha\beta}{\alpha+\beta} = \frac{11\times 2}{10} = \frac{11}{5}$

103. $$\text{HM} = \frac{2\left(\frac{a^2}{1-a^2b^2}\right)}{\frac{a}{1-ab}+\frac{a}{1+ab}} = \frac{2a^2}{2a} = a$$

104. $x_n = \frac{(n+1)ab}{na+b}$

Sixth HM, $x_6 = \frac{7\cdot 3\cdot 6/13}{\left(6\cdot 3+\frac{6}{13}\right)} = \frac{126}{240} = \frac{63}{120}$

105. We have $\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \frac{2ab}{a+b}$

$$\Rightarrow a^{n+2}+ab^{n+1}+ba^{n+1}+b^{n+2} = 2a^{n+1}b+2b^{n+1}a$$

$$\Rightarrow a^{n+1}(a-b) = b^{n+1}(a-b)$$

$$\Rightarrow \left(\frac{a}{b}\right)^{n+1} = (1) = \left(\frac{a}{b}\right)^0$$

Hence, $n = -1$

106. Putting $H = \frac{2ab}{a+b}$

$$\frac{H+a}{H-a}+\frac{H+b}{H-b} = \frac{2(H^2-ab)}{(H-a)(H-b)} = \frac{2\left[\frac{4ab}{(a+b)^2}-ab\right]}{\left[\frac{4ab}{(a+b)^2}-ab\right]} = 2$$

Trick: Let $a = 1$, $H = \frac{1}{2}$ and $b = \frac{1}{3}$. Then

$$\frac{H+a}{H-a}+\frac{H+b}{H-b} = \frac{3/2}{-1/2}+\frac{5/6}{1/6} = 2$$

107. From the section of inequality, we know that

AM of n^{th} powers > n^{th} power of AM

That is, $\frac{1}{2}(a^n+c^n) > \left(\frac{1}{2}(a+c)\right)^n$

Considering two quantities a and c, we have

$$\frac{1}{2}(a^n+c^n) > (A)^n \text{ or } \frac{1}{2}(a^n+c^n) > (H)^n$$

Since AM > HM

$$\frac{1}{2}(a^n+c^n) > (b)^n \Rightarrow a^n+c^n > 2b^n$$

Putting $n = 2$, we have $a^2+c^2 > 2b^2$

108. As a,b,c,d are in HP, b is the HM between a and c. Also the GM between a and $c = \sqrt{ac}$. Now, GM > HM so that

$$\sqrt{ac} > b \Rightarrow ac > b^2 \quad (1)$$

Again a,b,c,d are in HP, so c is the HM between b and d. Therefore,

$$bd > c^2 \quad (2)$$

Now multiplying Eqs. (1) and (2), we get

$$abcd > b^2c^2 \text{ or } ad > bc$$

Hence, answer (B) is true.

Now AM between a and $c = \frac{1}{2}(a+c)$

Now as AM > HM

$$a+c > 2b \quad (3)$$

And c is HM between b and d so

$$b+d>2c \quad (4)$$

Adding Eqs. (3) and (4), we get

$$(a+c)+(b+d)>2(b+c) \Rightarrow a+d>b+c$$

Hence answer (A) is true. So both (A) and (B) are correct.

109. It is a fundamental concept.

110. Let $a^{1/x}=b^{1/y}=c^{1/z}=k \Rightarrow a=k^x, b=k^y, c=k^z$

Now, a,b,c are in GP. So

$$b^2=ac \Rightarrow k^{2y}=k^x \cdot k^z=k^{x+z} \Rightarrow 2y=x+z$$

$\Rightarrow x,y,z$ are in AP

111. Given that a,b,c are in GP. So

$$b^2=ac \quad (1)$$

$$x=\frac{a+b}{2} \quad (2)$$

$$y=\frac{b+c}{2} \quad (3)$$

Now $$\frac{a}{x}+\frac{c}{y}=\frac{2a}{a+b}+\frac{2c}{b+c}=\frac{2(ab+bc+2ca)}{ab+ac+b^2+bc}$$

$$=\frac{2(ab+bc+2ca)}{(ab+ac+ac+bc)}=2 \quad \{\text{Since, } b^2=ac\}$$

Trick: Let $a=1$, $b=2$, $c=4$, then obviously $x=\frac{3}{2}$ and $y=3$, then $\frac{1}{3/2}+\frac{4}{3}=2$.

112. a^2,b^2,c^2 are in AP. Therefore, $a^2+(ab+bc+ca)$, $b^2+(ab+bc+ca)$, $c^2+(ab+bc+ca)$ will be in AP. Hence

$\{a(a+b)+c(c+b)\}$, $\{b(b+a)+c(b+a)\}$, $c(c+b)+a(b+c)$ will be in AP

$\Rightarrow (a+b)(a+c)$, $(b+a)(b+c)$, $(c+a)(c+b)$ will be in AP

$\Rightarrow \frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ will be in AP

{Dividing each term by $(a+b)(b+c)(c+a)$}

113. $x,1,z$ are in AP, then

$$2=x+z \quad (1)$$

and $$4=xz \quad (2)$$

Dividing Eqs. (2) by (1), we get

$$\frac{x\cdot z}{x+z}=\frac{4}{2} \text{ or } \frac{2xz}{x+z}=4$$

Hence, $x,4,z$ will be in HP.

114. Given that a,A_1,A_2,b are in AP. Therefore

$$A_1=\frac{a+A_2}{2},\ A_2=\frac{A_1+b}{2}$$

$$\Rightarrow A_1+A_2=\frac{1}{2}(a+b+A_1+A_2)$$

$$\Rightarrow \frac{1}{2}(A_1+A_2)=\frac{1}{2}(a+b) \text{ or } A_1+A_2=a+b \quad (1)$$

a,G_1,G_2,b are in GP. Therefore

$$G_1^2=aG_2,\ G_2^2=bG_1 \quad (2)$$

$$\Rightarrow G_1^2G_2^2=abG_1G_2 \Rightarrow G_1G_2=ab$$

Hence, $\frac{A_1+A_2}{G_1G_2}=\frac{a+b}{ab}$.

Trick: Let $a=1$, $b=2$, then $A_1+A_2=1+2=3$ and $G_1\cdot G_2=2\times1=2$. Therefore,

$$\frac{A_1+A_2}{G_1G_2}=\frac{3}{2}$$

115. Let a be the first term and d be the common difference of the given AP. Then as given the $(m+1)^{th}$, $(n+1)^{th}$ and $(r+1)^{th}$ terms are in GP. So, $a+md$, $a+nd$, $a+rd$ are in GP. This gives

$$(a+md)^2=(a+md)(a+rd)$$

$$\Rightarrow a(2n-m-r)=d(mr-n^2)$$

$$\Rightarrow \frac{d}{a}=\frac{2n-(m+r)}{mr-n^2} \quad (1)$$

Next, m,n,r in HP. So

$$n=\frac{2mr}{m+r} \quad (2)$$

From Eqs. (1) and (2)

$$\frac{d}{a}=\frac{2n-(m+r)}{mr-n^2}=\frac{2}{n}\left(\frac{2n-(m+r)}{(m+r)-2n}\right)=-\frac{2}{n}$$

116. Given AM = 2 (GM) or $\frac{1}{2}(a+b)=2\sqrt{ab}$

or $$\frac{a+b}{2\sqrt{ab}}=\frac{2}{1} \Rightarrow \frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}}=\frac{2+1}{2-1}=\frac{3}{1}$$

$$\Rightarrow \frac{(\sqrt{a}+\sqrt{b})^2}{(\sqrt{a}-\sqrt{b})}=\frac{3}{1} \Rightarrow \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}=\frac{\sqrt{3}}{1}$$

$$\Rightarrow \frac{a}{b}=\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)^2$$

$$\Rightarrow \frac{a}{b}=\frac{2+\sqrt{3}}{2-\sqrt{3}} \text{ or } a:b=(2+\sqrt{3}):(2-\sqrt{3})$$

117. $x+y+z=15$. If $9,x,y,z,a$ are in AP, then

$$\text{Sum}=9+15+a=\frac{5}{2}(9+a) \Rightarrow 24+a=\frac{5}{2}(9+a)$$

$$\Rightarrow 48+2a=45+5a \Rightarrow 3a=3 \Rightarrow a=1 \quad (1)$$

and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{5}{3}$. If $9, x, y, z, a$ are in HP, then

$$\text{Sum} = \frac{1}{9}+\frac{5}{3}+\frac{1}{a}=\frac{5}{2}\left[\frac{1}{9}+\frac{1}{a}\right] \Rightarrow a=1$$

118. Let the numbers be

$$\frac{a}{r},\ a,\ ar,\ 2ar-a \quad (1)$$

where first three numbers are in GP and last three are in AP. Given that the common difference of AP is 6, so

$$ar-a=6 \quad (2)$$

Also given

$$\frac{a}{r}=2ar-a \Rightarrow \frac{a}{r}=2(ar-a)+a$$

$$\Rightarrow \frac{a}{r}=2(6)+a, \text{ from Eq. (2)}$$

$$\Rightarrow \left(\frac{a}{r}\right)-a=12 \Rightarrow a(1-r)=12r \Rightarrow r=-\frac{1}{2}$$

From Eq. (1) we get

$$a\left[\left(-\frac{1}{2}\right)-1\right]=6a\left[\left(-\frac{1}{2}\right)-1\right]=6 \Rightarrow a=-4$$

Required numbers from Eq. (1) are $8, -4, 2, 8$

119. If a, b, c are in HP, then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are also in AP. So

$$\frac{a+b+c}{a}, \frac{a+b+c}{b}, \frac{a+b+c}{c} \text{ are in AP}$$

$$\Rightarrow \frac{b+c}{a}, \frac{a+c}{b}, \frac{a+b}{c} \text{ are in AP}$$

$$\Rightarrow \frac{a}{b+c}, \frac{b}{a+c}, \frac{c}{a+b} \text{ are in HP}$$

120. Let α, β be the first and $(2n-1)^{\text{th}}$ terms of the AP, the GP and the HP, respectively. Then we have

For AP: $\beta=\alpha+(2n-2)d \Rightarrow d=\frac{\beta-\alpha}{2n-2}$

$$n^{\text{th}} \text{ term } = a=\alpha+(n-1)d=\frac{1}{2}(\alpha+\beta) \quad (1)$$

Again for GP: $\beta=\alpha r^{2n-2} \Rightarrow r=\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2n-2}}$

Therefore, n^{th} term $=b=ar^{n-1}=\alpha\left(\frac{\beta}{\alpha}\right)^{\frac{n-1}{2n-2}}=\alpha\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}}$

or $$b=(\alpha\beta)^{1/2}=\sqrt{\alpha\beta} \quad (2)$$

Again for HP: $\frac{1}{\beta}=\frac{1}{\alpha}+(2n-2)d'$

$$\frac{1}{c}=\frac{1}{\alpha}+(n-1)d'=\frac{1}{\alpha}+\frac{\alpha-\beta}{2\alpha\beta}=\frac{\alpha+\beta}{2\alpha\beta}$$

$$\Rightarrow c=\frac{2\alpha\beta}{\alpha+\beta} \quad (3)$$

Now, more than one of the alternative answers may be correct. We try for option (A):

$$a-b=\frac{\alpha+\beta}{2}-\sqrt{\alpha\beta}=\frac{1}{2}(\sqrt{\alpha}-\sqrt{\beta})^2 \geq 0 \Rightarrow a \geq b$$

$$b-c=\sqrt{\alpha\beta}-\frac{2\alpha\beta}{\alpha+\beta}=\frac{\sqrt{\alpha\beta}}{\alpha+\beta}(\alpha+\beta-2\sqrt{\alpha\beta})$$

$$=\frac{\sqrt{\alpha\beta}}{(\alpha+\beta)}(\sqrt{\alpha}-\sqrt{\beta})^2 \geq 0 \Rightarrow b \geq c$$

Therefore,

$$\Rightarrow a \geq b \geq c \quad (4)$$

Now we try for option (C):

$$ac=\frac{\alpha+\beta}{2}\cdot\frac{2\alpha\beta}{\alpha+\beta}=\alpha\beta=b^2$$

$$\Rightarrow ac-b^2=0 \quad (5)$$

Obviously, it can be seen that $a+c \neq b$ (6)

Hence, (A) and (C) both hold good.

121. Let three terms of a GP are $\frac{a}{r}, a, ar$. So

$$\frac{a}{r}\cdot a\cdot ar=512 \Rightarrow a^3=8^3 \Rightarrow a=8$$

From second condition, we get $\frac{a}{r}+8, a+6$ will be in AP. So

$$2(a+6)=\frac{a}{r}+8+ar \Rightarrow 28=8\left\{\frac{1}{r}+1+r\right\}$$

$$\Rightarrow \frac{1}{r}+r+1=\frac{7}{2} \Rightarrow \frac{1}{r}+r-\frac{5}{2}=0$$

$$\Rightarrow r^2-\frac{5}{2}r+1=0 \Rightarrow 2r^2-5r+2=0$$

$$\Rightarrow (2r-1)(r-2)=0 \Rightarrow r=\frac{1}{2}, r=2 \ \ (\text{since } r>1) \ \Rightarrow r=2$$

Hence, the required numbers are 4, 8, 16, 4, 8, 16

Trick: Check for (a) $2+8,\ 4+6,\ 8$ are not in AP

(b) $4+8,\ 8+6,\ 16$ that is $12,\ 14,\ 16$ are in AP.

122. We have $\text{HM}=\frac{2ab}{a+b}$ and $\text{GM}=\sqrt{ab}$. So

$$\frac{\text{HM}}{\text{GM}}=\frac{4}{5} \Rightarrow \frac{2ab/(a+b)}{\sqrt{ab}}=\frac{4}{5}$$

$$\Rightarrow \frac{2\sqrt{ab}}{(a+b)}=\frac{4}{5} \Rightarrow \frac{a+b}{2\sqrt{ab}}=\frac{5}{4}$$

$$\Rightarrow \frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}}=\frac{5+4}{5-4} \Rightarrow \frac{(\sqrt{a}+\sqrt{b})^2}{(\sqrt{a}-\sqrt{b})^2}=\frac{9}{1}$$

$$\Rightarrow \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}=\frac{3}{1}\Rightarrow \frac{(\sqrt{a}+\sqrt{b})+(\sqrt{a}-\sqrt{b})}{(\sqrt{a}+\sqrt{b})-(\sqrt{a}-\sqrt{b})}=\frac{3+1}{3-1}$$

$$\Rightarrow \frac{2\sqrt{a}}{2\sqrt{b}}=\frac{4}{2}\Rightarrow \left(\frac{a}{b}\right)=2^2=4$$

$$\Rightarrow a:b=4:1 \text{ or } b:a=1:4$$

Aliter: Let the numbers be in the ratio $\lambda:1$ and let they be λa and a. Then

$$\frac{2(\lambda a)a}{\lambda a+a}\cdot\frac{1}{\sqrt{\lambda a\cdot a}}=\frac{4}{5}\Rightarrow \frac{\sqrt{\lambda}}{\lambda+1}=\frac{2}{5}$$

$$\Rightarrow 25\lambda=4(\lambda^2+2\lambda+1)\Rightarrow(\lambda-4)(4\lambda-1)=0$$

$$\Rightarrow \lambda=4 \text{ or } \lambda=\frac{1}{4}$$

Hence, both (C) and (D) are correct answers.

123. Let T_n be the n^{th} term and S the sum upto n terms.

$$S=1+3+7+15+31+\cdots+T_n$$

Again $S=1+3+7+15+\cdots+T_{n-1}+T_n$

Subtracting, we get $0=1+\{2+4+8+\cdots(T_n-T_{n-1})\}-T_n$

Therefore,

$$T_n=1+2+2^2+2^3+\cdots \text{ upto } n \text{ terms}$$

$$=\frac{1(2^n-1)}{2-1}=2^n-1$$

Now $\quad S=\sum T_n=\sum 2^n-\sum 1$

$$=(2+2^2+2^3+\cdots+2^n)-n$$

$$=2\left(\frac{2^n-1}{2-1}\right)-n=2^{n+1}-2-n.$$

Aliter: $\quad 1+3+7+\cdots+T_n$

$$=2-1+2^2-1+2^3-1+\cdots+2^n-1$$

$$=(2+2^2+\cdots+2^n)-n=2^{n+1}-2-n$$

Trick: Check the options for $n=1, 2$

124. We have $S=2+4+7+11+\cdots+T_{n-1}+T_n$

Again $S=2+4+7+11+\cdots+T_{n-1}+T_n$

Subtracting, we get

$$0=2+\{2+3+4+5+\cdots(T_n-T_{n-1})\}-T_n$$

$$T_n=2+\frac{1}{2}(n-1)(4+\{n-2)1\}=\frac{1}{2}(n^2+n+2)$$

Now, $\quad S=\Sigma T_n=\frac{1}{2}\Sigma(n^2+n+2)=\frac{1}{2}(\Sigma n^2+\Sigma n+2\Sigma 1)$

$$=\frac{1}{2}\left\{\frac{1}{6}n(n+1)(2n+1)+\frac{1}{2}n(n+1)+2n\right\}$$

$$=\frac{n}{12}\{(n+1)(2n+1+3)+12\}$$

$$=\frac{n}{6}\{(n+1)(n+2)+6\}=\frac{n}{6}(n^2+3n+8)$$

125. Let $\quad S=2+4+7+11+16+\cdots+T_n$

$$S=2+4+7+11+16+\cdots+T_{n-1}+T_n$$

Subtracting, we get

$$0=2+\{2+3+4+\cdots+(T_n-T_{n-1})\}-T_n$$

$$\Rightarrow T_n=1+(1+2+3+4+\cdots \text{ upto } n \text{ terms})$$

$$\Rightarrow 1+\frac{1}{2}n(n+1)=\frac{2+n^2+n}{2}=\frac{n^2+n+2}{2}$$

126. When n is odd, the last term, i.e. the n^{th} term will be n^2. In this case $n-1$ is even and so the sum of the first $n-1$ terms of the series is obtained by replacing n by $n-1$ in the given formula and so is $\frac{1}{2}(n-1)n^2$.

Hence, the sum of the n terms

= (the sum of $n-1$ terms) + the n^{th} term

$$=\frac{1}{2}(n-1)n^2+n^2=\frac{1}{2}(n+1)n^2$$

Trick: Check for $n=1,3$. Here, $0S_1=1, S_3=18$

127. Here T_n of the AP $1, 2, 3, \cdots=n$

and T_n of the AP $3, 5, 7, \cdots=2n+1$

Therefore,

T_n of given series $=n(2n+1)^2=4n^3+4n^2+n$

Hence, $S=\sum_1^{20}T_n=4\sum_1^{20}n^3+4\sum_1^{20}n^2+\sum_1^{20}n$

$$=4\cdot\frac{1}{4}20^2\cdot 21^2+4\cdot\frac{1}{6}20\cdot 21\cdot 41+\frac{1}{2}20\cdot 21=188090$$

128. $S=\frac{2}{1!}+\frac{2+4}{2!}+\frac{2+4+6}{3!}+\cdots+\frac{\frac{n}{2}(2+2n)}{n!}+\cdots\infty$

Here, $T_n=\frac{n(n+1)}{n!}=\frac{n-1+2}{(n-1)!}=\frac{1}{(n-2)!}+\frac{2}{(n-1)!}$

$$\Rightarrow S=\sum_{n=1}^{\infty}T_n=\sum_{n=1}^{\infty}\frac{1}{(n-2)!}+2\sum_{n=1}^{\infty}\frac{1}{(n-1)!}=e+2e=3e$$

129. $\quad S=\frac{2}{3!}+\frac{4}{5!}+\frac{6}{7!}+\cdots+\frac{2n}{(2n+1)}+\cdots$

Here, $T_n=\frac{(2n+1)-1}{(2n+1)!}=\frac{1}{(2n)!}-\frac{1}{(2n+1)!}$

$$\Rightarrow S=\sum_{n=1}^{\infty}T_n=\left(\frac{1}{2!}+\frac{1}{4!}++\frac{1}{6!}+\cdots\right)-\left(\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+\cdots\right)$$

$$=\left(\frac{e+e^{-1}}{2}-1\right)-\left(\frac{e-e^{-1}}{2}-1\right)=e^{-1}=\frac{1}{e}$$

130. $$S=\frac{1^2\cdot 2}{1!}+\frac{2^2\cdot 3}{2!}+\frac{3^2\cdot 4}{3!}+\cdots$$

Here, $T_n=\frac{n^2\cdot(n+1)}{n!}=\frac{n(n+1)}{(n-1)}=\frac{(n-1)(n-2)+4n-2}{(n-1)!}$

$$=\frac{1}{(n-3)!}+\frac{4(n-1)+2}{(n-1)!}$$

$$\frac{1}{(n-3)!}+\frac{4}{(n-2)!}+\frac{2}{(n-1)!}$$

$$\Rightarrow S=\sum T_n=e+4e+2e=7e$$

131. $T_n=\frac{\sum n}{n!}=\frac{n(n+1)}{2(n)!}=\frac{1}{2}\left[\frac{(n+1)}{(n-1)!}\right]=\frac{1}{2}\left[\frac{n-1}{(n-1)!}+\frac{2}{(n-1)!}\right]$

$$=\frac{1}{2}\left[\frac{1}{(n-2)!}+\frac{2}{(n-1)!}\right]=\frac{(e+2e)}{2}=\frac{3e}{2}$$

132. Sum of series $=1+2+\frac{1}{2!}+\frac{2}{3!}+\frac{1}{4!}+\frac{2}{5!}+\cdots$

$$=\left(1+\frac{1}{2!}+\frac{1}{4!}+\cdots\right)+2\left(1+\frac{1}{3!}+\frac{1}{5!}+\cdots\right)=\frac{e+e^{-1}}{2}+2\cdot\frac{e-e^{-1}}{2}$$

$$=\frac{3e-e^{-1}}{2}$$

133. The n^{th} term of given series is

$$T_n=\frac{1\cdot 3\cdot 5\cdot 7\cdots(2n-1)}{1\cdot 2\cdot 3\cdot 4\cdots(2n)}$$

$$T_n=\frac{1\cdot 2\cdot 3\cdot 4\cdots(2n-2)(n-1)(2n)}{1\cdot 2\cdot 3\cdot 4\cdots(2n-1)(2n)}\times\frac{1}{2\cdot 4\cdot 6\cdots(2n-2)(2n)}$$

$$T_n=\frac{1}{(2^n n!)}$$

Therefore,

$$S=\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)}{n!}=e^{\frac{1}{2}}-1$$

134. $$1+\frac{\log_e x}{1!}+\frac{(\log_e x)^2}{2!}+\frac{(\log_e x)^3}{3!}+\cdots+\frac{(\log_e x)^n}{n!}+\cdots=e^{(\log_e x)}=x$$

135. $$S=\log_e 3+\frac{(\log_e 3)^2}{2!}+\frac{(\log_e 3)^3}{3!}+\cdots+3\log_e 3+\frac{(3\log_e 3)^2}{2!}+\frac{(3\log_e 3)^3}{3!}+\cdots$$

$$=(e^{\log_e 3}-1)+(e^{3\log_e 3}-1)=(3-1)+(3^3-1)=28$$

136. $$3^x=e^{\log 3^x}=e^{x\log 3}=1+\frac{x\log 3}{1!}+\frac{x^2(\log 3)^2}{2!}+\frac{x^3(\log 3)^3}{3!}+\cdots$$

Coefficient of $x^3=\frac{(\log 3)^3}{3!}=\frac{(\log 3)^3}{6}$

137. Therefore, $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$

Putting $x=\frac{1}{2}$ on both the sides we get

$$\sqrt{e}=e^{1/2}=1+\frac{1}{2}+\frac{\left(\frac{1}{2}\right)^2}{2!}+\frac{\left(\frac{1}{2}\right)^3}{3!}+\frac{\left(\frac{1}{2}\right)^4}{4!}+\cdots$$

$$=1+\frac{1}{2}+\frac{1}{2^2 2}+\frac{1}{2^3 6}+\frac{1}{2^4\cdot 24}+\cdots$$

$1+0.5+0.1250+0.0208+0.0026=1.648$ (approximately).

138. $$\frac{1}{2\cdot 3}+\frac{1}{4\cdot 5}+\frac{1}{6\cdot 7}+\cdots\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{6}-\frac{1}{7}\right)+\cdots\infty=1-\log_e 2$$

$$=\log_e e-\log_e 2=\log\left(\frac{e}{2}\right)$$

139. Sum of $\frac{1}{2}+\frac{1}{3}\cdot\frac{1}{2^3}+\frac{1}{5}\cdot\frac{1}{2^5}+\cdots\infty$

$$=\frac{1}{2}\left[1+\frac{1}{3}\cdot\frac{1}{2^2}+\frac{1}{5}\cdot\frac{1}{2^4}+\cdots\infty\right]=\frac{1}{2}\cdot\log_e\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right)=\frac{1}{2}\cdot\log_e\left(\frac{3/2}{1/2}\right)$$

$$=\log_e\sqrt{3}$$

140. $\log_a x$ is defined for all positive real $x\neq 0$. Hence, the correct answer is option (C).

141. Since $\log_{y^n} x^m=\frac{m}{n}\log_y x$ and $\log_x x=1$, therefore

$$S=\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\cdots$$

Also, $\log_e(1+x)=x=\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^2}{2}+\cdots$

Putting $x=1$ we have, $S=1-\log_e 2$.

142. We have $\log(1+3x+2x^2)=\log(1+x)+\log(1+2x)$

$$=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(2x)^n}{n}=\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{n}+\frac{2^n}{n}\right)x^n$$

$$=\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1+2^n}{n}\right)x^n$$

So, coefficient of $x^n=(-1)^{n-1}\left(\frac{2^n+1}{n}\right)$

$$=\frac{(-1)^{n+1}(2^n+1)}{n}\text{, since, }[(-1)^n=(-1)^{n+2}=\cdots]$$

143. $$\sum_{x=1}^{1999}\log_n x$$

$$=\log_{(1999)!}1+\log_{(1999)!}2+\cdots+\log_{(1999)!}1999$$

$$=\log_{(1999)!}(1\cdot 2\cdot 3\cdots 999)=\log_{(1999)!}(1999)!=1$$

144. $e^{\left(x-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\dots\right)}$

$$e^{\left((x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\dots\right)^{+1}} = e^{\log(1+x-1)}e = e^{\log x}\cdot e = xe$$

145. Clearly, $2|x+1| = x+|x-1|$ give

$$x = \frac{-1}{2}, \frac{-3}{2}$$

Therefore, series are $\frac{-1}{2}, \frac{1}{2}, \frac{3}{2}, \cdots$ and $\frac{-3}{2}, \frac{1}{2}, \frac{5}{2}, \cdots$. So

$$S_{20} = 180 \text{ or } 350$$

146. Using AM ≥ GM

$$\frac{1}{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \ge \left[\left(\frac{a}{b}\right)\left(\frac{b}{c}\right)\left(\frac{c}{a}\right)\right]^{1/3} \Rightarrow \frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge 3$$

147. Using AM ≥ GM,

$$\frac{1}{3}(a+b+c) \ge (abc)^{1/3} \text{ and } \frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \ge \left(\frac{1}{abc}\right)^{1/3}$$

$$\Rightarrow \frac{1}{3}(a+b+c)\cdot\frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \ge 1 \Rightarrow (a+b+c)\cdot\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \ge 9$$

Equality will hold when $a = b = c$

148. Let the sides be $a-d$, a, $a+d$. Then

$$(a+d)^2 = (a-d)^2 + a^2$$

$$\Rightarrow 4ad = a^2$$

$$\Rightarrow a = 4d$$

Therefore, the sides are $3d, 4d, 5d$

So, sine of acute angle is $\frac{3}{5}$ or $\frac{4}{5}$.

149. The given inequality can be re-written as

$$(ap-b)^2 + (bp-c)^2 + (cp-d)^2 \le 0$$

$$\Rightarrow ap-b = bp-c = cp-d = 0$$

$$\Rightarrow \frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{1}{p} \Rightarrow a, b, c, d \text{ are in GP}$$

150. Let GP be $a, ar, ar^2, \cdots, ar^{n-1}$. Then

$$S = \frac{a(1-r^n)}{1-r}, P = a^n r^{\frac{n(n-1)}{2}}, R = \frac{1}{a}\frac{(1-r^n)}{1-r}\times\frac{1}{r^{n-1}}$$

$$\left(\frac{S}{R}\right)^n = (a^2r^{n-1})^n = \left[a^n r^{\frac{n(n-1)}{2}}\right]^2 = P^2$$

151.

$$t_r = \frac{1}{r\cdot 2^r} - \frac{1}{(r+1)2^{r+1}}$$

$$\Rightarrow S_n = \frac{1}{2} - \frac{1}{2^{n+1}(n+1)} = \frac{(n+1)2^n - 1}{2^{n+1}(n+1)}$$

152. $\displaystyle\sum_{j=1}^{n}\sum_{i=1}^{n} i = \sum_{j=1}^{n}\frac{n(n+1)}{2} = \frac{n^2(n+1)}{2}$

153. $\frac{1}{a}+\frac{1}{a-2b}+\frac{1}{c}+\frac{1}{c-2b} = (a+c-2b)\left(\frac{1}{a(c-2b)}+\frac{1}{c(a-2b)}\right) = 0$

As $a+c-2b \ne 0 \Rightarrow \frac{1}{b} = \frac{1}{a}+\frac{1}{c}$

154. $\dfrac{n[6+(n-1)2]}{\frac{10}{2}[10+27]} = 7 \Rightarrow n(n+2) = 35\times 37 \Rightarrow n = 35$

155. $\frac{a}{1-r} = 5$; $\frac{a^2}{1-r^2} = 5 \Rightarrow \frac{a}{1+r} = 1$

Now,

$$1+r = 5(1-r) \Rightarrow r = \frac{2}{3}$$

Hence $a = \frac{5}{3}$.

156. $\displaystyle\lim_{x\to\infty}\sum\frac{1}{(3n-2)(3n+1)} = \lim_{x\to\infty}\sum\frac{1}{3}\left(\frac{1}{3n-2}-\frac{1}{3n+1}\right)$

$$= \lim_{x\to\infty}\frac{n}{3n+1} = \frac{1}{3}$$

157. Given that $\dfrac{a^n+b^n}{a^{n-1}+b^{n-1}} = \sqrt{ab}$

Obviously, $n = 1/2$ is satisfying this relation.

158. $\dfrac{a^n+b^n}{a^{n-1}+b^{n-1}} = \dfrac{a+b}{2}$. By hit and trial we have $n = 1$.

159.

b = AM of a and c

≥ GM of a and c

$$\Rightarrow b \ge \sqrt{ac} \Rightarrow b^{3/2} \ge \sqrt{abc} = 2$$

$$\Rightarrow b \ge 2^{2/3} = 4^{1/3}$$

Hence, minimum value of b is $4^{1/3}$.

160. $\frac{a+b}{c}+1\ \frac{a+b}{c}+1+\frac{c+b}{a}+1-3 = \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)(a+b+c)-3$

Using AM > GM, we get $\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)(a+b+c) \ge 9$

Hence, minimum value is 6.

161. Using AM ≥ GM,

$$\frac{1}{2}(b+c-a+c+-b) \ge \sqrt{(b+c-a)(c+a-b)}$$

$$c \ge [(b+c-a)(a+c-b)]^{1/2}$$

Similarly, $b \ge [(a+b-c)(b+c-a)]^{1/2}$

$$a \ge [(a+b-b)(a+b-c)]^{1/2}$$

$$abc \ge (a+b-c)(c+a-b)(b+c-a)$$

$$\Rightarrow (a+b-c)(c+a-b)(b+c-a) - abc \le 0$$

162. Hint: $\dfrac{(n+1)-1}{(n+1)!} = \left[\dfrac{1}{n!}-\dfrac{1}{(n+1)!}\right]$

163. Let n (being even) AMs be inserted between a and b. Then $a, A_1, A_2, \cdots, A_n, b$ are in AP and $(n + 2)$ terms are there altogether.

Now since $a+b=A_1+A_n=A_2+A_{n-1}=\cdots$ constant

Also, $A_1+A_2+\cdots+A_n=n+1$ (given)

$$\Rightarrow n\left(\frac{a+b}{2}\right)=n+1 \Rightarrow n\left(\frac{13}{6\cdot 2}\right)=n+1$$

$$\Rightarrow 13n=12n+12 \Rightarrow n=12$$

164. Given that $T_p=(T_{p+1}+T_{p+2}+\cdots\infty)$

or $ar^{p-1}=ar^p+ar^{p+1}+\cdots\infty$, but $a=1$.

Therefore, $r^{p-1}=\frac{r^p}{1-r}$ [sum of an infinite GP]

$$1-r=r \Rightarrow r=1/2$$

Hence, the series is 1, $\frac{1}{2},\frac{1}{4},\frac{1}{8},\cdots\infty$

165. Let us consider $S=\frac{1}{3}+\frac{2}{9}+\frac{3}{27}+\cdots\infty$

$$\frac{S}{3}=\frac{1}{9}+\frac{2}{27}+\cdots\infty$$

On subtraction we get

$$\frac{2S}{3}=\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots\infty \Rightarrow S=\frac{3}{4}$$

Therefore, $p=3^{3/4} \Rightarrow p^{1/3}=3^{1/4}$.

166. We have $(x_1+x_2+x_3+\cdots+x_{50})\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots\frac{1}{x_{50}}\right)\geq(50)^2$

[since AM $\geq$ HM]

$$\Rightarrow\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots\frac{1}{x_{50}}\right)\geq(50)$$

167. $\frac{1}{H_1}=\frac{1}{a}+\left(\frac{1}{b}-\frac{1}{a}\right)\frac{1}{n+1}, \frac{1}{H_n}=\frac{1}{b}-\left(\frac{1}{b}-\frac{1}{a}\right)\frac{1}{n+1}$

$$\Rightarrow \frac{a}{H_1}=\frac{a+bn}{b+bn}, \quad \frac{b}{H_n}=\frac{an+b}{an+a}$$

$$\Rightarrow \frac{H_1+a}{H_1-a}=\frac{a+b+2nb}{b-a}, \quad \frac{H_n+b}{H_n-b}=\frac{a+b\_2na}{a-b}$$

Adding, we get

$$\frac{H_1+a}{H_1-a}+\frac{H_n+b}{H_n-b}=2n$$

168. For sum of infinite series

$$1+(x-1)+(x-1)^2+\cdots\infty$$

to exist we must have common ratio $(x - 1) \in (-1, 1) \Rightarrow x \in (0, 2)$.

169. $\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\infty\right)+\left(\frac{1}{2^2}+\frac{1}{4^2}+\cdots\right)=\frac{\pi^2}{6}$

$$\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots \text{ to } \infty\right)=\frac{\pi^2}{6}-\frac{1}{4}\left(\frac{\pi^2}{6}\right)=\frac{\pi^2}{8}=\frac{\pi^2}{8}$$

170. $S_n-S_{n-2}=2 \Rightarrow T_n+T_{n-1}=2$

Also

$$T_n+T_{n-1}=\left(\frac{1}{n^2}+1\right)T_{n-1}=2$$

$$\Rightarrow T_{n-1}\frac{2}{1+\frac{1}{n^2}}=\frac{2n^2}{1+n^2}$$

So, $$T_m=\frac{2(m+1)^2}{1+(m+1)^2}$$

171. Given $a_3-a_2=a_2-a_1$ and $\frac{a_r}{a_q}=\frac{a_q}{a_p}=\frac{a_q-a_r}{a_p-a_q}$

$$\Rightarrow \frac{a_q}{a_p}=\frac{a_1+(q-1)d-(a_1+(r-1)d)}{a_1+(p-1)d-(a_1+(q-1)d)}=\frac{q-r}{p-q}$$

172. Let $\frac{1}{H_{i+1}}-\frac{1}{H_i}=k$

$$\sum_{i=1}^{2n}(-1)^i\left(\frac{H_i+H_{i+1}}{H_i-H_{i+1}}\right)=\sum_{i=1}^{2n}\frac{(-1)^i}{k}\left(\frac{1}{H_{i+1}}+\frac{1}{H_i}\right)=2n$$

173. Let 2^n start from r^{th} term. By observation, we can see that $r=2^n$. Also, 2^n ends at $(r+2^n)^{\text{th}}$ term.

So, $2^n \leq 1025 \leq 2\cdot 2^n$

$$\Rightarrow 2^n \leq 2^{10}+1 \leq 2\cdot 2^n \Rightarrow n=10$$

Therefore, 2^{10} is 1025^{th} term.

174. $T_n=S_n-S_{n-1}=n(n+1)(n+2)(n+3)-(n-1)n(n+1)(n+2)$
$=4n(n+1)(n+2)$

$$\frac{1}{T_r}=\frac{1}{4r(r+1)(r+2)}=\frac{r+2-r}{8r(r+)(r+2)}$$

$$=\frac{1}{8}\left[\frac{1}{r(r+1)}-\frac{1}{r(r+1)(r+2)}\right]$$

$$\frac{1}{T_1}=\frac{1}{8}\left[\frac{1}{1\cdot 2}-\frac{1}{2\cdot 3}\right]$$

$$\frac{1}{T_2}=\frac{1}{8}\left[\frac{1}{2\cdot 3}-\frac{1}{3\cdot 4}\right]$$

$\vdots \quad \vdots \quad \vdots$

$\vdots \quad \vdots \quad \vdots$

$$\frac{1}{T_{10}}=\frac{1}{8}\left[\frac{1}{10\cdot 11}-\frac{1}{11\cdot 12}\right]$$

$$\sum_{r=1}^{10}\frac{1}{T_r}=\frac{1}{8}\left[\frac{1}{2}-\frac{1}{132}\right]=\frac{65}{1056}$$

175. Given that $2b = a + c,\ d = \dfrac{2ec}{e+c}$ and $c^2 = bd$

$$\Rightarrow c^2 = \frac{a+c}{2} \times \frac{2ec}{e+c}$$

$$\Rightarrow c(e+c) = (a+c)e \Rightarrow c^2 = ae$$

$\Rightarrow a, c, e$ are in GP

176. x^{15} will be obtained when we choose x from 15 factors and constant from the remaining factor.

$S = -(1 \cdot 2 \cdot 2^2 \cdot 2^3 \cdots 2^{14}) - (1 \cdot 2 \cdot 2^2 \ \ldots 2^{13} \cdot 2^{15}) - \cdots - (2 \cdot 2^2 \cdot 2^3 \cdots 2^{15})$

(Note that in the first term 2^{15} is missing, in the second 2^{14} is missing and so on)

$$S = -(1 \cdot 2 \cdot 2^2 \cdot 2^3 \ldots 2^{15})\left(\frac{1}{2^{15}} + \frac{1}{2^{14}} + \frac{1}{2^{13}} + \cdots \frac{1}{2^1} + \frac{1}{2^0}\right)$$

$$= -2^{120}\left(\frac{1-\left(\frac{1}{2}\right)^{16}}{1-\frac{1}{2}}\right) = -2^{121}\frac{(2^{16}-1)}{2^{16}} = 2^{105} - 2^{121}$$

177. $\sum r(2n - (2r-1)) = 2n\sum r - 2\sum r^2 + \sum r = \dfrac{n(n+1)(2n+1)}{6}$

178. $t_n = \dfrac{2n+3}{n+(n+1)} \cdot \dfrac{1}{3^n} + \dfrac{2n+3}{n(n+1)} = \dfrac{A}{n} + \dfrac{B}{n+1} = \dfrac{3}{n} - \dfrac{1}{n+1}$

Therefore,

$$t_n = \left(\frac{3}{n} - \frac{1}{n+!}\right) \cdot \frac{1}{3^n} = \frac{1}{n} \cdot \frac{1}{3^{n-1}} - \frac{1}{n+1} \cdot \frac{1}{3n}$$

$$S_n = \sum t_n = 1 - \frac{1}{n+1} \cdot \frac{1}{3^n}$$

179. $S = \displaystyle\sum_{r=1}^{10}(2^r + 2^{-r} + 2) = \frac{2(2^{10}-1)}{2-1} + \frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^2\right)}{1-\frac{1}{2}} + 20$

$$= 2^{11} - 2 + \frac{2^{10}-1}{2^{10}} + 20 = \frac{2^{21}-1}{2^{10}} + 19$$

180. Let the sequence is $a, a+2, a+4, \ldots, a+4n$,

$$\frac{a+4n}{2}, \frac{a+4n}{4}, \ldots, \frac{a+4n}{2^{2n}}$$

It is given that

$$a + 2n = \frac{a+4n}{2^n}$$

$$\Rightarrow a\left(1 - \frac{1}{2^n}\right) = \frac{4n}{2^n} - 2n \Rightarrow a = \frac{4n - 2n \cdot 2^n}{2^n - 1}$$

Therefore, middle term is $a + 4n = \dfrac{2n - (2-2^n)}{2^n - 1} + 4n = \dfrac{2n.2^n}{2^n - 1}$

181. 6, 12, 20, 30, 42, …, n^{th} term $= n^2 + 3n + 2$

$$= (n+1)(n+2)$$

9, 21, 37, 57, 81, …, n^{th} term $= 2n^2 + 6n + 1$

$$= 2n(n+3) + 1$$

Therefore, $u_n = (n+1)(n+2)\{2n(n+3) + 1\}$

$$= 2n(n+1)(n+2)(n+3) + (n+1)(n+2)$$

$$\Rightarrow S_n = \frac{2}{5}n(n+1)(n+2)(n+3)(n+4) + \frac{1}{3}(n+1)(n+2)(n+3)$$

182. The successive orders of differences are

13, 37, 109, 335, …

24, 72, 216, …

Thus, the second order of differences is a geometrical progression in which the common ratio is 3. Hence we may assume for the general term

$$u_n = a \cdot 3^{n-1} + bn + c$$

To determine the constants a, b, c make n equal to 1, 2, 3 successively. Then

$$a + b + c = 10,\ 3a + 2b + c = 23,\ 9a + 3b + c = 60$$

$$\Rightarrow a = 6, b = 1, c = 3$$

Hence

$$u_n = 6 \cdot 3^{n-1} + n + 3 = 2 \cdot 3^n + n + 3$$

183. $S = \log\dfrac{a^{-1}}{b^{m-1}} + \log\dfrac{a^1}{b^{m-1}} + \log\dfrac{a^3}{b^{m-1}} + \cdots + \log\dfrac{a^{2m-1}}{b^{m-1}}$

$$= \log\frac{a^{3+5+\cdots+2m-1}}{(b^{m-1})^{m+1}} = \log\frac{a^{m+1}}{(b^{m-1})^{m+1}} = (m+1)\log\frac{a}{b^{m-1}}$$

184. $x_n = 1 \cdot 2^{n-1}$

Now (x_n, y_n) lies on the parabola $y^2 = 4x$

$$\Rightarrow y_n^2 = 4x_n \Rightarrow y_n = 2\sqrt{x_n} = 2 \cdot 2^{\frac{n-1}{2}} = 2^{\frac{n+1}{2}}$$

185. $S_k = S_k = \displaystyle\lim_{x\to\infty}\left(\frac{1}{k+1} + \frac{1}{(k+1)^2} + \cdots + \frac{1}{(k+1)^n}\right) = \frac{1}{1-\frac{1}{k+1}} = \frac{k+1}{k}$

So, $\displaystyle\sum_{k=1}^{n} k\frac{(k+1)}{k} = \frac{(n+1)(n+2)}{2} - 1 = \frac{n(n+3)}{2}$

186. $S = 1 \cdot 2 + 1 \cdot 3 + 1 \cdot 4 + \cdots + 2 \cdot 3 + 2 \cdot 4 + \cdots + 3 \cdot 4 + 3 \cdot 5 + \cdots + (n-1)n$.

$[1 + 2 + 3 + \cdots (n-1) + n]^2 = 1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 + 2(1 \cdot 2 + 1 \cdot 3 + 1 \cdot 4 + \cdots + 2 \cdot 3 + 2 \cdot 4 + \cdots + 3 \cdot 4 + \cdots + (n-1)n)$

$$\Rightarrow \left(\sum n\right)^2 = \sum n^2 + 2S$$

$$\Rightarrow S = \frac{\left(\sum n\right)^2 - \sum n^2 + 2S}{2} = \frac{\{n(n+1)\}^2 - \frac{n(n+1)(2n+1)}{6}}{2}$$

$$= \frac{\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6}}{2}$$

$$= \frac{n(n+1)(3n^2 - n - 2)}{24}$$

187. Distance between vertex and diagonal is $\frac{n}{\sqrt{2}}$ cm. So the sum of distance of all such lines is

$$2(\sqrt{2}+2\sqrt{2}+3\sqrt{2}+\cdots+(n-1)\sqrt{2}+n\sqrt{2})=[n(n-1)+n]\sqrt{2}$$

$$=n^2\sqrt{2}$$

188. General term of series $= r(r+1)(r+2)$

Sum of the series $= \sum_{r=1}^{n} r(r+1)(r+2)$

$$=\frac{1}{4}\sum_{r=1}^{n} r(r+1)(r+2)\left[(r+3)-(r-1)\right]$$

$$=\frac{1}{4}n(n+1)(n+2)(n+3)$$

189. $f(k)=\sum_{r=1}^{n} a_r - a_k = s_n - a_k \Rightarrow \frac{f(k)}{a_k}-1 \forall k=1,2,\cdots n$

Given $a_1, a_2, \ldots, a_n$ are in HP $\Rightarrow \frac{1}{a_1},\frac{1}{a_2},\cdots,\frac{1}{a_n}$ are in AP

$$\Rightarrow \frac{s_n}{a_1}-1\frac{s_n}{a_2}-1,\cdots,\frac{s_n}{a_n}-1 \text{ are in AP}$$

$$\Rightarrow \frac{f(1)}{a_1},\frac{f(2)}{a_2},\cdots\frac{f(n)}{a_n} \text{ are in AP}$$

190. $|a_i|=|a_{i-1}+1|$ for $i=2,\ldots,n$

Squaring we have $a_i^2=a_{i-1}^2+2a_{i-1}+1 \Rightarrow a_i^2-a_{i-1}^2=2a_{i-1}+1$

$$\Rightarrow \sum_{i=2}^{n+1}\left(a_i^2-a_{i-1}^2\right)=2\sum_{i-2}^{n+1}a_{i-1}+\sum_{i-2}^{n+1}1$$

$$\Rightarrow a_{n+1}^2+a_1^2=2\sum_{i=1}^{n}a_i+n$$

$$\Rightarrow a_{n+1}^2-n=2\sum_{i=1}^{n}a_i$$

$$\Rightarrow a_{n+1}^2-n\ge -n \qquad \left(as\ a_{n+1}^2\ge 0\right)$$

$$\Rightarrow 2\sum_{i=1}^{n}a_i\ge -n$$

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n}a_i\ge -\frac{1}{2}$$

191. We have $2(\log 2b-\log 3c)=\log a-\log 2b+\log 3c-\log a$

$\Rightarrow 3(\log 2b-\log 3c)=0\Rightarrow 2b=3c$

Further, $b^2=ac\Rightarrow \frac{9c^2}{4}=ac\Rightarrow c=\frac{4a}{9}$

Thus, $a=\frac{9c}{4}$ and $b=\frac{3c}{2}\Rightarrow a:b:c=\frac{9}{4}:\frac{3}{2}:1=9:6:4$

Clearly, sum of any two are greater than third, so they form a triangle.

Also, $\cos A=\frac{4^2+6^2-9^2}{2\cdot 4\cdot 6}=\frac{29}{48}\Rightarrow$ is obtuse angle.

192. $$\frac{\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots\infty}{1+\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+\cdots\infty}=\frac{2\left[\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots\infty\right]}{2\left[1+\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+\cdots\infty\right]}=\frac{e+e^{-1}-2}{e-e^{-1}}$$

$$=\frac{(e-1)^2}{e^2-1}=\frac{e-1}{e+1}$$

193. $S=\frac{1}{1!}+\frac{4}{2!}+\frac{7}{3!}+\frac{10}{4!}+\cdots+\frac{3n-2}{n!}+\cdots\infty$

Here,

$$T_n=\frac{3}{(n-1)!}-\frac{2}{n!}$$

$$\Rightarrow S=\sum_{n=1}^{\infty}T_n=3\sum_{n=1}^{\infty}\frac{1}{(n-1)!}-2\sum_{n=1}^{\infty}\frac{1}{n!}=3e-2(e-1)=e+2$$

194. $\frac{1+x}{1!}+\frac{(1+x)^2}{2!}+\frac{(1+x)^3}{3!}+\cdots\infty=e^{1+x}-1=e\cdot e^x-$

$$1+e\left\{1+x+\frac{x^2}{2!}+\frac{x^2}{3!}+\cdots\right\}$$

Therefore, the coefficient of $x^n=e\frac{1}{n!}$

195. $$\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)^2=\left(\frac{e^x+e^{-x}}{2}\right)^2=\frac{1}{4}(e^{2x}+e^{-2x}+2)$$

$$=\frac{1}{4}\left\{2\left(1+\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}+\cdots\right)+2\right\}$$

Therefore, the coefficient of x^n(n even)

$$\frac{1}{2}\left\{\frac{2^n}{n!}\right\}=\frac{2^{n-1}}{n!}$$

196. $\frac{1}{0!}+\frac{1+2}{1!}+\frac{1+2+3}{2!}+\cdots\infty$

n^{th} term $T_n=\frac{1+2+3+4+\cdots+n}{(n-1)!}=\frac{n(n+1)}{2(n-1)!}$

$$T_n=\frac{1}{2}\left[\frac{1}{(n-3)!}+\frac{4}{(n-2)!}+\frac{2}{(n-1)!}\right]$$

Therefore, sum $S_\infty=\frac{7e}{2}$

197. $1\cdot 5+\frac{2\cdot 6}{1!}+\frac{3\cdot 7}{2!}+\frac{4\cdot 8}{3!}+\cdots$

$$T_n=\frac{n(n+4)}{(n-1)!}=\frac{(n-1)(n+4)}{(n-1)!}+\frac{(n+4)}{(n-1)!}$$

$$=\frac{n+4}{(n-2)!}+\frac{1}{(n-2)!}+\frac{5}{(n-1)!}$$

$$=\frac{1}{(n-3)!}+\frac{7}{(n-2)!}+\frac{5}{(n-1)!}$$

$$S_\infty=\sum_{n=1}^{\infty}\frac{1}{(n-3)!}+7\sum_{n=1}^{\infty}\frac{1}{(n-2)!}+5\sum_{n=1}^{\infty}\frac{1}{(n-1)!}=e+7e+5e=13e$$

198. We have

$$S=\sum_{n=0}^{\infty}\frac{(\log x)^{2n}}{(2n)!}=\left(\frac{e^{\log x}+e^{-\log x}}{2}\right)=\frac{x+x^{-1}}{2}$$

199. Let $S=4+11+22+37+\cdots+T_{n-1}+T_n$

or $S=4+11+22+37+\cdots+T_{n-1}+T_n$

Therefore, on subtracting we get

$$0=4+[7+11+15+19+\cdots+(T_n-T_{n-1})]-T_n$$

$$0=4+\frac{n-1}{2}[14+(n-2)4]-T_n$$

Therefore, $T_n=2n^2+n+1$

Thus, n^{th} term of given series are

$$T_n=\frac{2n^2+n+1}{(n)!}=\frac{2n}{(n-1)!}+\frac{1}{(n-1)!}+\frac{1}{n!}=\frac{2(n-1+1)}{(n-1)!}+\frac{1}{(n-1)!}$$

$$+\frac{1}{n!}=\frac{2}{(n-2)!}+\frac{3}{(n-1)!}+\frac{1}{n!}$$

Therefore, sum $\sum_{n=1}^{\infty}T_n=2e+3e+e-1=6e-1$

200. $(a+bx+cx^2)e^{-x}=(a+bx+cx^2)\left\{1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots\right\}$

Thus, coefficient of $x^n=a\cdot\frac{(-1)^n}{n!}+b\frac{(-1)^{n-1}}{(n-1)!}+c\frac{(-1)^{n-2}}{(n-2)!}$

201. Numerator $N=e^{m\log_e 3}\times e^{n\log_e 3}=e^{\log_e 3^m}\times e^{\log_e 3^n}$

$$=3^m\times 3^n=3^{m+n}$$

Denominator $D=e^{mn\log_e 3}=3^{mn}$

whereas given $m+n=1, mn=-1$

Therefore, $\frac{N}{D}=\frac{3^{m+n}}{3^{mn}}=\frac{3^1}{3^{-1}}=3^2=9$

202. $\frac{1}{3}+\frac{1}{2\cdot 3^2}+\frac{1}{3\cdot 3^3}+\frac{1}{4\cdot 3^4}+\cdots\infty=\left(\frac{1}{3}\right)+\frac{(1/3)^2}{2}+\frac{(1/3)^3}{3}$

$$+\frac{(1/3)^4}{4}+\cdots\infty$$

$$=-\log_e\left(1-\frac{1}{3}\right)=-\log_e\left(\frac{2}{3}\right)=\log_e\left(\frac{3}{2}\right)=\log_e 3-\log_e 2$$

203. $(1-x)\log_e(1-x)=(1-x)\left[-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5}-\cdots\infty\right]$

Hence, coefficient of $x^5=\frac{1}{4}-\frac{1}{5}=\frac{1}{20}$

204. $S=1+3+7+15+\ldots+T_n$ (1)

$S=1+3+7+15+\ldots+T_n$ (2)

Subtracting Eq. (2) from (1), we get

$$0=1+2+4+8+\ldots\ \text{upto } n \text{ terms}-T_n$$

$$T_n=2^n-1$$

Therefore,

$$1+\frac{3}{2!}+\frac{7}{3!}+\frac{15}{4!}+\cdots\infty=\frac{2^n-1}{n!}=\frac{2^n}{n!}-\frac{1}{n!}=\left(e^2-1\right)-\left(e-1\right)$$

$$=e(e-1)$$

205. $S=1+2\left\{\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots\right\}=\left(2-\frac{2}{2}\right)+2\left\{\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots\right\}$

$$2\left\{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\right\}=2\log_e 2=\log_e 4$$

206. $\log_e\left[1+ax^2+a^2+\frac{a}{x^2}\right]=\log_e(1+ax^2)\left(1+\frac{a}{x^2}\right)$

$$=\log_e(1+ax^2)+\log_e\left(1+\frac{a}{x^2}\right)$$

$$\left[ax^2-\frac{1}{2}a^2x^4+\frac{1}{3}a^3x^6-\cdots\right]+\left[\frac{a}{x^2}-\frac{1}{2}a^2\left(\frac{1}{x^4}\right)\right.$$

$$\left.+\frac{1}{3}a^3\left(\frac{1}{x^6}\right)-\cdots\right]$$

$$=a\left(x^2+\frac{1}{x^2}\right)-\frac{1}{2}.a^2\left(x^4+\frac{1}{x^4}\right)+\frac{1}{3}a^3\left(x^6+\frac{1}{x^6}\right)-\cdots$$

Practice Exercise 2

1. $S_{(1)}+S_{(2)}+\cdots+S_{(n-1)}$

$$1+$$

$$=1+\frac{1}{2}+$$

$$1+\frac{1}{2}+\frac{1}{3}$$

$$\cdots\cdots\cdots$$

$$\cdots\cdots\cdots$$

$$1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n-1}$$

Adding vertically:

$$=(n-1)+\frac{(n-2)}{2}+\frac{(n-3)}{2}+\cdots+\left(\frac{n-(n-1)}{(n-1)}\right)$$

$$=n\left[1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n-1}\right]-[1+1+1+\cdots1]=nS_{(n-1)}-(n-1)=nS_n-n$$

2. $T_r=\frac{r(\sqrt{r+1})-(r+1)\sqrt{r}}{r^2(r+1)-(r+1)^2r}=\frac{r\sqrt{r+1}-(r+1)\sqrt{r}}{-r^2-r}$

$$=\frac{(r+1)\sqrt{r}}{r(r+1)}\frac{r\sqrt{r+1}}{r(r+1)}=\frac{1}{\sqrt{r}}\frac{1}{\sqrt{r+1}}$$

$$\Rightarrow \sum_{r=1}^{99} T_r = \frac{1}{1} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdots - \frac{1}{\sqrt{100}} = 1 - \frac{1}{\sqrt{100}} = \frac{9}{10}$$

3. If n is even, then

$$f(n) = 2\left(1 + 2 + \cdots \frac{n-2}{2}\right) + \frac{n}{2} = \frac{n^2}{4}$$

If n is odd, then

$$f(n) = f(n-1) + \frac{n-1}{2} = \frac{(n-1)^2}{4} + \frac{n-1}{2} = \frac{n^2-1}{4}$$

$$f(n+m) - f(n-m) = \frac{(n+m)^2 - (n-m)^2}{4} \quad \text{if both are even}$$

$$f(n+m) = \frac{(n+m)^2 - 1 - (n-m)^2 + 1}{4} \quad \text{if both are odd}$$

$$= nm$$

4. $N = \underbrace{111\ldots1}_{n \text{ times}} = 1 + 10 + 100 + \cdots + 10^{n-1} = \dfrac{10^n - 1}{10 - 1}$

If $n = 5a$ (where a is an integer),

$$N = \frac{10^{5a} - 1}{10 - 1} = \frac{(10^5)^a - 1}{10^5 - 1} \cdot \frac{10^5 - 1}{10 - 1} = \left(\frac{(10^5)^a - 1}{10^5 - 1}\right) \cdot (11111)$$

Here,

$$\frac{N}{41} = \left(\frac{(10^5)^a - 1}{10^5 - 1}\right) \cdot \frac{(11111)}{41} = \left(\frac{(10^5)^a - 1}{10^5 - 1}\right) \cdot 271$$

So, N is divisible by 41 if $n = 5a$.

If $n = 6a$ (where a is an integer),

$$N = \frac{10^{6a} - 1}{10 - 1} = \frac{(10^6)^a - 1}{10^6 - 1} \cdot \frac{10^6 - 1}{10 - 1} = \left(\frac{(10^6)^a - 1}{10^6 - 1}\right) \cdot (111111)$$

Here,

$$\frac{N}{91} = \left(\frac{(10^6)^a - 1}{10^6 - 1}\right) \cdot \frac{(111111)}{91} = \left(\frac{(10^6)^a - 1}{10^6 - 1}\right) \cdot 1221$$

So, N is divisible by 91 if $n = 6a$.

5. $x = \dfrac{1}{1\cdot 2} + \dfrac{1}{3\cdot 4} + \dfrac{1}{5\cdot 6} + \cdots + \text{to } \infty$ (1)

$$y = \frac{1}{2\cdot 3} + \frac{1}{4\cdot 5} + \frac{1}{6\cdot 7} + \cdots + \text{to } \infty \qquad (2)$$

$$z = \frac{1}{1\cdot 2\cdot 3} + \frac{1}{3\cdot 4\cdot 5} + \frac{1}{5\cdot 6\cdot 7} + \cdots + \text{to } \infty \qquad (3)$$

Therefore,

$$\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \text{to } \infty$$

$$= \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \cdots + \text{to } \infty \qquad (4)$$

Let $\log_e 2 = a$. Therefore, $x = a$ from Eq. (4).

Also,

$$\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \text{to } \infty$$

$$= 1 - \frac{1}{2\cdot 3} + \frac{1}{4\cdot 5} + \frac{1}{6\cdot 7} + \cdots = 1 - \log_e 2$$

$$\Rightarrow y = 1 - a$$

On adding Eqs. (3) and (4), we get

$$2\log_e 2 = 1 + \left(\frac{1}{1\cdot 2} - \frac{1}{2\cdot 3}\right) + \left(\frac{1}{3\cdot 4} - \frac{1}{4\cdot 5}\right) + \ldots e \text{ to } \infty$$

$$2a = 1 + \frac{2}{1\cdot 2\cdot 3} + \frac{2}{3\cdot 4\cdot 5} + \ldots \text{to } \infty \Rightarrow 2a = 1 + 2z$$

Therefore, $z = a - \dfrac{1}{2}$.

6. $a + b = 50$

Hence, $A = 25$, $G = \sqrt{ab}$, $H = \dfrac{ab}{25}$

Now, $G = H + 4 \Rightarrow \sqrt{ab} = \dfrac{ab}{25} + 4 \Rightarrow G = \dfrac{G^2}{25} + 4 \Rightarrow G = 20, 5$

Hence, $H = 16$ or 1

Since $H > 1$, hence $H = 16$

But since $G^2 = AH$, hence $G = 20$, $H = 16$ is the only possibility.

7. $S = 1 + \dfrac{1}{(1+3)}(1+2)^2 + \dfrac{1}{(1+3+5)}(1+2+3)^2 + \dfrac{1}{(1+3+5+7)}(1+2+3+4)^2 + \cdots$

r^{th} term $T_r = \dfrac{1}{r^2}(1 + 2 + \cdots + r)^2 = \dfrac{1}{r^2}\left\{\dfrac{r(r+1)}{2}\right\}^2 = \dfrac{r^2 + 2r + 1}{4}$

Therefore, $T_7 = 16$ and

$$S_{10} = \sum_{r=1}^{10} T_r$$

$$= \frac{1}{4}\left\{\frac{(10)(10+1)(20+1)}{6} + (10)(10+1) + 10\right\} = \frac{505}{4}$$

8. $\displaystyle\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} = \frac{1}{6} n\cdot(n+1)\,[2n+1+3]$

$$= \frac{n(n+1)(n+2)}{3}$$

9. $a\cdot ar \cdots ar^{n-1}$

$$\frac{\dfrac{a[r^{11}-1]}{[r-1]}}{\dfrac{ar^{n-1}\left[\left(\dfrac{1}{r}\right)^{11} - 1\right]}{\left[\dfrac{1}{r} - 1\right]}} = \frac{1}{8}$$

$$\Rightarrow 8a\,\frac{(r^{11}-1)}{r-1} = \frac{ar^{n-1}}{r^{10}}\,\frac{(1-r^{11})}{(1-r)}$$

$$8 = r^{n-11}$$

$$\Rightarrow \frac{\dfrac{ar^9[r^{n-9}-1]}{(r-1)}}{\dfrac{(ar^{n-1})\left(\dfrac{1}{r}\right)^9\left[\left(\dfrac{1}{r}\right)^{n-9}-1\right]}{\left[\dfrac{1}{r}-1\right]}} = 2$$

$$\Rightarrow ar^9 = \frac{(r^{n-9}-1)}{(r-1)} = \frac{2ar^{n-1-9}}{r^{n-9}}\ \frac{[1-r^{n-9}]}{\dfrac{[1-r]}{r}}$$

$$\Rightarrow r^9 = \frac{2}{r}\times r \quad \Rightarrow r = 2^{1/9}$$

So,

$$8 = 2^{\frac{n-11}{9}}$$

$$\Rightarrow \frac{n-11}{9} = 3 \Rightarrow n = 11 + 27 = 38$$

10. Since, a_{912}, a_{951} and a_{480} are divisible by 3 then a_{91} is not a prime.

$$a_{91} = \frac{10^{91}-1}{10-1} = \frac{10^{91}-1}{10^7-1}\times\frac{10^7-1}{10-1}$$

$$= (1 + 107 + \cdots + 10^{84})\ (1 + 10 + \cdots + 10^6)$$

$\Rightarrow a_{91}$ is not a prime.

11. $G_1\, G_2 \ldots G_n = (\sqrt{1\times 1024})^n = 2^{5n} \Rightarrow 2^{5n} = 245 \Rightarrow n = 9$

12. $A_1 + A_2 + A_3 + \cdots + A_{m-1} + A_m = 1025 \times 171$

Therefore, $m\left(\dfrac{-2+1027}{2}\right) = 1025 \times 171 \Rightarrow m = 342$

13. Since $n = 9$, therefore

$$r = (1024)^{\frac{1}{9+1}} = 2$$

So, $\quad G_1 = 2, r = 2$

$$G_1 + G_2 + \ldots + G_n = \frac{2\cdot(2^9-1)}{2-1}e = 1024 - 2 = 1022$$

14. The common difference of sequence A_1, A_2,…, A_m is $\dfrac{1027+2}{342+1} = 3$.

Therefore, the common difference of sequence A_1, A_3, A_5,…, A_{m-1} is 6.

15. We have $A_{171} + A_{172} = -2 + 1027 = 1025$. Therefore

$$\frac{2A_{171}+2A_{172}}{2} = 1025$$

Also $\quad G_5 = 1 \times 2^5 = 32$. Therefore

$$G_5^2 = 1024 \Rightarrow G_5^2 + 1 = 1025$$

So, $2\,A_{171}$, $G_5^2 + 1$, $2A_{172}$ are in AP.

16. Let the numbers in set A be $a - D$, a, $a + D$ and in set B be $b - d$, b, $b + d$

$$3a = 3b = 15 \quad \Rightarrow \quad a = b = 5$$

Set $A = \{5 - D, 5, 5 + D\}$
Set $B = \{5 - d, 5, 5 + d\}$

where $D = d + 1$

$$\frac{p}{q} = \frac{5(25-D^2)}{5(25-d^2)} = \frac{7}{8}$$

$$25(8 - 7) = 8\,(d + 1)^2 - 7d^2$$

$\Rightarrow d = -17, 1$ but $d > 0$

$\Rightarrow d = 1$

So, numbers in set A are 3, 5, 7 and numbers in set B are 4, 5, 6.

Now, $p = 3 \times 5 \times 7 = 105$

Hence, (C) is the correct answer.

17. Value of $q = 4 \times 5 \times 6 = 120$

Hence, (B) is the correct answer.

18. Value of $D + d = 3$

Hence, (C) is the correct answer.

19. Let the four integers be $a - d$, a, $a + d$ and $a + 2d$ where, a and d are integers and $d > 0$.

Since

$$a + 2d = (a - d)^2 + a^2 + (a + d)^2$$

$$\Rightarrow 2d^2 - 2d + 3a^2 - a = 0 \qquad (1)$$

Therefore,

$$d = \frac{1}{2}[1 \pm \sqrt{1+2a-6a^2}\,] \qquad (2)$$

Since, d is positive integer

Therefore,

$$1 + 2a - 6a^2 > 0$$

$$\Rightarrow 6a^2 - 2a - 1 < 0$$

$$\Rightarrow \frac{1-\sqrt{7}}{6} < a < \frac{1+\sqrt{7}}{6}$$

Since, a is an integer, therefore $a = 0$. Put in Eq. (2) we get $d = 1$ or 0.

But, since $d > 0$, therefore, $d = 1$.

The smallest number is -1

Therefore, the four numbers are: -1, 0, 1, 2

Hence, (C) is the correct answer.

20. The common difference of the four numbers is $d = 1$.

Hence, (B) is the correct answer.

21. The sum of all the four numbers is $= -1 + 0 + 1 + 2 = 2$.

Hence, (C) is the correct answer.

22. (A) $\rightarrow$ (r), (B) $\rightarrow$ (p, t), (C) $\rightarrow$ (s), (D) $\rightarrow$ (q)

(A) $\quad F(n + 1) = \dfrac{2F(n)+1}{2} = F(n) + \dfrac{1}{2}$

$n = 1, 2, 3, \cdots$ and $f(1) = 2$

$$F(2) = F(1) + \frac{1}{2}$$

$$F(3) = F(2) + \frac{1}{2}$$

$$F(4) = F(3) + \frac{1}{2}$$

Therefore, $F(1)$, $F(2)$, $F(3)$, $\cdots$ is an AP with common difference $\dfrac{1}{2}$.

$$F(101) = a+(n-1)d$$
$$= 2 + (101-1)\times\frac{1}{2} = 2 + \frac{100}{2}$$
$$= 52$$

(B) $a_1 + 2d + a_1 + 4d + a_1 + 10d + a_1 + 16d + a_1 + 18d = 5a_1 + 50d = 5(a_1 + 10d) = 10$

That is, $a_1 + 10d = 2$

Now, $\sum_{i=1}^{21} a_i = \frac{21}{2}[2a_1 + 20d] = 21(a_1 + 10d) = 42$

(C) $S = 1 + 5 + 13 + 29 + \cdots + t_{10}$

$S = 1 + 5 + 13 + \cdots + t_9 + t_{10}$

Subtracting

$t_{10} = 1 + 4 + 8 + 16 + \cdots$ up to 10 terms

$= 1 + (4 + 8 + 16 + \cdots$ up to 9 terms)

$= 2045$

(D) Sum of all two digit numbers $= \frac{90}{2}(10 + 99) = (45)\ (109)$

Sum of all two digit numbers divisible by 2

$= \frac{45}{2}(10 + 98) = (45)\ (54)$

Sum of all two digit numbers divisible by 3

$= \frac{30}{2}(12 + 99) = 15\ (111)$

Sum of all two digit numbers divisible by 6

$= \frac{15}{2}(12 + 96) = 15\ (54)$

The required sum is 45(109) + 15(54) – (45) (54) – 15 (111) = 1620

23. (A) → (q), (B) → (r), (C) → (p), (D) → (t)

(A) $a + b = 12$

$$ab + \frac{6ab}{a+b} = 48$$

$$ab + \frac{ab}{2} = 48$$

Therefore, $ab = 32$

(B) $S = \frac{5}{1^2\cdot4^2} + \frac{11}{4^2\cdot7^2} + \frac{17}{7^2\cdot10^2} + \cdots$

$$\Rightarrow 3S = \frac{3\cdot5}{1^2\cdot4^2} + \frac{3\cdot11}{4^2\cdot7^2} + \frac{3\cdot17}{7^2\cdot10^2} + \ldots$$

$$\Rightarrow 3S = \frac{(4-1)\cdot(4+1)}{1^2\cdot4^2} + \frac{(7-4)(7+4)}{4^2\cdot7^2} + \frac{(10-7)(10+7)}{7^2\cdot10^2} + \cdots$$

$$\Rightarrow 3S = \frac{4^2-1^2}{1^2\cdot4^2} + \frac{7^2-4^2}{4^2\cdot7^2} + \frac{10^2-7^2}{7^2\cdot10^2} + \ldots$$

$$\Rightarrow 3S = 1 - \frac{1}{4^2} + \frac{1}{4^2} - \frac{1}{7^2} + \frac{1}{7^2} - \frac{1}{10^2} + \ldots$$

$$\Rightarrow 3S = 1 \Rightarrow S = \frac{1}{3}$$

(C) HM of $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ is $\frac{4}{2+3+4+5} = \frac{2}{7}$

(D) Since GM lies between the numbers, GM $= -\sqrt{(-4)\times(-9)} = -6$

24. In the first round all integers, which leave remainder 1 when divided by 15, will be marked; last number of this category is 991. Next number will be 91 + 15 – 1000 = 6. That means in second round all integers, which leave remainder 6 when divided by 15, will be marked.
In short, numbers of the form $5k + 1$ will be marked. Therefore,

$$A = 200 \text{ and } \frac{A}{50} = 4$$

25. Since, $[\sqrt{2046} = [\sqrt{2047}] = [\sqrt{2048}] = [\sqrt{2049} = 45$

Therefore, 2003[rd] term is $2003 + 45 = 2048$

Hence, remainder is 0

26. To exhaust all single-digit numbers he must have written $\sum_{i=1}^{9} i^2 = 285$ digits. To exhaust 10 he must write 2×10^2 more digits. That is, 485 digits.

So, the 500[th] digit will occur when he is writing 11.

27. Let the series be $a, ax, ax^2, ax^3, \cdots$ given that $|x| < 1$ and $x \neq 0$. Also,

$$\frac{T_4}{T_2} = \frac{ax^3}{ax} = \frac{1}{16} \Rightarrow x^2 = \frac{1}{16} \Rightarrow x = \pm\frac{1}{4}$$

But, since it is a decreasing GP $\Rightarrow x = \frac{1}{4}$

Also,

$$\frac{T_3}{T_2{}^2} = \frac{ax^2}{(ax)^2} = \frac{1}{9} \Rightarrow \frac{1}{a} = \frac{1}{9} \Rightarrow a = 9$$

$$S_\infty = \frac{a}{1-r} = \frac{9}{1-\frac{1}{4}} = \frac{9\times4}{3} = 12$$

28. $\sum_{\alpha=4}^{n+3} 4(\alpha - 3) = An^2 + Bn + C \Rightarrow \sum_{\alpha=1}^{n} 4\alpha = An^2 + Bn + C$

$\Rightarrow 2n\,(n + 1) = An^2 + Bn + C \Rightarrow A = 2, B = 2, C = 0$

Therefore, $A + B - C = 4$

29. $(1 - P)\,(1 + 3x + 9x^2 + 27x^3 + 81x^4 + 243x^5) = 1 - P^6$

$\Rightarrow (1 - P)\,\frac{1-(3x)^6}{1-3x} = 1 - P^6$

which is possible only, if $P = 3x$ or $\frac{P}{x} = 3$.

30. $(1^2 + 2^2 + \cdots + n^2) - (a_1 + a_2 + \cdots + a_n) = \frac{1}{3}n(n^2 - 1)$ (1)

Replacing n by $(n - 1)$, then

$(1^2 + 2^2 + \cdots + (n-1)^2) - (a_1 + a_2 + \cdots + a_{n-1})$

$= \frac{1}{3}(n - 1)\,((n - 1)^2 - 1)$ (2)

Subtracting Eq. (2) from Eq. (1)

$$n^2 - a_n = n^2 - n$$

$$\Rightarrow a_n = n$$

Therefore, $a_7 = 7$.

31. f is increasing. So, its greatest value is $f(3) = 27$.

Let the GP be $a, ar, ar^2, \ldots$ with $-1 < r < 1$.

$$\frac{a}{1-r} = 27 \text{ and } a - ar = 3 \Rightarrow r = \frac{4}{3} \text{ or } r = \frac{2}{3}$$

But, $-1 < r < 1$

So, $r = \frac{2}{3} \Rightarrow 27r = 18$

Solved JEE 2017 Questions

JEE Main 2017

1. If, for a positive integer n, the quadratic equation

$$x(x+1)+(x+1)(x+2)+\ldots+(x+\overline{n-1})(x+n)=10n$$

has two consecutive integral solutions, then n is equal to

(A) 9 **(B)** 10
(C) 11 **(D)** 12

(OFFLINE)

Solution: The given quadratic equation is

$$x(x+1)+(x+1)(x+2)+\cdots+\left[x+(n-1)\right](x+n)=10n$$

After simplifying, we get

$$nx^2+\{1+3+5+7+\cdots+(2n-1)x+[(0\cdot 1)+(1\cdot 2)$$
$$+(2\cdot 3)+\cdots+(n-1)n\}=10n$$

$$nx^2+n^2x+\left[\frac{n(n^2-1)}{3}\right]-10n=0$$

$$x^2+nx+\left(\frac{n^2-1-30}{3}\right)=0$$

$$x^2+nx+\left(\frac{n^2-31}{3}\right)=0$$

Using $n=11$ (where $n\in$ I), we get

$$x^2+11x+\left(\frac{121-31}{3}\right)=0$$

$$x^2+11x+30=0$$

$$(x+6)(x+5)=0$$

Therefore, $x=-5,-6$ (i.e., two consecutive integral solutions).
Thus, $n=11$.
Hence, the correct answer is option (C).

2. For any three positive real numbers a, b and c,

$$9(25a^2+b^2)+25(c^2-3ac)=15b(3a+c)$$

Then
(A) b, c and a are in AP
(B) a, b and c are in AP
(C) a, b and c are in GP
(D) b, c and a are in GP

(OFFLINE)

Solution: We have

$$(15a)^2+(3b)^2+(5c)^2-(15a)(5c)-(15a)(3b)-(3b)(5c)=0$$

$$\frac{1}{2}[(15a-3b)^2+(3b-5c)^2+(5c-15a)^2]=0$$

It is possible when $15a=3b=5c$.

Therefore, $b=\frac{5c}{3}$ and $a=\frac{c}{3}$.

That is, $a+b=2c$.
Thus, b, c and a are in AP.
Hence, the correct answer is option (A).

3. Let $a,b,c\in\mathbb{R}$. If $f(x)=ax^2+bx+c$ is such that $a+b+c=3$ and $f(x+y)=f(x)+f(y)+xy$, $\forall\, x,y\in\mathbb{R}$, then $\sum_{n=1}^{10} f(n)$ is equal to:

(A) 165 **(B)** 190
(C) 255 **(D)** 330

(OFFLINE)

Solution: We have

$$f(x)=x^2+bx+c$$

$$f(1)=a+b+c=3$$

Now,

$$f(x+y)=f(x)+f(y)+xy$$

Substituting $y=1$, we get

$$f(x+1)=f(x)+f(1)+x$$

$$f(x+1)=f(x)+x+3$$

Now,

$$f(2)=7 \text{ and } f(3)=12$$

Therefore,

$$S_n=3+7+12+\cdots+t_n \quad (1)$$

$$S_n=3+7+\cdots+t_{n-1}+S_n \quad (2)$$

On subtracting Eq. (2) from Eq. (1), we get

$$t_n=3+4+5+\cdots \text{ upto } n \text{ terms.}$$

Therefore,

$$t_n=\frac{(n^2+5n)}{2}$$

$$S_n=\sum t_n=\sum\frac{(n^2+5n)}{2}$$

$$S_n=\frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6}+\frac{S_n(n+1)}{2}\right]$$

On further simplification, we get $S_n=330$.
Hence, the correct answer is option (D).

4. If the sum of the first n terms of the series $\sqrt{3}+\sqrt{75}+\sqrt{243}+\sqrt{507}+\cdots$ is $435\sqrt{3}$, then n equals

(A) 29 **(B)** 18
(C) 15 **(D)** 13

(ONLINE)

Solution: Rewriting the given equation, we get

$$\sqrt{3}+5\sqrt{3}+9\sqrt{3}+13\sqrt{3}+\cdots=435\sqrt{3}$$

The given series is AP where $a=\sqrt{3}$ and $d=4\sqrt{3}$.

Now, the sum of AP is expressed as

$$S_n=\frac{n}{2}[2a+(n-1)d]$$

It is given that $S_n=435\sqrt{3}$. Therefore,

$$\frac{n}{2}[2a+(n-1)d]=435\sqrt{3}$$

Substituting the values of a and d, we get

$$\frac{n}{2}(2\sqrt{3}+(n-1)4\sqrt{3})=435\sqrt{3}$$

$$\Rightarrow \frac{n}{2}\times 2\sqrt{3}(1+(n-1)2)=435\sqrt{3}$$

$$\Rightarrow n(1+(n-1)2)=435$$

$$\Rightarrow n(1+2n-2)=435$$

$$\Rightarrow n(2n-1)=435$$

$$\Rightarrow 2n^2-n-435=0$$

$$\Rightarrow 2n^2-30n+29n-435=0$$

$$\Rightarrow 2n(n-15)+29(n-15)=0$$

$$\Rightarrow (2n+29)(n-15)=0$$

Thus,

$$2n+29=0\Rightarrow n=\frac{-29}{2}$$

and $\quad n-15=0\Rightarrow n=15$

It is obvious that $n=\frac{-29}{2}$ cannot be correct value and hence the correct value is $n=15$.

Hence, the correct answer is option (C).

5. If the arithmetic mean of two numbers a and b, $a>b>0$, is five times their geometric mean, then $\frac{a+b}{a-b}$ is equal to

(A) $\frac{\sqrt{6}}{2}$ **(B)** $\frac{3\sqrt{2}}{4}$

(C) $\frac{5\sqrt{6}}{12}$ **(D)** $\frac{7\sqrt{3}}{12}$

(ONLINE)

Solution: The arithmetic mean of two numbers a and b is

$$\frac{a+b}{2}$$

The geometric mean of two numbers a and b is

$$\sqrt{ab}$$

It is given that the arithmetic mean of two numbers a and b is five times their geometric mean. That is,

$$\frac{a+b}{2}=5\sqrt{ab}$$

$$\Rightarrow a+b=10\sqrt{ab} \qquad (1)$$

Squaring on both sides, of Eq. (1), we get

$$(a+b)^2=100ab$$

$$\Rightarrow a^2+b^2+2ab=100ab$$

$$\Rightarrow a^2+b^2=98ab$$

Subtracting $-2ab$ from both sides of this equation, we get

$$a^2+b^2-2ab=98ab-2ab$$

$$\Rightarrow (a-b)^2=96ab$$

$$\Rightarrow (a-b)=\sqrt{96}\sqrt{ab} \qquad (2)$$

Now,

$$\frac{a+b}{a-b}=\frac{10\sqrt{ab}}{\sqrt{96}\sqrt{ab}}=\frac{10}{\sqrt{96}}=\frac{10}{\sqrt{16\times 6}}$$

$$\Rightarrow \frac{a+b}{a-b}=\frac{10}{4\sqrt{6}}=\frac{5}{2\sqrt{6}}$$

Multiplying and dividing RHS by $\sqrt{6}$, we get

$$\frac{a+b}{a-b}=\frac{5}{2\sqrt{6}}\times\frac{\sqrt{6}}{\sqrt{6}}=\frac{5\sqrt{6}}{2\times 6}=\frac{5\sqrt{6}}{12}$$

Hence, the correct answer is option (C).

6. Let $S_n=\frac{1}{1^3}+\frac{1+2}{1^3+2^3}+\frac{1+2+3}{1^3+2^3+3^3}+\cdots+\frac{1+2+\cdots+n}{1^3+2^3+\cdots+n^3}$. If $100S_n=n$, then n is equal to

(A) 99 **(B)** 19

(C) 200 **(D)** 199

(ONLINE)

Solution: It is given that

$$S_n=\frac{1}{1^3}+\frac{1+2}{1^3+2^3}+\frac{1+2+3}{1^3+2^3+3^3}+\cdots+\frac{1+2+\cdots+n}{1^3+2^3+\cdots+n^3}$$

Also,

$$t_n=\frac{\sum_{k=1}^{r}k}{\sum_{k=1}^{k}k^2}=\frac{\left[\frac{r(r+1)}{2}\right]}{\left[\frac{r(r+1)}{2}\right]^2}=\frac{2}{r(r+1)}=\frac{2(r+1-r)}{r(r+1)}=\frac{2}{r}-\frac{2}{r+1}$$

Therefore,

$$S_n=\sum_{r=1}^{n}\frac{2}{r}-\frac{2}{r+1}=\frac{2}{1}-\frac{2}{2}+\frac{2}{2}-\frac{2}{3}+\frac{2}{3}-\frac{2}{4}+\cdots+\frac{2}{n}-\frac{2}{n+1}$$

It is given that $100S_n=n$. Therefore,

$$100\left(2-\frac{2}{n+1}\right)=n$$

$$\Rightarrow 100(2n+2-2)=n(n+1)$$

$$\Rightarrow 200n=n(n+1)$$

$$\Rightarrow x+1=200$$

$$\Rightarrow n=199$$

Hence, the correct answer is option (D).

7. If three positive numbers a, b and c are in AP such that $abc = 8$, then the minimum possible value of b is

(A) $4^{2/3}$ **(B)** $4^{1/3}$
(C) 4 **(D)** 2

(ONLINE)

Solution: For three positive numbers, we have

$$\text{AP} \geq \text{GP}$$

$$\frac{a+b+c}{3} \geq (abc)^{1/3}$$

Since a, b, c are in AP, we have

$$\frac{a+b+c}{3} = b$$

$$\Rightarrow b \geq (abc)^{1/3}$$

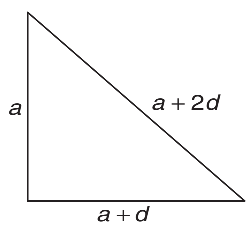
Therefore, the minimum possible value of b is obtained as $b \geq 2$ (since it is given that $abc = 8$).

Hence, the correct answer is option (D).

JEE Advanced 2017

1. The sides of a right-angled triangle are in arithmetic progression. If the triangle has area 24, then what is the length of its smallest side?

Solution: Let a be the first term and d be the common difference, then the sides of triangle are a, $a + d$ and $a + 2d$.



$$\text{Area of triangle} = \frac{1}{2} \times \text{Base} \times \text{Height}$$

Substituting the values, we get

$$24 = \frac{1}{2} \times (a+d) \times a$$

$$a(a+d) = 2 \times 24$$

$$a(a+d) = 48 \qquad (1)$$

Applying Pythagoras theorem, we get

$$a^2 + (a+d)^2 = (a+2d)^2$$

$$\Rightarrow a^2 + a^2 + d^2 + 2ad = a^2 + 4d^2 + 4ad$$

$$\Rightarrow a^2 + 4d^2 + 4ad - a^2 - a^2 - d^2 - 2ad = 0$$

$$\Rightarrow 3d^2 - a^2 + 2ad = 0 \Rightarrow 3d^2 + 3ad - ad - a^2 = 0$$

$$\Rightarrow 3d(d+a) - a(d+a) = 0$$

$$\Rightarrow (3d - a)(a + d) = 0$$

From Eq. (1), we know that $a + d \neq 0$. Therefore,

$$3d - a = 0 \Rightarrow 3d = a$$

Substituting in Eq. (1), we get

$$3d(3d + d) = 48$$

$$\Rightarrow 3d \times 4d = 48$$

$$\Rightarrow d^2 = \frac{48}{12} \Rightarrow d^2 = 4 \Rightarrow d = 2$$

From $3d = a$, we get

$$a = 2 \times 3 = 6 \Rightarrow a = 6$$

Therefore, the sides of the triangle are 6, (6 + 2), (6 + 2 × 2), that is, 6, 8, 10.

Hence, the correct answer is (6).

10 Cartesian Coordinates and Straight Lines

10.1 Cartesian Coordinates

10.1.1 Cartesian System of Coordinates

To locate a point in a two-dimensional plane, we use Cartesian system of coordinates (Fig. 10.1). The axes *OX* and *OY* together are called coordinate axes where *OX* is *x*-axis and *OY* is *y*-axis. The distance between point *Q* and the origin *O* is called 'abscissa' of point *P* (i.e. $OQ = a$) and the distance between point *P* and point *Q* is called 'ordinate' of point *P* (i.e. $PQ = b$).

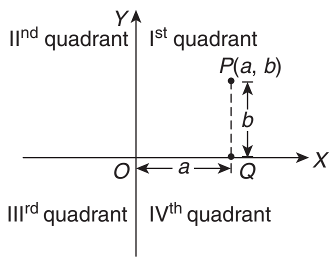


Figure 10.1

10.1.2 Distance Formula

See Fig. 10.2. The distance between points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is named as 'distance formula' which is expressed as

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

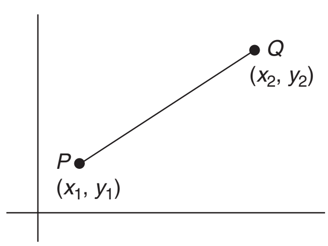


Figure 10.2

> **Key Points**
> 1. The order of the point does not matter.
> 2. Formula is applicable for all four quadrants.

See Fig. 10.3. The distance between the point (x_1, y_1) from the origin is expressed as

$$D = \sqrt{x_1^2 + y_1^2}$$

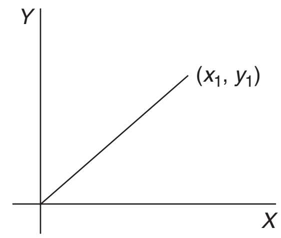


Figure 10.3

Illustration 10.1 Prove that the points $A(3, 4)$, $B(5, 7)$, $C(7, 10)$ are collinear.

Solution: To prove that the given points are collinear, we should prove that the sum of two sides is equal to the third side.

$$AB = \sqrt{(5-3)^2 + (7-4)^2} = \sqrt{4+9} = \sqrt{13}$$
$$BC = \sqrt{(7-5)^2 + (10-7)^2} = \sqrt{4+9} = \sqrt{13}$$
$$AC = \sqrt{(7-3)^2 + (10-4)^2} = \sqrt{4^2 + 6^2} = \sqrt{52} = 2\sqrt{13}$$

Therefore,

$$AB + BC = AC$$

Hence, the given points *A*, *B* and *C* are collinear.

1. **Area of a triangle:** If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ form a triangle, the area of the triangle is

$$\Delta = \left| \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|$$

The area of a triangle can also be determined using 'stair method':

$$\Delta = \left| \frac{1}{2} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} \right|$$

or $$\Delta = \frac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}$$

> **Key Point**
> When points *A*, *B* and *C* are taken in anticlockwise order, the area is positive and if the points are taken in clockwise direction, the area results as a negative quantity. As area needs to be necessarily a positive quantity, care should be taken in finding. Whenever area of a triangle is provided, consider '±' signs.

2. **Condition for collinearity of three points:** If three points A, B and C are collinear, then the area of the triangle ABC has to be zero, that is

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

or
$$x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$$

3. **Area of a convex polygon:** Let $A_1, A_2, \ldots, A_n$ are the vertices of an n-sided plane polygon, its area is given by 'stair method'

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ \cdot & \cdot \\ \cdot & \cdot \\ x_n & y_n \\ x_1 & y_1 \end{vmatrix} = \frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_ny_1 - x_1y_n)]$$

10.1.3 Section Formula

(i) See Fig. 10.4(a). The coordinates of point $R(\bar{x}, \bar{y})$ which divides points $P(x_1, y_1)$ and $Q(x_2, y_2)$ internally, in the ratio $m_1{:}m_2$, are

$$\bar{x} = \frac{m_1x_2 + m_2x_1}{m_1 + m_2};\ \bar{y} = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}$$

(ii) See Fig. 10.4(b). The coordinates of the point $R(\bar{x}, \bar{y})$ which divides the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ externally, in the ratio $m_1{:}m_2$, are

$$\bar{x} = \frac{m_1x_2 - m_2x_1}{m_1 - m_2};\ \bar{y} = \frac{m_1y_2 - m_2y_1}{m_1 - m_2}$$

In both cases, $m_1 \neq m_2$.

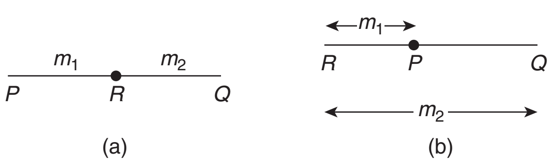


Figure 10.4

Corollaries:

1. Midpoint (i.e. when $m_1 = m_2$) of PQ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
2. If point $R(\bar{x}, \bar{y})$ divides points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the ratio $\lambda : 1\ (\lambda > 0)$, then

$$\bar{x} = \frac{\lambda x_2 \pm x_1}{\lambda \pm 1};\ \bar{y} = \frac{\lambda y_2 \pm y_1}{\lambda \pm 1}$$

Here, '+' sign is considered for internal division and '–' sign is considered for external division.

3. For finding the ratio of division, use $\lambda : 1$. If λ is positive, it indicates the internal division and if λ is negative, it indicates the external division.
4. Line $ax + by + c = 0$ divides join of $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the ratio

$$\left[-\frac{(ax_1 + by_1 + c)}{(ax_2 + by_2 + c)}\right]$$

10.1.4 Centroid, Incentre and Excentre of a Triangle

If vertices of ΔABC have coordinates $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, then the following definitions are considered:

(i) The coordinates of the 'centroid' of the triangle are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

(ii) The coordinates of the 'incentre' of the triangle are

$$\left(\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c}\right)$$

(iii) The coordinates of the 'excentre' I_a (Fig. 10.5) are

$$\left(\frac{-ax_1 + bx_2 + cx_3}{-a + b + c}, \frac{-ay_1 + by_2 + cy_3}{-a + b + c}\right)$$

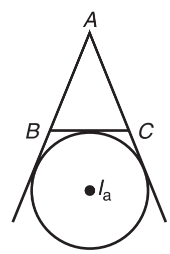


Figure 10.5

In above cases, $a = BC$, $b = AC$ and $c = AB$. If G, C and H denote the centroid, circumcentre and orthocentre, respectively, of ΔABC, then, G, C and H are collinear and G divides CH internally in the ratio 1:2.

Note: 'Incentre' is the point of intersection of internal bisectors of angles of triangle. Its distance from all three sides is same and called 'in-radius' (r) of circle.

10.1.5 Circumcentre of a Triangle

Circumcentre is the point of intersection of perpendicular bisectors of sides, so its distance from all three vertices is same. If $O(x, y)$ (See Fig. 10.6) be the circumcentre of ΔABC, [$A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$], then

$$(OA)^2 = (OB)^2 = (OC)^2$$

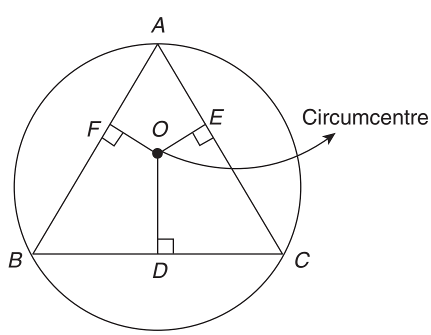


Figure 10.6

That is,

$$(x - x_1)^2 + (y - y_1)^2 = (x - x_2)^2 + (y - y_2)^2 = (x - x_3)^2 + (y - y_3)^2$$

which will give (x, y) when solved.

Aliter

1. If D, E and F are midpoints of BC, AC and AB, respectively, then

$$\text{Slope of } BC \times \text{Slope of } OD = -1$$
$$\text{Slope of } AC \times \text{Slope of } OE = -1$$
$$\text{Slope of } AB \times \text{Slope of } OF = -1$$

On solving any two of this set of equations, we get x and y.

2. Circumcentre is given by

$$\left[\frac{x_1\sin 2A + x_2\sin 2B + x_3\sin 2C}{\sin 2A+\sin 2B+\sin 2C}, \frac{y_1\sin 2A + y_2\sin 2B + y_3\sin 2C}{\sin 2A+\sin 2B+\sin 2C}\right]$$

Illustration 10.2 A triangle ABC has its vertices $(5\sqrt{2}, 5)$, $(3\sqrt{2}, 9)$ and $(-\sqrt{2}, 2)$, respectively. At what distance the centroid is located from the origin?

(A) $\sqrt{\dfrac{108}{5}}$ **(B)** $\sqrt{\dfrac{118}{3}}$

(C) $\dfrac{\sqrt{118}}{3}$ **(D)** $\dfrac{\sqrt{108}}{5}$

Solution: By using formula, we have

$$\text{Centroid (G)} = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$
$$= \left(\frac{7\sqrt{2}}{3}, \frac{16}{3}\right)$$

Therefore,

$$OG = \sqrt{\left(\frac{7\sqrt{2}}{3}\right)^2 + \left(\frac{16}{3}\right)^2}$$
$$= \frac{1}{3}\sqrt{98+256} = \frac{\sqrt{354}}{3} = \sqrt{\frac{118}{3}}$$

10.1.6 Locus and its Equation

When a point moves so as to satisfy a given condition or conditions, the path it traces out is called its 'locus' under the given condition or conditions.

Equation to the locus (or curve) is the relation between the coordinates of an arbitrarily chosen point on the curve and this relation holds for no other points except those lying on the curve.

1. Given two points, $A(x_1, y_1)$ and $P(x, y)$ move such that the slope corresponding to points P and A is a constant m for all positions of point P. The locus of point P is a straight line through point A and with slope m. The equation to the locus is

$$\frac{y-y_1}{x-x_1} = m$$

2. If $O(x_1, y_1)$ is a fixed point and a point $P(x, y)$ moves such that its distance from point O is a constant, say, a. The locus of point P is a circle with centre O and radius a. The equation of the locus is

$$\sqrt{(x-x_1)^2+(y-y_1)^2} = a$$

3. If A and B are two fixed points and a point P moves such that $PA = PB$, then the locus of point P is the perpendicular bisector of AB.
4. Given a line L and a point S, which is not located on line L, a point P moves such that its distance from point S is e (> 0) times its distance from line L. The locus of point P is
   - **(i)** an ellipse, if $e < 1$
   - **(ii)** a parabola, if $e = 1$
   - **(iii)** a hyperbola, if $e > 1$.

10.1.7 Standard Method for Finding Equation of a Locus

1. Assume the point whose locus (path) is to be found as (h, k).
2. Make the equation involving (h, k) as per the given conditions.
3. Simplify the equation.
4. Substitute h with x and k with y in the simplified form of equation and you get the equation of locus.

Illustration 10.3 Find the equation of the locus of a point so that the sum of its distance from two given points $P(3, 2)$ and $Q(4, 3)$ is 4.

Solution: Let the required variable point be $R(h, k)$. We have $PR + QR = 4$; hence

$$\sqrt{(h-3)^2+(k-2)^2} + \sqrt{(h-4)^2+(k-3)^2} = 4$$

That is,

$$h^2+k^2+13-6h-4k$$
$$=16+h^2+k^2+25-8h-6k-8\sqrt{(h-4)^2+(k-3)^2}$$
$$\Rightarrow 2h+2k-28 = -8\sqrt{(h-4)^2+(k-3)^2}$$
$$\Rightarrow h+k-14 = -4\sqrt{(h-4)^2+(k-3)^2}$$

Squaring both the sides, we have

$$h^2+k^2+196-28h-28k+2hk = 16h^2+16k^2+400-128h-96k$$
$$\Rightarrow 15h^2+15k^2-2hk-100h-68k+204=0$$

Hence, the locus is

$$15x^2+15y^2-2xy-100x-68y+204=0$$

10.1.8 Shifting of Origin

'Shifting of origin' implies the meaning that the change of axes by changing the origin, but the direction of axes remaining the same.

Let OXY and $O'X'Y'$ be two rectangular Cartesian system of axes. Let P be any point in the plane of the axes and let point P and origin O' have coordinates (x, y) and (h, k), respectively, with respect to OXY system. Then the coordinates (x', y') of point P with respect to the system $OX'Y'$ are shown in Fig. 10.7.

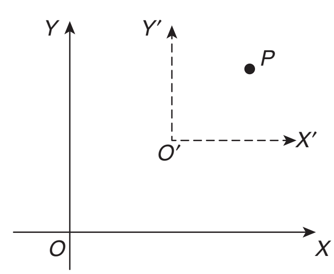


Figure 10.7

10.1.9 Rotation of Axes

'Rotation of axes' implies the meaning that the change of axes (without changing the origin) by changing the direction of axes, both systems of coordinates being rectangular. If a point P in the plane OXY has coordinates (x, y) and (x', y') with respect to the system OXY and $OX'Y'$ (Fig. 10.8), respectively, then

$$x = x'\cos\theta - y'\sin\theta$$
$$y = x'\sin\theta - y'\cos\theta$$

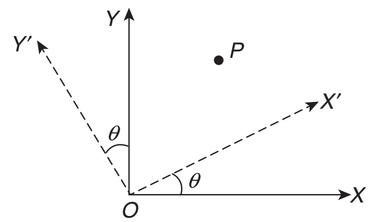


Figure 10.8

10.2 Slope of a Line

If $A(x_1, y_1)$ and $B(x_2, y_2)$ are any two points, then the slope between points A and B is defined as

$$m = \frac{y_1 - y_2}{x_1 - x_2} = \tan\theta,\quad 0 \le \theta < \pi;\ \theta \ne \frac{\pi}{2}$$

where θ is the angle of inclination of the line joining points A and B with positive direction of x-axis.

1. If $x_1 = x_2$, then the slope is not defined.
2. Points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear (i.e. the points lie on a straight line) if slope between A and B is equal to slope between B and C.
3. Slope of line $ax + by + c = 0$ is $-(a/b)$.
4. Two lines with slopes m_1 and m_2 are
   (i) parallel, if $m_1 = m_2$.
   (ii) perpendicular, if $m_1 m_2 = -1$.

10.3 Intercepts of a Line

Let a line $L \equiv ax + by + c = 0$ intersects OX-axes at point A and OY-axes at point B, then OA and OB are called x-intercept and y-intercept of line, respectively. For x-intercept, substitute $y = 0$ in the equation

$$ax + c = 0$$

Therefore, $x = -(c/a)$ is the x-intercept. Similarly, for y-intercept, substitute $x = 0$

$$by + c = 0$$

Therefore, $y = -(c/b)$ is the y-intercept.

Note: See Fig. 10.9. The equation of line having its x- and y-intercepts as a and b, respectively, is

$$\frac{x}{a} + \frac{y}{b} = 1$$

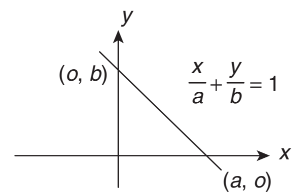


Figure 10.9

which is called the intercept form of line.

Illustration 10.4 If O be the origin and if points Q_1 and Q_2 have their coordinates (x_1, y_1) and (x_2, y_2), respectively, show that $OQ_1(OQ_2)\cos\angle Q_1OQ_2 = x_1x_2 + y_1y_2$.

Solution: See Fig. 10.10.

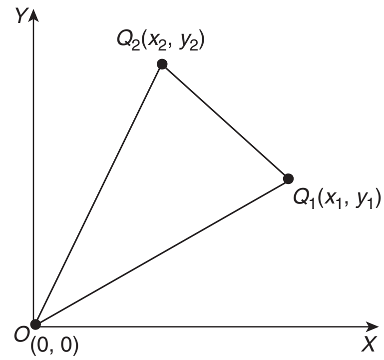


Figure 10.10

The cosine formula applied to triangle Q_1OQ_2 gives

$$\cos\angle Q_1OQ_2 = \frac{OQ_1^2 + OQ_2^2 - Q_1Q_2^2}{2(OQ_1)OQ_2} \qquad \text{(cosine rule)}$$

$$= \frac{[(x_1-0)^2 + (y_1-0)^2] + [(x_2-0)^2 + (y_2-0)^2] - [(x_1-x_2)^2 + (y_1-y_2)^2]}{2(OQ_1)OQ_2}$$

$$= \frac{2(x_1x_2) + 2(y_1y_2)}{2(OQ_1)OQ_2}$$

Hence,

$$OQ_1(OQ_2)\cos\angle Q_1OQ_2 = x_1x_2 + y_1y_2$$

Illustration 10.5 Find the points which divide the line joining the points (–3, –4) and (–8, –7) (a) internally in the ratio 7:5 and (b) externally in the ratio 7:5.

Solution: See Fig. 10.11. Let us write $A(-3, -4)$ and $B(-8, -7)$.

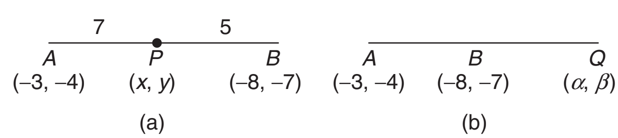


Figure 10.11

(a) See Fig. 10.11(a). If point $P(x, y)$ divides AB internally in the ratio 7:5, then

$$x = \frac{7(-8) + 5(-3)}{7+5} = -\frac{71}{12}$$
$$y = \frac{7(-7) + 5(-4)}{7+5} = -\frac{69}{12}$$

(b) See Fig. 10.11(b). If point $Q(\alpha, \beta)$ divides AB externally in the ratio 7:5, then

$$\alpha = \frac{7(-8)-5(-3)}{7-5} = -\frac{41}{2}$$
$$\beta = \frac{7(-7)-5(-4)}{7-5} = -\frac{29}{2}$$

Illustration 10.6 The coordinates of points A, B, C and P are (6, 3), (–3, 5), (4, –2) and (x, y), respectively. Show that

$$\frac{\text{Area of } \Delta PBC}{\text{Area of } \Delta ABC} = \frac{|x+y-2|}{7}$$

Solution: The area of ΔABC is

$$\frac{1}{2}\begin{vmatrix} -3 & 5 & 1 \\ 4 & -2 & 1 \\ 6 & 3 & 1 \end{vmatrix} = \frac{49}{2}$$

The area of ΔPBC is modulus of the determinant

$$\frac{1}{2}\begin{vmatrix} x & y & 1 \\ -3 & 5 & 1 \\ 4 & -2 & 1 \end{vmatrix} = \frac{7}{2}(x+y-2)$$

Therefore,

$$\frac{\text{Area of } \Delta PBC}{\text{Area of } \Delta ABC} = \frac{|x+y-2|}{7}$$

Illustration 10.7 For what value of k are the points (k, 2 – 2k), (1 – k, 2k) (–4 –k, 6 – 2k) collinear?

Solution: If the points are collinear, then

$$\frac{1}{2}\begin{vmatrix} k & 2-2k & 1 \\ 1-k & 2k & 1 \\ -4-k & 6-2k & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} k & 2-2k & 1 \\ 1-2k & 4k-2 & 0 \\ -4-2k & 4 & 0 \end{vmatrix} = 0$$

$$\Rightarrow 4(1-2k)-(4k-2)(-4-2k)=0$$
$$\Rightarrow 4-8k+16k-8+8k^2-4k=0$$
$$\Rightarrow 8k^2+4k-4=0$$
$$\Rightarrow 2k^2+k-1=0$$

Therefore,

$$k = \frac{-1\pm\sqrt{1+8}}{4} = \frac{-1\pm 3}{4}$$

or $$k = \frac{1}{2}, -1$$

Illustration 10.8 The ends of a rod of length l move on two mutually perpendicular lines. Find the locus of the point on the rod which divides it in the ratio 1:2.

Solution: Let the two mutually perpendicular lines be x and y-axes and $P(x_0, y_0)$ be any point on the locus. Let AB denote the corresponding position of the rod such that $AP/BP = 2$, where we have $A(a, 0)$ and $B(0, b)$. Then

$$l^2 = AB^2 = a^2 + b^2$$

Now,

$$y_0 = \frac{2b+0}{3}; \quad x_0 = \frac{2(0)+1(a)}{3}$$

Therefore,

$$\left(\frac{3y_0}{2}\right)^2 + (3x_0)^2 = l^2$$

$$y_0^2 + 4x_0^2 = \frac{4l^2}{9}$$

The equation of locus is

$$4x^2 + y^2 = \frac{4l^2}{9}$$

Note: If $AP/BP = 1/2$, then the equation of locus will be $4y^2 + x^2 = 4l^2/9$.

Illustration 10.9 $ABCD$ is a variable rectangle having its sides parallel to fixed directions. The vertices B and D lie on $x = a$ and $x = -a$ and A lies on the line $y = 0$. Find the locus of point C.

Solution: See Fig. 10.12. Let A be $(x_1, 0)$, B be (a, y_2) and D be $(-a, y_3)$. We are given that AB and AD have fixed directions and hence their slopes are constant, say, m_1 and m_2. Therefore,

$$\frac{y_2}{a-x_1} = m_1 \text{ and } \frac{y_3}{-a-x_1} = m_2$$

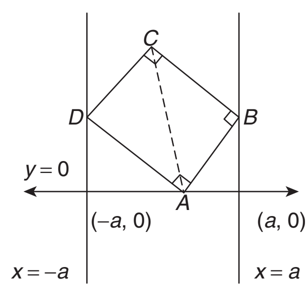


Figure 10.12

Further, $m_1 m_2 = -1$ since $ABCD$ is rectangle.

$$\frac{y_2}{a-x_1} = m_1 \text{ and } \frac{y_3}{a+x_1} = \frac{1}{m_1} \quad (1)$$

Let the coordinates of C be (α, β). Now,

Midpoint of $BD \equiv$ Midpoint of AC

This implies that

$$\frac{x_1+\alpha}{2} = 0 \text{ and } \frac{y_2+y_3}{2} = \frac{\beta}{2}$$

That is,

$$\alpha = -x_1 \text{ and } \beta = y_2 + y_3 \quad (2)$$

By Eqs. (1) and (2), we have

$$-(m_1^2-1)\alpha + m_1\beta = (m_1^2+1)a$$

Therefore, the locus of point C is

$$m_1 y = (m_1^2+1)a + (m_1^2-1)x$$

10.4 Slope of a Straight Line

Let l be a line, which is not parallel to y-axis (Fig. 10.13). Let it make an angle $\theta\,(0 \le \theta < \pi,\ \theta \ne \pi/2)$ with positive direction of x-axis in anticlockwise direction, then tan θ is called the slope of the line l. It is usually denoted by m. Slope m of the straight line passing through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

where $x_1 \ne x_2$.

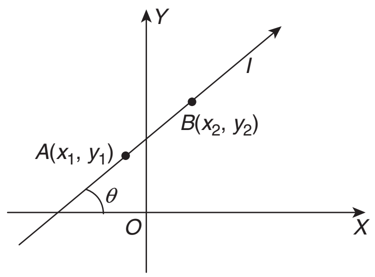


Figure 10.13

Note: If $x_1 = x_2$, the slope of AB is not defined.

Illustration 10.10 Using section formula find the foot of perpendicular drawn from the point (2, 3) to the line joining the points (2, 0) and $\left(\frac{8}{13}, \frac{12}{13}\right)$.

Solution: Let A, B and C be the points (2, 3), (2, 0) and $\left(\frac{8}{13}, \frac{12}{13}\right)$, respectively (Fig. 10.14).

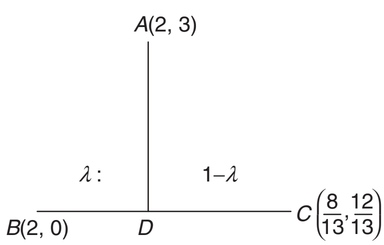


Figure 10.14

Let the foot D of perpendicular AD to BC divides BC in the ratio λ:(1–λ). Then the coordinates of D are given as follows:

$$x\text{-coordinate} = \lambda\left(\frac{8}{13}\right) + (1-\lambda)2 = \frac{-18\lambda + 26}{13}$$

$$y\text{-coordinate} = \frac{12\lambda + (1-\lambda)\times 0}{13} = \frac{12\lambda}{13}$$

The slope of AD is

$$\frac{3 - (12/13)\lambda}{2 - [(-18\lambda + 26)/13]} = \frac{39 - 12\lambda}{18\lambda}$$

The slope of BC is

$$\frac{12/13}{(8/13) - 2} = \frac{12}{-18} = -\frac{2}{3} \qquad \text{(since AD} \perp \text{BC)}$$

Therefore,

$$\frac{39 - 12\lambda}{18\lambda} \times \left(-\frac{2}{3}\right) = -1$$

That is, $\lambda = 1$. Therefore, the coordinates of D are

$$\left(\frac{8}{13}, \frac{12}{13}\right)$$

10.5 Standard Forms of Equation of a Straight Line

1. **General equation of a straight line:** An equation of first degree, namely, $ax + by + c = 0$, where a and b are not zero, represents a straight line. Its slope is $-a/b$.
2. **Slope–intercept form:** $y = mx + c$, where m is the slope and c is the y-intercept of the line.
   Note: Equation of any line parallel to y-axis cannot be expressed in this form.
3. **Intercept form:**
   $$\frac{x}{a} + \frac{y}{b} = 1$$
   where $a \ne 0$ and $b \ne 0$ are x-intercept and y-intercept, respectively.
   Note: Equation of any line passing through origin cannot be expressed in this form.
4. **Point slope form:** Equation of the straight line passing through the point $A(x_1, y_1)$ and whose slope is m is given by
   $$y - y_1 = m(x - x_1)$$
   Note: Equation of any straight line parallel to y-axis cannot be expressed in this form.
5. **Two point form:** Equation of the straight line passing through points $A(x_1, y_1)$ and $B(x_2, y_2)$ where $x_1 \ne x_2$ is given by
   $$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$
6. **Normal form or perpendicular form:** See Fig. 10.15. Here, $x\cos\alpha + y\sin\alpha = p$, where p is the length of the perpendicular drawn from the origin to the line and α is the angle which the perpendicular drawn from the origin to the line makes with the positive direction of x-axis. Here, $p > 0$ and $0 \le \alpha < 2\pi$. A line passing through origin cannot be written in this form.

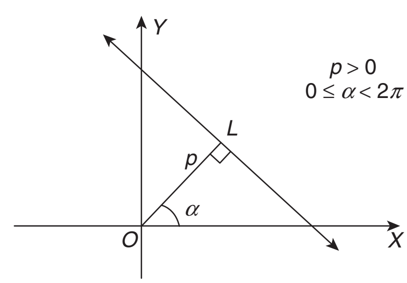


Figure 10.15

7. **Symmetric form or distance form or parametric form:** See Fig. 10.16. Equation of a straight line passing through point $A(x_1, y_1)$ and having slope tan θ is given by
   $$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r \qquad \text{(say)}$$

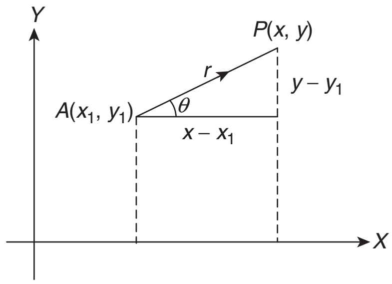


Figure 10.16

Here, $|r|$ is the distance between the points $A(x_1, y_1)$ and $P(x, y)$ which implies that

$$x = x_1 + r\cos\theta$$
$$y = y_1 + r\sin\theta$$

If r is positive, then point $P(x, y)$ lies above point $A(x_1, y_1)$ and if r is negative, then point $P(x, y)$ lies below point $A(x_1, y_1)$.

Illustration 10.11 Find the slope of the line $2x - 5y - 4 = 0$. Also express the equation in intercept form.

Solution: The slope of the line is

$$-\frac{\text{coefficient of } x}{\text{coefficient of } y} = \frac{2}{5}$$

The intercept form of the line $2x - 5y - 4 = 0$ is

$$\frac{x}{2} + \frac{y}{4/-5} = 1$$

Therefore, 2 is the x-intercept and $-4/5$ is the y-intercept made by the given line $(2x - 5y - 4 = 0)$ on the coordinate axes.

Illustration 10.12 Write the equation in normal form for the line $3x - 4y + 5 = 0$.

Solution: Given

$$3x - 4y + 5 = 0$$
$$\Rightarrow 3x - 4y = -5 \Rightarrow -3x + 4y = 5$$
$$\Rightarrow \frac{-3}{\sqrt{(-3)^2 + 4^2}}x + \frac{4}{\sqrt{(-3)^2 + 4^2}}y = \frac{5}{\sqrt{(-3)^2 + 4^2}}$$
$$\Rightarrow \frac{-3}{5}x + \frac{4}{5}y = 1$$

Here, $\cos\alpha = -3/5$, $\sin\alpha = 4/5$ and $p = 1$.

10.6 Position of Two Points w.r.t. Straight Line

Two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on the same side or on the opposite side of the line $ax + by + c = 0$ according as $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of the same sign or of opposite signs, respectively.

Illustration 10.13 Check the position of points (1, 2) and (–5, –1) with respect to the line $x + y + 2 + 0$.

Solution: Using the inequality $(ax_1 + by_1 + c)\cdot(ax_2 + by_2 + c) > 0$ or < 0, we can say the point lies on the same side and the opposite side of the line.

$$(1+2+2)\cdot(-5-1+2) < 0$$

Hence, the points lie on the opposite sides of the given line.

Illustration 10.14 If the point (1, 1) and (a^2, a) lies on the same side of line $x + 3y + 2 + 0$, then find the range of a?

Solution: If the points are lying on the same side of line, then

$$(1+3+2)(a^2+3a+2)>0$$
$$\Rightarrow 6(a+1)(a+2)>0 \Rightarrow a \in (-\infty, -2) \cup (-1, \infty)$$

10.7 Angle between Two Straight Lines

Let θ be the angle between two straight lines whose slopes are m_1 and m_2. Then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

where $m_1 m_2 \neq -1$.

(i) If $m_1 m_2 = -1$, then the two lines are perpendicular to each other.

(ii) If $m_1 = m_2$, then the two lines are parallel.

(iii) If l_1 and l_2 are lines of slopes m_1 and m_2, then the angle $\theta \in [0, \pi)$ given by

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

is the angle by which line l_2 should be rotated in the anticlockwise direction to coincide with l_1.

Note:

1. For finding the equation of any line parallel to a given line, leave the terms of x and y as they are and replace the constant c by another constant k.
2. For finding the equation of any line perpendicular to a given line, interchange the coefficients of x and y and change the sign of any one of them and replace the constant c by another constant k.

Illustration 10.15 Find the angle between the lines $x + 2y + 2 = 0$ and $2x + 3y + 4 = 0$.

Solution: Slope of line $x + 2y + 2 = 0$ is $m_1 = -\frac{1}{2}$ and slope of line $2x + 3y + 4 = 0$ is $m_2 = -\frac{2}{3}$.

Therefore, angle between the given line is

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{-\frac{1}{2} + \frac{2}{3}}{1 + \frac{1}{2}\cdot\frac{2}{3}}\right| = \left|\frac{1}{6}\cdot\frac{3}{4}\right| = \frac{1}{8}$$

10.8 Distance between Two Parallel Straight Lines

Let $ax + by + c = 0$ and $ax + by + c' = 0$ be the parallel straight lines. Then the distance between them is given by

$$\left|\frac{c - c'}{\sqrt{a^2 + b^2}}\right|$$

Illustration 10.16 Find the distance between the lines $x + 2y + 2 = 0$ and $x + 2y + 5 = 0$.

Solution: Distance between two parallel lines is given by

$$\left|\frac{c-c'}{\sqrt{a^2+b^2}}\right| = \left|\frac{5-2}{\sqrt{1+4}}\right| = \frac{3}{\sqrt{5}}$$

Illustration 10.17 Find the distance between the lines $2x + 3y + 2 = 0$ and $4x + 6y + 5 = 0$.

Solution: Distance between two parallel lines is given by

$$\left|\frac{c-c'}{\sqrt{a^2+b^2}}\right|$$

But this formula is applied only when coefficients of x and y of both the lines are same.

Now, distance between two parallel lines is

$$\left|\frac{\frac{5}{2}-2}{\sqrt{4+9}}\right| = \left|\frac{\frac{1}{2}}{\sqrt{4+9}}\right| = \frac{1}{2\sqrt{13}}$$

Your Turn 1

1. The larger of the two angles made with x-axis of a straight line drawn through point (1, 2) so that it intersects $x + y = 4$ at a distance $\sqrt{6}/3$ from (1, 2) is
   (A) 105° **(B)** 75° **(C)** 60° **(D)** 15° **Ans.** (B)
2. A straight line through point (2, 2) intersects the lines $\sqrt{3}x + y = 0$ and $\sqrt{3}x - y = 0$ at the points A and B. The equation to the line AB, so that the triangle OAB is equilateral, is
   (A) $x - 2 = 0$ **(B)** $y - 2 = 0$
   (C) $x + y - 4 = 0$ **(D)** None of these
   Ans. (B)
3. If two vertices of a triangle are (5, –1) and (–2, 3) and if its orthocentre lies at the origin, then the coordinates of the third vertex are
   (A) (4, 7) **(B)** (–4, –7)
   (C) (2, –3) **(D)** (5, –1) **Ans.** (B)
4. Consider the equation $y - y_1 = m(x - x_1)$. If m and x_1 are fixed and different lines are drawn for different values of y_1, which of the following statements is true?
   (A) The lines will pass through a single point
   (B) There will be one possible line only
   (C) There will be a set of parallel lines
   (D) None of these **Ans.** (C)
5. All points that are lying inside a triangle formed by the points (1, 3), (5, 0) and (–1, 2) satisfy
   (A) $3x + 2y \geq 0$ **(B)** $2x + y - 13 \geq 0$
   (C) $2x - 3y + 12 \leq 0$ **(D)** $-2x + y \geq 0$
   Ans. (A)
6. The distance of the point (3, 5) from the line $2x + 3y - 14 = 0$ measured parallel to the line $x - 2y = 1$ is
   (A) $7/\sqrt{5}$ **(B)** $7/\sqrt{13}$ **(C)** $\sqrt{5}$ **(D)** $\sqrt{13}$ **Ans.** (C)
7. The equation of a straight line passing through (1, 2) and having intercept of length three between the straight lines $3x + 4y = 24$ and $3x + 4y = 12$ is
   (A) $7x + 24y - 55 = 0$ **(B)** $24x + 7y - 38 = 0$
   (C) $24x - 7y - 10 = 0$ **(D)** $7x - 24y + 41 = 0$
   Ans. (D)
8. A point $P(1, 1)$ is translated parallel to $2x = y$ in the first quadrant through a unit distance. The coordinates of the new position of point P are
   (A) $\left(1\pm\frac{2}{\sqrt{5}}, 1\pm\frac{1}{\sqrt{5}}\right)$ **(B)** $\left(1\pm\frac{1}{\sqrt{5}}, 1\pm\frac{2}{\sqrt{5}}\right)$
   (C) $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$ **(D)** $\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$
   Ans. (B)
9. Find the equations of the line on which the perpendiculars from the origin make 30° angle with x-axis and which form a triangle of area $50/\sqrt{3}$ with axes.
   (A) $x + \sqrt{3}y \pm 10 = 0$ **(C)** $\sqrt{3}x + y \pm 10 = 0$
   (C) $x \pm \sqrt{3}y - 10 = 0$ **(D)** None of these
   Ans. (B)
10. If each of the points $(x_1, 4)$ and $(-2, y_1)$ lies on the line joining the points (2, –1) and (5, –3), then the point $P(x_1, y_1)$ lies on the line
    (A) $x = 3y$ **(B)** $x = -3y$
    (C) $y = 2x + 1$ **(D)** $2x + 6y + 1 = 0$
    Ans. (D)

Note: *The 'condition of concurrency' of three straight lines*: Let the equations of three given lines be

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$
$$a_3x + b_3y + c_3 = 0$$

Then the condition is

$$a_3(b_1c_2 - b_2c_1) + b_3(c_1a_2 - c_2a_1) + c_3(a_1b_2 - a_2b_1) = 0$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

10.9 Perpendicular Distance of a Point From a Straight Line

1. The distance of a point (x_1, y_1) from the straight line $ax + by + c = 0$ is

$$\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}}\right|$$

2. The coordinates of the foot of perpendicular drawn from $P(x_1, y_1)$ to the line $ax + by + c = 0$ are given by

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = -\frac{ax_1 + by_1 + c}{a^2 + b^2}$$

3. The coordinates of the image of the point $P(x_1, y_1)$ in the line $ax + by + c = 0$ are given by

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{-2(ax_1 + by_1 + c)}{a^2 + b^2}$$

10.10 Slope of Straight Line that Makes Angle α with Line

Let L be a given straight line which makes an angle θ with positive x-axis. Then, there are two straight lines making angle α with L (in fact, passing through a given point on L). Thus, the two lines make angles $\theta - \alpha$ and $\theta + \alpha$ with positive x-axis. Therefore, their slopes are given by

$$\tan(\theta-\alpha)=\frac{\tan\theta-\tan\alpha}{1+\tan\theta\tan\alpha}\ [\text{provided } (\tan\theta)(\tan\alpha)\neq -1]$$

and $$\tan(\theta-\alpha)=\frac{\tan\theta+\tan\alpha}{1-\tan\theta\tan\alpha}\ [\text{provided } (\tan\theta)(\tan\alpha)\neq 1]$$

Illustration 10.18 Find the equations of the straight lines passing through the point (2, 3) and inclined at $\pi/4$ radians to the line $2x + 3y = 5$.

Solution: Let the line $2x + 3y = 5$ make an angle θ with positive x-axis. Then, $\tan\theta = 2/3$.

Now,

$$(\tan\theta)\tan(\pi/4) = -\frac{2}{3}\times\tan\left(\frac{\pi}{4}\right) = -\frac{2}{3}\neq \pm 1$$

The slopes of the required lines are

$$\tan[\theta+(\pi/4)] = \frac{\tan\theta+\tan(\pi/4)}{1-[\tan\theta\tan(\pi/4)]} = \frac{-(2/3)+1}{1-[-(2/3)]} = \frac{1}{5}$$

and $$\tan[\theta-(\pi/4)] = -5$$

Therefore, the equations of the lines are

$$y-3=\frac{1}{5}(x-2) \quad \text{or} \quad x-5y+13=0$$

$$y-3=-5(x-2) \quad \text{or} \quad 5x+y-13=0$$

Note: Let a line L makes an angle θ with positive x-axis. Let the lines L_1 and L_2 are equally inclined to L and let they have slopes m_1 and m_2, respectively, and are such that the value of $\tan\theta$ (say, m) lies between m_1 and m_2. Then

$$\frac{m-m_1}{1+m\,m_1}=\frac{m_2-m}{1+m_2m}$$

Illustration 10.19 A ray of light travelling along the line $2x - 3y + 5 = 0$ after striking a plane mirror lying along the line $x + y = 2$ gets reflected. Find the equation of the straight line containing the reflected ray.

Solution: The point of intersection of the lines $2x - 3y + 5 = 0$ and $x + y = 2$ is $\left(\frac{1}{5},\frac{9}{5}\right)$ which is also the point of incidence. Slope m of the normal to the mirror (i.e. normal to the line $x + y = 2$) is 1. Now, the incident ray and reflected ray both are equally inclined to the normal and are on opposite side of it. The slope of incident ray is

$$m_1=\frac{2}{3}$$

Let the slope of the reflected ray be m_2. Then

$$\frac{m_1-m}{1+m_1m}=\frac{m-m_2}{1+m_2m}$$

$$=\frac{(2/3)-1}{1+(2/3)\times 1}=\frac{1-m_2}{1+m_2\times 1}$$

Therefore,

$$m_2=\frac{3}{2}$$

Therefore, the equation of the straight line that contains the reflected ray is

$$y-\frac{9}{5}=\frac{3}{2}\left(x-\frac{1}{5}\right)$$

$$\Rightarrow 3x-2y+3=0$$

Your Turn 2

1. The straight line $x + 2y - 9 = 0$, $3x + 5y - 5 = 0$ and $ax + by - 1 = 0$ are concurrent if the straight line $35x - 22y + 1 = 0$ passes through the point
   (A) (a, b) **(B)** (b, a)
   (C) $(a, -b)$ **(D)** $(-a, b)$
   Ans. (A)
2. The image of the point (3, 8) in the line $x + 3y = 7$ is
   (A) (1, 4) **(B)** (4, 1)
   (C) (–1, –4) **(D)** (–4, –1)
   Ans. (C)
3. If $A(1, 1)$, $B(\sqrt{3}+1, 2)$ and $C(\sqrt{3}, \sqrt{3}+2)$ be three vertices of a square, then find the diagonal through B.
   (A) $y = (\sqrt{3}-2)x + (3-\sqrt{3})$ **(B)** $y = 0$
   (C) $y = x$ **(D)** None of these
   Ans. (D)
4. The algebraic sum of the perpendicular distances from $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ to a variable line is zero, then the line passes through which among the following?
   (A) The orthocentre of ΔABC
   (B) The centroid of ΔABC
   (C) The circumcentre of ΔABC
   (D) None of these
   Ans. (B)
5. The equations of the lines through (–1, –1) and making angle 45° with the line $x + y = 0$ are given by
   (A) $x + 1 = 0, y + 2 = 0$
   (B) $x - 1 = 0, y - 2 = 0$
   (C) $x - 1 = 0, y - 1 = 0$
   (D) $x + 1 = 0, y + 1 = 0$
   Ans. (D)

10.11 Angle Bisectors

Let $L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_2 \equiv a_2x + b_2y + c_2 = 0$ be two intersecting lines. The equations of the lines bisecting the angles between L_1 and L_2 are given by

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

If $a_1a_2 + b_1b_2 = 0$, then the given lines are perpendicular to each other or else they will have acute and obtuse angles. Now, we shall be interested in the case when

$$a_1a_2+b_1b_2\neq 0$$

See Fig. 10.17. Let θ be the angle between L_1 and L_2 which is bisected by one of the bisectors say L_3. Then the angle between L_1 and L_3 is θ, $\theta/2$. Now, find $\tan(\pi/2)$.

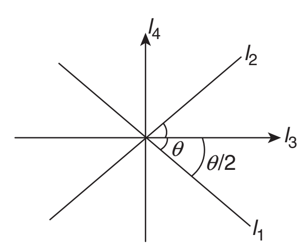


Figure 10.17

Here, two cases arise as follows:

(i) If $|\tan(\theta/2)| < 1$, then $\theta < \pi/2$. Thus, L_3 will be bisecting the acute angles between L_1 and L_2.

(ii) If $|\tan(\theta/2)| > 1$, then $\theta > \pi/2$. Thus, L_3 will be bisecting the obtuse angle between L_1 and L_2.

1. To find the equation of that bisector of the angle between the two lines which contain a given point (α, β): Let the equations of the two lines be $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$. The equation of the bisector of the angle between the two lines containing the point (α, β) will be

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \frac{a_2x+b_2x+c_2}{\sqrt{a_2^2+b_2^2}}$$

or

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = -\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

Accordingly, $a_1\alpha + b_1\beta + c_1$ and $a_2\alpha + b_2\beta + c_2$ are of same signs or of opposite signs.

1. If $c_1 \neq 0$, $c_2 \neq 0$, then the origin must lie in one of the angles between L_1 and L_2. Now, assuming $c_1c_2 > 0$, we have the equation of the bisector, which bisects the angle in which origin lies, is

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}} \quad \text{(considering positive sign)}$$

Also, if $a_1a_2 + b_1b_2 > 0$, the origin lies in obtuse angle or else the origin lies in acute angle.

Illustration 10.20 Find the bisector (a) of acute angle (b) of the angle containing the point (1, –2) between the lines $3x - 4y = 0$ and $5x + 12y + 7 = 0$.

Solution: The equations of the bisectors are

$$\frac{3x-4y}{5} = \pm\frac{5x+12y+7}{13}$$

That is,

$$2x - 16y - 5 = 0 \text{ and } 64x + 8y + 35 = 0$$

Now, suppose θ be the angle between the given lines which is bisected by the bisector

$$2x - 16y - 5 = 0$$

The angle between $3x - 4y = 0$ and $2x - 16y - 5 = 0$ is $\theta/2$ which is certainly acute. Therefore,

$$|\tan(\theta/2)| = \left|\frac{(3/4)-(1/8)}{1+(3/4)\times(1/8)}\right| = \left|\frac{24-4}{32+3}\right| = \frac{20}{35} = \frac{4}{7} < 1$$

Therefore,

$$\frac{\theta}{2} < \frac{\pi}{4} \text{ and so } \theta < \frac{\pi}{2}$$

Hence, $2x - 16y - 5 = 0$ is the required bisector. Substituting (1, –2) in both given line, we get positive and negative values and so the required angle bisector is

$$64x + 8y + 35 = 0$$

Illustration 10.21 If the origin lies in the acute angle between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, show that $(a_1a_2 + b_1b_2)c_1c_2 < 0$.

Solution: See Fig. 10.18. In ΔOM_1M_2, $\angle M_1OM_2$ is obtuse, that is,

$$\cos\angle M_1OM_2 < 0$$

So that

$$OM_1^2 + OM_2^2 - M_1M_2^2 < 0 \quad (1)$$

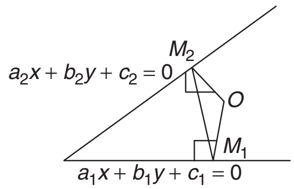


Figure 10.18

Now, the coordinates of M_1 and M_2 are

$$\left(-\frac{a_1c_1}{a_1^2+b_1^2}, -\frac{b_1c_1}{a_1^2+b_1^2}\right) \text{ and } \left(-\frac{a_2c_2}{a_2^2+b_2^2}, -\frac{b_2c_2}{a_2^2+b_2^2}\right)$$

Also,

$$OM_1 = \left|\frac{-c_1}{\sqrt{a_1^2+b_1^2}}\right| \text{ and } OM_2 = \left|\frac{-c_2}{\sqrt{a_2^2+b_2^2}}\right|$$

Substituting these values in Eq. (1), we get

$$\frac{c_1^2}{a_1^2+b_1^2} + \frac{c_2^2}{a_2^2+b_2^2} - \frac{a_1^2c_1^2}{(a_1^2+b_1^2)^2} - \frac{a_2^2c_2^2}{(a_2^2+b_2^2)^2}$$

$$-\frac{b_1^2c_1^2}{(a_1^2+b_1^2)^2} - \frac{b_2^2c_2^2}{(a_2^2+b_2^2)^2} + \frac{2(a_1a_2+b_1b_2)c_1c_2}{(a_1^2+b_1^2)(a_2^2+b_2^2)} < 0$$

$$\Rightarrow (a_1a_2 + b_1b_2)c_1c_2 < 0$$

Your Turn 3

1. The equation of the bisector of the acute angle between the lines $2x - y + 4 = 0$ and $x - 2y = 1$ is

(A) $x + y + 5 = 0$ **(B)** $x - y + 1 = 0$
(C) $x - y = 5$ **(D)** $x - y + 5 = 0$

Ans. (C)

2. The equation of the bisector of that angle between the lines $x + y = 3$ and $2x - y = 2$ which contains the point (1, 1) is

(A) $(\sqrt{5}-2\sqrt{2})x + (\sqrt{5}+\sqrt{2})y = 3\sqrt{5}-2\sqrt{2}$
(B) $(\sqrt{5}+2\sqrt{2})x + (\sqrt{5}-\sqrt{2})y = 3\sqrt{5}+2\sqrt{2}$
(C) $3x = 10$
(D) $3x - 5y + 2 = 0$

Ans. (A)

3. The equation of the straight line which bisects the intercepts made by the axes on the lines $x + y = 2$ and $2x + 3y = 6$ is

(A) $2x = 3$ **(B)** $y = 1$
(C) $2y = 3$ **(D)** $x = 1$

Ans. (B)

4. The line $x + 3y - 2 = 0$ bisects the angle between a pair of straight lines of which one has the equation $x - 7y + 5 = 0$. The equation of other line is

(A) $3x + 3y - 1 = 0$ **(B)** $x - 3y + 2 = 0$
(C) $5x + 5y + 3 = 0$ **(D)** $5x + 5y - 3 = 0$

Ans. (D)

10.12 Family of Straight Lines

1. If $L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_2 \equiv a_2x + b_2y + c_2 = 0$ are two straight lines (not parallel), then $L_1 + \lambda L_2 \equiv a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0$, $\lambda \in R$, represents a family of lines passing through the point of intersection of $L_1 = 0$ and $L_2 = 0$. Here, λ is a parameter.

Note: $ax + by + c = 0$ with $a\alpha + b\beta + 1 = 0$ is the families of lines through (α, β).

2. The family of straight lines parallel to the line $ax + by + c = 0$ is given by $ax + by + k = 0$, where k is a parameter.
3. The family of straight lines perpendicular to the line $ax + by + c = 0$ is given by $bx - ay + k = 0$, where k is a parameter.
4. The family of lines not passing through origin is $ax + by = 1$; where a and b are parameters.
5. The family of lines at a distance of p from origin is $x\cos\alpha + y\sin\alpha = p$, where α is a parameter.

Illustration 10.22 Find the equation of the straight line passing through the point (2, 0) and through the point of intersection of the lines $x + 2y = 3$ and $2x - 3y = 4$.

Solution: The equation of any straight line passing through the intersection of the lines

$$x + 2y = 3 \text{ and } 2x - 3y - 4 = 0$$

is written as

$$\lambda(x + 2y - 3) + (2x - 3y - 4) = 0 \qquad (1)$$

Since the line passes through the point (2, 0), we have

$$\lambda(2 + 0 - 3) + (4 - 0 - 4) = 0$$

That is, $\lambda = 0$. Now substituting this value of λ in Eq. (1), we get

$$2x - 3y - 4 = 0$$

which is the equation of the given line.

10.13 Locus of a Point

1. The path traced by a point moving under a given condition (or a given set of conditions) is called the 'locus' of the point. If an equation is satisfied by the coordinates of every point on the path and any point whose coordinates satisfy the equation lies on the path, then the equation is called the 'equation of the locus'.
2. **Equation of locus:** To find the equation of locus of a point under given condition(s), we proceed as follows:
   (i) Assign the coordinates (h, k) [or (x, y)] to the point whose locus is to be determined.
   (ii) Properly conceive the given geometrical condition(s) which the point, whose locus is to be determined, is to satisfy.
   (iii) Express the said condition(s) in an analytical relation in h and k (or in x and y).
   (iv) Solve to eliminate the parameter(s) so that the resulting expression contains known quantities and (h, k) or (x, y).
   (v) Replace (h, k) by (x, y) if taken. The resulting equation will be the required locus.

Illustration 10.23 A variable straight line drawn through the point of intersection of the lines $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{b}+\frac{y}{a}=1$ meets the coordinate axes at points A and B. Show that the locus of the mid-point of AB is the curve $2xy(a + b) = ab(x + y)$.

Solution: Any line through the point of intersection of given lines is

$$\left(\frac{x}{a}+\frac{y}{b}-1\right)+\lambda\left(\frac{x}{b}+\frac{y}{a}-1\right)=0$$

The straight line meets x-axis at point $A \equiv \left[\frac{ab(1+\lambda)}{a\lambda+b}, 0\right]$

The straight line meets the y-axis at point $B \equiv \left[0, \frac{ab(1+\lambda)}{a+b\lambda}\right]$

Let the midpoint of AB be $P(h, k)$. Then

$$h = \frac{ab(1+\lambda)}{2(a\lambda+b)};\ k = \frac{ab(1+\lambda)}{2(a+b\lambda)}$$

$$\Rightarrow \frac{1}{h} = \frac{2(a\lambda+b)}{ab(1+\lambda)};\ \frac{1}{k} = \frac{2(a+b\lambda)}{ab(1+\lambda)}$$

$$\Rightarrow \frac{1}{h}+\frac{1}{k} = \frac{2}{ab(1+\lambda)}[a\lambda+b+a+b\lambda]$$

$$= \frac{2}{ab(1+\lambda)}[a(\lambda+1)+b(1+\lambda)]$$

$$= \frac{2(a+b)}{ab}$$

$$\Rightarrow 2hk\,(a + b) = ab(h + k)$$

The locus of $P(h, k)$ is $2xy(a + b) = ab(x + y)$.

Your Turn 4

1. A and B are fixed points. The vertex C of ΔABC moves such that $\cot A + \cot B$ = constant. The locus of C is a straight line that satisfies which among the following?
   (A) Perpendicular to AB
   (B) Parallel to AB
   (C) Inclined at an angle $(A - B)$ to AB
   (D) None of the above **Ans.** (B)
2. The position of a moving point in an xy-plane at time t is given by $[u\sin\alpha(t), u\sin\alpha[t - (1/2)gt^2]$, where u, α and g are constants. What is the locus of the moving point?
   (A) A circle (B) A parabola
   (C) An ellipse (D) None of these **Ans.** (B)
3. A ray of light coming from the point (1, 2) is reflected at a point A on x-axis and then passes through the point (5, 3). The coordinates of point A are
   (A) $\left(\frac{13}{5}, 0\right)$ **(B)** $\left(\frac{5}{13}, 0\right)$
   (C) (–7, 0) **(D)** None of these
   Ans. (A)
4. Find the equation of straight line which passes through the intersection of the straight lines $3x - 4y + 1 = 0$ and $5x + y - 1 = 0$ and cuts off equal intercepts from the axes.
   Ans. $23x + 23y = 11$
5. Find the equation of the line passing through the point of intersection of the lines $x + 5y + 7 = 0$ and $3x + 2y - 5 = 0$ and (a) parallel to the line $7x + 2y - 5 = 0$; (b) perpendicular to the line $7x + 2y - 5 = 0$.
   Ans. (a) $7x + 2y - 17 = 0$; (b) $2x - 7y - 20 = 0$

10.14 Shifting of Origin

See Fig. 10.19. If origin is shifted from $O(0, 0)$ to $O'(h, k)$, then if (x, y) are the coordinates of point P in the old system (when origin was O) and (X, Y) are the coordinates of the same point P in the new system (when origin is O'), then

$$x = X + h \text{ and } y = Y + k$$

Thus, if $f(x, y) = 0$ is the equation of a curve in a coordinates system and if origin is shifted to a point (h, k), then the equation changes to

$$f(X + h, Y + k) = 0$$

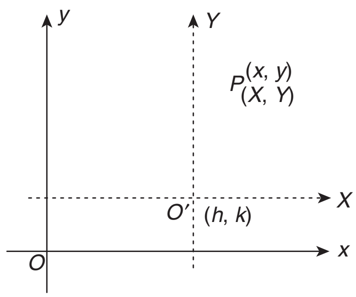


Figure 10.19

For example, if $u \equiv ax^2 + 2hxy + by^2 + 2gx + 2gy + c = 0$ represents a pair of straight lines, which intersect at (α, β), then on shifting the origin to (α, β) (i.e. x is replaced by $x + \alpha$ and y is replaced by $y + \beta$) the new equation should be homogenous of degree two, that is, in $a(x + \alpha)^2 + 2h(x + \alpha)(y + \beta) + b(y + \beta)^2 + 2g(x + \alpha) + 2f(y + \beta) + c = 0$

The coefficients of x, y and constant term should be zero, that is,

$$2a\alpha + 2h\beta + 2g = 0 \quad (1)$$

$$2h\alpha + 2b\beta + 2f = 0 \quad (2)$$

and $$a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c = 0 \quad (3)$$

Equations (1) and (2) can be solved to find α and β and if these values are replaced in Eq. (3), then we get

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

which is a necessary condition for $u = 0$ to represent a pair of straight lines.

Remark: Equations (1) and (2) are in fact $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0$ at $x = \alpha$, $y = \beta$, where $\frac{\partial u}{\partial x}$ represent the partial derivative of u with respect to x.

Illustration 10.24 The straight lines $3x + 4y = 5$ and $4x - 3y = 15$ intersect at the point A. On these lines the points B and C are chosen so that $AB = AC$. Find the possible equations of the line BC passing through (1, 2).

Solution: Two straight lines are given at right angles. Since $AB = AC$, the triangle is an isosceles right-angled triangle. The required equation is of the form

$$y - 2 = m(x - 1) \quad (1)$$

with $$\tan 45° = \pm\frac{m + (3/4)}{1 - (3m/4)} = \pm\frac{m - (4/3)}{1 + (4m/3)}$$

$$\Rightarrow 1 = \pm\frac{m + (3/4)}{1 - (3m/4)} \text{ and } 1 = \pm\frac{m - (4/3)}{1 + (4m/3)}$$

$$\Rightarrow m = -7, \frac{1}{7}$$

Substituting the value of m in Eq. (1), we get the required equations.

Equation of lines are $y - 2 = -7(x - 1)$ and $y - 2 = \frac{1}{7}(x - 1)$.

Illustration 10.25 Find the range of θ in the interval $(0, \pi)$ such that the points (3, 5) and $(\sin\theta, \cos\theta)$ lie on the same side of the line $x + y - 1 = 0$.

Solution: We have

$$3 + 5 - 1 = 7 > 0$$

Therefore,

$$\sin\theta + \cos\theta - 1 > 0$$

$$\Rightarrow \sin\left(\frac{\pi}{4} + \theta\right) > \frac{1}{\sqrt{2}}$$

$$\Rightarrow \frac{\pi}{4} < \frac{\pi}{4} + \theta < \frac{3\pi}{4}$$

$$\Rightarrow 0 < \theta < \frac{\pi}{2}$$

Illustration 10.26 The vertices of a triangle are $[at_1t_2, a(t_1 + t_2)]$, $[at_2t_3, a(t_2 + t_3)]$ and $[at_3t_1, a(t_3 + t_1)]$. Find the orthocentre of the triangle.

Solution: See Fig. 10.20.

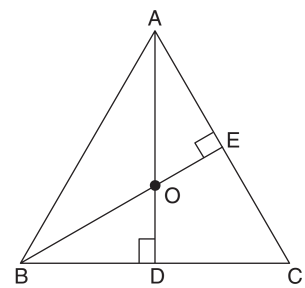


Figure 10.20

Let A, B and C be the vertices

$$A \equiv [at_1t_2, a(t_1 + t_2)]$$

$$B \equiv [at_2t_3, a(t_2 + t_3)]$$

$$C \equiv [at_3t_1, a(t_3 + t_1)]$$

The slope of BC is

$$\frac{a(t_3 + t_1) - a(t_2 + t_3)}{at_3t_1 - at_2t_3} = \frac{1}{t_3}$$

The slope of altitude $AD = -t_3$. The equation of AD is

$$a(t_1 + t_2) = -t_3(x - at_1t_2) \quad (1)$$

Similarly, the equation of altitude BE is

$$y - a(t_2 + t_3) = -t_1(x - at_2t_3) \quad (2)$$

The coordinates of orthocentre O are obtained by simultaneously solving Eqs. (1) and (2). Subtracting Eq. (2) from Eq. (1), we get

$$a(t_3 - t_1) = -x(t_3 - t_1) \Rightarrow x = -a$$

Subtracting $x = -a$ in Eq. (1), we get

$$y = a(t_1 + t_2) - t_3(-a - at_1t_2)$$

$$y = a(t_1 + t_2 + t_3 + t_1t_2t_3)$$

The coordinates of the orthocentre does not depend on the values of t_1, t_2 and t_3.

Illustration 10.27 The base BC of a triangle ABC contains the points $P(p_1, q_1)$ and $Q(p_2, q_2)$ and the equation of sides AB and AC

are $p_1x + q_1y = 1$ and $q_2x + p_2y = 1$, respectively. Prove that the equations of AP and AQ respectively, are

$$(p_1q_2 + q_1p_2 - 1)(p_1x + q_1y - 1) = (p_1^2 + q_1^2 - 1)(q_2x + p_2y - 1)$$

and $$2(p_2q_2 - 1)(p_1x + q_1y - 1) = (p_1p_2 + q_1q_2 - 1)(q_2x + p_2y - 1)$$

Solution: See Fig. 10.21.

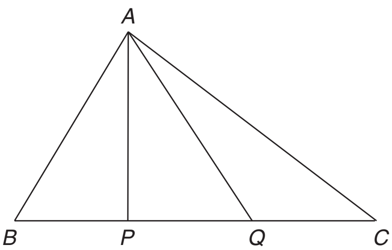


Figure 10.21

Since both AP and AQ pass through A the intersection of the two given lines, their equation is

$$(p_1x + q_1y - 1) + \lambda(q_2x + p_2y - 1) = 0 \quad (1)$$

In order to find the equation of AP, we find the value of λ by using the fact that Eq. (1) passes through $P(p_1, q_1)$ as follows:

$$(p_1^2 + q_1^2 - 1) + \lambda_1(p_1q_2 + q_1p_2 - 1) = 0$$

$$\Rightarrow \lambda_1 = -\frac{p_1^2 + q_1^2 - 1}{p_1q_2 + q_1p_2 - 1}$$

Thus, the equation of AP is

$$(p_1q_2 + q_1p_2 - 1)(p_1x + q_1y - 1) = (p_1^2 + q_1^2 - 1)(q_2x + p_2y - 1)$$

Next, to find the equation of AQ, we find the value of λ by using the fact that Eq. (1) passes through $Q(p_2, q_2)$ as follows:

$$(p_1p_2 + q_1q_2 - 1) + \lambda(2p_2q_2 - 1) = 0$$

$$\Rightarrow \lambda_2 = -\frac{p_1p_2 + q_1q_2 - 1}{2p_2q_2 - 1}$$

Therefore, the equation of AQ is

$$(2p_2q_2 - 1)(p_1x + q_1y - 1) = (p_1p_2 + q_1q_2 - 1)$$
$$\Rightarrow q_2x + p_2y - 1$$

Illustration 10.28 Prove that all lines represented by the equation

$$(2\cos\theta + 3\sin\theta)x + (3\cos\theta - 5\sin\theta)y = 5\cos\theta - 2\sin\theta \quad (1)$$

pass through a fixed point for all θ. What are the coordinates of this fixed point and its reflection in the line $x + y = \sqrt{2}$? Prove that all lines through reflection point can be represented by equation

$$(2\cos\theta + 3\sin\theta)x + (3\cos\theta - 5\sin\theta)y = (\sqrt{2} - 1)(5\cos\theta - 2\sin\theta) \quad (2)$$

Solution: Equation (1) can be expressed as

$$(2x + 3y - 5)\cos\theta + (3x - 5y + 2)\sin\theta = 0$$

or $$(2x + 3y - 5) + \tan\theta(3x - 5y + 2) = 0$$

It is clear that these lines will pass through the point of intersection of the following lines (for all values of θ)

$$2x + 3y - 5 = 0 \quad (3)$$

$$3x - 5y + 2 = 0 \quad (4)$$

Solving Eqs. (3) and (4), we get the coordinates of the required fixed points as $P(1, 1)$. Let $Q(\alpha, \beta)$ be the reflection of $P(1, 1)$ in the line $x + y = \sqrt{2}$. The line PQ is perpendicular to the line $x + y = \sqrt{2}$ and the midpoint of segment PQ lies on the line $x + y = \sqrt{2}$. Thus

$$\left(\frac{\beta - 1}{\alpha - 1}\right)(-1) = -1 \Rightarrow \beta = \alpha$$

and $$\frac{\beta + 1}{2} + \frac{\alpha + 1}{2} = \sqrt{2} \Rightarrow \beta = \alpha = \sqrt{2} - 1$$

Hence, the coordinates of the reflection Q of P in the line $x + y = \sqrt{2}$ are

$$Q \equiv (\sqrt{2} - 1, \sqrt{2} - 1)$$

If the required family of lines is

$$(2\cos\theta + 3\sin\theta)x + (3\cos\theta - 5\sin\theta)y = \lambda$$

In order that each member of the family passes through Q, we have

$$\lambda = (\sqrt{2} - 1)\left[2\cos\theta + 3\sin\theta + 3\cos\theta - 5\sin\theta\right]$$
$$= (\sqrt{2} - 1)\left[5\cos\theta - 2\sin\theta\right]$$

Hence, the equation of the family is

$$(2\cos\theta + 3\sin\theta)x + (3\cos\theta - 5\sin\theta)y = (\sqrt{2} - 1)\left[5\cos\theta - 2\sin\theta\right]$$

Illustration 10.29 One diagonal of a square is the intercept of the line $\frac{x}{a} + \frac{y}{b} = 1$ between the axes. Find the coordinates of other two vertices and hence, prove that if two opposite vertices of a square move on two perpendicular lines, the other two vertices also move on perpendicular lines.

Solution: See Fig. 10.22. The coordinates of centre E are $\left(\frac{a}{2}, \frac{b}{2}\right)$.

The slope of $AC = -\frac{b}{a}$

$$\text{The slope of } DB = \frac{a}{b} \Rightarrow \tan\theta = \frac{a}{b} \Rightarrow \sin\theta = \frac{a}{\sqrt{a^2 + b^2}}$$

$$\Rightarrow \cos\theta = \frac{b}{\sqrt{a^2 + b^2}}$$

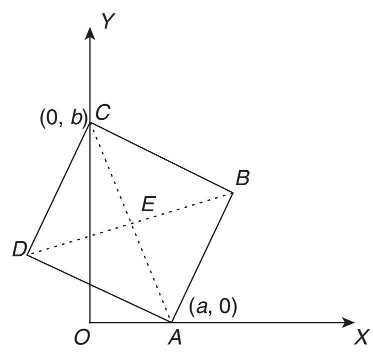


Figure 10.22

$$\text{Length } AE = EC = EB = ED = \frac{1}{2}\text{AC} = \frac{1}{2}\sqrt{a^2 + b^2}$$

Using the distance form for DB, we get

$$\frac{x - (a/2)}{b/\sqrt{a^2 + b^2}} = \frac{y - (b/2)}{a/\sqrt{a^2 + b^2}} = \pm\frac{1}{2}\sqrt{a^2 + b^2} \Rightarrow x = \frac{a \pm b}{2},\ y = \frac{b \pm a}{2}$$

Thus, the coordinates of other two vertices are thus

$$B = \left(\frac{a+b}{2}, \frac{b+a}{2}\right) \quad \text{and} \quad D = \left(\frac{a-b}{2}, \frac{b-a}{2}\right)$$

Now, given that A and C move on perpendicular lines (axes). For the coordinates $\left(\frac{a+b}{2}, \frac{b+a}{2}\right)$, we have the locus $x = y$ and for $\left(\frac{a-b}{2}, \frac{b-a}{2}\right)$ the locus is $x = -y$. These are also perpendicular lines.

Additional Solved Examples

1. If O is the origin and A and B are, respectively, (a_1, b_1) and (a_2, b_2), then $OA \cdot OB \sin(AOB)$ is equal to

(A) $a_1a_2 + b_1b_2$ **(B)** $a_1b_2 + a_2b_1$
(C) $a_1b_2 - a_2b_1$ **(D)** $a_1a_2 - b_1b_2$

Solution: See Fig. 10.23

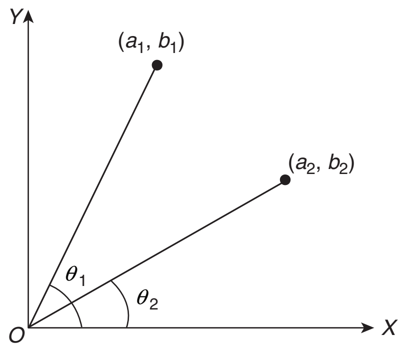


Figure 10.23

$$\sin(\theta_1 - \theta_2) = \sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2$$

$$= \frac{a_1}{\sqrt{a_1^2+b_1^2}} \cdot \frac{b_2}{\sqrt{a_2^2+b_2^2}} - \frac{a_2}{\sqrt{a_2^2+b_2^2}} \cdot \frac{b_1}{\sqrt{a_1^2+b_1^2}}$$

$$\Rightarrow \sqrt{a_1^2+b_1^2}\sqrt{a_2^2+b_2^2}\sin(\theta_1 - \theta_2) = a_1b_2 - a_2b_1$$

Now,

$$OA \cdot OB \sin(AOB) = a_1b_2 - a_2b_1$$

Hence, the correct answer is option (C).

2. If three equations are consistent

$$(a + 1)^3x + (a + 2)^3y = (a + 3)^3$$
$$(a + 1)x + (a + 2)y = a + 3$$
$$x + y = 1,$$

then $a =$

(A) 1 **(B)** 2
(C) −2 **(D)** 3

Solution: Since the equations are consistent,

$$D = 0$$

$$\Rightarrow \begin{vmatrix} (a+1)^3 & (a+2)^3 & -(a+3)^3 \\ (a+1) & (a+2) & -(a+3) \\ 1 & 1 & -1 \end{vmatrix} = 0$$

Put $u = (a + 1)$, $v = a + 2$, $w = a + 3$, we get

$$u - v = -1,\ v - w = -1,\ w - u = 2$$
$$\Rightarrow u + v + w = 3a + 6$$

Therefore,

$$\begin{vmatrix} u^3 & v^3 & -w^3 \\ u & v & -w \\ 1 & 1 & -1 \end{vmatrix} = 0$$

$$\Rightarrow (u - v)(v - w)(w - u)(u + v + w) = 0$$
$$\Rightarrow (-1)(-1)(2)(3a + 6) = 0$$
$$\Rightarrow a = -2$$

Hence, the correct answer is option (C).

3. ABC is a right-angled triangle, right angled at B. The triangle rotates about B such that C and A always lie on two perpendicular lines OX, OY, respectively. The locus of centroid of ΔABC is

(A) A straight line parallel to the perpendicular bisector of OB
(B) A straight line perpendicular to OA
(C) A circle centered at midpoint of OB
(D) None of these

Solution: See Fig. 10.24. Quadrilateral $OCBA$ is cyclic for which AC is diameter. Therefore, midpoint B' of CA lies on perpendicular bisector of fixed line segment OB. The centroid G of ΔABC is on BB' dividing it in the ratio 2:1. Now the locus of B' is the perpendicular bisector of OB implies that the locus of G is straight line parallel to perpendicular bisector of OB.

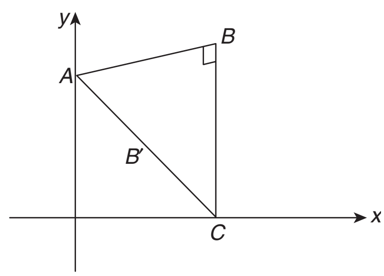


Figure 10.24

Hence, the correct answer is option (A).

4. If the line $y = 2x$ intersects the curve $y = x^2 - 1$ at A and B, then $OA + OB$ and $OA \cdot OB$ is (where O is origin)

(A) $\sqrt{30}, 5$ **(B)** $\sqrt{40}, -5$

(C) $\sqrt{40}, 4$ **(D)** $\sqrt{40}, 5$

Solution: Let $x = r\cos\theta$, $y = r\sin\theta$. Then

$$r^2\cos^2\theta - r\sin\theta - 1 = 0$$

Now,

$$r_1 + r_2 = \frac{\sin\theta}{\cos^2\theta} = 2\sqrt{5}$$

$$r_1r_2 = -\frac{1}{\cos^2\theta} = -5 < 0$$

Therefore, r_1 and r_2 are of opposite sign.

Now,

$$OA \cdot OB = |r_1| \cdot |r_2| = 5$$

$$OA + OB = |r_1| + |r_2| = |r_1 - r_2| = \sqrt{(r_1 + r_2)^2 - 4r_1r_2} = \sqrt{40}$$

Hence, the correct answer is option (D).

5. The number of ordered doubles (m, n) such that the identity $|mx + ny| + |nx + my| = |x| + |y|$ holds for all $x, y \in R$

(A) 1 **(B)** 2
(C) 3 **(D)** 4

Solution: On substituting $x = y$ and $x = -y$, we get

$$|m + n| = |m - n| = 1$$

$\Rightarrow m \cdot n = 0 \Rightarrow$ either $m = \pm 1, n = 0$ or $m = 0, n = \pm 1$

Hence, 4 ordered pairs are possible.

Hence, the correct answer is option (D).

6. If one side of a rhombus has end point (4, 5) and (1, 1), then maximum area of the rhombus is

(A) 5 **(B)** 25
(C) 12.5 **(D)** $\frac{61}{2}$

Solution: See Fig. 10.25.

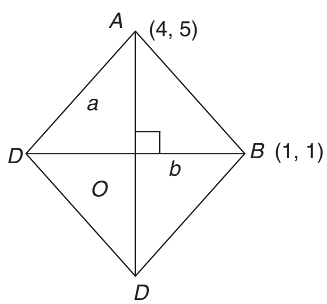


Figure 10.25

Let $OA = a$ and $OB = b$. Then

$$a^2 + b^2 = 25$$

Area of rhombus $ABCD = 2ab$

Let $a = 5\sin\theta, b = 5\cos\theta$. Then

$$2ab = 25 \sin 2\theta$$

Hence, maximum area = 25 sq. units for $\theta = \frac{\pi}{4}$.

Hence, the correct answer is option (B).

7. The number of positive integral solution of $4x + 5y = 625$ is

(A) 30 **(B)** 31
(C) 28 **(D)** None of these

Solution: From the given equation, we have

$$y = \frac{625 - 4x}{5}$$

So, x should be multiple of 5, that is,

$$5, 10, 15, 20, 25, 30, \ldots, 155$$

Hence, the correct answer is option (B).

8. The diagonals of a rhombus $ABCD$ intersect at the point (1, 2) and its sides are parallel to the lines $x - \sqrt{3}y + 2\sqrt{3} = 0$ and $\sqrt{3}x - y + 3 = 0$. If the vertex A be situated on x-axis, then possible co-ordinates of vertex C are

(A) (1, 4) and (– 3, 4) **(B)** (–1, –4) and (– 3, –4)
(C) (– 1, 4) and (3, 4) **(D)** None of these

Solution: Equation of diagonal A may be

$$y - 2 = \pm (x - 1) \Rightarrow y = x + 1 \text{ or } y = -x + 3$$

$\Rightarrow$ Vertex A may be (–1, 0) or (3, 0)

Also, length of diagonal AC in both cases is $4\sqrt{2}$.

Now, if $A \equiv (-1, 0)$, then vertex C will be

$$\frac{x+1}{1/\sqrt{2}} = \frac{y-0}{1/\sqrt{2}} = 4\sqrt{2} \text{ or } C \equiv (3, 4)$$

Also, if $A \equiv (3, 0)$, then vertex C will be

$$\frac{x-3}{-1/\sqrt{2}} = \frac{y-0}{1/\sqrt{2}} = 4\sqrt{2} \text{ or } C \equiv (-1, 4)$$

Hence, the correct answer is option (C).

9. If L_1 and L_2 are two lines belonging to family of lines $(3 + 2\lambda)x + (4 + 3\lambda)y - 7 - 5\lambda = 0$ (λ is parameter) such that it is at maximum and minimum distance from (2, 3), respectively, then the equation of lines passing through (1, 2) and making equal angles with L_1 and L_2 is/are

(A) $x + 2y = 7$ **(B)** $3x + y = 5$
(C) $x - 3y = -5$ **(D)** None of these

Solution: Given family of lines can be written as

$$3x + 4y - 7 + \lambda(2x + 3y - 5) = 0$$

The point of intersection is (1, 1).

Let $Q \equiv (2, 3)$

L_1 will be perpendicular to PQ through P and L_2 will be passing through Q. Therefore,

$$L_1 \equiv 2y + x = 3$$
$$L_2 \equiv 2x - y = 1$$

As angle between the line is same for L_1 & L_2. Therefore,

$$m_1 = \frac{1+2}{1-2} = -3$$

Other slope will be perpendicular to m_1, that is,

$$m_2 = \frac{1}{3}$$

So, equation of lines passing through (1, 2) are

$$3x + y = 5$$
$$x - 3y = -5$$

Hence, the correct answers are options (B) and (C).

10. If the lines joining the origin to the points of intersection of the line $3y = mx + 3$ and the curve $x^2 + y^2 = 1$ are at right angle, then value of $m + 3$ is equal to

(A) 5 **(B)** 6
(C) 1 **(D)** Zero

Solution: See Fig. 10.26.

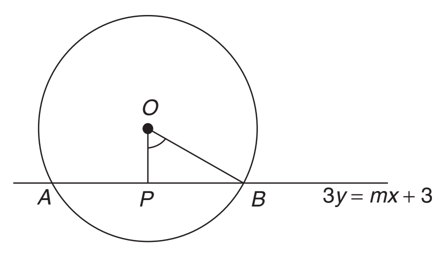


Figure 10.26

$$OP = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \left|\frac{0-0+3}{\sqrt{9+m^2}}\right| = \frac{1}{\sqrt{2}}$$

$$\Rightarrow 9 + m^2 = 18$$
$$\Rightarrow m^2 = 9 \Rightarrow m = \pm 3$$
$$\Rightarrow m + 3 = 6, 0$$

Hence, the correct answers are options (B) and (D).

Paragraph for Questions 11–13: Diagonal *AC* of rhombus *ABCD* is a member of both the family of lines $L_1 + lL_2 = 0$ and $L_3 + mL_4 = 0$ and vertex *B* of rhombus is (3, 2). Suppose
$L_1 \equiv x + y - 1 = 0; L_2 \equiv 2x + 3y - 2 = 0; L_3 \equiv x - y + 2 = 0; L_4 \equiv 2x - 3y + 5 = 0$

11. The equation of diagonal *AC* is

(A) $2x + y + 1 = 0$ **(B)** $x + 2y + 3 = 0$
(C) $x + 2y - 1 = 0$ **(D)** $2x + y - 7 = 0$

Solution: Since diagonal is a member of both the family of lines, it passes through (1, 0) and (–1, 1).

Therefore, equation of diagonal *AC* is $x + 2y - 1 = 0$.
Hence, the correct answer is option (C).

12. The equation of diagonal *BD* is

(A) $2x + y + 1 = 0$ **(B)** $2x - y - 4 = 0$
(C) $x - 2y + 7 = 0$ **(D)** $x + 2y - 1 = 0$

Solution: As point (3, 2) does not lie on *AC*, it lies on *BD*. Hence, equation of *BD* is $2x - y = 4$.

Hence, the correct answer is option (B).

13. If the area of rhombus *ABCD* is $12\sqrt{5}$ sq. units, then coordinates of centroid of ΔACD is

(A) $\left(\frac{6}{5}, \frac{-7}{5}\right)$ **(B)** $\left(\frac{-7}{5}, \frac{6}{5}\right)$
(C) $\left(\frac{7}{5}, \frac{-6}{5}\right)$ **(D)** $\left(\frac{-6}{5}, \frac{7}{5}\right)$

Solution: See Fig. 10.27.

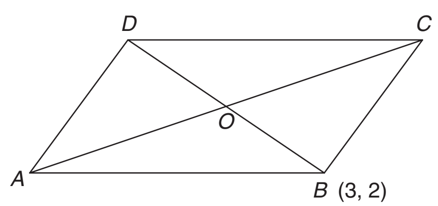


Figure 10.27

Point of intersection of diagonal *AC* and *BD* is $\left(\frac{9}{5}, \frac{-2}{5}\right)$

Therefore, Vertex *D* is $\left(\frac{3}{5}, \frac{-14}{5}\right)$.

Length of $BD = \frac{6\sqrt{5}}{5} \Rightarrow BD = \frac{12\sqrt{5}}{5} = \frac{12}{\sqrt{5}}$

Area (rhombus *ABCD*) $= \frac{1}{2} \times d_1 \times d_2 = 12\sqrt{5}$
$$\Rightarrow d_2 = 10 \text{ units}$$
$$\Rightarrow AC = 10$$

Using parametric form *SL* and *AC*, we get

$$A \equiv \left(\frac{9}{5} - 2\sqrt{5}, \frac{-2}{5} + \sqrt{5}\right)$$

$$C \equiv \left(\frac{9}{5} + 2\sqrt{5}, \frac{-2}{5} - \sqrt{5}\right)$$

Therefore, centroid of ΔACD is $\left(\frac{7}{5}, \frac{-6}{5}\right)$.

Hence, the correct answer is option (C).

Paragraph for Questions 14–16: *ABC* is an isosceles triangle with $AB = AC = 5$ and $BC = 6$. Let P be a point inside the triangle *ABC* such that the distance from *P* to the base *BC* equals the geometric mean of the distance to the sides *AB* and *AC*.

14. The locus of the point *P* is

(A) a semicircle **(B)** a minor arc of a circle
(C) major arc of a circle **(D)** a complete circle

Solution: Let the triangle *ABC* has vertices *A*(0, 4), *B*(–3, 0) and *C*(3, 0).

Let the point *P* be (α, β).
Equation of line *AC* is $4x + 3y - 12 = 0$ and the equation of line *AB* is $4x - 3y + 12 = 0$.

$$|\beta| = \sqrt{\frac{|(4\alpha + 3\beta - 12)(-4\alpha + 3\beta - 12)|}{\sqrt{25} \times \sqrt{25}}}$$

$$\Rightarrow 2(\alpha^2 + \beta^2) + 39\beta - 18 = 0 \quad (1)$$

Since point *P* lies inside triangle *ABC*, its locus is the minor arc of circle.
Hence, the correct answer is option (B).

15. The minimum distance of the point *A* from the locus of the point *P* is

(A) $\frac{25 + 2\sqrt{117}}{4}$ **(B)** $\frac{25 - 2\sqrt{117}}{4}$
(C) $\frac{25 + \sqrt{117}}{4}$ **(D)** $\frac{25 - \sqrt{117}}{4}$

Solution: Radius (*R*) of the circle represented by Eq. (1) is $\frac{\sqrt{117}}{2}$ and the distance of the point *A* from the centre *Q* of the circle is $\frac{25}{4}$.

Minimum distance $= AQ - R = \frac{25}{4} - \frac{\sqrt{117}}{2}$

Hence, the correct answer is option (B).

16. If the tangents to the locus at *B* and *C* intersect at point *P*, then the area of the triangle *PBC* is

(A) 10 **(B)** 12
(C) 14 **(D)** 18

Solution: Equation of the tangents to Eq. (1) at *C* is obtained by $T = 0$

The tangent is
$$2(3x + 0y) + \frac{3}{2}y - 18 = 0$$
$$\Rightarrow 4x + 3y - 12 = 0$$

which is same as line *AC*. Hence, tangents at *B* and *C* intersect at *A*

So, Area of triangle *PBC* is $\frac{1}{2} \cdot 6 \cdot 4 = 12$.

Hence, the correct answer is option (B).

Paragraph for Questions 17–19: Two straight lines rotate about two fixed points (–*a*, 0) and (*a*, 0). If they start from their position of coincidence such that one rotates at the rate double that of the other, then

17. The point (–*a*, 0) always lies

(A) Inside the curve **(B)** Outside the curve
(C) On the curve **(D)** None of these

Solution: From the given statement, PB makes an angle θ and PA makes an angle 2θ with $X'OX$.

Let the coordinate be $P(h, k)$. Then

$$\tan\theta = \text{slope of } BP = \frac{k}{h+a}$$

$$\tan 2\theta = \text{slope of } AP = \frac{k}{h-a}$$

As

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

Therefore,

$$\frac{k}{h-a} = \frac{\frac{2k}{h+a}}{1-\left(\frac{k}{h+a}\right)^2}$$

$$\Rightarrow \frac{2k(h+a)}{(h+a)^2-k^2} = \frac{k}{h-a}$$

$$\Rightarrow 2k(h^2 - a^2) = k((h + a)^2 - k^2)$$

$$\Rightarrow 2(h^2 - a^2) = h^2 + a^2 + 2ah - k^2$$

$$\Rightarrow h^2 + k^2 - 2ah - 3a^2 = 0$$

Now, locus of (h, k) is

$$x^2 + y^2 - 2ax - 3a^2 = 0 \quad (1)$$

Hence, the correct answer is option (C).

18. Locus of the curve is

(A) circle **(B)** straight line

(C) parabola **(D)** ellipse

Solution: Point $(-a, 0)$ lies on the curve as $a^2 + 0^2 + 2a^2 - 3a^2 = 0$.

Hence, the correct answer is option (A).

19. Distance of the point $(a, 0)$ from the variable point on the curve is

(A) 0 **(B)** $2a$

(C) $3a$ **(D)** $4a$

Solution: From Eq. (1), we have

$x^2 + y^2 - 2ax - 3a^2 = 0$ represents a circle having centre $(a, 0)$ and radius $2a$.

Thus distance of any point from $(a, 0)$ is radius $= 2a$.

Hence, the correct answer is option (B).

Paragraph for Questions 20–22: $A(1, 1)$ is vertex and $H(2, 4)$ is the orthocentre of the triangle ABC. For a, b, c in A.P., the sides AB and BC are represented by the family of lines $ax + by + c = 0$. Then

20. The coordinates of vertex C are

(A) $\left(-8, -\frac{1}{2}\right)$ **(B)** (4, 8)

(C) (17, 4) **(D)** (17, – 4)

Solution: Since a, b, c are in AP, the equation of the lines becomes

$$2ax + (a + c)y + 2c = 0$$

$$\Rightarrow 2x + y + \frac{c}{a}(y + 2) = 0$$

Now, $A(1, 1)$ lies on the line AB, so $\frac{c}{a} = -1$ and the equation of the line AB becomes $x = 1$.

Also AH is perpendicular to the line BC and hence

$$\left(\frac{-2}{1+c/a}\right)\left(\frac{3}{1}\right) = -1$$

$$\Rightarrow \frac{c}{a} = 5,$$

So, the equation of line BC is $x + 3y + 5 = 0 \Rightarrow B$ is (1, – 2).

The side AC through $A(1, 1)$ is perpendicular to BH. Therefore, its equation is $x + 6y = 7$.

The intersection of AC and BC gives the coordinates of C as (17, 4).

Hence, the correct answer is option (C) .

21. The coordinates of the centroid of the triangle ABC are

(A) (–5, 1) **(B)** (5, –1)

(C) (5, 1) **(D)** (– 5, –1)

Solution: The coordinates of the centroid G are (–5, 1).

Hence, the correct answer is option (A).

22. The coordinates of the circumcentre of the triangle ABC are

(A) (6, 1) **(B)** $\left(\frac{17}{2}, \frac{1}{4}\right)$

(C) $\left(-\frac{17}{2}, -\frac{1}{2}\right)$ **(D)** None of these

Solution: Let $O(h, k)$ be the circumcentre of triangle ABC. Since G divides OH in the ratio 1:2, we have

$$-5 = \frac{2h+2}{3},\ 1 = \frac{2k+4}{3}$$

$$\Rightarrow O \text{ is } \left(-\frac{17}{2}, -\frac{1}{2}\right)$$

Hence, the correct answer is option (C).

Previous Years' Solved JEE Main/AIEEE Questions

1. Let $A(h, k)$, $B(1, 1)$ and $C(2, 1)$ be the vertices of a right angled triangle with AC as its hypotenuse. If the area of the triangle is 1, then the set of values which 'k' can take is given by

(A) {1, 3} **(B)** {0, 2}

(C) {–1, 3} **(D)** {–3, –2}

[AIEEE 2007]

Solution: We have

$$\frac{1}{2}\times 1(k-1) = \pm 1$$

Therefore,

$$k - 1 = \pm 2 \Rightarrow k = 3$$

or

$$k = -1$$

Hence, the correct answer is option (C).

2. The perpendicular bisector of the line segment joining $P(1, 4)$ and $Q(k, 3)$ has y-intercept –4. Then a possible value of k is

(A) 1 **(B)** 2

(C) –2 **(D)** –4

[AIEEE 2008]

Solution: See Fig. 10.28.

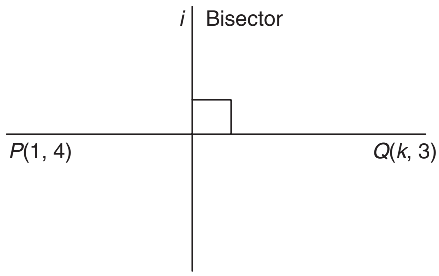


Figure 10.28

Slope of $l = \dfrac{1}{\dfrac{3-4}{k-1}} = (k-1)$

Middle point $= \left(\dfrac{k+1}{2}, \dfrac{7}{2}\right)$

Equation of bisector is

$$y - \frac{7}{2} = (k-1)\left(x - \frac{(k+1)}{2}\right)$$

Substituting $x = 0$ and $y = -4$, we get, $k = \pm 4$.

Hence, the correct answer is option (D).

3. The shortest distance between line $y - x = 1$ and curve $x = y^2$ is

(A) $\dfrac{3\sqrt{2}}{8}$ **(B)** $\dfrac{8}{3\sqrt{2}}$

(C) $\dfrac{4}{\sqrt{3}}$ **(D)** $\dfrac{\sqrt{3}}{4}$

[AIEEE 2011]

Solution: We have

$$P = (y^2, y)$$

The perpendicular distance of $P = (t^2, t)$ on $y^2 = x$ from $y - x - 1 = 0$ is $\dfrac{|t - t^2 - 1|}{\sqrt{2}} = \dfrac{t^2 - t + 1}{\sqrt{2}}$. This is minimum if $t = 1/2$.

Thus, the shortest distance is calculated as

$$\frac{t^2 - t + 1}{\sqrt{2}} = \frac{(1/2)^2 - (1/2) + 1}{\sqrt{2}} = \frac{3}{4\sqrt{2}} = \frac{3\sqrt{2}}{8}$$

Hence, the correct answer is option (A).

4. A line is drawn through the point (1, 2) to meet the coordinate axes at P and Q such that it forms a triangle OPQ, where O is the origin. If the area of the triangle OPQ is least, then the slope of the line PQ is

(A) $-\dfrac{1}{4}$ **(B)** -4

(C) -2 **(D)** $-\dfrac{1}{2}$

[AIEEE 2012]

Solution: Equation of line passing through (1, 2) with slope m is $y - 2 = m(x - 1)$.

$$\text{Area of } \Delta OPQ = \frac{(m-2)^2}{2|m|} \Rightarrow \Delta = \frac{m^2 + 4 - 4m}{2m} \Rightarrow \Delta = \frac{m}{2} + \frac{2}{m} - 2$$

Δ is least if

$$\frac{m}{2} = \frac{2}{m} \Rightarrow m^2 = 4 \Rightarrow m = \pm 2 \Rightarrow m = -2$$

Hence, the correct answer is option (C).

5. The x-coordinate of the incentre of the triangle that has the coordinates of midpoints of its sides as (0, 1) (1, 1) and (1, 0) is

(A) $2 - \sqrt{2}$ **(B)** $1 + \sqrt{2}$

(C) $1 - \sqrt{2}$ **(D)** $2 + \sqrt{2}$

[JEE MAIN 2013]

Solution: From Fig. 10.29 of the given triangle, the x-coordinate of the incentre is obtained as follows:

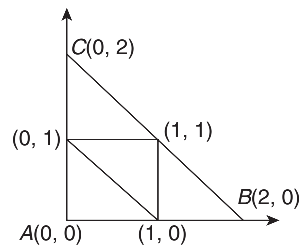


Figure 10.29

$$\frac{ax_1 + bx_2 + cx_3}{a+b+c} = \frac{2\times 2 + 2\sqrt{2}\times 0 + 2\times 0}{2 + 2 + 2\sqrt{2}} = \frac{4}{4 + 2\sqrt{2}} = \frac{2}{2+\sqrt{2}} = 2 - \sqrt{2}$$

Hence, the correct answer is option (A).

6. Let a, b, c and d be non-zero numbers. If the point of intersection of the lines $4ax + 2ay + c = 0$ and $5bx + 2by + d = 0$ lies in the fourth quadrant and is equidistant from the two axes, then

(A) $3bc - 2ad = 0$ **(B)** $3bc + 2ad = 0$

(C) $2bc - 3ad = 0$ **(D)** $2bc + 3ad = 0$

[JEE MAIN 2014 (OFFLINE)]

Solution:

$$4ax + 2ay + c = 0$$
$$5bx + 2by + d = 0$$
$$\frac{x}{2ad - abc} = \frac{y}{5bc - 4ad} = \frac{1}{8ab - 10ab}$$

Now,

$$x = \frac{\cancel{2}(ad - bc)}{-\cancel{2}(ab)} = \frac{bc - ad}{ab}$$

$$y = \frac{5bc - 4ad}{-2ab}$$

Now according to question point being in 4th quadrant and equidistance from axes, lies on $y = -x$

Therefore,

$$\frac{5bc - 4ad}{-2\cancel{ab}} = -\frac{(bc - ad)}{\cancel{ab}}$$

$$\Rightarrow 5bc - 4ad = 2(bc - ad)$$
$$\Rightarrow 5bc - 4ad = 2bc - 2ad$$
$$\Rightarrow 3bc - 2ad = 0$$

Hence, the correct answer is option (A).

7. Let a and b be any two numbers satisfying $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{4}$. Then, the foot of perpendicular from the origin on the variable line, $\frac{x}{y}+\frac{y}{b}=1$, lies on

(A) a hyperbola with each semi-axis $=\sqrt{2}$.
(B) a hyperbola with each semi-axis = 2.
(C) a circle of radius = 2.
(D) a circle of radius = $\sqrt{2}$.

[JEE MAIN 2014 (ONLINE SET-1)]

Solution: See Fig. 10.30. Given

$$\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{4}$$

For the foot of $\perp (\alpha, \beta)$

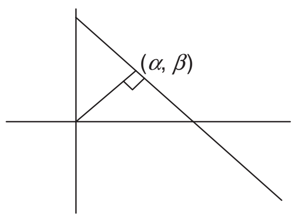


Figure 10.30

$$\frac{\alpha-0}{1/a}=\frac{\beta-0}{1/b}=\frac{-(0+0-1)}{1/a^2+1/b^2}$$

Therefore,

$$\alpha=\frac{1/a}{(1/a^2+1/b^2)}, \beta=\frac{1/b}{(1/a^2+1/b^2)}$$

$$\Rightarrow \alpha^2+\beta^2=\frac{1/a^2+1/b^2}{(1/a^2+1/b^2)^2}=\frac{1}{1/4}=4=2^2$$

Thus, locus of (α, β) is a circle with radius 2.

Hence, the correct answer is option (C).

8. Given three points P, Q, R with $P(5, 3)$ and R lies on the x-axis. If equation of RQ is $x-2y=2$ and PQ is parallel to the x-axis, then the centroid of ΔPQR lies on the line

(A) $2x+y-9=0$ (B) $x-2y+1=0$
(C) $5x-2y=0$ (D) $2x-5y=0$

[JEE MAIN 2014 (ONLINE SET-1)]

Solution: See Fig. 10.31. $(\alpha, 0)$ satisfies $x-2y=2$.

Therefore, $\alpha=2$

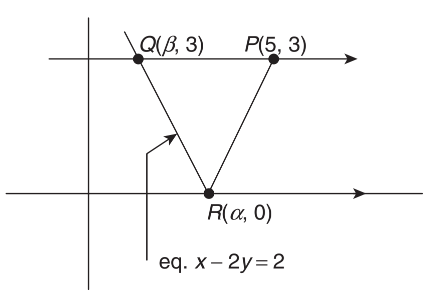


Figure 10.31

Also $(\beta, 3)$ lies on $x-2y=2$. So,

$$\beta-6=2\Rightarrow\beta=8$$

Therefore,

$$\text{centroid}=\left(\frac{\alpha+\beta+5}{3},\frac{0+3+3}{3}\right)=\left(\frac{2+8+5}{3},\frac{6}{3}\right)=(5,2)$$

It satisfies $2x-5y=0$.

Hence, the correct answer is option (D).

9. A stair-case of length l rests against a vertical wall and a floor of a room. Let P be a point on the stair-case, nearer to its end on the wall, that divides its length in the ratio 1:2. If the stair-case begins to slide on the floor, then the locus of P is

(A) an ellipse of eccentricity $\frac{1}{2}$
(B) an ellipse of eccentricity $\frac{\sqrt{3}}{2}$
(C) a circle of radius $\frac{l}{2}$
(D) a circle of radius $\frac{\sqrt{3}}{2}l$

[JEE MAIN 2014 (ONLINE SET-2)]

Solution: See Fig. 10.32.

$$AP:PB=2:1$$

Therefore,

$$h=\frac{2\times0+1\times x}{2+1}=\frac{x}{3}\Rightarrow x=3h \quad (1)$$

$$k=\frac{2\times y+1\times0}{2+1}=\frac{2y}{3}\Rightarrow y=\frac{3k}{2} \quad (2)$$

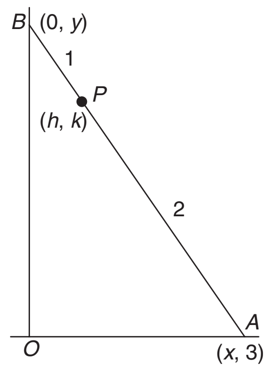


Figure 10.32

Now in triangle OAB,

$$P^2=x^2+y^2 \text{ (Pythagoras theorem)}$$

$$\Rightarrow(3h)^2+\left(\frac{3k}{2}\right)^2=P^2\Rightarrow 9h^2+\frac{9k^2}{4}=P^2$$

$$\Rightarrow\frac{h^2}{\left(\frac{P^2}{9}\right)}+\frac{k^2}{\left(\frac{4P^2}{9}\right)}=1\Rightarrow h^2$$

Therefore, Locus of (h, k) is $\frac{x^2}{\left(\frac{P}{3}\right)^2}+\frac{y^2}{\left(\frac{2P}{3}\right)^2}=1$

$$\text{Ellipse}=\left(\frac{P}{3}\right)^2=\left(\frac{2P}{3}\right)^2(1-e^2)\Rightarrow 1-e^2=\frac{P^2}{9}\times\frac{9}{4P^2}=\frac{1}{4}$$

Therefore,

$$1-e^2=1-\frac{1}{4}=\frac{3}{4}\Rightarrow e=\frac{\sqrt{3}}{2}$$

Hence, the correct answer is option (B).

10. If a line intercepted between the coordinate axis is trisected at a point $A(4, 3)$, which is nearer to x-axis, then its equation is

(A) $4x-3y=7$
(B) $3x+2y=18$
(C) $3x+8y=36$
(D) $x+3y=13$

[JEE MAIN 2014 (ONLINE SET-3)]

Solution: See Fig. 10.33. Let equation of the required line be

$$\frac{x}{\alpha}+\frac{y}{\beta}=1$$

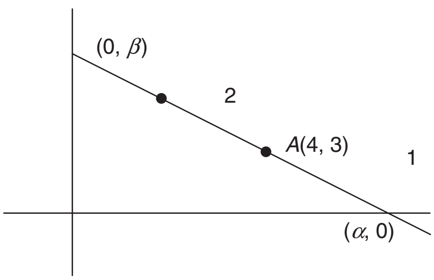


Figure 10.33

Now,

$$4=\frac{2\alpha+0}{3}\Rightarrow\alpha=6,\ 3=\frac{2(0)+1(0)}{3}\Rightarrow\beta=9$$

Therefore, line is

$$\frac{x}{6}+\frac{x}{9}=1 \text{ or } 9x+6y=54\Rightarrow 3x+3y=18$$

Hence, the correct answer is option (B).

11. The circumcentre of a triangle lies at the origin and its centroid is the midpoint of the line segment joining the points (a^2+1, a^2+1) and $(2a, -2a)$, $a\neq 0$. Then for any a, the orthocentre of this triangle lies on the line

(A) $y-2ax=0$
(B) $y-(a^2+1)x=0$
(C) $y+x=0$
(D) $(a-1)^2x-(a+1)^2y=0$

[JEE MAIN 2014 (ONLINE SET-4)]

Solution: See Fig. 10.34

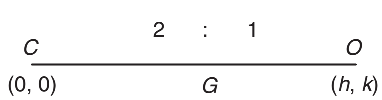


Figure 10.34

We know the above ratio in the case of a triangle. Therefore,

$$\frac{h(2)+0(1)}{3}=\frac{a^2+1+2a}{2}$$

By section formula,

$$4h=3a^2+3+6a \quad (1)$$

Again,
$$\frac{k(2)+0(1)}{3}=\frac{a^2+1-2a}{2}\Rightarrow 4k=3a^2+3-6a \quad (2)$$

Dividing Eq. (1) by (2), we get

$$\frac{h}{k}=\frac{a^2+1+2a}{a^2+1-2a}=\frac{(a+1)^2}{(a-1)^2}\Rightarrow h(a-1)^2=k(a+1)^2$$

Thus, Locus of (h, k) is $x(a-1)^2-y(a+1)^2=0$.

Hence, the correct answer is option (D).

12. If a line L is perpendicular to the line $5x-y=1$, and the area of the triangle formed by the line L and the coordinate axes is 5, then the distance of line L from the line $x+5y=0$ is

(A) $\frac{7}{\sqrt{5}}$ **(B)** $\frac{5}{\sqrt{13}}$

(C) $\frac{7}{\sqrt{13}}$ **(D)** $\frac{5}{\sqrt{7}}$

[JEE MAIN 2014 (ONLINE SET-4)]

Solution: See Fig. 10.35.

Given line: $5x-y=1$, that is, $y=5x-1$

Therefore, L is

$$-x-5y=\lambda \text{ or } \frac{x}{-\lambda}+\frac{y}{\frac{\lambda}{-5}}=1$$

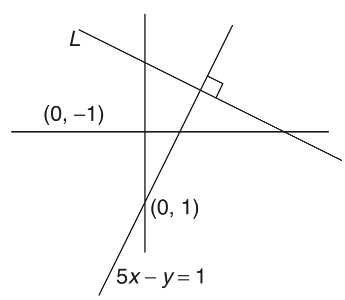


Figure 10.35

Area of triangle is

$$\frac{1}{2}(\lambda)\left(\frac{\lambda}{5}\right)=5\Rightarrow\lambda^2=50\Rightarrow\lambda=\pm5\sqrt{2}$$

Therefore,

$$P=\pm5\sqrt{2}\Rightarrow L \text{ is either } -x-5y=5\sqrt{2} \text{ or } -x-5y=-5\sqrt{2}$$

Now, distance of the line $x+5y+5\sqrt{2}=0$ is same as distance of $x+5y+5\sqrt{2}=0$ from origin. Therefore,

$$\text{distance}=\left|\frac{0+0+5\sqrt{2}}{\sqrt{1+25}}\right|=\frac{5\sqrt{2}}{\sqrt{26}}=\frac{5\sqrt{2}}{\sqrt{13}\sqrt{2}}=\frac{5}{\sqrt{13}}$$

Hence, the correct answer is option (B).

13. The number of points, having both co-ordinates as integers that lie in the interior of the triangle with vertices (0, 0) (0, 41) and (41, 0), is

(A) 861 **(B)** 820 **(C)** 780 **(D)** 901

[JEE MAIN 2015 (OFFLINE)]

Solution: See Fig. 10.36.

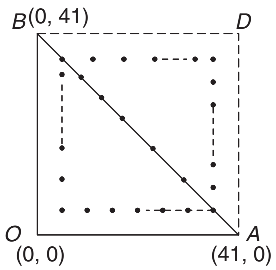


Figure 10.36

Number of points of desired type $= 1 + 2 + 3 + \cdots + 39$

$$= \frac{40 \times 39}{2} = 780$$

Hence, the correct answer is option (C).

14. The points $\left(0, \frac{8}{3}\right)$, (1, 3) and (82, 30)

(A) form an obtuse-angled triangle.
(B) form an acute-angled triangle.
(C) form a right-angled triangle.
(D) lie on a straight line.

[JEE MAIN 2015 (ONLINE SET-1)]

Solution:

$$A \equiv \left(0, \frac{8}{3}\right),\ B \equiv (1,3),\ C \equiv (82,30)$$

$$\text{Slope of } AB = \frac{3-8/3}{1-0} = \frac{1}{3},\ \text{Slope of } BC = \frac{30-3}{82-1} = \frac{27}{81} = \frac{1}{3}$$

Therefore, *A*, *B*, *C* lie on a straight line.

Hence, the correct answer is option (D).

15. If in a regular polygon the number of diagonals is 54, then the number of sides of this polygon is:

(A) 10 **(B)** 12 **(C)** 9 **(D)** 9

[JEE MAIN 2015 (ONLINE SET-2)]

Solution: Let '*n*' be the number of sides of regular polygon. Then

$$\text{No. of diagonals} = ({}^nC_2 - n) = 54 \Rightarrow n(n-1) - 2n = 108$$

$$\Rightarrow n^2 - 3n - 108 = 0 \Rightarrow (n-12)(n+9) = 0$$

$$\Rightarrow n = 12 = \text{no. of sides}$$

Hence, the correct answer is option (B).

16. Two sides of a rhombus are along the lines, $x - y + 1 = 0$ and $7x - y - 5 = 0$. If its diagonals intersect at $(-1, -2)$, then which one of the following is a vertex of this rhombus?

(A) $\left(-\frac{10}{3}, \frac{7}{3}\right)$ **(B)** $(-3, -9)$

(C) $(-3, -8)$ **(D)** $\left(\frac{1}{3}, -\frac{8}{3}\right)$

[JEE MAIN 2016 (OFFLINE)]

Solution: See Fig. 10.37. We have

$$\begin{array}{l} 7x - y - 5 = 0 \\ x - y + 1 = 0 \\ -\ +\ - \\ \hline 6x - 6 = 0 \\ \Rightarrow x = 1 \end{array}$$

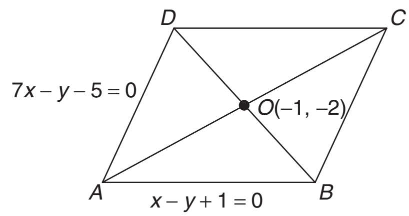


Figure 10.37

$$x - y + 1 = 0$$

$$1 - y + 1 = 0 \Rightarrow y = 2$$

Therefore, point (1, 2), that is, point *A*(1, 2).
$C(\alpha, \beta)$; *O* is midpoint of points *A* and *C*.

$$\frac{\alpha+1}{2} = -1 \Rightarrow \alpha + 1 = -2 \Rightarrow \alpha = -3$$

$$\frac{\beta+2}{2} = -2 \Rightarrow \beta + 2 = -4 \Rightarrow \beta = -6$$

Therefore, we get point *C*(–3, –6).

Point *B* is (x_1, y_1) and point *D* is (x_2, y_2). That is,

$$x_1 - y_1 + 1 = 0 \text{ and } 7x_2 - y_2 - 5 = 0$$

$$\frac{x_1 + x_2}{2} = -1 \text{ and } \frac{y_1 + y_2}{2} = -2$$

$$x_2 = -2 - x_1 \text{ and } y_2 = -4 - 2y_1$$

Therefore,

$$7(-2 - x_1) + 4 + y_1 - 5 = 0$$

$$-14 - 7x_1 + y_1 - 1 = 0$$

$$-7x_1 + y_1 - 15 = 0$$

$$-7x_1 + x_1 + 1 - 15 = 0$$

$$-6x_1 - 14 = 0$$

$$\Rightarrow x_1 = -\frac{7}{3} \text{ and } y_1 = \frac{-7}{3} + 1 = \frac{-4}{3}$$

Therefore, we get point $B\left(-\frac{7}{3}, -\frac{4}{3}\right)$.

Now,

$$x_2 = -2 + \frac{7}{3} = +\frac{1}{3}$$

$$y_2 = -4 + \frac{4}{3} = -\frac{8}{3}$$

Therefore, we get point $D\left(\frac{1}{3}, \frac{-8}{3}\right)$.

Hence, the correct answer is option (D).

17. If a variable line drawn through the intersection of the lines $\frac{x}{3}+\frac{y}{4}=1$ and $\frac{x}{4}+\frac{y}{3}=1$ meets the coordinate axes at A and B, $(A \neq B)$, then the locus of the midpoint of AB is

(A) $7xy = 6(x+y)$ **(B)** $4(x+y)^2 - 28(x+y)+49=0$
(C) $6xy = 7(x+y)$ **(D)** $14(x+y)^2 - 97(x+y)+168=0$

[JEE MAIN 2016 (ONLINE SET-1)]

Solution: Equation of line which is passing through the point of intersection of the given line is

$$\left(\frac{x}{3}+\frac{y}{4}-1\right)+\lambda\left(\frac{x}{4}+\frac{y}{3}-1\right)=0$$

$$x\left(\frac{4+3\lambda}{12}\right)+y\left(\frac{3+4\lambda}{12}\right)-(1+\lambda)=0$$

$$\frac{x}{\left(\frac{12(1+\lambda)}{4+3\lambda}\right)}+\frac{y}{\left(\frac{12(1+\lambda)}{3+4\lambda}\right)}=1$$

Therefore, the points are obtained as

$$A\left(\frac{12(1+\lambda)}{(4+3\lambda)},0\right) \text{ and } B\left(0,\frac{12(1+\lambda)}{3+4\lambda}\right)$$

The midpoint of AB is (h, k), that is,

$$h=\frac{6(1+\lambda)}{4+3\lambda} \text{ and } k=\frac{6(1+\lambda)}{3+4\lambda}$$

Therefore,

$$\frac{1}{k}+\frac{1}{h}=\frac{4+3\lambda}{6(1+\lambda)}+\frac{3+4\lambda}{6(1+\lambda)}=\frac{7}{6}$$

Hence, the locus is $6(x+y) = 7xy$.

Hence, the correct answer is option (A).

18. The distance of the point (1, –2, 4) from the plane passing through the point (1, 2, 2) and perpendicular to the planes $x - y + 2z = 3$ and $2x - 2y + z + 12 = 0$ is

(A) 2 **(B)** $\sqrt{2}$
(C) $2\sqrt{2}$ **(D)** $\frac{1}{\sqrt{2}}$

[JEE MAIN 2016 (ONLINE SET-1)]

Solution: The direction ratio of the required plane is

$$\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 2\\ 2 & -2 & 1\end{vmatrix}=\hat{i}(-1+4)-\hat{j}(1-4)+\hat{k}(-2+2)$$

$$=3\hat{i}+3\hat{j}+\hat{k}(0)$$

The equation of plane is

$$3(x-1)+3(y-2)+0(z-2)=0$$

$$\Rightarrow x+y-3=0$$

Therefore, the distance of the given point from the plane is

$$\frac{|1-2-3|}{\sqrt{2}}=2\sqrt{2}$$

Hence, the correct answer is option (C).

Previous Years' Solved JEE Advanced/ IIT-JEE Questions

1. Let $ABCD$ be a quadrilateral with area 18, with side AB parallel to the side CD and $AB = 2CD$. Let AD be perpendicular to AB and CD. If a circle is drawn inside the quadrilateral $ABCD$ touching all the sides, then its radius is

(A) 3 **(B)** 2
(C) $\frac{3}{2}$ **(D)** 1

[IIT-JEE 2007]

Solution: See Fig. 10.38. In the quadrilateral $ABCD$, let $CD = x$ and $AD = 2r$. Therefore, the area of the $ABCD = 18$. That is,

$$\frac{1}{2}(x+2x)(2r)=18$$

$$xr=6$$

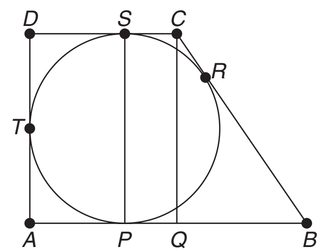


Figure 10.38

Now, $AP = AT = DS = DT = r$

$$BP = BR = 2x - r$$
$$CS = CR = x - r$$

In ΔBCQ, we get

$$CQ^2 + BQ^2 = BC^2$$

$$\Rightarrow (2r)^2 + x^2 = (3x-2r)^2$$
$$\Rightarrow 4r^2 + x^2 = 9x^2 + 4r^2 - 12xr$$
$$\Rightarrow 8x^2 = 12xr$$
$$\Rightarrow 8x^2 = 72$$
$$\Rightarrow x = 3$$
$$\Rightarrow r = 2$$

Hence, the correct answer is option (B).

2. Consider three points $P=(-\sin(\beta-\alpha), -\cos\beta)$, $Q=(\cos(\beta-\alpha), \sin\beta)$ and $R=(\cos(\beta-\alpha+\theta), \sin(\beta-\theta))$, where $0<\alpha, \beta, \theta<\frac{\pi}{4}$. Then

(A) P lies on the line segment RQ
(B) Q lies on the line segment PR
(C) R lies on the line segment QP
(D) P, Q, R are non-collinear

[IIT-JEE 2008]

Solution: We have

$$P = (-\sin(\beta - \alpha), -\cos\beta) = (x_1, y_1)$$
$$Q = (\cos(\beta - \alpha), \sin\beta) = (x_2, y_2)$$
$$R = (\cos(\beta - \alpha + \theta), \sin(\beta - \theta))$$
$$= (x_2 \cos\theta + x_1 \sin\theta), y_2 \cos\theta + y_1 \sin\theta)$$

Let T is a point on PQ which divides PQ in $\cos\theta$: $\sin\theta$. Then

$$T = \left(\frac{x_2\cos\theta + x_1\sin\theta}{\cos\theta + \sin\theta}, \frac{y_2\cos\theta + y_1\sin\theta}{\cos\theta + \sin\theta}\right)$$

$\Rightarrow P, Q, T$ are collinear

Therefore, P, Q, R are non-collinear.

Hence, the correct answer is option (D).

3. A line with positive direction cosines passes through the point $P(2, -1, 2)$ and makes equal angles with the coordinate axes. The line meets the plane $2x + y + z = 9$ at point Q. The length of the line segment PQ equals

(A) 1 **(B)** $\sqrt{2}$ **(C)** $\sqrt{3}$ **(D)** 2

[IIT-JEE 2009]

Solution: DC of the line are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.
Any point on the line at a distance t from $P(2, -1, 2)$ is

$$\left(2 + \frac{t}{\sqrt{3}}, -1 + \frac{t}{\sqrt{3}}, 2 + \frac{t}{\sqrt{3}}\right)$$

which lies on $2x + y + z = 9$, therefore, $t = \sqrt{3}$.

Hence, the correct answer is option (C).

4. The straight line $2x - 3y = 1$ divides the circular region $x^2 + y^2 \le 6$ into two parts. If

$$S = \left\{\left(2, \frac{3}{4}\right), \left(\frac{5}{2}, \frac{3}{4}\right), \left(\frac{1}{4}, -\frac{1}{4}\right), \left(\frac{1}{8}, \frac{1}{4}\right)\right\},$$

then the number of point(s) in S lying inside the smaller part is ______.

[IIT-JEE 2011]

Solution: See Fig. 10.39.

$$L: 2x - 3y - 1$$
$$S: x^2 + y^2 - 6$$

If $L_1 > 0$ and $S_1 < 0$, then point lies in the smaller part. Therefore, $\left(2, \frac{3}{4}\right)$ and $\left(\frac{1}{4}, -\frac{1}{4}\right)$ lies inside.

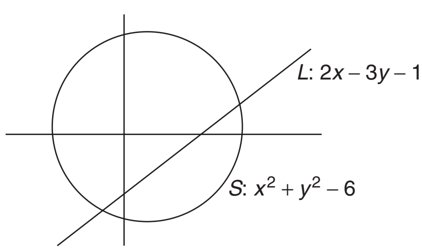


Figure 10.39

Hence, the correct answer is (2).

Practice Exercise 1

1. The number of integral values of m, for which the x-coordinate of the point of intersection of the lines $3x + 4y = 9$ and $y = mx + 1$ is also an integer is

(A) 2 **(B)** 0 **(C)** 4 **(D)** 1

2. A ray of light coming from the point (1, 2) is reflected at point A on the x-axis and then passes through the point (5, 3). The coordinates of the point A are

(A) (13/5, 0) **(B)** (5/13, 0)
(C) (–7, 0) **(D)** None of these

3. If the coordinates of the middle point of the portion of a line intercepted between coordinate axes (3, 2), then the equation of the line will be

(A) $2x + 3y = 12$ **(B)** $3x + 2y = 12$
(C) $4x - 3y = 6$ **(D)** $5x - 2y = 10$

4. A line through point $A(-5, -4)$ meets the lines $x + 3y + 2 = 0$, $2x + y + 4 = 0$ and $x - y - 5 = 0$ at points B, C and D, respectively. If $\left(\frac{15}{AB}\right)^2 + \left(\frac{10}{AC}\right)^2 = \left(\frac{6}{AD}\right)^2$, then the equation of the line is

(A) $2x + 3y + 22 = 0$ **(B)** $5x - 4y + 7 = 0$
(C) $3x - 2y + 3 = 0$ **(D)** None of these

5. The equation of perpendicular bisectors of sides AB and AC of a triangle ABC are $x - y + 5 = 0$ and $x + 2y = 0$, respectively. If point A is $(1, -2)$, then the equation of line BC is

(A) $23x + 14y - 40 = 0$ **(B)** $14x - 23y + 40 = 0$
(C) $23x - 14y + 40 = 0$ **(D)** $14x + 23y - 40 = 0$

6. The medians AD and BE of a triangle with vertices $A(0, b)$, $B(0, 0)$ and $C(a, 0)$ are perpendicular to each other if

(A) $a = \sqrt{2}b$ **(B)** $a = -\sqrt{2}b$
(C) Both (A) and (B) **(D)** None of these

7. Let PS be the median of the triangle with vertices $P(2, 2)$, $Q(6, -1)$ and $R(7, 3)$. Then the equation of the line passing through $(1, -1)$ and parallel to PS is

(A) $2x - 9y - 7 = 0$ **(B)** $2x - 9y - 11 = 0$
(C) $2x + 9y - 11 = 0$ **(D)** $2x + 9y + 7 = 0$

8. The equation of straight line passing through $(-a, 0)$ and making the triangle with axes of area T is

(A) $2Tx + a^2y + 2aT = 0$ **(B)** $2Tx - a^2y + 2aT = 0$
(C) $2Tx - a^2y - 2aT = 0$ **(D)** None of these

9. The equations of two equal sides of an isosceles triangle are $7x - y + 3 = 0$ and $x + y - 3 = 0$ and the third side passes through the point $(1, -10)$. The equation of the third side is

(A) $y = \sqrt{3}x + 9$ but not $x^2 - 9y^2 = 0$
(B) $3x + y + 7 = 0$ but not $x - 3y - 31 = 0$

(C) $3x+y+7=0$ or $x-3y-31=0$

(D) Neither $3x+y+7$ nor $x-3y-31=0$

10. The graph of the function $\cos x\cos(x+2)-\cos^2(x+1)$ is

(A) A straight line passing through $(0,-\sin^2 1)$ with slope 2
(B) A straight line passing through (0, 0)
(C) A parabola with vertex $(1-\sin^2 1)$
(D) A straight line passing through the point $\left(\frac{\pi}{2},-\sin^2 1\right)$ and parallel to the x-axis

11. If the equation of base of an equilateral triangle is $2x-y=1$ and the vertex is (–1, 2), then the length of the side of the triangle is

(A) $\sqrt{\frac{20}{3}}$ **(B)** $\frac{2}{\sqrt{15}}$

(C) $\sqrt{\frac{8}{15}}$ **(D)** $\sqrt{\frac{15}{2}}$

12. If x_1,x_2,x_3 and y_1,y_2,y_3 are both in GP, with the same common ratio, then the points (x_1,y_1), (x_2, y_2) and (x_3, y_3)

(A) Lie on a straight line
(B) Lie on an ellipse
(C) Lie on a circle
(D) Are vertices of a triangle

13. A line $4x+y=1$ passes through the point $A(2, -7)$ meets the line BC whose equation is $3x-4y+1=0$ at point B. The equation to the line AC so that $AB=AC$, is

(A) $52x+89y+519=0$ **(B)** $52x+89y-519=0$

(C) $89x+52y+519=0$ **(D)** $89x+52y-519=0$

14. In what direction a line be drawn through the point (1, 2) so that its points of intersection with the line $x+y=4$ is at a distance $\sqrt{6}/3$ from the given point

(A) 30° **(B)** 45°

(C) 60° **(D)** 75°

15. If straight lines $ax+by+p=0$ and $x\cos\alpha+y\sin\alpha-p=0$ include an angle $\pi/4$ between them and meet the straight line $x\sin\alpha-y\cos\alpha=0$ in the same point, then the value of a^2+b^2 is equal to

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4

16. The sides AB, BC, CD and DA of a quadrilateral are $x+2y=3$, $x=1$, $x-3y=4$, $5x+y+12=0$, respectively. The angle between diagonals AC and BD is

(A) 45° **(B)** 60°

(C) 90° **(D)** 30°

17. Given vertices $A(1, 1)$, $B(4,-2)$ and $C(5,5)$ of a triangle, then the equation of the perpendicular dropped from C to the interior bisector of the angle A is

(A) $y-5=0$ **(B)** $x-5=0$

(C) $y+5=0$ **(D)** $x+5=0$

18. If the straight line through the point $P(3, 4)$ makes an angle $\pi/6$ with the x-axis and meets the line $12x+5y+10=0$ at Q, then the length PQ is

(A) $\frac{132}{12\sqrt{3}+5}$ **(B)** $\frac{132}{12\sqrt{3}-5}$

(C) $\frac{132}{5\sqrt{3}+12}$ **(D)** $\frac{132}{5\sqrt{3}-12}$

19. The vertices of a triangle are (2, 1), (5, 2) and (4, 4). The lengths of the perpendicular from these vertices on the opposite sides are

(A) $\frac{7}{\sqrt{5}},\frac{7}{\sqrt{13}},\frac{7}{\sqrt{6}}$ **(B)** $\frac{7}{\sqrt{6}},\frac{7}{\sqrt{8}},\frac{7}{\sqrt{10}}$

(C) $\frac{7}{\sqrt{5}},\frac{7}{\sqrt{8}},\frac{7}{\sqrt{15}}$ **(D)** $\frac{7}{\sqrt{5}},\frac{7}{\sqrt{13}},\frac{7}{\sqrt{10}}$

20. The equation of the line joining the point (3, 5) to the point of intersection of the lines $4x+y-1=0$ and $7x-3y-35=0$ is equidistant from the points (0, 0) and (8, 34)

(A) True **(B)** False
(C) Nothing can be said **(D)** None of these

21. A variable line passes through a fixed point P. The algebraic sum of the perpendicular drawn from (2, 0), (0, 2) and (1, 1) on the line is zero, then the coordinates of point P are

(A) (1, –1) **(B)** (1, 1)
(C) (2, 1) **(D)** (2, 2)

22. Given the four lines with equations $x+2y=3$, $3x+4y=7$, $2x+3y=4$ and $4x+5y=6$, then these lines are

(A) Concurrent **(B)** Perpendicular
(C) The sides of a rectangle **(D)** None of these

23. The line $3x+2y=24$ meets y-axis at point A and x-axis at point B. The perpendicular bisector of AB meets the line through $(0,-1)$ parallel to x-axis at point C. The area of the triangle ABC is

(A) 182 sq. units **(B)** 91 sq. units

(C) 48 sq. units **(D)** None of these

24. A pair of straight lines drawn through the origin form with the line $2x+3y=6$ an isosceles right-angled triangle, then the lines and the area of the triangle thus formed are

(A) $x-5y=0;\ 5x+y=0;\ \Delta=36/13$

(B) $3x-y=0;\ x+3y=0;\ \Delta=12/17$

(C) $5x-y=0;\ x+5y=0;\ \Delta=13/5$

(D) None of these

25. The diagonals of a parallelogram $PQRS$ are along the lines $x+3y=4$ and $6x-2y=7$. Then, what is the shape of $PQRS$?

(A) Rectangle **(B)** Square
(C) Cyclic quadrilateral **(D)** Rhombus

26. The area enclosed within the curve $|x|+|y|=1$ is

(A) $\sqrt{2}$ **(B)** 1 **(C)** $\sqrt{3}$ **(D)** 2

27. The area of triangle formed by the lines $x=0, y=0$ and $\frac{x}{a}+\frac{y}{b}=1$, is

(A) ab **(B)** $ab/2$ **(C)** $2ab$ **(D)** $ab/3$

28. A line L passes through the points (1, 1) and (2, 0) and another line L' passes through $\left(\frac{1}{2}, 0\right)$ and perpendicular to L. Then the area of the triangle formed by the lines L, L' and y-axis, is

(A) 15/8 **(B)** 25/4 **(C)** 25/8 **(D)** 25/16

29. The image of the point (4, –3) with respect to the line $y = x$ is

(A) (–4, –3) **(B)** (3, 4)
(C) (–4, 3) **(D)** (–3, 4)

30. The locus of point P which divides the line joining (1, 0) and $(2\cos\theta, 2\sin\theta)$ internally in the ratio 2:3 for all θ, is a

(A) Straight line **(B)** Circle
(C) Pair of straight lines **(D)** Parabola

31. The equation of the locus of foot of perpendiculars drawn from the origin to the line passing through a fixed point (a, b) is

(A) $x^2+y^2-ax-by=0$ **(B)** $x^2+y^2+ax+by=0$
(C) $x^2+y^2-2ax-2by=0$ **(D)** None of these

32. The orthocentre of the triangle formed by the lines $xy = 0$ and $x + y = 1$ is

(A) $(0,0)$ **(B)** $\left(\frac{1}{2},\frac{1}{2}\right)$ **(C)** $\left(\frac{1}{3},\frac{1}{3}\right)$ **(D)** $\left(\frac{1}{4},\frac{1}{4}\right)$

33. The product of perpendiculars drawn from the origin to the lines represented by the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ will be

(A) $\frac{ab}{\sqrt{a^2-b^2+4h^2}}$ **(B)** $\frac{bc}{\sqrt{a^2-b^2+4h^2}}$
(C) $\frac{ca}{\sqrt{(a^2+b^2)+4h^2}}$ **(D)** $\frac{c}{\sqrt{(a-b)^2+4h^2}}$

34. The area of the triangle formed by the lines $y^2-9xy+18x^2=0$ and $y=9$ is

(A) (27/4) sq. unit **(B)** 27 sq. unit
(C) (27/2) sq. unit **(D)** None of these

35. The locus of the point $P(x, y)$ satisfying the relation $\sqrt{(x-3)^2+(y-1)^2}+\sqrt{(x+3)^2+(y-1)^2}=6$ is

(A) straight line **(B)** pair of straight lines
(C) circle **(D)** ellipse

36. The square of distance between the point of intersection of the lines represented by the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ and origin, is

(A) $\frac{c(a+b)-f^2-g^2}{ab-h^2}$ **(B)** $\frac{c(a-b)+f^2+g^2}{\sqrt{ab-h^2}}$
(C) $\frac{c(a+b)-f^2-g^2}{ab+h^2}$ **(D)** None of these

37. The lines joining the origin to the points of intersection of the line $y=mx+c$ and the circle $x^2+y^2=a^2$ will be mutually perpendicular if

(A) $a^2(m^2+1)=c^2$ **(B)** $a^2(m^2-1)=c^2$
(C) $a^2(m^2+1)=2c^2$ **(D)** $a^2(m^2-1)=2c^2$

38. Two of the lines represented by the equation $ay^4+bxy^3+cx^2y^2+dx^3y+ex^4=0$ will be perpendicular when

(A) $(b+d)(ad+be)+(e-a)^2(a+c+e)=0$
(B) $(b+d)(ad+be)+(e+a)^2(a+c+e)=0$
(C) $(b-d)(ad-be)+(e-a)^2(a+c+e)=0$
(D) $(b-d)(ad-be)+(e+a)^2(a+c+e)=0$

Practice Exercise 2

Single/Multiple Correct Choice Type Questions

1. Points on the line $x + y = 4$ that lie at a unit distance from the line $4x + 3y - 10 = 0$ are

(A) (3, 1) **(B)** (–7, 11)
(C) (–3, 7) **(D)** (–7, –11)

2. If a line is perpendicular to the line $5x - y = 0$ and forms a triangle, with the coordinate axes, of area 5 sq. units, then its equation is

(A) $x+5y+5\sqrt{2}=0$ **(B)** $x+5y-5\sqrt{2}=0$
(C) $5x+y-5\sqrt{2}=0$ **(D)** $5x-y-5\sqrt{2}=0$

3. Let ABC be a triangle with equations of the sides AB, BC and CA, respectively, $x - 2 = 0$, $y - 5 = 0$ and $5x + 2y - 10 = 0$. Then the orthocentre of the triangle lies on the line

(A) $x - y = 0$ **(B)** $3x - y = 1$
(C) $4x + y = 13$ **(D)** $x - 2y = 1$

4. If $a^2 + b^2 - c^2 - 2ab = 0$, then the family of straight lines $ax + by + c = 0$ is concurrent at the points

(A) (–1, 1) **(B)** (1, –1)
(C) (1, 1) **(D)** (–1, –1)

5. Two sides of a rhombus $OABC$ (lying entirely in the first quadrant or fourth quadrant) of the area equal to 2 sq. units, are $y=\frac{x}{\sqrt{3}}$, $y=\sqrt{3}x$. Then the possible coordinates of B is/are ('O' being the origin)

(A) $(1+\sqrt{3}, 1+\sqrt{3})$ **(B)** $(-1-\sqrt{3}, -1-\sqrt{3})$
(C) $(\sqrt{3}-1, \sqrt{3}-1)$ **(D)** None of these

6. The sides of a triangle are the straight lines $x + y = 1$; $7y = x$ and $\sqrt{3}y + x = 0$. Then which of the following is an interior point of the triangle?

(A) Circumcentre **(B)** Centroid
(C) Incentre **(D)** Orthocentre

7. If one diagonal of a square is the portion of the line $\frac{x}{a}+\frac{y}{b}=1$ intercepted by the axes, then the extremities of the other diagonal of the square are

(A) $\left(\frac{a+b}{2}, \frac{a+b}{2}\right)$ **(B)** $\left(\frac{a-b}{2}, \frac{a+b}{2}\right)$

(C) $\left(\frac{a-b}{2}, \frac{b-a}{2}\right)$ **(D)** $\left(\frac{a+b}{2}, \frac{b-a}{2}\right)$

8. Two straight lines $u=0$ and $v=0$ passes through the origin and angle between them is $\tan^{-1}(7/9)$. If the ratio of the slope of $v=0$ and $u=0$ is 9/2, then their equations are

(A) $y=3x$ and $3y=2x$ **(B)** $2y=3x$ and $3y=x$

(C) $y+3x=0$ and $3y+2x=0$ **(D)** $2y+3x=0$ and $3y+x=0$

9. A and B are two fixed points whose coordinates are (3, 2) and (5, 4), respectively. The coordinates of a point P, if ABP is an equilateral triangle, are

(A) $(4-\sqrt{3}, 3+\sqrt{3})$ **(B)** $(4+\sqrt{3}, 3-\sqrt{3})$

(C) $(3-\sqrt{3}, 4+\sqrt{3})$ **(D)** $(3+\sqrt{3}, 4-\sqrt{3})$

10. The points $A(0, 0)$, $B(\cos\alpha, \sin\alpha)$ and $C(\cos\beta, \sin\beta)$ are the vertices of a right angled triangle if

(A) $\sin\frac{\alpha-\beta}{2}=\frac{1}{\sqrt{2}}$ **(B)** $\cos\frac{\alpha-\beta}{2}=-\frac{1}{\sqrt{2}}$

(C) $\cos\frac{\alpha-\beta}{2}=\frac{1}{\sqrt{2}}$ **(D)** $\sin\frac{\alpha-\beta}{2}=-\frac{1}{\sqrt{2}}$

11. If $x-2y+4=0$ and $2x+y-5=0$ are the sides of an isosceles triangle having area of 10 sq. units, then equation of third side is

(A) $x+3y+10=0$ **(B)** $3x-y+9=0$

(C) $x+3y-19=0$ **(D)** $3x-y-11=0$

Comprehension Type Questions

Paragraph for Questions 12–14: Suppose we define the distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ as $d(P, Q)=\max.\{|x_2-x_1|, |y_2-y_1|\}$, then

12. The area of the region bounded by the locus of a point P satisfying $d(P, A)=4$, where A is (1, 2) is

(A) 64 sq. units **(B)** 54 sq. units

(C) 16π sq. units **(D)** None of these

13. Suppose that points A and B have coordinates (1, 0) and (–1, 0), respectively, then for a variable point P on this plane the equation $d(P, A)+d(P, B)=2$ represents

(A) a line segment joining A and B

(B) an ellipse with foci at A and B

(C) region lying inside a square of area 2

(D) region inside a semicircle with AB as diameter

14. Suppose that points A and B have coordinates (1, 0) and (–1, 0), respectively, then the area of the region bounded by the curves on which P lies, with $\{d(A, P)\}^2+\{d(B, P)\}^2=4$, is

(A) 4π **(B)** $\frac{2}{3}[4\pi-3(\sqrt{3}+1)]$

(C) 16 **(D)** $\frac{2\pi+\sqrt{3}}{4}$

Paragraph for Questions 15–17: Let $P(x_1, y_1)$ be a point not lying on the line ℓ: $ax+by+c=0$. Let L be a point on line ℓ, such that PL is perpendicular to the line ℓ. Let $Q(x, y)$ be a point on the line passing through P and L. Let the absolute distance between P and Q is n times ($n \in R^+$) the absolute distance between P and L. If L and Q lie on the same side of P, then coordinates of Q are given by the formula $\frac{x-x_1}{a}=\frac{y-y_1}{b}=-n\frac{ax_1+by_1+c}{a^2+b^2}$ and if L and Q lie on the opposite sides of P, then the coordinates of Q are given by the formula $\frac{x-x_1}{a}=\frac{y-y_1}{b}=n\frac{ax_1+by_1+c}{a^2+b^2}$

15. Let (2, 3) be the point P and $3x-4y+1=0$ be the straight line ℓ, if the sum of the coordinates of a point Q lying on PL, where L and Q lie on the same side of P and $n=15$ is α, then $\alpha=$_____.

(A) 0 **(B)** 1

(C) 2 **(D)** 3

16. Let (1, 1) be the point P and $-5x+12y+6=0$ be the straight line ℓ, if the sum of the coordinates of a point Q lying on PL, where L and Q are on the opposite sides of P and $n=13\alpha$ is β, then $\beta=$ _____. (α is as obtained in the above question)

(A) –9 **(B)** 25

(C) 12 **(D)** 16

17. Let (2, –1) be the point P and $x-y+1=0$ be the straight line ℓ, if a point Q lies on PL, where L and Q are on the same side of P for which $n=\beta$, then the coordinates of the image Q' of the point Q in the line ℓ, are _____. (β is as obtained in the above question)

(A) (14, 28) **(B)** (30, –29)

(C) (26, –27) **(D)** (–26, 27)

Paragraph for Questions 18–20: See Fig. 10.40. Let us consider the situation when axes are inclined at an angle 'ω'. If coordinates of a point P are (x_1, y_1) then $PN=x_1$, $PM=y_1$. Where PM is parallel to y-axis and PN is parallel to x-axis.

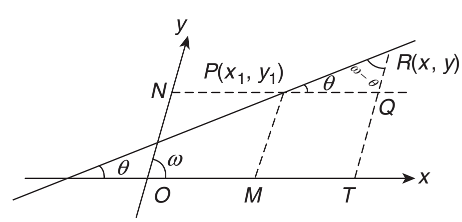


Figure 10.40

Now $RQ=y-y_1$, $PQ=x-x_1$

From ΔPQR, we have

$$\frac{PQ}{\sin(\omega-\theta)}=\frac{RQ}{\sin\theta}$$

The equation of straight line through P and makes an angle θ with x-axis is

$$y-y_1=\frac{\sin\theta}{\sin(\omega-\theta)}(x-x_1)$$

written in the form of

$y-y_1=m(x-x_1)$ where $m=\frac{\sin\theta}{\sin(\omega-\theta)}$ (m is called of slope of line)

The angle of inclination of line with x-axis is given by

$$\tan\theta = \left(\frac{m\sin\omega}{1+m\cos\omega}\right)$$

18. The axes being inclined at an angle of 60°, then the inclination of the straight line $y = 2x + 5$ with the axis of x is

(A) 30° **(B)** $\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$

(C) $\tan^{-1}2$ **(D)** 60°

19. The axes being inclined at an angle of 60°, then angle between the two straight lines $y = 2x + 5$ and $2y + x + 7 = 0$ is

(A) 90° **(B)** $\tan^{-1}\left(\frac{5}{3}\right)$

(C) $\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$ **(D)** $\tan^{-1}\left(\frac{5}{\sqrt{3}}\right)$

20. The axes being inclined at an angle of 30°, then the equation of the straight line which makes an angle of 60º with the positive direction of x-axis and x-intercept equal to 2, is

(A) $y - \sqrt{3}x = 0$ **(B)** $\sqrt{3}y = x$

(C) $y + \sqrt{3}x = 2\sqrt{3}$ **(D)** $y + 2x = 0$

Paragraph for Questions 21–23: $A(1, 3)$ and $C\left(-\frac{2}{5}, -\frac{2}{5}\right)$ are the vertices of a triangle ABC and the equation of the angle bisector of $\angle ABC$ is $x + y = 2$.

21. Equation of the side BC is

(A) $7x + 3y - 4 = 0$ **(B)** $7x + 3y + 4 = 0$

(C) $7x - 3y + 4 = 0$ **(D)** $7x - 3y - 4 = 0$

22. Coordinates of the vertex B are

(A) $\left(\frac{3}{10}, \frac{17}{10}\right)$ **(B)** $\left(\frac{17}{10}, \frac{3}{10}\right)$

(C) $\left(-\frac{5}{2}, \frac{9}{2}\right)$ **(D)** (1, 1)

23. Equation of the side AB is

(A) $3x + 7y = 24$ **(B)** $3x + 7y + 24 = 0$

(C) $13x + 7y + 8 = 0$ **(D)** $13x - 7y + 8 = 0$

Matrix Match Type Questions

24. Match the following:

List I	List II
(A) Two vertices of a triangle are (5, –1) and (–2, 3). If orthocentre is the origin, then the coordinates of the third vertex are	**(p)** (–4, –7)
(B) A point on the line $x + y = 4$ which lies at a unit distance from the line $4x + 3y = 10$, is	**(q)** (–7, 11)
(C) Orthocentre of the triangle made by the lines $x + y - 1 = 0$, $x - y + 3 = 0$, $2x + y = 7$ is	**(r)** (1, –2)
(D) If a, b, c are in AP, then lines $ax + by = c$ are concurrent at	**(s)** (–1, 2)
	(t) (4, –7)

25. Match the following:

List I	List II
(A) Lines $x - 2y - 6 = 0$, $3x + y - 4 = 0$ and $\lambda x + 4y + \lambda^2 = 0$ are concurrent, then the value of λ is	**(p)** 2
(B) The points $(\lambda+1, 1)$, $(2\lambda+1, 3)$ and $(2\lambda+2, 2\lambda)$ are collinear, then the value of λ is	**(q)** 4
(C) If line $x + y - 1 - \lambda = 0$, passing through the intersection of $x - y + 1 = 0$ and $3x + y - 5 = 0$ is perpendicular to one of them, then the value of λ is	**(r)** –1/2
(D) If line $y - x - 1 + \lambda = 0$ is equally inclined to axes and equidistant from the points (1, –2) and (3, 4), then λ is	**(s)** – 4
	(t) 3

26. Match the following:

List I	List II
(A) The number of integral values of 'a' for which the point $P(a, a^2)$ lies completely inside the triangle formed by the lines $x = 0$, $y = 0$ and $x + 2y = 3$	**(p)** 1
(B) Triangle ABC with $AB = 13$, $BC = 5$ and $AC = 12$ slides on the coordinate axis with A and B on the positive x-axis and positive y-axis, respectively, the locus of vertex C is a line $12x - ky = 0$, then the value of k is	**(q)** 4
(C) The reflection of the point $(t - 1, 2t + 2)$ in a line is $(2t + 1, t)$, then the line has a slope equal to	**(r)** 3
(D) In a triangle ABC the bisector of angles B and C lie along the lines $x = y$ and $y = 0$. If A is (1, 2) then $\sqrt{10}d(A, BC)$ where $d(A, BC)$ represents the distance of point A from the side BC	**(s)** 5
	(t) 0

27. Match the following:

List I	List II
(A) Area of the region enclosed by $2\|x\| + 3\|y\| \le 6$ is	**(p)** 12
(B) The extremities of the base of an isosceles triangle ABC are the points $A(2, 0)$ and $B(0, 1)$. If the equation of the side AC is $x = 2$ and 'm' be the slope of side BC, then '$4m$' equals to	**(q)** 4
(C) Area of ΔABC is 20 sq. units where points A, B and C are (4, 6), (10, 14) and (x, y), respectively. If AC is perpendicular to BC, then the number of positions of C is	**(r)** 5
(D) In a ΔABC coordinates of orthocentre, centroid and vertex A are (2, 2), (2, 1) and (0, 2), respectively. Then x-coordinate of the vertex B is	**(s)** 3
	(t) 2

Integer Type Questions

28. The vertices B and C of a triangle ABC lie on the lines $3y = 4x$ and $y = 0$, respectively, and the side BC passes through the point $\left(\frac{2}{3}, \frac{2}{3}\right)$. If $ABOC$ is a rhombus, O being the origin. If coordinates of the vertex A is (α, β), then find the value of $\frac{5}{2}(\alpha + \beta)$.

29. If the portion of the line $ax + by - 1 = 0$, intercepted between the lines $ax + y + 1 = 0$ and $x + by = 0$ subtends a right angle at the origin, then find the value of $4a + b^2 + (b + 1)^2$.

30. If two rays in the first quadrant $x + y = |a|$ and $ax - y = 1$ intersect each other in the interval $a \in (a_0, \infty)$, then the value of a_0 is ______.

Answer Key

Practice Exercise 1

1. (A)	**2.** (A)	**3.** (A)	**4.** (A)	**5.** (D)	**6.** (C)
7. (D)	**8.** (B)	**9.** (C)	**10.** (D)	**11.** (A)	**12.** (A)
13. (A)	**14.** (D)	**15.** (B)	**16.** (C)	**17.** (B)	**18.** (A)
19. (D)	**20.** (A)	**21.** (B)	**22.** (D)	**23.** (B)	**24.** (A)
25. (D)	**26.** (D)	**27.** (B)	**28.** (D)	**29.** (D)	**30.** (B)
31. (A)	**32.** (A)	**33.** (D)	**34.** (A)	**35.** (B)	**36.** (A)
37. (C)	**38.** (A)				

Practice Exercise 2

1. (A), (B)	**2.** (A, B)	**3.** (B), (C)	**4.** (A), (B)	**5.** (A), (B)	**6.** (C)
7. (A), (C)	**8.** (A), (B), (C), (D)	**9.** (A), (B)	**10.** (A), (C)	**11.** (B), (C), (D)	**12.** (A)
13. (C)	**14.** (B)	**15.** (C)	**16.** (D)	**17.** (B)	**18.** (B)
19. (B)	**20.** (C)	**21.** (B)	**22.** (C)	**23.** (A)	

24. (A) → (p), (B) → (q), (C) → (s), (D) → (s)

25. (A) → (p), (s), (B) → (p, r), (C) → (p), (D) → (p)

26. (A)→ (t), (B) → (s), (C) → (p), (D) → (q)

27.(A) → (p), (B) → (s), (C) → (q), (D) → (s)

28. 6 **29.** 1 **30.** 1

Solutions

Practice Exercise 1

1. Solving $3x + 4y = 9,\ \ y = mx + 1$, we get

$$x = \frac{5}{3 + 4m}$$

Here, x is an integer if $3 + 4m = 1,\ -1,\ 5,\ -5$. Therefore,

$$m = \frac{-2}{4}, \frac{-4}{4}, \frac{2}{4}, \frac{-8}{4}$$

Hence, m has two integral values.

2. Let the coordinates of A be $(a, 0)$. Then, the slope of the reflected ray is

$$\frac{3-0}{5-a} = \tan\theta \qquad \text{(say)}$$

The slope of the incident ray is

$$\frac{2-0}{1-a} = \tan(\pi - \theta)$$

Since $\tan\theta + \tan(\pi - \theta) = 0$,

$$\frac{3}{5-a} + \frac{2}{1-a} = 0$$

$$\Rightarrow 13 - 5a = 0 \Rightarrow a = \frac{13}{5}$$

Thus, the coordinates of A are $\left(\frac{13}{5}, 0\right)$.

3. For obvious reasons, the coordinates of the points A and B are (6, 0) and (0, 4), respectively (Fig. 10.41).

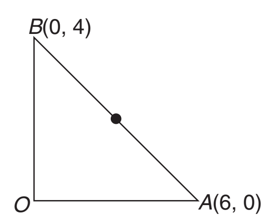


Figure 10.41

Therefore, the equation of line AB is

$$\frac{x}{6}+\frac{y}{4}=1$$

$$\Rightarrow 2x+3y=12$$

4. We have

$$\frac{x+5}{\cos\theta}=\frac{y+4}{\sin\theta}=\frac{r_1}{AB}=\frac{r_2}{AC}=\frac{r_3}{AD}$$

Now, $(r_1\cos\theta-5, r_1\sin\theta-4)$ lies on $x+3y+2=0$. Therefore,

$$r_1=\frac{15}{\cos\theta+3\sin\theta}\Rightarrow\frac{15}{AB}=\cos\theta+3\sin\theta$$

Similarly,

$$\frac{10}{AC}=2\cos\theta+\sin\theta \text{ and } \frac{6}{AD}=\cos\theta-\sin\theta$$

Substituting in the given relation, we get

$$(2\cos\theta+3\sin\theta)^2=0$$

Therefore,

$$\tan\theta=-\frac{2}{3}\Rightarrow y+4=-\frac{2}{3}(x+5)\Rightarrow 2x+3y+22=0$$

5. Let the equation of perpendicular bisector FN of AB is

$$x-y+5=0 \quad (1)$$

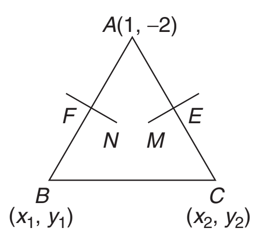


Figure 10.42

See Fig. 10.42. The midpoint F of AB is $\left(\frac{x_1+1}{2},\frac{y_1-2}{2}\right)$ that lies on the line given in Eq. (1).

Therefore,

$$x_1-y_1=-13 \quad (2)$$

Also, AB is perpendicular to FN. So the product of their slopes is –1. That is,

$$\frac{y_1+2}{x_1-1}\times 1=-1 \text{ or } x_1+y_1=-1 \quad (3)$$

On solving Eqs. (2) and (3), we get $B(-7, 6)$. Similarly, we get $C\left(\frac{11}{5},\frac{2}{5}\right)$.

Hence, the equation of BC is

$$14x+23y-40=0$$

6. From Fig. 10.43, we get

$$\left(\frac{b/2}{a/2}\right)\left(\frac{b}{-a/2}\right)=-1\Rightarrow a^2=2b^2\Rightarrow a=\pm\sqrt{2}b$$

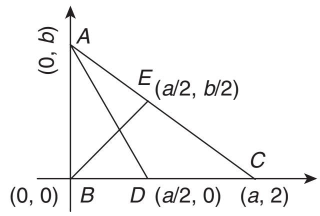


Figure 10.43

7. We have

$$S=\text{Midpoint of } QR=\left(\frac{6+7}{2},\ \frac{-1+3}{2}\right)=\left(\frac{13}{2},\ 1\right)$$

Therefore, m of PS is

$$PS=\frac{2-1}{2-(13/2)}=-\frac{2}{9}$$

Hence, the required equation is

$$y+1=\frac{-2}{9}(x-1)$$

That is,

$$2x+9y+7=0$$

8. If the line cuts off the axes at points A and B, then the area of triangle is

$$\frac{1}{2}\times OA\times OB=T$$

$$\Rightarrow\ \frac{1}{2}(a)OB=T\Rightarrow OB=\frac{2T}{a}$$

Hence, the equation of line is

$$\frac{x}{-a}+\frac{y}{2T/a}=1$$

$$\Rightarrow 2Tx-a^2y+2aT=0$$

9. Any line through point (1, –10) is given by

$$y+10=m(x-1)$$

Since it makes equal angle, say, α, with the given lines $7x-y+3=0$ and $x+y-3=0$, we have

$$\tan\alpha=\frac{m-7}{1+7m}=\frac{m-(-1)}{1+m(-1)}\Rightarrow m=\frac{1}{3}\text{ or }-3$$

Hence, the two possible equations of third side are

$$3x+y+7=0 \text{ and } x-3y-31=0$$

10. We have

$$y=\cos(x+1-1)\cos(x+1+1)-\cos^2(x+1)$$

$$=\cos^2(x+1)-\sin^2 1-\cos^2(x+1)=-\sin^2 1$$

which represents a straight line parallel to x-axis with $y=-\sin^2 1$ for all x and for $x=\pi/2$.

11. From Fig. 10.44, we notice that

$$AD=\left|\frac{-2-2-1}{\sqrt{(2)^2+(-1)^2}}\right|=\left|\frac{-5}{\sqrt{5}}\right|=\sqrt{5}$$

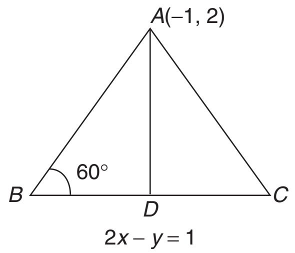


Figure 10.44

Also

$$\tan 60^\circ = \frac{AD}{BD} \Rightarrow \sqrt{3} = \frac{\sqrt{5}}{BD} \Rightarrow BD = \sqrt{\frac{5}{3}}$$

Therefore,

$$BC = 2BD = 2\sqrt{\frac{5}{3}} = \sqrt{\frac{20}{3}}$$

12. Taking coordinates as $\left(\frac{x}{r}, \frac{y}{r}\right)$, (x, y) and (xr, yr), we see that the coordinates satisfy the relation $y = mx$ and it is concluded that they lie on a straight line.

13. See Fig. 10.45. Slopes of AB and BC are –4 and $3/4$, respectively. If α be the angle between AB and BC, then

$$\tan\alpha = \frac{-4-(3/4)}{1-4(3/4)} = \frac{19}{8} \quad (1)$$

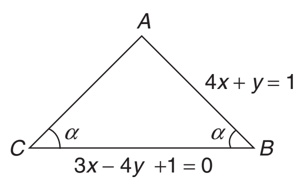


Figure 10.45

Since $AB = AC$, we have

$$\angle ABC = \angle ACB = \alpha$$

Thus, line AC also makes an angle α with line BC. If m be the slope of the line AC, then its equation is

$$y + 7 = m(x-2) \quad (2)$$

Now,

$$\tan\alpha = \pm\left[\frac{m-(3/4)}{1+m(3/4)}\right] \Rightarrow \frac{19}{8} = \pm\frac{4m-3}{4+3m} \Rightarrow m = -4 \text{ or } -\frac{52}{89}$$

However, the slope of AB is –4, so the slope of AC is $-(52/89)$. Therefore, the equation of line AC given by Eq. (2) is

$$52x + 89y + 519 = 0$$

14. Let the required line through the point (1, 2) be inclined at an angle θ to the axis of x. Then its equation is

$$\frac{x-1}{\cos\theta} = \frac{y-2}{\sin\theta} = r \quad (1)$$

where r is the distance of any point (x, y) on the line from the point (1, 2). The coordinates of any point on the line given in Eq. (1) are $(1 + r\cos\theta,\ 2 + r\sin\theta)$. If this point is at a distance $\sqrt{6}/3$ from the point (1, 2), then $r = \sqrt{6}/3$. Therefore, the point is

$$\left(1 + \frac{\sqrt{6}}{3}\cos\theta,\ 2 + \frac{\sqrt{6}}{3}\sin\theta\right)$$

However, this point lies on the line $x + y = 4$. So,

$$\frac{\sqrt{6}}{3}(\cos\theta + \sin\theta) = 1 \text{ or } \sin\theta + \cos\theta = \frac{3}{\sqrt{6}}$$

$$\Rightarrow \frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta = \frac{\sqrt{3}}{2},$$

(Dividing both sides by $\sqrt{2}$)

$$\Rightarrow \sin(\theta + 45^\circ) = \sin 60^\circ \text{ or } \sin 120^\circ$$

$$\Rightarrow \theta = 15^\circ \text{ or } 75^\circ$$

15. It is given that the lines $ax + by + p = 0$ and $x\cos\alpha + y\sin\alpha = p$ are inclined at an angle $\pi/4$. Therefore,

$$\tan\frac{\pi}{4} = \frac{-(a/b) + (\cos\alpha/\sin\alpha)}{1 + (a\cos\alpha/b\sin\alpha)}$$

$$\Rightarrow a\cos\alpha + b\sin\alpha = -a\sin\alpha + b\cos\alpha \quad (1)$$

It is given that the lines $ax + by + p = 0$, $x\cos\alpha + y\sin\alpha - p = 0$ and $x\sin\alpha - y\cos\alpha = 0$ are concurrent. Therefore,

$$\begin{vmatrix} a & b & p \\ \cos\alpha & \sin\alpha & -p \\ \sin\alpha & -\cos\alpha & 0 \end{vmatrix} = 0$$

$$\Rightarrow -ap\cos\alpha - bp\sin\alpha - p = 0 \Rightarrow -a\cos\alpha - b\sin\alpha = 1$$

$$\Rightarrow a\cos\alpha + b\sin\alpha = -1 \quad (2)$$

From Eqs. (1) and (2), we have

$$-a\sin\alpha + b\cos\alpha = -1$$

From Eqs. (2) and (3), we get

$$(a\cos\alpha + b\sin\alpha)^2 + (-a\sin\alpha + b\cos\alpha)^2 = 2$$

$$\Rightarrow a^2 + b^2 = 2$$

16. The four vertices on solving are $A(-3, 3)$, $B(1, 1)$, $C(1, -1)$ and $D(-2, -2)$.

$$m_1 = \text{slope of } AC = -1$$

$$m_2 = \text{slope of } BD = 1$$

Therefore, $m_1 m_2 = -1$. Hence, the angle between diagonals AC and BD is 90°.

17. The internal bisector of the angle A divides the opposite side BC at point D in the ratio of arms of the angle, that is, $AB = 3\sqrt{2}$ and $AC = 4\sqrt{2}$. Hence, by 'ratio formula', the point D is $\left(\frac{31}{7}, 1\right)$. The slope of AD by $\frac{y_2 - y_1}{x_2 - x_1} = 0$. Therefore, the slope of a line perpendicular to AD is ∞.

Any line through C perpendicular to this bisector is

$$\frac{y-5}{x-5} = m = \infty$$

Therefore, $x - 5 = 0$.

18. The equation of any line passing through the given point $P(3, 4)$ and making an angle $\pi/6$ with x-axis is

$$\frac{x-3}{\cos 30^\circ} = \frac{y-4}{\sin 30^\circ} = r \text{ (say)} \qquad (1)$$

where r represents the distance of any point Q on this line from the given point $P(3, 4)$. The coordinates (x, y) of any point Q on line (1) are

$$(3 + r\cos 30^\circ,\ 4 + r\sin 30^\circ)$$

That is,

$$\left(3 + \frac{r\sqrt{3}}{2},\ 4 + \frac{r}{2}\right)$$

If the point lies on the line $12x + 5y + 10 = 0$, then

$$12\left(3 + \frac{r\sqrt{3}}{2}\right) + 5\left(4 + \frac{r}{2}\right) + 10 = 0 \Rightarrow r = \frac{132}{12\sqrt{3}+5}$$

19. We have

$$L_{12} \equiv x - 3y + 1 = 0$$

$$L_{23} \equiv 2x + y - 12 = 0$$

$$L_{13} \equiv 3x - 2y - 4 = 0$$

Therefore, the required distances are

$$D_3 = \left|\frac{4 - 3\times 4 + 1}{\sqrt{10}}\right| = \frac{7}{\sqrt{10}}$$

$$D_1 = \left|\frac{4 + 1 - 12}{\sqrt{5}}\right| = \frac{7}{\sqrt{5}}$$

$$D_2 = \left|\frac{3\times 5 - 2\times 2 - 4}{\sqrt{9+4}}\right| = \frac{7}{\sqrt{13}}$$

20. From $P + \lambda Q = 0$, the required line is $12x - y - 31 = 0$ and its distance from both points is $31/\sqrt{145}$.

21. Let $P(x_1, y_1)$. Then the equation of line passing through P and whose gradient is m is

$$y - y_1 = m(x - x_1)$$

According to the condition

$$\frac{-2m + (mx_1 - y_1)}{\sqrt{1+m^2}} + \frac{2 + (mx_1 - y_1)}{\sqrt{1+m^2}} + \frac{1 - m + (mx_1 - y_1)}{\sqrt{1+m^2}} = 0$$

we can write as

$$3 - 3m + 3mx_1 - 3y_1 = 0 \Rightarrow y_1 - 1 = m(x_1 - 1)$$

Since it is a variable line, so hold for every value of m. Therefore,

$$y_1 = 1, x_1 = 1 \Rightarrow P(1,\ 1)$$

22. These lines cannot be the sides of a rectangle as none of these are parallel nor they are perpendicular. Now, let us check for concurrency

$$\begin{vmatrix} 1 & 2 & -3 \\ 3 & 4 & -7 \\ 2 & 3 & -4 \end{vmatrix} = 1(-16+21) - 2(2) - 3(1) \neq 0$$

Hence, neither is concurrent.

23. The coordinates of points A and B are $(0,\ 12)$ and $(8, 0)$, respectively. The equation of the perpendicular bisector of AB is

$$y - 6 = \frac{2}{3}(x - 4) \text{ or } 2x - 3y + 10 = 0 \qquad (1)$$

Equation of a line passing through $(0, -1)$ and parallel to x-axis is $y = -1$. This meets the line [Eq. (1)] at point C. Therefore, the coordinates of point C are $\left(-\frac{13}{2}, -1\right)$. Hence, the area of the triangle ABC is

$$\Delta = \frac{1}{2}\begin{vmatrix} 0 & 12 & 1 \\ 8 & 0 & 1 \\ -\frac{13}{2} & -1 & 1 \end{vmatrix} = 91 \text{ sq. unit}$$

24. We have $y = mx$. It makes an angle of $\pm 45^\circ$ (Fig. 10.46) with $2x + 3y = 6$.

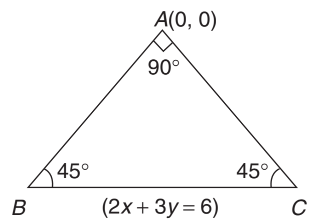


Figure 10.46

Therefore,

$$\tan(\pm 45^\circ) = \frac{m - (-2/3)}{1 + m(-2/3)} = \pm 1$$

$$\Rightarrow 3m + 2 = \pm(3 - 2m) \Rightarrow m = \frac{1}{5}, -5$$

Hence, the sides are $x - 5y = 0$, $5x + y = 0$ and $2x + 3y = 6$. Solving in pairs, the vertices are

$$(0, 0), \left(\frac{6}{13}, \frac{30}{13}\right), \left(\frac{30}{13}, -\frac{6}{13}\right)$$

Therefore,

$$\Delta = \left|\frac{1}{2}(x_1y_2 - x_2y_1)\right| = \frac{1}{2} \times \frac{936}{169} = \frac{36}{13}$$

25. We have $m_1 = -1/3$ and $m_2 = 3$. Hence, the lines $x + 3y = 4$ and $6x - 2y = 7$ are perpendicular to each other. Therefore, the parallelogram is rhombus.

26. The given lines are

$$\pm x \pm y = 1$$

That is,

$$x + y = 1, x - y = 1, x + y = -1 \text{ and } x - y = -1$$

These lines form a quadrilateral whose vertices are $A(-1, 0)$, $B(0, -1)$, $C(1, 0)$ and $D(0, 1)$ Obviously, $ABCD$ is a square. The length of each side of this square is

$$\sqrt{1^2 + 1^2} = \sqrt{2}$$

Hence, the area of square is

$$\sqrt{2} \times \sqrt{2} = 2 \text{ sq. unit}$$

Trick: The required area is

$$\frac{2c^2}{|ab|}=\frac{2\times 1^2}{|1\times 1|}=2$$

27. The area of the right-angled triangle is

$$\frac{1}{2}\text{ (Perpendicular)}\times\text{(Base)}=\frac{1}{2}ab$$

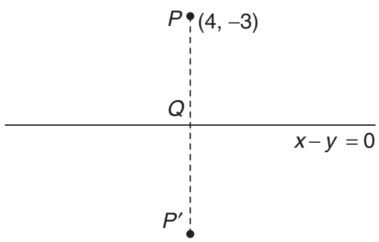
28. Here, $L\equiv x+y=2$ and $L'\equiv 2x-2y=1$. The equation of y-axis is $x=0$. Hence, the vertices of the triangle are

$$A(0,2),\ B\left(0,-\frac{1}{2}\right)\text{ and }C\left(\frac{5}{4},\frac{3}{4}\right)$$

Therefore, the area of the triangle is

$$\frac{1}{2}\begin{vmatrix} 0 & 2 & 1\\ 0 & -\frac{1}{2} & 1\\ \frac{5}{4} & \frac{3}{4} & 1\end{vmatrix}=\frac{25}{16}$$

29.



We have

$$x-y=0 \quad (1)$$

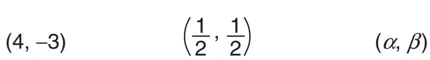
Slope of given line is $m_1=1$
Slope of PQ is $m_2=-1$
Equation of PQ is

$$y+3=-1(x-4)$$
$$\Rightarrow x+y=1 \quad (2)$$

Now point of intersection of (1) and (2) is

$$x-y=0$$
$$x+y=1$$
$$2x=1$$
$$x=\frac{1}{2}$$
$$y=\frac{1}{2}$$

Now let the co-ordinates of P' is (α,β)



$$\frac{\alpha+4}{2}=\frac{1}{2}$$
$$\alpha+4=1$$
$$\alpha=-3$$
$$\frac{-3+\beta}{2}=\frac{1}{2}$$
$$\beta=4$$

So, coordinates of P' is $(-3, 4)$.

30. Let the coordinates of point P which divides the line joining $(1, 0)$ and $(2\cos\theta,\ 2\sin\theta)$ in the ratio $2:3$ be (h,k). Then

$$h=\frac{4\cos\theta+3}{5}\text{ and }k=\frac{4\sin\theta}{5}$$

$$\Rightarrow \cos\theta=\frac{5h-3}{4}\text{ and }\sin\theta=\frac{5k}{4}\Rightarrow\left(\frac{5h-3}{4}\right)^2+\left(\frac{5k}{4}\right)^2=1$$

$$\Rightarrow (5h-3)^2+(5k^2)=16$$

Therefore, the locus of (h,k) is $(5x-3)^2+(5y)^2=16$, which is a circle.

31. The equation of line is $\lambda(x-a)+(y-b)=0$. Also

$$r=-\left(\frac{-a\lambda-b}{\lambda^2+1}\right)$$

The coordinates of point $\equiv\left\{-\lambda\left(\frac{-a\lambda-b}{\lambda^2+1}\right),-\left(\frac{-a\lambda-b}{\lambda^2+1}\right)\right\}$

Now,

$$h=\lambda\left(\frac{a\lambda+b}{\lambda^2+1}\right),\ k=\frac{a\lambda+b}{\lambda^2+1},\ \lambda=\frac{h}{k}$$

Therefore,

$$h=h\left(\frac{ah+kb}{h^2+k^2}\right)\Rightarrow x^2+y^2=ax+by$$

32. Since the triangle is right-angled at $O(0,0)$, it is obvious that $(0, 0)$ is its orthocentre.

33. The product of perpendiculars is

$$\frac{c}{\sqrt{(a-b)^2+4h^2}}$$

34. The lines represented by $y^2-9xy+18x^2=0$ are $6x-y=0$ and $3x-y=0$ and a third line is $y=9$. Therefore, the coordinates of the vertices of the triangle are given by

$$A(0,0);\ B(3,9)\text{ and }C\left(\frac{3}{2},9\right)$$

Hence, the area of ΔABC is

$$\frac{1}{2}\begin{vmatrix} 0 & 0 & 1\\ 3 & 9 & 1\\ 3/2 & 9 & 1\end{vmatrix}=\frac{27}{4}\text{ sq. units}$$

Aliter: Applying the formula discussed in this chapter, the required area is

$$\frac{(-9)^2\sqrt{(9/2)^2-18}}{18\times 1+9\times 0\times 1+1\times 0}=\frac{81}{18}\sqrt{\frac{81}{4}-18}$$

$$=\frac{81}{18}\times\frac{3}{2}=\frac{27}{4}\text{ sq.units}$$

35. We have

$$\sqrt{(x-3)^2+(y-1)^2}+\sqrt{(x+3)^2+(y-1)^2}=6$$
$$\sqrt{(x-3)^2+(y-1)^2}=6-\sqrt{(x+3)^2+(y-1)^2}$$

Squaring on both sides, we get

$$12x+36=12\sqrt{(x+3)^2+(y-1)^2}$$

On squaring once again, we get the given equation is pair of straight lines.

36. Let the lines represented by given equation be

$$y = m_1x + c_1 \text{ and } y = m_2x + c_2$$

Then

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$
$$= b(y - m_1x - c_1)(y - m_2x - c_2) = 0$$

On comparing the coefficients of x^2, xy, x, y and constant term, we get

$$m_1m_2 = \frac{a}{b},\ m_1 + m_2 = \frac{-2h}{b},\ m_1c_2 + m_2c_1 = \frac{2g}{b},$$
$$c_1 + c_2 = -\frac{2f}{b}$$

and $$c_1c_2 = \frac{c}{b}$$

Also, the point of intersection of $y = m_1x + c_1$ and $y = m_2x + c_2$ is

$$\left(\frac{c_2 - c_1}{m_1 - m_2}, \frac{m_1c_2 - m_2c_1}{m_1 - m_2}\right)$$

Therefore, the square of distance of this point from origin is

$$\left(\frac{c_2 - c_1}{m_1 - m_2}\right)^2 + \frac{(m_1c_2 - m_2c_1)^2}{(m_1 - m_2)^2}$$
$$= \frac{[(c_1 + c_2)^2 - 4c_1c_2] + [(m_1c_2 + m_2c_1)^2 - 4m_1m_2c_1c_2]}{(m_1 + m_2)^2 - 4m_1m_2}$$

Now, substituting the value defined above, we get the required distance, that is,

$$\frac{-c(a+b) + f^2 + g^2}{h^2 - ab}$$

37. By making the equation of circle homogeneous with the help of line $y = mx + c$, we get

$$x^2 + y^2 - a^2\left(\frac{y - mx}{c}\right)^2 = 0$$
$$\Rightarrow c^2x^2 + c^2y^2 - a^2y^2 - a^2m^2x^2 + 2a^2mxy = 0$$
$$\Rightarrow (c^2 - a^2m^2)x^2 + (c^2 - a^2)y^2 - 2a^2mxy = 0 \quad (1)$$

Hence, lines represented by Eq. (1) are perpendicular if

$$c^2 - a^2m^2 + c^2 - a^2 = 0 \Rightarrow 2c^2 = a^2(1 + m^2)$$

38. Let $ay^4 + bxy^3 + cx^2y^2 + dx^3y + ex^4$

$$= (ax^2 + pxy - ay^2)(x^2 + qxy + y^2)$$

On comparing the coefficient of similar terms, we get

$$b = aq - p,\ c = -pq,\ d = aq + p,\ e = -a$$
$$b + d = 2aq,\ e - a = -2a$$
$$ad + be = 2ap,\ a + c + e = -pq$$
$$(b + d)(ad + be) = -(e - a)^2(a + c + e)$$

Therefore,

$$(b + d)(ad + eb) + (e - a)^2(a + c + e) = 0$$

Practice Exercise 2

1. Any point on the line $x + y = 4$ is of the form $(t, 4 - t)$. So

$$\left|\frac{4t + 3(4 - t) - 10}{\sqrt{4^2 + 3^2}}\right| = 1 \Rightarrow t = -7, 3$$

The required points are (3, 1) and (–7, 11).

2. Line perpendicular to $5x - y = 0$ will be $x + 5y + \lambda = 0$

$$\frac{1}{2}\left|\lambda \cdot \frac{\lambda}{5}\right| = 5 \Rightarrow \lambda^2 = 50 \Rightarrow \lambda = \pm 5\sqrt{2}.$$

3. The given triangle is a right angled triangle. Hence, the orthocentre is the vertex containing the right angle, therefore orthocentre is (2, 5) which lies on the lines $3x - y = 1$ and $4x + y = 13$.

4. $a^2 + b^2 - c^2 - 2ab = 0 \Rightarrow (a - b)^2 - c^2 = 0 \Rightarrow (a - b - c)(a - b + c) = 0$

$\Rightarrow (-a + b + c)(a - b + c) = 0 \Rightarrow -a + b + c = 0$ or $a - b + c = 0$

$\Rightarrow a(-1) + b(1) + c = 0$ or $a(1) + b(-1) + c = 0$

Hence, the points of concurrency are (–1, 1) or (1, –1).

5. See Fig 10.47.

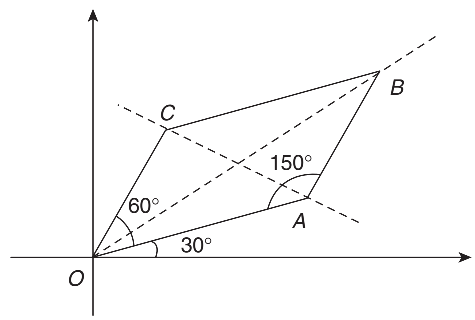


Figure 10.47

$$\text{Area } = l^2 \sin 30° = 2 \Rightarrow l = 2 \text{ unit}$$
$$OB^2 = OA^2 + AB^2 - 2OA \cdot AB\cos 150°$$
$$= 4 + 4 - 2(4)\left(-\frac{\sqrt{3}}{2}\right) = 4(2 + \sqrt{3})$$
$$OB = 2\sqrt{2 + \sqrt{3}}$$
$$B \equiv \left(\sqrt{4 + 2\sqrt{3}}, \sqrt{4 + 2\sqrt{3}}\right) \equiv (1 + \sqrt{3}, 1 + \sqrt{3})$$

Hence, coordinates of B can be

$$(1 + \sqrt{3}, 1 + \sqrt{3}) \text{ or } (-1 - \sqrt{3}, -1 - \sqrt{3})$$

6. Slope of the lines are $-1, -\frac{1}{\sqrt{3}}$ and $\frac{1}{7}$

Therefore,

$$\tan\alpha = \frac{-1 + \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} < 0$$

Therefore, it is an obtuse angled triangle. In an obtuse angle triangle, orthocentre and circumcentre are exterior to the triangle.

7. See Fig. 10.48.

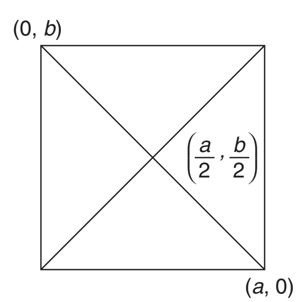


Figure 10.48

$$\frac{x-\frac{a}{2}}{\frac{b}{\sqrt{a^2+b^2}}}=\frac{y-\frac{b}{2}}{\frac{a}{\sqrt{a^2+b^2}}}=\pm\frac{\sqrt{a^2+b^2}}{2}$$

$$x=\frac{a}{2}+\frac{b}{2}, y=\frac{b}{2}+\frac{a}{2}$$

and $$x=\frac{a}{2}-\frac{b}{2}, y=\frac{b}{2}-\frac{a}{2}$$

Therefore, the required points are $\left(\frac{a+b}{2},\frac{a+b}{2}\right)$ and $\left(\frac{a-b}{2},\frac{b-a}{2}\right)$.

8. Let the slope of $u=0$ be m. Then the slope of $v=0$ is $\frac{9m}{2}$.

Therefore,

$$\frac{7}{9}=\left|\frac{m-\frac{9m}{2}}{1+m\cdot\frac{9m}{2}}\right|=\left|\frac{-7m}{2+9m^2}\right|$$

$$\Rightarrow 9m^2-9m+2=0 \text{ or } 9m^2+9m+2=0$$

$$m=\frac{9\pm\sqrt{81-72}}{18}=\frac{9\pm3}{18}=\frac{2}{3},\frac{1}{3} \text{ or } m=\frac{-9\pm3}{18}=-\frac{2}{3},-\frac{1}{3}$$

Therefore, equations of the lines are

(i) $3y=x$ and $2y=3x$
(ii) $3y=2x$ and $y=3x$
(iii) $x+3y=0$ and $3x+2y=0$
(iv) $2x+3y=0$ and $3x+y=0$.

9. Length of side of triangle ABP and altitude are $2\sqrt{2}$ and $\sqrt{6}$, respectively.

Midpoint of AB is (4, 3), slope of line AB and slope of altitude from P are 1 and –1, respectively.

Using parametric form, the coordinate of P is

$$(4\pm\sqrt{6}\cos135°, 3\pm\sqrt{6}\sin135°)=(4\mp\sqrt{3}, 3\pm\sqrt{3})$$

10.
$$\tan\alpha\tan\beta=-1$$
$$\Rightarrow \cos(\alpha-\beta)=0$$
$$\Rightarrow \alpha-\beta=\frac{\pi}{2}$$

11. See Fig. 10.49. Since given lines are perpendicular and intersect at $\left(\frac{6}{5},\frac{13}{5}\right)$.

Equations of angle bisectors of the given lines are

$$x+3y=9 \text{ and } 3x-y=1$$

Third side (let BC) will be parallel to these bisectors.
Let $AD=a$. Then $AB=a\sqrt{2}$.
Since area of $\Delta ABC=10$.

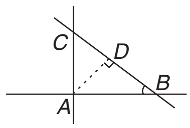


Figure 10.49

So,

$$\frac{1}{2}(a\sqrt{2})^2=10$$
$$\Rightarrow a=\sqrt{10}$$

Let the equation of BC is $x+3y=k$. Then

$$\left|\frac{\frac{6}{5}+\frac{39}{5}-k}{\sqrt{1+9}}\right|=\sqrt{10}\Rightarrow k=-1, 19$$

Thus, equation of BC is $x+3y+1=0$ or $x+3y-19=0$.
If the equation of BC is $3x-y=k_1$, then

$$\left|\frac{\frac{18}{5}-\frac{13}{5}-k_1}{\sqrt{10}}\right|=\sqrt{10}\Rightarrow k_1=-9, 11$$

Hence, equation of BC is $3x-y+9=0$ or $3x-y-11=0$.

12. We have $\max\{|x-1|, |y-2|\}=4$.

If $|x-1|\geq|y-2|$, then $|x-1|=4$.
That is, if $(x+y-3)(x-y+1)\geq 0$, then $x=-3$ or 5.
If $|y-2|\geq|x-1|$, then $|y-2|=4$.
That is, $(x+y-3)(x-y+1)\leq 0$, then $y=-2$ or 6.
So, the locus of P bounds a square, the equation of whose sides are

$$x=-3, x=5, y=-2, y=6$$

Thus, the area is $(8)^2=64$.

13. Consider a square with vertices at (–1, 0), (1, 0), (0, 1) and (0, –1). (See Fig. 10.50.)

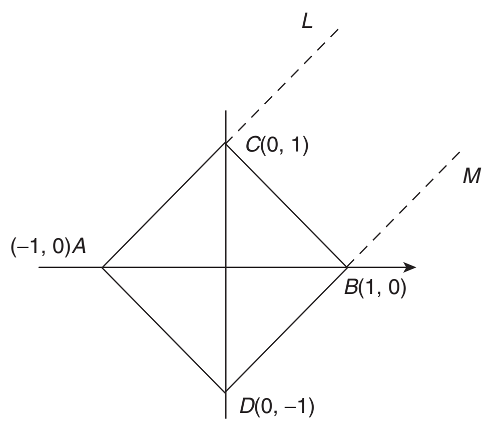


Figure 10.50

If we select a point P_1 in the region $BCLM$, then

$$d(A, P)=1+x$$
$$D(B, P)=y$$

Therefore, $d(A, P)+d(B, P)=1+x+y>2$.
However, for points P_2 and P_3 lying respectively on the line BC and below the line BC,

$$d(P, A)+d(P, B)=2$$

The same argument holds for other quadrants also.
Hence, the $d(P, A)+d(P, B)=2$ represents the region lying inside the square $ABCD$.

14. See Fig. 10.51. For the same square *ABCD*, for any point below the line *BC* (in the first quadrant)

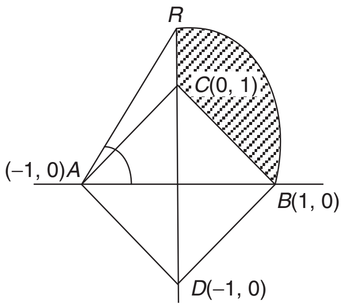


Figure 10.51

$$d(A, P) = 1 + x$$
$$d(B, P) = 1 - x$$

So, the equation is not satisfied (since $x < 1$).
Hence, *P* lies above or on the line *BC*.
Now, $d(A, P) = 1 + x$ and $d(B, P) = y$ and the equation becomes $(x + 1)^2 + y^2 = 4$, which represents a part of circle with centre (–1, 0) and radius 2 lying in the first quadrant.
Extending the argument to all four quadrants we find the desired locus as shown.

Clearly, $\angle RAB = \frac{\pi}{3}$.

Therefore,

$$\text{required area} = 4\left[\frac{\pi}{3}\cdot\frac{4}{2} - \frac{1}{2}\times 1\times 1 - \frac{1}{2}\sqrt{3}\right]$$
$$= \frac{2}{3}[4\pi - 3(\sqrt{3}+1)]$$

15. $\frac{x-2}{3} = \frac{y-3}{-4} = -15\frac{6-12+1}{25} = 3$

Therefore, $x = 11,\ y = -9$.
Hence, $\alpha = 2$.

16. $\frac{x-1}{-5} = \frac{y-1}{12} = 26\frac{-5+12+6}{169} = 2$

$$x = -9, y = 25$$

Therefore, $\beta = 16$.

17. See Fig. 10.52.

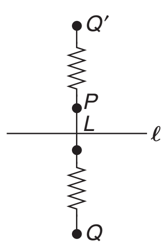


Figure 10.52

Since, $PQ = 16PL$,
Therefore, $LQ = 15\ PL$ and so $PQ' = 14PL$.
Thus, $n = 14$ for the point *Q*′.
Since, *L* and *Q*′ are on opposite sides of *P*. Therefore,

$$\frac{x-2}{1} = \frac{y+1}{-1} = 14\cdot\frac{2+1+1}{2} = 28$$

Therefore, *Q*′ is (30, –29).

18. We have

$$\omega = 60°, m = 2$$

$$\tan\theta = \frac{m\sin\omega}{1+m\cos\omega} = \frac{2\sin 60°}{1+2\cos 60°} = \frac{2\times\sqrt{3}/2}{1+2\times 1/2} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

19. We have

$$\omega = 60°, m_1 = 2, m_2 = -\frac{1}{2}$$

$$\tan\theta_1 = \frac{m_1\sin\omega}{1+m_1\cos\omega} = \frac{2\times\sqrt{3}/2}{1+2\times 1/2} = \frac{\sqrt{3}}{2}$$

$$\tan\theta_2 = \frac{-1/2\times\sqrt{3}/2}{1-1/2\times 1/2} = \frac{-\sqrt{3}}{4}\times\frac{4}{3} = -\frac{1}{\sqrt{3}}$$

Let angle between the lines be ϕ. Then

$$\tan\phi = \left|\frac{\tan\theta_1 - \tan\theta_2}{1+\tan\theta_1\tan\theta_2}\right|$$

$$= \left|\frac{\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{3}}}{1-\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{3}}}\right|$$

$$\Rightarrow \phi = \tan^{-1}\left(\frac{5}{\sqrt{3}}\right)$$

20. We have

$$m = \frac{\sin 60°}{\sin(30° - 60°)} = -\sqrt{3}$$

Therefore, equation of the line is

$$y - 0 = -\sqrt{3}(x-2)$$
$$\Rightarrow \sqrt{3}x + y = 2\sqrt{3}$$

21. See Fig. 10.53.

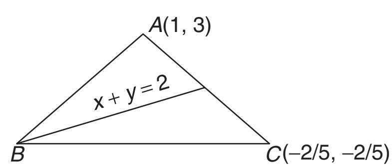


Figure 10.53

Image of *A*(1, 3) in line $x + y = 2$ is $\left(1 - \frac{2(2)}{2}, 3 - \frac{2(2)}{2}\right) \equiv (-1, 1)$

So, line *BC* passes through (–1, 1) and $\left(-\frac{2}{5}, -\frac{2}{5}\right)$.

Equation of the line *BC* is

$$y - 1 = \frac{-2/5-1}{-2/5+1}(x+1)$$
$$\Rightarrow 7x + 3y + 4 = 0$$

22. Vertex B is point of intersection of $7x+3y+4=0$ and $x+y=2$. That is, $B=(-5/2, 9/2)$.

23. Line AB is

$$y-3=\frac{3-9/2}{1+5/2}(x-1) \Rightarrow 3x+7y=24$$

24. See Fig. 10.54.

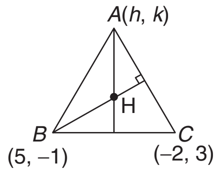


Figure 10.54

(A) $AH \perp BC \Rightarrow \left(\frac{k}{h}\right)\left(\frac{3+1}{-2-5}\right)=-1$

$$4k=7h \quad (1)$$

$$BH \perp AC \Rightarrow \left(\frac{0+1}{0-5}\right)\left(\frac{k-3}{h+2}\right)=-1$$

$$k-3=5(h+2) \quad (2)$$

$$\Rightarrow 7h-12=20h+40$$

$$13h=-52$$

$$h=-4$$

Therefore, $k=-7$.

Hence, $A(-4, -7)$.

(B)

$$x+y-4=0 \quad (3)$$

$$4x+3y-10=0 \quad (4)$$

Let $(h, 4-h)$ be the point on Eq. (3). Then

$$\left|\frac{4h+3(4-h)-10}{5}\right|=1$$

$$\Rightarrow h+2=\pm 5$$

$$\Rightarrow h=3;\ h=-7$$

Therefore, required point is either (3, 1) or (–7, 11).

(C) Orthocentre of the triangle is the point of intersection of the lines.

$$x+y-1=0 \text{ and } x-y+3=0$$

That is, (–1, 2).

(D) Since a, b, c are in AP. So,

$$b=\frac{a+c}{2}$$

Therefore, the family of lines is

$$ax+\frac{a+c}{2}y=c$$

That is,

$$a\left(x+\frac{y}{2}\right)+c\left(\frac{y}{2}-1\right)=0$$

Therefore, point of concurrency is (–1, 2).

25. (A) Lines are concurrent, so

$$\begin{vmatrix} 1 & -2 & -6 \\ 3 & 1 & -4 \\ \lambda & 4 & \lambda^2 \end{vmatrix}=0$$

$$\Rightarrow \lambda^2+2\lambda-8=0 \Rightarrow \lambda=2, -4$$

(B) Points are collinear, so

$$\begin{vmatrix} \lambda+1 & 1 & 1 \\ 2\lambda+1 & 3 & 1 \\ 2\lambda+2 & 2\lambda & 1 \end{vmatrix}=0$$

$$\Rightarrow 2\lambda^2-3\lambda-2=0 \Rightarrow \lambda=2, -1/2$$

(C) Point of intersection of $x-y+1=0$ and $3x+y-5=0$ is (1, 2), it will satisfy $x+y-1-\lambda=0$, so $\lambda=2$.

(D) Midpoint of (1, –2) and (3, 4) will satisfy $y-x-1+\lambda=0$

So, $\lambda=2$.

26. See Figs. 10.55 and 10.56.

(A) For point (a, a^2) to lie inside the triangle must satisfy

$$a>0 \quad (1)$$

$$a^2>0 \quad (2)$$

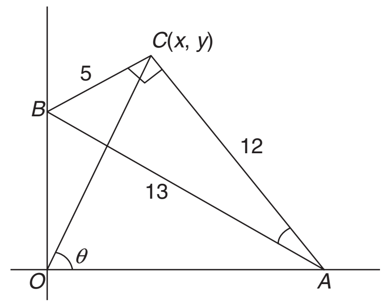


Figure 10.55

and

$$a+2a^2-3<0 \quad (3)$$

$$(2a+3)(a-1)<0$$

$$\Rightarrow a<1$$

$$\Rightarrow a \in (0, 1)$$

(B) Since $\angle BCA=90^\circ$

Points A, O, B, C are concyclic.

Let $\angle AOC=\theta$. Then

$$\angle BOC=\angle BAC$$

$$\tan\left(\frac{\pi}{2}-\theta\right)=\frac{5}{12}$$

$$\frac{x}{y}=\frac{5}{12} \Rightarrow 12x-5y=0$$

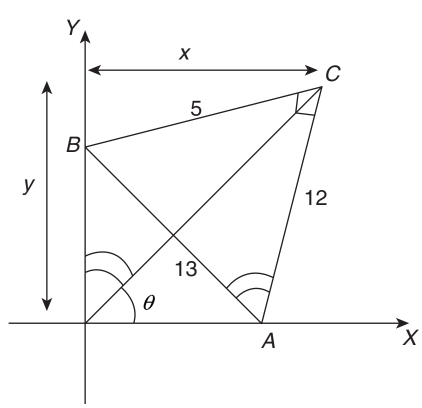


Figure 10.56

(C) Slope of the line joining the point $(t-1, 2t+2)$ and its image $(2t+1, t)$ is

$$\frac{(2t+2)-t}{t-1-2t-1} = \frac{t+2}{-(t+2)} = -1$$

So, slope of line is 1.

(D) Image of point $A(1, 2)$ in bisector of angles B and C lie on the line BC.

Image of A in $x = y$ is $(2, 1)$ and image of A in $y = 0$ is $(1, -2)$.

So, equation of line BC is $y = 3x - 5$. So,

$$d(A, BC) = \frac{4}{\sqrt{10}}$$

$$\Rightarrow \sqrt{10}\, d(A, BC) = 4$$

27. (A) $2|x| + 3|y| \le 6$ will represent the shaded region as shown in Fig. 10.57.

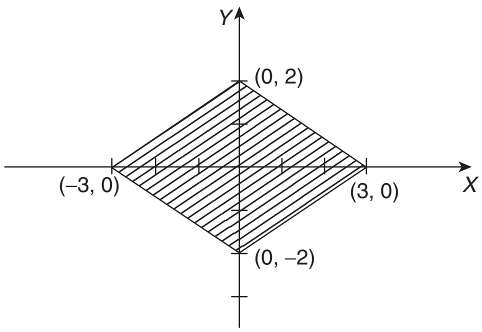


Figure 10.57

Therefore, required area $= 4 \times \frac{1}{2} \times 3 \times 2 = 12$ sq. units.

(B) See Fig. 10.58. Since,

$$D\left(1, \frac{1}{2}\right)$$

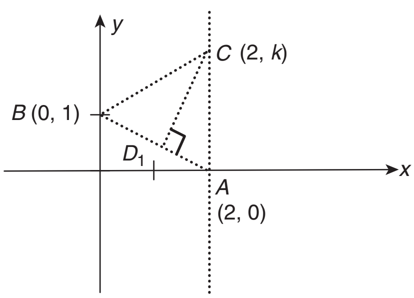


Figure 10.58

Since, CD is perpendicular AB. Therefore,

$$\frac{k-\frac{1}{2}}{2-1} = 2$$

$$\Rightarrow k = \frac{5}{2}$$

Therefore, slope of $BC = \frac{3}{4}$.

Therefore, $4m = 3$.

(C) See Fig. 10.59.

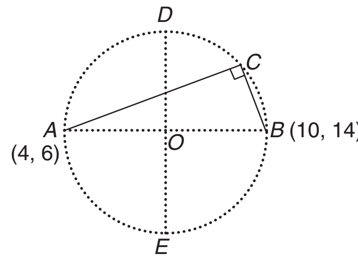


Figure 10.59

Since, area of triangle $ABC = 20$ square units

Therefore, C cannot be at D and E.

Therefore, four positions are possible two above AB and two below AB.

(D) See Fig. 10.60.

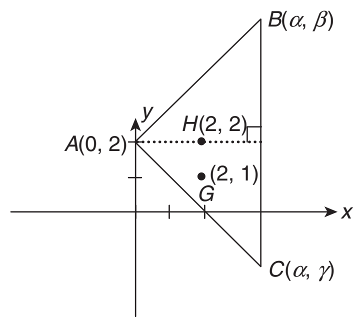


Figure 10.60

Therefore,

$$\frac{\alpha + \alpha + 0}{3} = 2$$

$$\alpha = 3$$

Therefore, x-coordinate of $B = 3$.

28. See Fig. 10.61.

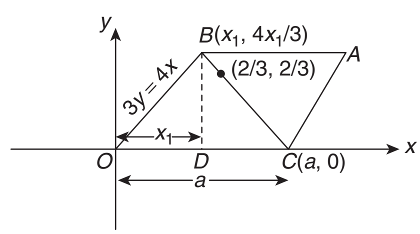


Figure 10.61

Let $OC = a$. Then

$$OC = CA = AB = BO = a$$

Let $\left(x_1, \frac{4x_1}{3}\right)$. Then

$$A\left(a + x_1, \frac{4x_1}{3}\right)$$

Since, $x_1^2 + \frac{16x_1^2}{9} = a^2$ (since ODB is a right angled triangle)

Therefore,

$$a=\frac{5x_1}{3}$$

Since equation of BC is

$$y-0=\frac{\frac{4x_1}{3}-0}{x_1-a}(x-a)$$

Since,

$$a=\frac{5x_1}{3}$$

Thus,

$$y=-2x+\frac{10x_1}{3}$$

And BC passes through $\left(\frac{2}{3},\frac{2}{3}\right)$. Therefore,

$$x_1=3/5$$

Now $a=1$. So,

$$A\left(1+\frac{3}{5},\frac{4}{3}\times\frac{3}{5}\right)=A\left(\frac{8}{5},\frac{4}{5}\right)$$

Therefore, $\frac{5}{2}(\alpha+\beta)=6$.

29. Given lines are

$$ax+y+1=0 \quad (1)$$

$$x+by=0 \quad (2)$$

$$ax+by=1 \quad (3)$$

Joint equation of Eqs. (1) and (2) is

$$(ax+y+1)(x+by)=0$$

$$\Rightarrow ax^2+by^2+(ab+1)xy+x+by=0 \quad (4)$$

Making Eq. (4) homogeneous with the help of Eq. (1), we have

$$ax^2+by^2+(ab+1)xy+x(ax+by)+by(ax+by)=0$$

Since angle between these two lines is 90°, we have

$$\text{Coefficient of } x^2+\text{Coefficient of } y^2=0$$

$$\Rightarrow 4a+b^2+(b+1)^2=1$$

30. Solving the two equations of ray, that is,

$$x+y=|a| \text{ and } ax-y=1$$

we get

$$x=\frac{|a|+1}{a+1}>0 \text{ and } y=\frac{|a|-1}{a+1}>0$$

When $a+1>0$; we get $a>1$

Therefore, $a_0=1$.

11 Pair of Straight Lines

11.1 Pair of Straight Lines – Fundamentals

1. A pair of straight lines is represented by the multiplication of equation of two straight lines that forms a second degree equation.
2. **Homogeneous equation of second degree:** An algebraic expression in x and y in which the sum of powers of x and y in every term is the same, say n, is called a homogeneous expression of degree n. If $f(x, y)$ is a homogeneous expression of degree n in x and y, then

$$f(x, y) = x^n F\left(\frac{y}{x}\right)$$

where F is a function of (y/x). Here, $f(x, y) = 0$ is called a homogeneous equation of degree n. A general homogeneous equation of second degree in x and y is expressed as

$$ax^2 + 2hxy + by^2 = 0 \tag{11.1}$$

Equation (11.1) represents a pair of straight lines (which can be real and distinct or coincident or imaginary) that are passing through the origin. If $y = m_1x$ and $y = m_2x$ be the pair of lines represented by Eq. (11.1), then

$$m_1m_2 = \frac{a}{b}$$

and $$m_1 + m_2 = -\frac{2h}{b}$$

Key Point: Two lines represented by Eq. (11.1) are parallel if $h^2 = ab$ and they are perpendicular if $a + b = 0$.

Illustration 11.1 Separate the straight lines represented by the pair of straight lines, $x^2 - xy - 6y^2 = 0$.

Solution: We have

$$x^2 - 3xy + 2xy - 6y^2 = 0$$
$$\Rightarrow x(x - 3y) + 2y(x - 3y) = 0$$
$$\Rightarrow (x - 3y)(x + 2y) = 0$$

Therefore, the lines are $x - 3y = 0$ and $x + 2y = 0$.

Illustration 11.2 Represent the two lines separately in the intercept form, which is given by $x^2 - y^2 - 2y - 1 = 0$.

Solution: We have

$$x^2 - y^2 - 2y - 1 = 0$$

Using $(a + b)^2 = a^2 + 2ab + b^2$, we get

$$x^2 - (y + 1)^2 = 0$$

Using $a^2 - b^2 = (a + b)(a - b)$, we get

$$(x - y - 1) \times (x + y + 1) = 0$$
$$\Rightarrow x - y = 1 \text{ and } x + y = -1$$
$$\Rightarrow \frac{x}{1} + \frac{y}{-1} = 1 \text{ and } \frac{x}{-1} + \frac{y}{-1} = 1$$

The intercepts are 1, –1 and –1, –1, respectively.

11.1.1 Angle between a Pair of Straight Lines

The angle between a pair of straight lines represented by $ax^2 + 2hxy + by^2 = 0$ is given by

$$\tan\theta = \pm\frac{2\sqrt{h^2 - ab}}{a + b}$$

The acute angle is given by

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{|a + b|}$$

and $$\cos\theta = \frac{a + b}{\sqrt{(a - b)^2 + 4h^2}}$$

Key Points:
1. Two lines are coincident if $h^2 = ab$.
2. Two lines are perpendicular if $a + b = 0$.
3. The equation of pair of straight lines through origin and perpendicular to the lines $ax^2 + 2hxy + by^2 = 0$ is
$$bx^2 - 2hxy + ay^2 = 0$$

Illustration 11.3 If the equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of mutually perpendicular straight lines, then the graph of b versus a (consider b on y-axis and a on x-axis) is

(A) a straight line with positive slope and positive y-intercept.
(B) a straight line with negative slope and positive y-intercept.
(C) a straight line with zero y-intercept.
(D) a straight line with positive slope.

Solution: For the lines to be perpendicular, they should satisfy

$$b + a = 0$$
$$\Rightarrow b = -a$$

Therefore, b versus a graph is a straight line with negative slope and passing through origin.
Hence, the correct answer is option (C).

Illustration 11.4 Find the equation of pair of straight lines formed when the pair of straight lines given by $x^2 - (\sqrt{3} + 1)xy + \sqrt{3}y^2 = 0$ is rotated in 15° anticlockwise direction about the origin.

Solution: On factorizing the given equation, we get

$$(x-y)(x-\sqrt{3}y)=0$$

Therefore,

$$y=x \text{ and } y=\frac{x}{\sqrt{3}}$$

Hence, the inclination of the two lines is 45° and 30°, respectively. On turning this in 15° anticlockwise direction, the inclinations become 60° and 45°. Thus, the new equations are

$$y=\sqrt{3}x \text{ and } y=x$$

Therefore, the combined equation is

$$(x-y)(\sqrt{3}x-y)=0$$

$$\Rightarrow \sqrt{3}x^2-(\sqrt{3}+1)xy+y^2=0$$

Your Turn 1

1. Find the equation of the lines perpendicular to the lines represented by $3x^2 - 4xy - 7y^2 = 0$. **Ans.** $-7x^2 + 4xy + 3y^2 = 0$
2. Find the value of λ if $3x^2 + 7xy + \lambda y^2 = 0$ represents the pair of perpendicular straight line. **Ans.** -3
3. Find the combined equation of lines that are passing through the origin and equally inclined to coordinate axes. **Ans.** $x^2 - y^2 = 0$
4. Find the condition for which $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight line passing through origin. **Ans.** $h^2 \ge ab$
5. Find the combined equation of coordinate axes. **Ans.** $xy = 0$

11.1.2 Angle Bisectors between a Pair of Straight Lines

If $ax^2 + 2hxy + by^2 = 0$ represents lines l_1, l_2, then the combined equation of bisector lines B_1, B_2 (Fig. 11.1) is represented by

$$\frac{x^2-y^2}{a-b}=\frac{xy}{h}$$

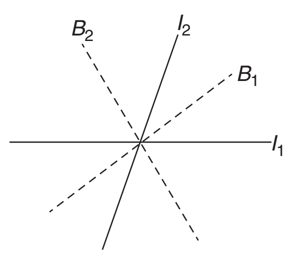


Figure 11.1

> **Key Points:**
> 1. If $h = 0$, the bisectors are $x = 0$ and $y = 0$.
> 2. If $a = b$, the bisectors are $x = y$ and $x = -y$.
> 3. If the condition,
>
> $$\text{coefficient of } x^2 + \text{coefficient of } y^2 = 0$$
>
> is satisfied by the equation of bisectors, then the two bisectors are perpendicular.

Illustration 11.5 For what real value of a do the pairs of straight lines $3x^2 + 8xy - 3y^2 = 0$ and $ax^2 + 6xy - ay^2 = 0$ bisect each other?

Solution: The equation of bisectors of $3x^2 + 8xy - 3y^2$ is

$$\frac{x^2-y^2}{6}=\frac{xy}{4}$$

$$\Rightarrow 4x^2 - 6xy - 4y^2 = 0$$

$$\Rightarrow -4x^2 + 6xy + 4y^2 = 0$$

On comparing with $ax^2 + 6xy - ay^2 = 0$, we get $a = -4$.

11.1.3 General Second-Degree Equation

The general equation of second degree in x and y is given by

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad (11.2)$$

The condition for Eq. (11.2) to represent a pair of straight lines is

$$abc+2fgh-af^2-bg^2-ch^2=0$$

or

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$$

and

$$h^2 \ge ab$$

Remarks: If Eq. (11.2) represents two straight lines, then the equation of the lines through the origin and parallel to them is $ax^2 + 2hxy + by^2 = 0$. This implies that if Eq. (11.2) represents pair of straight lines, then the angle θ between them is given by the same formula as for homogeneous equation.

> **Key Point:** The general second-degree equation [Eq. (11.2)] represents different conic sections under different conditions as listed in Table 11.1.

Table 11.1 Curves and their conditions

S. No.	Curve	Condition
1.	Two straight lines (real or imaginary)	$\Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$
2.	Two parallel straight lines	$\Delta=0$ and $h^2=ab$
3.	Circle	$\Delta\neq 0$ and $h=0, a=b\neq 0$
4.	Ellipse	$\Delta\neq 0$ and $h^2<ab$
5.	Parabola	$\Delta\neq 0$ and $h^2=ab$
6.	Hyperbola	$\Delta\neq 0$ and $h^2>ab$
7.	Rectangular hyperbola	$\Delta\neq 0$ and $h^2>ab, a+b=0$

Illustration 11.6 For what rational values of a, the equation $ax^2 + 8xy + ay^2 + 8x + 6y + 4 = 0$ represents **(a)** a pair of straight lines and **(b)** a parabola?

Solution:

(a) For the equation, to represent a pair of straight lines, we have

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0 = \begin{vmatrix} a & 4 & 4 \\ 4 & a & 3 \\ 4 & 3 & 4 \end{vmatrix} = 0$$

$$\Rightarrow 4ab^2 - 25a + 32 = 0$$

$$\Rightarrow a = \frac{+25 \pm \sqrt{113}}{8}$$

which is irrational. Hence, the equation does not represent pair of straight lines for any rational value of x.

(b) For the given equation to represent a parabola, we have $\Delta \neq 0$ and $h^2 = ab$. Here, $\Delta \neq 0$ for all rational values of a. Therefore,

$$16 = (a)(a) \Rightarrow a = \pm 4$$

Hence, for $a = 4$ and -4, it represents a parabola.

11.1.4 Angle between Lines Represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

We should note that the angle

$$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right| = \frac{2\sqrt{h^2 - ab}}{|a+b|}$$

is same as that the angle between the lines $ax^2 + 2hxy + by^2 = 0$. The two lines are parallel if $h^2 = ab$ and $bg^2 = af^2$.

11.1.5 Point of Intersection of Lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Point of intersection can be obtained by solving

$$\frac{\partial F}{\partial x} = 0 \text{ and } \frac{\partial F}{\partial y} = 0$$

$$\Rightarrow 2ax + 2hy + 2g = 0\text{, } 2hx + 2by + 2f = 0$$

Solving equations, we get

$$x = \frac{bg - hf}{h^2 - ab}, y = \frac{af - hg}{h^2 - ab}$$

Illustration 11.7 The equation $16x^2 + 2hxy + 25y^2 + 2gx + 16y + c = 0$ represents a pair of parallel straight line. Find the possible positive values of h, g and c.

Solution: We have $a = 16$, $b = 25$, $h = h$, $g = g$, $f = 8$ and $c = c$. For parallel lines, we have

$$h^2 = ab \text{ and } bg^2 = af^2$$

Therefore, $h = +20$ and hence

$$25g^2 = 16 \times 64$$

$$\Rightarrow g = \pm\left(\frac{4 \times 8}{5}\right) = 6.4$$

Therefore,

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Hence,

$$\begin{vmatrix} 16 & 20 & 6.4 \\ 20 & 25 & 8 \\ 6.4 & 8 & c \end{vmatrix} = 0$$

is true since $R_2 = 1.25R_1$ and hence, this is true for all values of c. Therefore, $h = 20$, $g = 6.4$ and c is any real, positive value.

11.1.6 Angle Bisectors of Lines Represented by General Second-Degree Equation

If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of intersecting lines, then the combined equation of two angle bisectors is given by

$$\frac{(x - x')^2 - (y - y')^2}{a - b} = \frac{(x - x')(y - y')}{h}$$

where (x', y') is the point of intersection of the lines given by general second-degree equation.

11.1.7 Distance between Parallel Lines

If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ which is general second-degree equation that represents two parallel lines, then the distance between these lines is given by

$$d = 2\sqrt{\frac{g^2 - ac}{a(a+b)}}$$

Illustration 11.8 The equation $9x^2 + y^2 - 6xy + 42x - 14y + 45 = 0$ represents

(A) a pair of intersecting lines at point (2, 3).
(B) a pair of parallel lines with distance 4 between them.
(C) a parabola.
(D) a pair of parallel lines at a separation of $2\sqrt{2/5}$.

Solution: Since $\Delta = 0$, this is a pair of straight lines. Also, $h^2 = ab$, that is, $9 = 9 \times 1$.
Hence, these are parallel lines. Therefore,

$$d = 2\sqrt{\frac{21^2 - 9(45)}{9(9+1)}}$$

$$= 2\sqrt{\frac{441 - 405}{90}} = 2\sqrt{\frac{36}{90}} = \frac{4}{\sqrt{10}} = \frac{2\sqrt{2}}{\sqrt{5}}$$

Hence, the correct answer is option (D).

11.1.8 Important Results

Some important results about pair of straight lines represented by general second-degree equation, $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are listed as follows:

1. The equation of pair of straight lines which are passing through and parallel to the lines represented by general second-degree equation is $ax^2 + 2hxy + by^2 = 0$.
2. The equation of pair of straight lines through (α, β) and perpendicular to the pair of lines $ax^2 + 2hxy + by^2 = 0$ is
$$b(x-\alpha)^2 - 2h(x-\alpha)(y-\beta) + a(y-\beta)^2 = 0$$
3. Two pairs of straight lines $ax^2 + 2hxy + by^2 = 0$ and $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ form
   - **(i)** a parallelogram if $(a-b)fg + h(f^2 - g^2) \neq 0$; $a + b \neq 0$.
   - **(ii)** a rhombus if $(a-b)fg + h(f^2 - g^2) \neq 0$; $a + b \neq 0$.
   - **(iii)** a rectangle if $(a-b)fg + h(f^2 - g^2) \neq 0$; $a + b = 0$.
   - **(iv)** a square if $(a-b)fg + h(f^2 - g^2) = 0$; $a + b = 0$.

Your Turn 2

1. Find the distance between the parallel lines $9x^2 - 6xy + y^2 + 18x - 6y + 8 = 0$. **Ans.** $2/\sqrt{10}$
2. Find the equation of the pair of lines joining origin to the points of intersection of $x^2 + y^2 = 9$ and $x + y = 3$. **Ans.** $xy = 0$
3. If the area of triangle formed by the lines $y^2 - 9xy + 18x^2 = 0$ and $y = 9$ is Δ, find the value of 4Δ. **Ans.** 27 sq. units
4. Find the angle between the angle bisectors of the lines represented by $x^2 - xy - 5y^2 = 0$. **Ans.** $\pi/2$

11.2 Separation of Equations of Straight Lines from their Joint Equation

1. Factorise the quadratic part into two parts:
$$ax^2 + 2hxy + by^2 = (px + qy)(p'x + q'y)$$
2. Add two constants r and r' to each part:
$$(px + qy + r) \text{ and } (p'x + q'y + r')$$
3. Now,
$(px + qy + r)(p'x + q'y + r') = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$.
Compare the coefficients of x and y on both sides and solve for r and r'. Thus, two lines are obtained as $px + qy + r = 0$ and $p'x + q'y + r' = 0$.

Illustration 11.9 Separate the equation of two lines from their combined equation $9x^2 + y^2 - 6xy + 42x - 14y + 45 = 0$.

Solution:

Step 1: Here, $\Delta = 0$, so the quadratic equation represents the pair of straight lines.
Step 2: $9x^2 - 6xy + y^2 = (3x - y)(3x - y)$
Step 3: $(3x - y + r)(3x - y + r') = 9x^2 - 6xy + y^2 + 42x - 14y + 45$
On comparing the coefficients of x, y and constant terms, we get
$$\left.\begin{array}{l} rr' = 456 \\ 3r' + 3r = 42 \\ -r - r' = -14 \end{array}\right\} \Rightarrow r = 9 \text{ and } r' = 5$$
Therefore, the straight lines are $3x - y + 9 = 0$ and $3x - y + 5 = 0$.

11.3 Combined Equation of Lines Joining Origin to Points of Intersections of a Line and a Curve

Let us consider a curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and a line $lx + my = n$.

1. We have $\dfrac{lx + my}{n} = 1$.
2. Now,
$$ax^2 + 2hxy + by^2 + (2gx + 2fy)\left(\frac{lx + my}{n}\right) + c\left(\frac{lx + my}{n}\right)^2 = 0$$
3. Make the equation of curve homogenous with the help of equation of line and this gives combined equation of OP and OQ (Fig. 11.2).

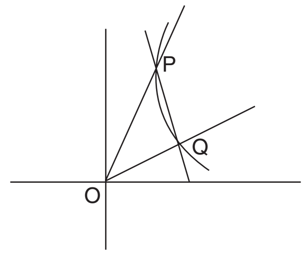


Figure 11.2

Note: A pair of lines is equally inclined to the other pair of lines if both the pairs have the same angle bisectors.

Illustration 11.10 Show that the equation $6x^2 - 5xy + y^2 = 0$ represents a pair of distinct straight lines, each passing through the origin. Find the separate equations of these lines.

Solution: The given equation is a homogeneous equation of second degree. Thus, it represents a pair of straight lines passing through the origin. On comparing the given equation with $ax^2 + 2hxy + by^2 = 0$, we obtain $a = 6$, $b = 1$ and $2h = -5$. Therefore,
$$h^2 - ab = \frac{25}{4} - 6 = \frac{1}{4} > 0 \Rightarrow h^2 > ab$$
Hence, the given equation represents a pair of distinct lines passing through the origin. Now,
$$6x^2 - 5xy + y^2 = 0$$
$$\Rightarrow \left(\frac{y}{x}\right)^2 - 5\left(\frac{y}{x}\right) + 6 = 0$$
$$\Rightarrow \left(\frac{y}{x}\right)^2 - 3\left(\frac{y}{x}\right) - 2\left(\frac{y}{x}\right) + 6 = 0$$
$$\Rightarrow \left(\frac{y}{x} - 3\right)\left(\frac{y}{x} - 2\right) = 0$$
$$\Rightarrow \frac{y}{x} - 3 = 0 \text{ or } \frac{y}{x} - 2 = 0$$
$$\Rightarrow y - 3x = 0 \text{ or } y - 2x = 0$$
Thus, the given equation represents the straight lines $y - 3x = 0$ and $y - 2x = 0$.

Illustration 11.11 Find the equations to the pair of lines through the origin which are perpendicular to the lines represented by $2x^2 - 7xy + 2y^2 = 0$.

Solution: We have
$$2x^2 - 7xy + 2y^2 = 0$$
$$\Rightarrow 2x^2 - 6xy - xy + 3y^2 = 0$$
$$\Rightarrow 2x(x - 3y) - y(x - 3y) = 0$$

$$\Rightarrow (x-3y)(2x-y)=0$$
$$\Rightarrow x-3y=0 \text{ or } 2x-y=0$$

Thus, the given equation represents the lines $x - 3y = 0$ and $2x - y = 0$. The equations of the lines which are passing through the origin and perpendicular to the given lines are

$$y-0=-3(x-0)$$

and $$y-0=-\frac{1}{2}(x-0)$$

Since, the slope of $x - 3y = 0$ is 1/3 and the slope of $2x - y = 0$ is 2, we have

$$\Rightarrow y+3x=0 \text{ and } 2y+x=0$$

Illustration 11.12 Find the angle between the pair of straight lines $4x^2 + 24xy + 11y^2 = 0$.

Solution: The given equation is

$$4x^2+24xy+11y^2=0$$

Here, we have

a = coefficient of $x^2 = 4$; b = coefficient of $y^2 = 11$

and

$2h$ = coefficient of $xy = 24$, that is, $h = 12$.

Now,

$$\tan\theta=\left|\frac{2\sqrt{h^2-ab}}{a+b}\right|=\left|\frac{2\sqrt{144-44}}{4+11}\right|=\frac{4}{3}$$

where θ is the acute angle between the lines. Therefore, the acute angle between the lines is $\tan^{-1}(4/3)$ and obtuse angle between them is $\pi-\tan^{-1}(4/3)$.

Illustration 11.13 Find the equation of the bisectors of the angle between the lines represented by $3x^2 - 5xy + 4y^2 = 0$.

Solution: The given equation is

$$3x^2-5xy+4y^2=0 \quad (1)$$

On comparing it with the equation

$$ax^2+2hxy+by^2=0 \quad (2)$$

we get,

$$a=3,\ 2h=-5 \text{ and } b=4$$

Now, the equation of the bisectors of the angle between the pair of lines [Eq. (1)] is

$$\frac{x^2-y^2}{a-b}=\frac{xy}{h}$$
$$\Rightarrow \frac{x^2-y^2}{3-4}=\frac{xy}{-(5/2)}$$
$$\Rightarrow \frac{x^2-y^2}{-1}=\frac{2xy}{-5}$$
$$\Rightarrow 5x^2-2xy-5y^2=0$$

Illustration 11.14 Prove that the equation $2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$ represents a pair of straight lines. Find the coordinates of their point of intersection and also the angle between them.

Solution: The given equation is

$$2x^2+5xy+3y^2+6x+7y+4=0 \quad (1)$$

On writing Eq. (1) as a quadratic equation, which is quadratic in x, we get

$$2x^2+(5y+6)x+3y^2+7y+4=0$$

Therefore,

$$x=\frac{-(5y+6)\pm\sqrt{(5y+6)^2-4(2)(3y^2+7y+4)}}{4}$$
$$=\frac{-(5y+6)\pm\sqrt{25y^2+60y+36-24y^2-56y-32}}{4}$$
$$=\frac{-(5y+6)\pm\sqrt{y^2+4y+4}}{4}=\frac{-(5y+6)\pm(y+2)}{4}$$

Therefore,

$$x=\frac{-5y-6+y+2}{4},\ \frac{-5y-6-y-2}{4}$$
$$\Rightarrow 4x+4y+4=0$$
and $$4x+6y+8=0$$
$$\Rightarrow x+y+1=0$$
and $$2x+3y+4=0$$

Hence, Eq. (1) represents a pair of straight lines whose equations are

$$x+y+1=0$$

and $$2x+3y+4=0$$

Solving these two equations, the point of intersection is obtained as (1, –2).

Also, the angle between the lines is given by

$$\theta=\tan^{-1}\left(\frac{2\sqrt{\left(\frac{5}{2}\right)^2-2\times 3}}{2+3}\right)=\tan^{-1}\left(\frac{1}{5}\right)$$

Illustration 11.15 Prove that the angle between the lines joining the origin to the points of intersection of the straight line $y = 3x + 2$ with the curve $x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0$ is $\tan^{-1}(2\sqrt{2}/3)$.

Solution: The equation of the given curve is

$$x^2+2xy+3y^2+4x+8y-11=0 \quad (1)$$

and the equation of the given straight line is

$$y=3x+2 \quad (2)$$

Therefore,

$$\frac{y-3x}{2}=1 \quad (3)$$

On making Eq. (1) as homogeneous equation of the second degree in x and y with the help of Eq. (3), we get

$$x^2+2xy+3y^2+4x\left(\frac{y-3x}{2}\right)+8y\left(\frac{y-3x}{2}\right)-11\left(\frac{y-3x}{2}\right)^2=0$$

$$\Rightarrow x^2+2xy+3y^2+\frac{1}{2}(4xy+8y^2-12x^2-24xy)-\frac{11}{4}(y^2-6xy+9x^2)=0$$

$$\Rightarrow 4x^2+8xy+12y^2+2(8y^2-12x^2-20xy)-11(y^2-6xy+9x^2)=0$$

$$\Rightarrow -119x^2+34xy+17y^2=0$$

or $$119x^2 - 34xy - 17y^2 = 0 \quad (4)$$

or $$7x^2 - 2xy - y^2 = 0$$

This is the equation of the lines joining the origin to the points of intersection of the curve [Eq. (1)] and the straight line [Eq. (2)]. On comparing Eq. (4) with the equation $ax^2 + 2hxy + by^2 = 0$, we get $a = 7$, $b = -1$ and $2h = -2$, that is, $h = -1$. If θ be the acute angle between the pair of lines [Eq. (4)], then

$$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{(a+b)}\right| = \left|\frac{2\sqrt{1+7}}{7-1}\right| = \frac{2\sqrt{8}}{6} = \frac{2\sqrt{2}}{3}$$

Therefore,

$$\theta = \tan^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

Illustration 11.16 Find the condition that the pair of straight lines joining the origin to the intersections of the line $y = mx + c$ and the circle $x^2 + y^2 = a^2$ may be at right angles.

Solution: The equations of the line and the circle, respectively, are

$$y = mx + c \quad (1)$$

and $$x^2 + y^2 = a^2 \quad (2)$$

The pair of straight line joining the origin to the intersections of the line [Eq. (1)] and the circle [Eq. (2)] is obtained by making Eq. (2) homogenous with the help of Eq. (1). Since $y = mx + c$, we get

$$\frac{y - mx}{c} = 1$$

Therefore,

$$x^2 + y^2 = a^2(1)^2$$

$$\Rightarrow x^2 + y^2 = a^2\left(\frac{y - mx}{c}\right)^2$$

$$\Rightarrow x^2(c^2 - a^2m^2) + 2ma^2xy + y^2(c^2 - a^2) = 0 \quad (3)$$

According to the question, the lines given in Eq. (3) do exist at right angles. Therefore,

$$\text{Coefficient of } x^2 + \text{Coefficient of } y^2 = 0$$

$$\Rightarrow c^2 - a^2m^2 + c^2 - a^2 = 0$$

$$\Rightarrow 2c^2 = a^2(1 + m^2)$$

which is the required condition.

Illustration 11.17 The straight lines $3x + 4y = 5$ and $4x - 3y = 15$ intersect at the point A. On these lines, two points B and C are chosen, so that AB = AC. Find the possible equations of the line BC which is passing through (1, 2).

Solution: The two given straight lines are at right angles as $m_1 = \frac{-3}{4}$ and $m_2 = \frac{4}{3}$. Since AB = AC, the triangle is an isosceles right-angled triangle. The equation which is required is of the form

$$y - 2 = m(x - 1) \quad (1)$$

with the angle

$$\tan 45° = \pm\frac{m + (3/4)}{1 - (3m/4)} = \pm\frac{m - (4/3)}{1 + (4m/3)}$$

$$\Rightarrow 1 = \pm\frac{m + (3/4)}{1 - (3m/4)}$$

and $$1 = \pm\frac{m - (4/3)}{1 + (4m/3)}$$

Therefore, $m = 1$ and 1/7. On substituting the value of m in Eq. (1), we get the required equations.

Additional Solved Examples

1. If the lines $x = k$ where $k = 1, 2, \ldots, n$ meet the line $y = 3x + 4$ at the points $A_k(x_k, y_k), k = 1, 2, \ldots, n$, then the ordinate of the centre of mean position of the points $A_k, k = 1, 2, \ldots, n$, is

(A) $\frac{n+1}{2}$ **(B)** $\frac{3n+11}{2}$

(C) $\frac{3(n+1)}{2}$ **(D)** None of these

Solution: We have $y_k = 3k + 4$, the ordinate of A_k the point of intersection of $x = k$ and $y = 3x + 4$. So, the ordinate of the centre of mean position of the points A_k, $k = 1, 2, \ldots, n$ is

$$\frac{1}{n}\sum_{k=1}^{n} y_k = \frac{1}{n}\sum_{k=1}^{n}(3k+4) = \frac{3}{n}\sum_{k=1}^{n} k + 4 = \frac{3n(n+1)}{n \cdot 2} + 4 = \frac{3n+11}{2}$$

Hence, the correct answer is option (B).

2. Joint equation of the diagonals of the square formed by the pair of lines $xy + 4x - 3y - 12 = 0$ and $xy - 3x + 4y - 12 = 0$ is

(A) $x^2 - y^2 + x - y = 0$ **(B)** $x^2 - y^2 + x + y = 0$

(C) $x^2 + 2xy + y^2 + x + y = 0$ **(D)** $x^2 - 2xy + y^2 + x - y = 0$

Solution:

$$xy + 4x - 3y - 12 = (x - 3)(y + 4) = 0$$
$$\Rightarrow x = 3, y = -4$$

and $$xy - 3x + 4y - 12 = (x + 4)(y - 3) = 0$$
$$\Rightarrow x = -4, y = 3$$

We find that the vertices of the square are

$$A(3,3), B(-4,3), C(-4,-4) \text{ and } D(3,-4)$$

Equation of the diagonal AC is $y = x$ and of BD is

$$y - 3 = \frac{3+4}{-4-3}(x + 4) \Rightarrow x + y + 1 = 0$$

Hence, the required joint equation of the diagonals is

$$(x - y)(x + y + 1) = 0 \Rightarrow x^2 - y^2 + x - y = 0$$

Hence, the correct answer is option (A).

3. The line $x + y = 1$ meets the lines represented by the equation $y^3 - xy^2 - 14x^2y + 24x^3 = 0$ at the points A, B, C. If O is the point of intersection of the lines represented by the given equation, then $OA^2+OB^2+OC^2$ is equal to

(A) 22/9 **(B)** 85/72
(C) 181/72 **(D)** 221/72

Solution: The given equation can be written in cubic form as

$$(y-2x)(y-3x)(y+4x)=0$$

Therefore, the three lines given by this equation are

$$y=2x,\ y=3x \quad \text{and} \quad y=-4x$$

They intersect at $O(0,0)$ and meet the line $x+y=1$ at the points

$$A(1/3,2/3), B(1/4,3/4) \text{ and } C(-1/3,4/3)$$

So,

$$OA=\sqrt{\frac{1}{3^2}+\frac{2^2}{3^2}}=\sqrt{\frac{5}{9}}$$

$$OB=\sqrt{\frac{1}{4^2}+\frac{3^2}{4^2}}=\sqrt{\frac{10}{16}}$$

$$OC=\sqrt{\frac{1}{3^2}+\frac{4^2}{3^2}}=\sqrt{\frac{17}{9}}$$

Therefore,

$$OA^2+OB^2+OC^2=\frac{5}{9}+\frac{10}{16}+\frac{17}{9}=\frac{221}{72}$$

Hence, the correct answer is option (D).

4. If p_1, p_2 denote the lengths of the perpendiculars from the point (2, 3) on the lines given by $15x^2+31xy+14y^2=0$, then if $p_1>p_2$, $p_1^2+\frac{1}{74}-p_2^2+\frac{1}{13}$ is equal to

(A) –2 **(B)** 0
(C) 2 **(D)** None of these

Solution: The lines given by $15x^2+31xy+14y^2=0$ are $5x+7y=0$ and $3x+2y=0$.

Length of the perpendiculars from (2, 3) on these lines are

$$p_1=\frac{31}{\sqrt{74}} \quad \text{and} \quad p_2=\frac{12}{\sqrt{13}} \quad [\text{since } p_1>p_2]$$

So that

$$p_1^2+\frac{1}{74}-p_2^2+\frac{1}{13}=\frac{961}{74}+\frac{1}{74}-\left(\frac{144}{13}-\frac{1}{13}\right)=2$$

Hence, the correct answer is option (C).

5. If a line passing through the point $P(2, 3)$ meets the lines represented by $x^2-2xy-y^2=0$ at the points A and B such that $PA\cdot PB=17$, then the equation of the line is

(A) $x=2$ **(B)** $y=3$
(C) $3x-2y=0$ **(D)** None of these

Solution: Let the equation of the line through $P(2,3)$ making an angle θ with the positive direction of x-axis be $\frac{x-2}{\cos\theta}=\frac{y-3}{\sin\theta}$. Then the coordinates of any point on this line at a distance r from P are $(2+r\cos\theta, 3+r\sin\theta)$.

General point $(2+r\cos\theta, 3+r\sin\theta)$ will satisfy the equation $x^2-2xy-y^2=0$, so

$$(2+r\cos\theta)^2-2(2+r\cos\theta)(3+r\sin\theta)-(3+r\sin\theta)^2=0$$

$$\Rightarrow r^2(\cos2\theta-\sin2\theta)-2r(\cos\theta+5\sin\theta)-17=0$$

If $PA=r_1$ and $PB=r_2$, then r_1, r_2 are the roots of the equation.

$$17=PA\cdot PB=\frac{17}{\cos2\theta-\sin2\theta}$$

and $$\Rightarrow \cos2\theta-\sin2\theta=1,$$

which is satisfied by $\theta=0$. Thus the equation of the line is $y=3$.

Hence, the correct answer is option (B).

6. Let PQR be a right-angled isosceles triangle, right-angled at P(2,1). If the equation of the line QR is $2x+y=3$, then the equation representing the pair of lines PQ and PR is

(A) $3x^2-3y^2+8xy+20x+10y+25=0$

(B) $3x^2-3y^2+8xy-20x-10y+25=0$

(C) $3x^2-3y^2+8xy+10x+15y+20=0$

(D) $3x^2-3y^2-8xy-10x-15y-20=0$

Solution: The equations of PQ and PR are given by

$$y-y_1=\frac{m\pm\tan\theta}{1\mp m\tan\theta}(x-x_1)$$

$$\Rightarrow y-1=\frac{-2\pm\tan45^\circ}{1\pm(-2)\tan45^\circ}(x-2)$$

$$\Rightarrow y-1=\left(\frac{-2\pm1}{1\pm2}\right)(x-2) \Rightarrow y-1=-\frac{1}{3}(x-2) \Rightarrow y-1=3(x-2)$$

$$\Rightarrow \quad x+3y-5=0 \text{ and } 3x-y-5=0$$

The combined equation of these two lines is

$$(x+3y-5)\,(3x-y-5)=0$$

$$\Rightarrow 3x^2-3y^2+8xy-20x-10y+25=0$$

Hence, the correct answer is option (B).

7. The equation $\sqrt{(x-2)^2+y^2}+\sqrt{(x+2)^2+y^2}=4$ represents a

(A) Circle **(B)** Pair of straight lines
(C) Parabola **(D)** Ellipse

Solution:

$$\sqrt{(x-2)^2+y^2}+\sqrt{(x+2)^2+y^2}=4$$

$$\Rightarrow \sqrt{x^2+y^2+4-4x}+\sqrt{x^2+y^2+4+4x}=4$$

$$\Rightarrow \sqrt{k-4x}+\sqrt{k+4x}=4, \text{ (where } k=x^2+y^2+4)$$

$$\Rightarrow \sqrt{k-4x}=4-\sqrt{k+4x}$$

Squaring both sides,

$$k-4x=16+k+4x-8\sqrt{k+4x} \Rightarrow x+2=\sqrt{k+4x}$$

Squaring again

$$x^2+4x+4=k+4x \Rightarrow y^2=0$$

Squaring, and on simplification, it reduces to $y^2=0$ and equation is a pair of two coincident straight lines.

Hence, the correct answer is option (B).

8. If the equation $x^2+y^2+2gx+2fy+1=0$ represents a pair of straight lines, then

(A) $g^2-f^2=1$ **(B)** $f^2-g^2=1$

(C) $g^2+f^2=1$ **(D)** $f^2+g^2=\frac{1}{2}$

Solution: Comparing the given equation with the standard equation, we get $a=1$, $b=1$, $h=0$ and $c=1$. We also know that the condition for the general equation, $ax^2+2hxy+by^2+2gx+2fy+c=0$ to represent a straight line is $abc+2fgh-af^2-bg^2-ch^2=0$

$$\Rightarrow (1\times1\times1)+(2\times f\times g\times 0)-(1\times f^2)-(1\times g^2)-(1\times 0)=0$$

$$\Rightarrow 1-f^2-g^2=0$$

$$\Rightarrow f^2+g^2=1$$

Hence, the correct answer is option (C).

9. If the pair of straight lines $xy-x-y+1=0$ and the line $ax+2y-3=0$ are concurrent, then the value of a is

(A) -1 **(B)** 0

(C) 3 **(D)** 1

Solution: Given that the equation of pair of straight lines $xy-x-y+1=0$

$$\Rightarrow (x-1)(y-1)=0$$

$$\Rightarrow x-1=0 \text{ or } y-1=0$$

Therefore, the intersection point is (1, 1).

Since lines $x-1=0$, $y-1=0$ and $ax+2y-3=0$ are concurrent. Therefore, the intersecting point of first two lines satisfies the third line.

Hence,

$$a+2-3=0 \Rightarrow a=1$$

Hence, the correct answer is option (D).

10. The area of the triangle formed by the line $4x^2-9xy-9y^2=0$ and $x=2$ is

(A) 2 **(B)** 3

(C) 10/3 **(D)** 20/3

Solution: We have

$$4x^2-9xy-9y^2=0$$

$$\Rightarrow 4x^2-12xy+3xy-9y^2=0$$

$$\Rightarrow 4x(x-3y)+3y(x-3y)=0$$

$$\Rightarrow (4x+3y)(x-3y)=0$$

$$\Rightarrow 4x+3y=0 \text{ or } x-3y=0$$

Thus, the lines represented by the given homogeneous equations are $4x+3y=0$ and $x-3y=0$. So, the sides of triangle are $4x+3y=0$, $x-3y=0$ and $x=2$.

Solving these equations, we obtain the vertices of triangle $A(2,-8/3)$, $B(2,2/3)$, $C(0,0)$.

Hence, the area of triangle ABC is

$$\frac{1}{2}\left|x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right|$$

$$=\frac{1}{2}\left[2\left(\frac{2}{3}\right)+2\left(\frac{8}{3}\right)+0\right]=\frac{10}{3}$$

Hence, the correct answer is option (C).

11. If the equations of opposite sides of a parallelogram are $x^2-7x+6=0$ and $y^2-14y+40=0$, then the equation of its one diagonal is

(A) $6x+5y+14=0$ **(B)** $6x-5y+14=0$

(C) $5x+6y+14=0$ **(D)** $5x-6y+14=0$

Solution: The lines represented by $x^2-7x+6=0$ are $x=6$ and $x=1$. Similarly, the lines represented by $y^2-14y+40=0$ are $y=10$ and $y=4$.

Therefore, the equation of one diagonal is

$$y-4=\frac{10-4}{6-1}(x-1)$$

$$\Rightarrow y-4=\frac{6}{5}(x-1)$$

$$\Rightarrow 6x-5y+14=0$$

Hence, the correct answer is option (B).

12. The image of the pair of lines represented by $ax^2+2hxy+by^2=0$ by the line mirror $y=0$ is

(A) $ax^2-2hxy-by^2=0$ **(B)** $bx^2-2hxy+ay^2=0$

(C) $bx^2+2hxy+ay^2=0$ **(D)** $ax^2-2hxy+by^2=0$

Solution: Let $y=m_1x$ and $y=m_2x$ be the lines represented by $ax^2+2hxy+by^2=0$. Then their images in $y=0$ are $y=-m_1x$ and $y=-m_2x$. So, their combined equation is

$$y^2+m_1m_2x^2+xy(m_1+m_2)=0$$

$$\Rightarrow y^2+\frac{a}{b}x^2+xy\left(-\frac{2h}{b}\right)=0$$

$$\left(\text{since } m_1+m_2=\frac{-2h}{b}, m_1m_2=\frac{a}{b}\right)$$

$$\Rightarrow ax^2-2hxy+by^2=0$$

Hence, the correct answer is option (D).

13. If the portion of the line $lx+my=1$ falling inside the circle $x^2+y^2=a^2$ subtends an angle of $45°$ at the origin, then

(A) $4[a^2(l^2+m^2)-1]=a^2(l^2+m^2)$

(B) $4[a^2(l^2+m^2)-1]=a^2(l^2+m^2)-2$

(C) $4[a^2(l^2+m^2)-1]=[a^2(l^2+m^2)-2]^2$

(D) None of these

Solution: Making the equation of circle homogeneous with the help of line $lx+my=1$, we get

$$x^2+y^2-a^2(lx+my)^2=0$$

$$\Rightarrow (a^2l^2-1)x^2+(a^2m^2-1)y^2+2a^2lmxy=0$$

Now,

$$\tan 45° = \frac{2\sqrt{(a^2lm)^2-(a^2l^2-1)(a^2m^2-1)}}{a^2l^2+a^2m^2-2}$$
$$\Rightarrow 2\sqrt{a^2l^2+a^2m^2-1} = a^2l^2+a^2m^2-2$$

On squaring both sides, we get

$$4[a^2(l^2+m^2)-1]=[a^2(l^2+m^2)-2]^2$$

Hence, the correct answer is option (C).

14. The angle between lines joining the origin to the points of intersection of the line $x\sqrt{3}+y=2$ and the curve $y^2+x^2=4$ is

(A) $\pi/6$ **(B)** $\pi/4$
(C) $\pi/3$ **(D)** $\pi/2$

Solution: The equation of pair of straight lines joining the origin to their points of intersection is

$$y^2+x^2=4\left[\frac{x\sqrt{3}+y}{2}\right]^2=4\left[\frac{(\sqrt{3}x+y)^2}{4}\right]$$
$$\Rightarrow y^2+x^2=3x^2+y^2+2\sqrt{3}xy \Rightarrow 2x^2+2\sqrt{3}xy=0$$

Therefore, if α be the required angle, then

$$\tan\alpha=\frac{2\sqrt{h^2-ab}}{a+b}=\frac{2\sqrt{(\sqrt{3})^2-0}}{2}=\sqrt{3} \Rightarrow \alpha=\frac{\pi}{3}$$

Hence, the correct answer is option (C).

15. Mixed term xy is to be removed from the general equation $ax^2+by^2+2hxy+2fy+2gx+c=0$. One should rotate the axes through an angle θ given by $\tan 2\theta$ equal to

(A) $\frac{a-b}{2h}$ **(B)** $\frac{2h}{a+b}$

(C) $\frac{a+b}{2h}$ **(D)** $\frac{2h}{(a-b)}$

Solution: Let (x', y') be the coordinates on new axes. Then

$$x=x'\cos\theta-y'\sin\theta$$
$$y=x'\sin\theta+y'\cos\theta$$

Put the value of x and y in

$$ax^2+by^2+2hxy+2gx+2fy+c=0$$

we get,

$$a(x'\cos\theta-y'\sin\theta)^2+b(x'\sin\theta+y'\cos\theta)^2$$
$$+2h(x'\cos\theta-y'\sin\theta)(x'\sin\theta+y'\cos\theta)$$
$$+2g(x'\cos\theta-y'\sin\theta)+2f(x'\sin\theta+y'\cos\theta)+c=0$$

Now,

Coefficient of $xy=0$
$$\Rightarrow 2(b-a)\sin\theta\cos\theta+2h\cos 2\theta=0$$
$$\Rightarrow \tan 2\theta=\frac{2h}{a-b}$$

Hence, the correct answer is option (D).

16. Locus of the points equidistant from the lines represented by $x^2\cos^2\theta-xy\sin^2\theta-y^2\sin^2\theta=0$ is

(A) $x^2+y^2+2xy\sec^2\theta=0$

(B) $x^2+y^2+2xy\,\mathrm{cosec}^2\theta=0$

(C) $x^2-y^2+2xy\sec^2\theta=0$

(D) $x^2-y^2+2xy\,\mathrm{cosec}^2\theta=0$

Solution: We know that the point lying on the bisector of angle between the lines represented by any curve is always equidistant from the lines. Therefore, equation of bisectors will be the required locus. Thus,

$$\frac{x^2-y^2}{xy}=\frac{a-b}{-h}$$
$$\Rightarrow \frac{x^2-y^2}{xy}=\frac{\cos^2\theta+\sin^2\theta}{\frac{-\sin^2\theta}{2}}$$
$$\Rightarrow \frac{x^2-y^2}{xy}=\frac{1}{\frac{-\sin^2\theta}{2}} \Rightarrow x^2-y^2+2xy\,\mathrm{cosec}^2\theta=0$$

Hence, the correct answer is option (D).

17. If the pair of lines $ax^2+2hxy+by^2+2gx+2fy+c=0$ intersect on the y-axis, then

(A) $2fgh=bg^2+ch^2$ **(B)** $bg^2\neq ch^2$
(C) $abc=2fgh$ **(D)** None of these

Solution:

$$f(x,y)=ax^2+by^2+2hxy+2gx+2fy+c=0$$

Points of intersection of lines

$$\frac{\partial f(x,y)}{\partial x}=0$$
$$\Rightarrow 2ax+2hy+2g=0$$

Since $x=0$ intersects on y-axis, we have

$$y=-g/h$$

Thus, putting $x=0$ and $y=-g/h$ in $f(x,y)$, we get

$$\frac{bg^2}{h^2}+2f(-g/h)+c=0$$
$$\Rightarrow bg^2+ch^2=2fgh$$

Hence, the correct answer is option (A).

Previous Years' Solved JEE Main/AIEEE Questions

1. If one of the lines of $my^2+(1-m^2)xy-mx^2=0$ is a bisector of the angle between the lines $xy=0$, then m is

(A) $-1/2$ **(B)** -2
(C) 1 **(D)** 2

[AIEEE 2007]

Solution: Equation of bisectors of lines $xy = 0$ are $y = \pm x$.

These lines are contained in the pair of lines,

$$my^2 + (1-m^2)xy - mx^2 = 0$$

Substituting, $y = \pm x$ in $my^2 + (1-m^2)xy - mx^2 = 0$, we get

$$(1-m^2)x^2 = 0$$
$$\Rightarrow m = \pm 1$$

Hence, the correct answer is option (C).

2. The lines $p(p^2+1)x - y + q = 0$ and $(p^2+1)^2x + (p^2+1)y + 2q = 0$ are perpendicular to a common line for

(A) no value of p
(B) exactly one value of p
(C) exactly two values of p
(D) more than two values of p

[AIEEE 2009]

Solution: If the lines are perpendicular to a common line, then lines must be parallel. Therefore, the slopes are equal. Thus,

$$p(p^2+1) = -(p^2+1) \Rightarrow p = -1$$

Hence, the correct answer is option (B).

3. The line L given by $\frac{x}{5} + \frac{y}{b} = 1$ passes through the point (13, 32). The line K is parallel to L and has the equation $\frac{x}{c} + \frac{y}{3} = 1$. Then the distance between L and K is

(A) $\sqrt{17}$ **(B)** $\frac{17}{\sqrt{15}}$

(C) $\frac{23}{\sqrt{17}}$ **(D)** $\frac{23}{\sqrt{15}}$

[AIEEE 2010]

Solution:

$$\text{Slope of line } L = -\frac{b}{5}$$

$$\text{Slope of line } K = -\frac{3}{c}$$

Line L is parallel to line K. Therefore,

$$\frac{b}{5} = \frac{3}{c} \Rightarrow bc = 15$$

(13, 32) is a point on L. Therefore,

$$\frac{13}{5} + \frac{32}{b} = 1 \Rightarrow \frac{32}{b} = -\frac{8}{5} \Rightarrow b = -20 \Rightarrow c = -\frac{3}{4}$$

Thus, equation of K is $y - 4x = 3$, and equation of L is $y - 4x = -20$. So, the distance between L and K is

$$\frac{|3-(-20)|}{\sqrt{17}} = \frac{23}{\sqrt{17}}$$

Hence, the correct answer is option (C).

4. The lines L_1, $y - x = 0$ and L_2, $2x + y = 0$ intersect the line L_3, $y + 2 = 0$ at P and Q, respectively. The bisector of the acute angle between L_1 and L_2 intersect L_3 at R.

Statement-1: The ratio $PR : RQ$ equals $2\sqrt{2} : \sqrt{5}$.

Statement-2: In any triangle, bisector of an angle divides the triangle into two similar triangles.

(A) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.
(B) Statement-1 is true, Statement-2 is false.
(C) Statement-1 is false, Statement-2 is true.
(D) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

[AIEEE 2011]

Solution: See Fig. 11.3.

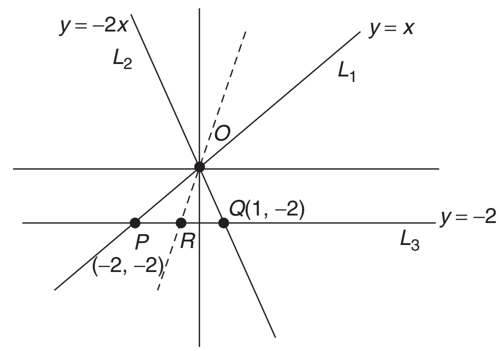


Figure 11.3

$\frac{|OP|}{|OQ|} = \frac{|PR|}{|RQ|}$ (Since bisector divides the side PQ in the ratio of the sides $OP : OQ$)

$$\Rightarrow \frac{|OP|}{|OQ|} = \frac{|PR|}{|RQ|} = \frac{\sqrt{8}}{\sqrt{5}} = \frac{2\sqrt{2}}{\sqrt{5}}$$

Hence, the correct answer is option (B).

5. If the line $2x + y = k$ passes through the point which divides the line segment joining the points (1, 1) and (2, 4) in the ratio 3:2, then k equals

(A) $\frac{29}{5}$ **(B)** 5

(C) 6 **(D)** $\frac{11}{5}$

[AIEEE 2012]

Solution:

$$x = \frac{3\times 2 + 2\times 1}{3+2}, y = \frac{3\times 4 + 2\times 1}{3+2}$$

$$\Rightarrow x = \frac{8}{5},\ y = \frac{14}{5}$$

Putting values of x and y, we get

$$2x + y = k \Rightarrow \frac{16}{5} + \frac{14}{5} = k \Rightarrow k = \frac{30}{5} = 6$$

Hence, the correct answer is option (C).

6. A ray of light along $x + \sqrt{3}y = \sqrt{3}$ gets reflected upon reaching x-axis, the equation of the reflected ray is

(A) $\sqrt{3}y = x - \sqrt{3}$ **(B)** $y = \sqrt{3}x - \sqrt{3}$

(C) $\sqrt{3}y = x - 1$ **(D)** $y = x + \sqrt{3}$

[JEE MAIN 2013]

Solution: Let us consider a point $(\sqrt{3}, 0)$ on the line, that is, on the ray of light. So, this point also lies on the image of this line (Fig. 11.4). Thus, the equation of the reflected ray is

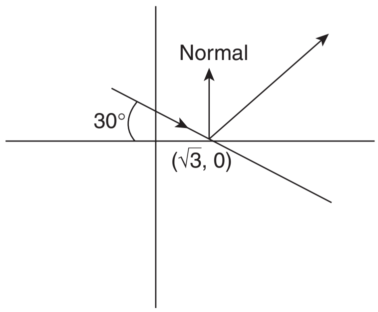


Figure 11.4

$$(y-0)=(\tan 30°)(x-\sqrt{3})$$
$$\Rightarrow y=\frac{1}{\sqrt{3}}(x-\sqrt{3})$$
$$\Rightarrow y\sqrt{3}=x-\sqrt{3}$$
$$\Rightarrow x-\sqrt{3}y=\sqrt{3}$$

Therefore, the equation of reflected ray is $\sqrt{3}y=x-\sqrt{3}$.

Hence, the correct answer is option (A).

7. The number of values of k, for which the system of equations

$$(k+1)x+8y=4k, \text{ and}$$
$$kx+(k+3)y=3k-1$$

has no solution is

(A) 1 **(B)** 2
(C) 3 **(D)** infinite

[JEE MAIN 2013]

Solution: For no solution the lines must be parallel for which, we have

$$\frac{k+1}{k}=\frac{8}{k+3}\neq\frac{4}{3k-1}\Rightarrow k^2+4k+3=8k\Rightarrow k^2-4k+3=0\Rightarrow k=1,3$$

$k=1$ is rejected since it gives coincident lines. Therefore, number of such values of k is just one.

Hence, the correct answer is option (A).

8. Let PS be the median of the triangle with vertices $P(2, 2)$, $Q(6, -1)$ and $R(7, 3)$. The equation of the line passing through $(1, -1)$ and parallel to PS is

(A) $4x + 7y + 3 = 0$ **(B)** $2x - 9y - 11 = 0$
(C) $4x - 7y - 11 = 0$ **(D)** $2x + 9y + 7 = 0$

[JEE MAIN 2014 (OFFLINE)]

Solution: See Fig. 11.5.

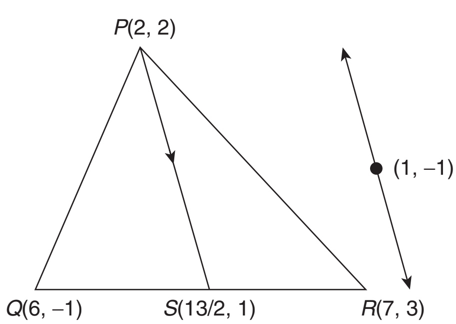


Figure 11.5

Slope of required line is

$$\frac{1}{(-9/2)}=-\frac{2}{9}$$

So, the equation is

$$y+1=-\frac{2}{9}(x-1)\Rightarrow 9y+9=-2x+2 \text{ or } 2x+9y+7=0$$

Hence, the correct answer is option (D).

9. The base of an equilateral triangle is along the line given by $3x + 4y = 9$. If a vertex of the triangle is $(1, 2)$, then the length of a side of the triangle is

(A) $\frac{2\sqrt{3}}{15}$ **(B)** $\frac{4\sqrt{3}}{15}$
(C) $\frac{4\sqrt{3}}{5}$ **(D)** $\frac{2\sqrt{3}}{5}$

[JEE MAIN 2014 (ONLINE SET-2)]

Solution: See Fig. 11.6.

Height of triangle, $h=\frac{|3(1)+4(2)-9|}{\sqrt{3^2+4^2}}=\frac{2}{\sqrt{25}}=\frac{2}{5}$

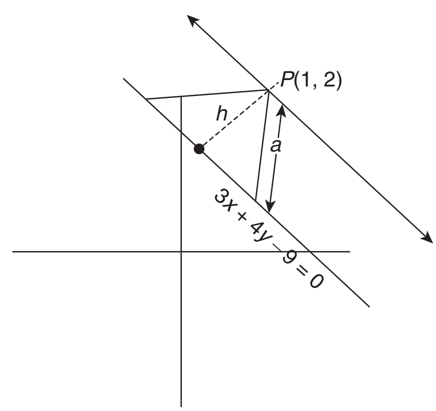


Figure 11.6

$$\text{Area}=\frac{1}{2}a\times h=\frac{\sqrt{3}a^2}{4}$$

Therefore,

$$\frac{a}{2}\times\frac{2}{5}=\frac{\sqrt{3}a^2}{4}\Rightarrow a=\frac{4}{5\sqrt{3}}=\frac{4}{5}\times\frac{\sqrt{3}}{3}=\frac{4\sqrt{3}}{15}$$

Hence, the correct answer is option (B).

10. If the three distinct lines $x + 2ay + a = 0$, $x + 3by + b = 0$ and $x + 4ay + a = 0$ are concurrent, then the point (a, b) lies on a

(A) circle **(B)** hyperbola
(C) straight line **(D)** parabola

[JEE MAIN 2014 (ONLINE SET-3)]

Solution:

Applying condition of concurrency, we have

$$\begin{vmatrix}1 & 2a & a\\ 1 & 3b & b\\ 1 & 4a & a\end{vmatrix}=0\Rightarrow 1(3ab-4ab)-2a(a-b)+a(4a-3b)=0$$

$\Rightarrow -ab - 2a^2 + 2ab + 4a^2 - 3ab = 0 \Rightarrow 2a^2 - 2ab = 0 \Rightarrow a(a-b) = 0$
Therefore, locus of (a, b) is a pair of lines $x = 0$ and $x = y$.
Hence, the correct answer is option (C).

11. Let L be the line passing through the point $P(1, 2)$ such that its intercepted segment between the coordinate axes is bisected at P. If L_1 is the line perpendicular to L and passing through the point $(-2, 1)$, then the point of intersection of L and L_1 is

(A) $\left(\frac{4}{5}, \frac{12}{5}\right)$ **(B)** $\left(\frac{11}{20}, \frac{29}{10}\right)$
(C) $\left(\frac{3}{10}, \frac{17}{5}\right)$ **(D)** $\left(\frac{3}{5}, \frac{23}{10}\right)$

[JEE MAIN 2015 (ONLINE SET-1)]

Solution:
L line passing through $P(1, 2)$ be
$$\frac{x}{a} + \frac{y}{b} = 1 \Rightarrow \frac{1}{a} + \frac{2}{b} = 1 \quad (1)$$
Also line intercepted between the axes is bisected at $P(1, 2)$. So,
$$\frac{a}{2} = 1 \text{ and } \frac{b}{2} = 2 \Rightarrow a = 2, b = 4$$
Therefore, the line is
$$\frac{x}{2} + \frac{y}{4} = 1 \quad (2)$$
Line L_1 perpendicular to (2) and passing through $(-2, 1)$ be
$$y = \frac{1}{2}x + c \Rightarrow 1 = -1 + c \Rightarrow c = 2$$
That is, L_1 is
$$y = \frac{1}{2}x + 2 \quad (3)$$
Therefore, point of intersection of Eqs. (2) and (3) is $\left(\frac{4}{5}, \frac{12}{5}\right)$.
Hence, the correct answer is option (A).

12. A straight line L through the point $(3, -2)$ is inclined at an angle of 60° to the line $\sqrt{3}x + y = 1$. If L also intersects the x-axis, then the equation of L is

(A) $y + \sqrt{3}x + 2 - 3\sqrt{3} = 0$ **(B)** $y - \sqrt{3}x + 2 + 3\sqrt{3} = 0$
(C) $\sqrt{3}y - x + 3 + 2\sqrt{3} = 0$ **(D)** $\sqrt{3}y + x - 3 + 2\sqrt{3} = 0$

[JEE MAIN 2015 (ONLINE SET-2)]

Solution: See Fig. 11.7.

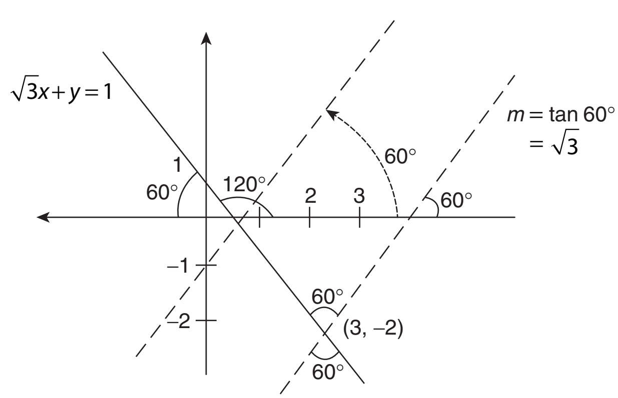


Figure 11.7

$\sqrt{3}x + y = 1$ has slope angle 120°. So, any line with inclination of 60° with above line has either slope angle = 180° (parallel to x-axis, not passing through origin, does not intersect x-axis) or has slope angle 60° which is required.
Therefore, its equation must be
$$(y + 2) = \tan 60° (x - 3) \Rightarrow y - \sqrt{3}x + 3\sqrt{3} + 2 = 0$$
Hence, the correct answer is option (B).

13. The point $(2, 1)$ is translated parallel to the line L, $x - y = 4$ by $2\sqrt{3}$ units. If the new point Q lies in the third quadrant, then the equation of the line passing through Q and perpendicular to L is

(A) $x + y = 2 - \sqrt{6}$ **(B)** $2x + 2y = 1 - \sqrt{6}$
(C) $x + y = 3 - 3\sqrt{6}$ **(D)** $x + y = 3 - 2\sqrt{6}$

[JEE MAIN 2016 (ONLINE SET-1)]

Solution: See Fig. 11.8. We have
$$\frac{x-2}{\cos(\pi/4)} = \frac{y-1}{\sin(\pi/4)} = -2\sqrt{3}$$
That is,
$$x = 2 - \sqrt{6}$$
$$y = 1 - \sqrt{6}$$

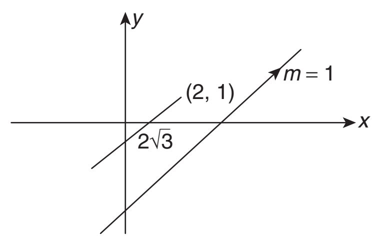


Figure 11.8

The perpendicular line is
$$x + y = \lambda$$
$$\Rightarrow 2 - \sqrt{6} + 1 - \sqrt{6} = \lambda$$
$$\Rightarrow x + y = 3 - 2\sqrt{6}$$
Hence, the correct answer is option (D).

14. A straight line through origin O meets the lines $3y = 10 - 4x$ and $8x + 6y + 5 = 0$ at points A and B, respectively. Then O divides the segment AB in the ratio

(A) 2:3 **(B)** 1:2
(C) 4:1 **(D)** 3:4

[JEE MAIN 2016 (ONLINE SET-2)]

Solution: (See Fig. 11.9). The line that passes through the origin is
$$y = mx$$
Now,
$$3y = 10 - 4x$$
$$\Rightarrow 3y + 4x = 10$$
Substituting, $y = mx$, we get
$$\Rightarrow 3mx + 4x = 10$$
$$\Rightarrow x(4 + 3m) = 10$$
$$\Rightarrow x = \frac{10}{3m+4}, y = \frac{10m}{3m+4}$$

Therefore, the point A is given by

$$A = \left(\frac{10}{3m+4}, \frac{10m}{3m+4}\right)$$

Similarly,

$$8x + 6mx + 5 = 0$$

$$\Rightarrow x(8 + 6m) = -5$$

$$\Rightarrow x = \frac{-5}{8+6m},\ y = \frac{-5m}{8+6m}$$

and the point B is given by

$$B = \left(\frac{-5}{8+6m}, \frac{-5m}{8+6m}\right)$$

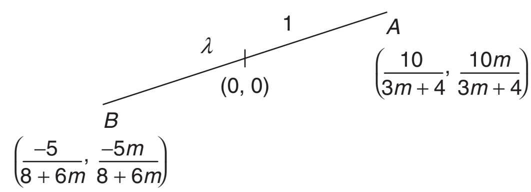


Figure 11.9

Now, by section formula

$$\frac{[10\lambda/(3m+4)] - [5/(8+6m)]}{\lambda+1} = 0$$

$$\Rightarrow \frac{10\lambda}{3m+4} - \frac{5}{8+6m} = 0$$

$$\Rightarrow \frac{20\lambda}{6m+8} - \frac{5}{8+6m} = 0 \Rightarrow \lambda = \frac{1}{4}$$

Hence, the origin divides the segment AB in the ratio 4:1.

Hence, the correct answer is option (C).

15. A ray of light is incident along a line which meets another line, $7x - y + 1 = 0$, at the point (0, 1). The ray is then reflected from this point along the line, $y + 2x = 1$. Then the equation of the line of incidence of the ray of light is

(A) $41x - 25y + 25 = 0$ **(B)** $41x + 25y - 25 = 0$
(C) $41x - 38y + 38 = 0$ **(D)** $41x + 38y - 38 = 0$

[JEE MAIN 2016 (ONLINE SET-2)]

Solution: See Fig. 11.10. We have

$$\frac{x}{(-1/7)} + \frac{y}{1} = 1$$

$$\Rightarrow \frac{x}{(1/2)} + y = 1$$

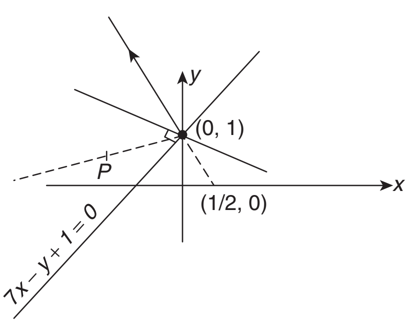


Figure 11.10

The image of the point $\left(\frac{1}{2}, 0\right)$ lies on the incident ray. Therefore,

$$\frac{x-(1/2)}{7} = \frac{y-0}{-1} = \frac{-2[(7/2)+1]}{50} = -\frac{9}{50}$$

Therefore,

$$x - \frac{1}{2} = \frac{-63}{50}$$

$$\Rightarrow x = \frac{1}{2} - \frac{63}{50} = \frac{25-63}{50} = \frac{-38}{50} = \frac{-19}{25}$$

and $$y = \frac{9}{50}$$

Thus, we get the point P as $P\left(\frac{-38}{50}, \frac{9}{50}\right)$. The equation of incident ray is

$$y - 1 = \left[\frac{(9/50)-1}{(-38/50)-0}\right] \times (x - 0) = \left(\frac{41}{38}\right)x$$

So, the equation of the line of incidence of the ray of light is $41x - 38y + 38 = 0$.

Hence, the correct answer is option (C).

Previous Years' Solved JEE Advanced/ IIT-JEE Questions

1. Consider the lines given by

$$L_1 : x + 3y - 5 = 0$$
$$L_2 : 3x - ky - 1 = 0$$
$$L_3 : 5x + 2y - 12 = 0$$

Match the statements/expressions in Column I with the values given in Column II.

Column I	Column II
(A) L_1, L_2, L_3 are concurrent, if	**(P)** $k = -9$
(B) One of L_1, L_2, L_3 is parallel to at least one of the other two, if	**(Q)** $k = -\frac{6}{5}$
(C) L_1, L_2, L_3 form a triangle, if	**(R)** $k = \frac{5}{6}$
(D) L_1, L_2, L_3 do not form a triangle, if	**(S)** $k = 5$

[IIT-JEE 2008]

Solution: We have

$$L_1 : x + 3y - 5 = 0$$
$$L_2 : 3x - ky - 1 = 0$$
$$L_3 : 5x + 2y - 12 = 0$$

Point of intersection of L_1 and L_3 is (2, 1).

If lines are concurrent, then (2, 1) will satisfy L_2

$$\Rightarrow 3(2) - k(1) - 1 = 0 \Rightarrow k = 5$$

(A) → (S)

If L_1 and L_2 are parallel

$$\frac{3}{1}=\frac{-k}{3}\neq\frac{-1}{-5}\Rightarrow k=-9$$

If L_2 and L_3 are parallel

$$\frac{3}{5}=\frac{-k}{2}\Rightarrow k=\frac{-6}{5}$$

(B) → (P, Q)

As lines form a triangle, they cannot be concurrent and no two of them are parallel.

$$\Rightarrow k\neq 5,-9,\frac{-6}{5}$$

$$\Rightarrow k=\frac{5}{6}$$

(C) → (R)

If the lines do not form a triangle, then

$$k=5,-9,\frac{-6}{5}$$

(D) → (P, Q, S)

Hence, the correct matches are (A)→(S); (B)→(P, Q); (C)→(R); (D)→(P, Q, S).

2. For $a > b > c > 0$, the distance between (1, 1) and the point of intersection of the lines $ax + by + c = 0$ and $bx + ay + c = 0$ is less than $2\sqrt{2}$. Then

(A) $a + b - c > 0$ **(B)** $a - b + c < 0$
(C) $a - b + c > 0$ **(D)** $a + b - c < 0$

[JEE ADVANCED 2013]

Solution: We know that

$$ax+by+c=0 \quad (1)$$

$$bx+ay+c=0 \quad (2)$$

Solving, we get

$$x=\frac{-c}{a+b}$$

From Eqs. (1) and (2), we get $y = x$. That is, the point of intersection lies on $y = x$. This implies that

$$y=\frac{-c}{a+b}$$

It is given that

$$\sqrt{\left(1+\frac{c}{a+b}\right)^2+\left(1+\frac{c}{a+b}\right)^2}<2\sqrt{2}$$

That is,

$$\sqrt{2}\left(1+\frac{c}{a+b}\right)<2\sqrt{2}$$

$$\Rightarrow \frac{a+b+c}{a+b}<2$$

$$\Rightarrow a+b+c<2a+2b$$

$$\Rightarrow a+b-c>0$$

Hence, the correct answer is option (A).

Practice Exercise 1

1. The value of λ so that $3x^2 + 7xy + \lambda y^2 + 5x + 5y + 2 = 0$ represents a pair of straight lines is

(A) 3 **(B)** –3
(C) 2 **(D)** –2

2. If the equation $4x^2 + 4xy + y^2 - 6x - 3y - 4 = 0$ represents a pair of parallel line, then the distance between these lines is

(A) $2\sqrt{5}$ **(B)** $\sqrt{5}$
(C) $2/\sqrt{5}$ **(D)** $3/\sqrt{5}$

3. The point of intersection of lines represented by $3x^2 + 10xy + 8y^2 + 14x + 22y + 15 = 0$ is

(A) (–1, –2) **(B)** (1, – 2)
(C) (1, 2) **(D)** (–2, 1)

4. The centroid of the triangle whose sides are of the equations $12x^2 - 20xy + 7y^2 = 0$ and $2x - 3y + 4 = 0$ is

(A) $\left(\frac{8}{3},\frac{8}{3}\right)$ **(B)** $\left(\frac{4}{3},\frac{4}{3}\right)$
(C) (2, 2) **(D)** (1, 1)

5. The equations of the diagonals of the square formed by the pairs of straight lines $3x^2 + 8xy - 3y^2 = 0$ and $3x^2 + 8xy - 3y^2 + 2x - 4y - 1 = 0$ are

(A) $x = 2y, 4x + 2y + 1 = 0$
(B) $2x + y = 0, 2x = 4y + 1$
(C) $x = 2y, 2x = 4y + 1$
(D) $2x + y = 0, 4x + 2y + 1 = 0$

6. The area of triangle formed by the lines $18x^2 - 9xy + y^2 = 0$ and the line $y = 9$ is

(A) 27/4 **(B)** 27/2
(C) 27/8 **(D)** 27

7. The equation of the lines which are passing through the origin and which are perpendicular to the pair of straight lines $x^2 - 3xy + 2y^2 = 0$ is

(A) $2x^2 - 3xy + y^2 = 0$ **(B)** $2x^2 + 3xy - y^2 = 0$
(C) $2x^2 + 3xy + y^2 = 0$ **(D)** $2x^2 - 3xy - y^2 = 0$

8. The value of h for which $3x^2 - 3hxy + 4y^2 = 0$ represents a pair of coincident lines are

(A) $\pm 3/\sqrt{3}$ **(B)** ±3
(C) $\pm 2/\sqrt{3}$ **(D)** $\pm 4/\sqrt{3}$

9. If the pairs of lines $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ are such that each pair bisects the angle between the other pair, then find the value of pq.

(A) –1 **(B)** 1
(C) 2 **(D)** –2

10. The line $ax + 3y = 5$ cuts the ellipse $5x^2 + 4y^2 = 10$ at points R and S. Find a so that OR is perpendicular to OS.

(A) $a = \pm 5\sqrt{3}/\sqrt{2}$ **(B)** $a = \pm 4\sqrt{3}/\sqrt{2}$
(C) $a = \pm 3\sqrt{3}/\sqrt{2}$ **(D)** None of these

11. The equation $x^2 - 4y^2 - 2x + 8y - 3 = 0$ and $4x^2 - y^2 - 16x + 4y + 12 = 0$ form only

(A) Square **(B)** Rectangle
(C) Rhombus **(D)** Cyclic quadrilateral

12. Without using the condition $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$, find the equation $9x^2 + y^2 - 6xy + 42x - 14y + 50 = 0$ represents
(A) A pair of straight lines
(B) An ellipse
(C) A circle
(D) None of these

13. If a general equation of second degree represents a pair of parallel straight lines, then the condition, that is not fulfilled, is
(A) $\Delta = 0$ **(B)** $h^2 = ab$
(C) $ag^2 = bf^2$ **(D)** $bg^2 = af^2$

14. The angle between the bisectors of the lines represented by $6x^2 - xy - 12y^2 - 8x + 29y - 14 = 0$ is
(A) $\tan^{-1}(6/17)$ **(B)** $\tan^{-1}(7/16)$
(C) $\tan^{-1}(17/6)$ **(D)** $\cot^{-1}(0)$

15. If the equation $12x^2 + 7xy - py^2 - 18x + qy + 6 = 0$ represents a pair of perpendicular straight lines, then
(A) $p = 12, q = 1$ **(B)** $p = 1, q = 12$
(C) $p = -1, q = 12$ **(D)** $p = 1, q = -12$

16. If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of parallel lines, then
(A) $\frac{a}{h} = \frac{b}{h} = \frac{f}{g}$ **(B)** $\frac{a}{h} = \frac{h}{b} = \frac{f}{g}$
(C) $\frac{a}{h} = \frac{h}{b} = \frac{g}{f}$ **(D)** None of these

17. The equation $x^2 + 2\sqrt{2}xy + 2y^2 + 4x + 4\sqrt{2}y + 1 = 0$ represents a pair of lines. The distance between them is
(A) 4 **(B)** $4/\sqrt{3}$
(C) 2 **(D)** None of these

18. The angle between the straight line $x^2 - y^2 - 2y - 1 = 0$ is
(A) 90° **(B)** 60°
(C) 75° **(D)** 36°

19. For what value of λ, the equation $\lambda x^2 + 2xy + \lambda y^2 + 4x + 4y + 3 = 0$ represents a pair of straight line?
(A) –1 **(B)** 1
(C) 0 **(D)** Any real value

20. The angle θ between the pairs of straight lines represented by $6x^2 - xy - 12y^2 - 8x + 29y - 14 = 0$ is such that $\tan\theta$ equals
(A) 6/17 **(B)** 7/16
(C) 17/6 **(D)** 16/7

21. The equation to the pair of straight lines through origin and perpendicular to lines $ax^2 + 2hxy + by^2 = 0$ is
(A) $bx^2 + 2hxy + ay^2 = 0$
(B) $ax^2 - 2hxy + by^2 = 0$
(C) $bx^2 - 2hxy + ay^2 = 0$
(D) Any of these

22. The equation $8x^2 + 8xy + 2y^2 + 26x + 13y + 15 = 0$ represents
(A) A pair of intersecting straight lines
(B) A pair of parallel straight lines
(C) A parabola
(D) A pair of skew straight lines

23. If one of the lines given by $ax^2 + 2hxy + by^2 = 0$ bisects the angle between coordinate axes, then
(A) $(a + b)^2 = 4h^2$ **(B)** $(a + b)^2 = h^2$
(C) $(a + b)^2 = 2h^2$ **(D)** $(a+b)^2 = h^2/y$

24. The distance between parallel lines $8x^2 + 8xy + 2y^2 + 26x + 13y + 15 = 0$ is
(A) $\frac{7\sqrt{5}}{2}$ **(B)** $7\sqrt{5}$
(C) 7/2 **(D)** $\frac{7\sqrt{5}}{10}$

25. The equations $3x^2 - 8xy - 3y^2 = 0$ and $x + 2y = 3$ together represent
(A) An equilateral triangle **(B)** A point
(C) An isosceles triangle **(D)** A scalene triangle

26. If slope of one line is λ times the other, when the lines are represented by $ax^2 + 2hxy + by^2 = 0$, then find the value of $(\lambda + 1)^2/\lambda$.
(A) h^2/ab **(B)** $2h^2/ab$
(C) $4h^2/ab$ **(D)** Any multiple of h^2/ab

27. The difference of the tangents of the angles which the lines $x^2(\sec^2\theta - \sin^2\theta) - 2xy\tan\theta + y^2\sin^2\theta = 0$ make with the x-axis
(A) $2\tan\theta$ **(B)** 2
(C) $2\cot\theta$ **(D)** $\sin 2\theta$

28. If the straight lines joining the points of intersection of the curve $3x^2 + 8xy + 2y^2 + 5x = 0$ and the curve $5x^2 + 16xy + 3y^2 + 2gx = 0$ to the origin are mutually perpendicular, then g equals
(A) 5 **(B)** 4
(C) 1 **(D)** 0

29. If the lines $2x^2 + 6xy + y^2 = 0$ are inclined at the same angle to the lines $4x^2 + 2hxy + y^2 = 0$, then find the value of h.
(A) 18 **(B)** 6
(C) 9 **(D)** Any real value

30. The combined equation of bisectors of the angle between the lines joining origin to the intersection points of line $x + y + 2 = 0$ with $x^2 + xy + y^2 + x + 3y + 1 = 0$ is
(A) $x^2 + y^2 = 4xy$ **(B)** $x^2 - y^2 = 4xy$
(C) $y^2 - x^2 = 4xy$ **(D)** $x^2 + y^2 + 4xy = 0$

31. Two lines are given by the equation $(3x - 4y)^2 + k(3x - 4y) = 0$. One of the values of k, so that the distance between the lines is 3 is
(A) 3 **(B)** 5
(C) –15 **(D)** None of these

32. The coordinates of the orthocentre of the triangle formed by the lines $2x^2 - 2y^2 + 3xy + 3x + y + 1 = 0$ and $3x + 2y + 1 = 0$ are
(A) $\left(\frac{4}{5}, \frac{3}{5}\right)$ **(B)** $\left(\frac{-3}{5}, \frac{-1}{5}\right)$
(C) $\left(\frac{1}{5}, \frac{-4}{5}\right)$ **(D)** $\left(\frac{2}{5}, \frac{1}{5}\right)$

33. The four lines represented by $12x^2 + 7xy - 12y^2 = 0$ and $12x^2 + 7xy - 12y^2 - x + 7y - 1 = 0$ form

(A) a square **(B)** a rectangle
(C) a rhombus **(D)** a parallelogram

34. The point of intersection of the lines given by the equation $x^2 - 5xy + 4y^2 + x + 2y = 2$ is (x_1, y_1) such that

(A) $y_1 = 2x_1$ **(B)** $x_1 = 2y_1$
(C) $x_1 = y_1$ **(D)** $x_1 = y_1^2$

35. The angle between straight lines passing through (1, 0) and parallel to straight lines $2x^2 - xy - 6y^2 + 7x + 21y = 15$ is

(A) $\tan^{-1}(2/7)$ **(B)** $\tan^{-1}(4/7)$
(C) $\tan^{-1}(7/4)$ **(D)** $\tan^{-1}(7/2)$

Practice Exercise 2

Single/Multiple Correct Choice Type Questions

1. The equation of the pair of straight lines parallel to the x-axis and touching the circle $x^2 + y^2 - 6x - 4y - 12 = 0$ is

(A) $y^2 - 4y - 21 = 0$ **(B)** $y^2 + 4y - 21 = 0$
(C) $y^2 - 4y + 21 = 0$ **(D)** $y^2 + 4y + 21 = 0$

2. If two of the lines represented by the equation $ay^4 + bxy^3 + cx^2y^2 + dx^3y + ex^4 = 0$ are perpendicular, then

(A) $(b+d)(ad+be)+(e-a)^2(a+c+e)=0$
(B) $(b+d)(ad+be)+(e+a)^2(a+c+e)=0$
(C) $(b-d)(ad-be)+(e-a)^2(a+c+e)=0$
(D) $(b-d)(ad-be)+(e+a)^2(a+c+e)=0$

3. The lines represented by the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ will be equidistant from the origin, if

(A) $f^2 + g^2 = c(b-a)$ **(B)** $f^4 + g^4 = c(bf^2 + ag^2)$
(C) $f^4 - g^4 = c(bf^2 - ag^2)$ **(D)** $f^2 + g^2 = af^2 + bg^2$

4. The circumcentre of the triangle formed by the lines $xy + 2x + 2y + 4 = 0$ and $x + y + 2 = 0$ is

(A) (0, 0) **(B)** (–2, –2)
(C) (–1, –1) **(D)** (–1, –2)

5. If the bisectors of the lines $x^2 - 2pxy - y^2 = 0$ be $x^2 - 2qxy - y^2 = 0$, then

(A) $pq + 1 = 0$ **(B)** $pq - 1 = 0$
(C) $p + q = 0$ **(D)** $p - q = 0$

6. The angle between the pair of straight lines $y^2\sin^2\theta - xy\sin^2\theta + x^2(\cos^2\theta - 1) = 1$ is

(A) $\frac{\pi}{3}$ **(B)** $\frac{\pi}{4}$
(C) $\frac{2\pi}{3}$ **(D)** None of these

7. The lines joining the origin to the point of intersection of the circle $x^2 + y^2 = 3$ and the line $x + y = 2$ are

(A) $y - (3+2\sqrt{2})x = 0$ **(B)** $x - (3+2\sqrt{2})y = 0$
(C) $x - (3-2\sqrt{2})y = 0$ **(D)** $y - (3-2\sqrt{2})x = 0$

Matrix Match Type Question

8. The line $y = x$ makes

Intercept	Curve
(A) $4\sqrt{2}$	**(i)** $x^2/3 + y^2 = 1$
(B) 4	**(ii)** $x^2 - y^2/3 = 1$
(C) $\sqrt{6}$	**(iii)** $x^2 + y^2 = 4$
(D) $2\sqrt{3}$	**(iv)** $y^2 = 4x$

Integer Type Questions

9. Find the area bounded by the angle bisectors of the lines $x^2 - y^2 + 2y = 1$ and the line $x + y = 3$.

10. If the equation $\lambda x^2 + 2y^2 - 5xy + 5x - 7y + 3 = 0$ represents two straight lines, then find the value of λ.

11. The equation $x^2 + ky^2 + 4xy = 0$ represents two coincident lines, if $k = n$. Find the value of n.

12. If $2x^2 + 7xy + 3y^2 + 8x + 14y + \lambda = 0$ represents a pair of straight lines, find λ.

13. If $Lx^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$ represents a pair of straight lines, find the value of L.

14. If the slope of one line of the pair of lines represented by $ax^2 + 4xy + y^2 = 0$ is 3 times the slope of the other line, then find a.

15. If the angle between the pair of straight lines represented by the equation $x^2 - 3xy + \lambda y^2 + 3x - 5y + 2 = 0$ is $\tan^{-1}\left(\frac{1}{3}\right)$, where '$\lambda$' is a non-negative real number, find λ.

Answer Key

Practice Exercise 1

1. (C)	**2.** (B)	**3.** (B)	**4.** (A)	**5.** (B)	**6.** (A)
7. (C)	**8.** (D)	**9.** (A)	**10.** (C)	**11.** (D)	**12.** (D)
13. (C)	**14.** (D)	**15.** (A)	**16.** (C)	**17.** (C)	**18.** (A)

19. (B)	**20.** (C)	**21.** (C)	**22.** (B)	**23.** (A)	**24.** (D)
25. (C)	**26.** (C)	**27.** (B)	**28.** (B)	**29.** (C)	**30.** (C)
31. (C)	**32.** (B)	**33.** (A)	**34.** (B)	**35.** (C)	

Practice Exercise 2

1. (A)	**2.** (A)	**3.** (C)	**4.** (C)	**5.** (A)	**6.** (D)
7. (A), (B), (C), (D)	**8.** (A)→(iv), (B)→(iii), (C)→(i), (D)→(ii)		**9.** 2	**10.** 2	**11.** 4
12. 8	**13.** 2	**14.** 3	**15.** 2		

Solutions

Practice Exercise 1

1. We know that $x^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Now, for the equation $3x^2 + 7xy + ly^2 + 5x + 5y + 2 = 0$, we have

$$\begin{vmatrix} 3 & 7/2 & 5/2 \\ 7/2 & \lambda & 5/2 \\ 5/2 & 5/2 & 2 \end{vmatrix} = 0$$

$$\Rightarrow 3\left(2\lambda - \frac{25}{4}\right) - \frac{7}{2}\left(7 - \frac{25}{4}\right) + \frac{5}{2}\left(\frac{35}{4} - \frac{5\lambda}{2}\right) = 0$$

$$\Rightarrow 6\lambda - \frac{75}{4} - \frac{49}{2} + \frac{175}{8} + \frac{175}{8} - \frac{25\lambda}{4} = 0$$

$$\Rightarrow 6\lambda - \frac{75}{4} - \frac{98}{4} + \frac{175}{4} - \frac{25\lambda}{4} = 0$$

$$\Rightarrow -\frac{\lambda}{4} + \frac{2}{4} = 0$$

$$\Rightarrow \frac{\lambda}{4} = \frac{1}{2}$$

$$\Rightarrow \lambda = 2$$

2. The formula for the distance between two parallel lines is

$$D = 2\sqrt{\frac{g^2 - ac}{a(a+b)}}$$

Now, for the equation $4x^2 + 4xy + y^2 - 6x - 3y - 4 = 0$, we have $a = 4$, $f = -3/2$, $b = 1$, $g = -3$, $c = -4$ and $h = 2$. So,

$$D = 2\sqrt{\frac{9 - 4\times(-4)}{4(4+1)}}$$

$$= 2\sqrt{\frac{9+16}{20}} = 2\sqrt{\frac{25}{20}} = \sqrt{5}$$

3. The point of intersections of the lines is the point of intersection of

$$\frac{\partial f}{\partial x} = 6x + 10y + 14 = 0; \quad \frac{\partial f}{\partial y} = 10x + 16y + 22 = 0$$

Now,

$$5(3x + 5y + 7 = 0) \quad (1)$$

$$3(5x + 8y + 11 = 0) \quad (2)$$

Using Eqs. (1) and (2) and simplifying further, we get

$$y + 2 = 0 \Rightarrow y = -2$$

Put $y = -2$ in Eq. (1)

$$3x - 10 + 7 = 0$$
$$3x = 3$$
$$x = 1$$

Thus, the point of intersection is (1, –2).

4. We have

$$12x^2 - 20xy + 7y^2 = 0$$
$$\Rightarrow 12x^2 - 14xy - 6xy + 7y^2 = 0$$
$$\Rightarrow 2x(6x - 7y) - y(6x - 7y) = 0$$
$$\Rightarrow (6x - 7y)(2x - y) = 0$$

Hence, the lines are

$$\begin{cases} L_1 & 6x - 7y = 0 \\ L_2 & 2x - y = 0 \\ L_3 & 2x - 3y + 4 = 0 \end{cases}$$

The points of intersections are as follows: L_1, L_2 is (0, 0); L_2, L_3 is (1, 2) and L_3, L_1 is (7, 6). The centroid is

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

$$= \left(\frac{0+1+7}{3}, \frac{0+2+6}{3}\right) = \left(\frac{8}{3}, \frac{8}{3}\right)$$

5. The pairs of straight lines are $3x^2 + 8xy - 3y^2 = 0$ and $3x^2 + 8xy - 3y^2 + 2x - 4y - 1 = 0$. That is,

$$3x^2 + 8xy - 3y^2 = 0$$
$$\Rightarrow 3x^2 + 9xy - xy - 3y^2 = 0$$
$$\Rightarrow 3x(x + 3y) - y(x + 3y) = 0$$
$$\Rightarrow (x + 3y)(3x - y) = 0 \quad (1)$$

and

$$3x^2 + 8xy - 3y^2 + 2x - 4y - 1 = 0$$
$$\Rightarrow (x + 3y + \lambda_1)(3x - y + \lambda_2) = 0$$
$$\Rightarrow (x + 3y + 1)(3x - y - 1) = 0 \quad (2)$$

From Eqs. (1) and (2), we get the four sides of the square (Fig. 11.11).

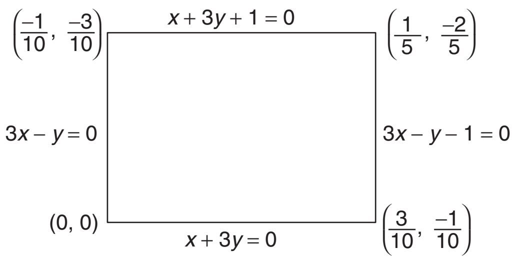


Figure 11.11

Now, we have

$$y-0=\left(\frac{-2/5}{1/5}\right)(x-0)$$

$$\Rightarrow y=-2x$$

$$\Rightarrow 2x+y=0$$

Also,

$$y+\frac{3}{10}=\frac{-(1/10)+(3/10)}{(3/10)+(1/10)}\left(x+\frac{1}{10}\right)$$

$$y+\frac{3}{10}=\frac{1}{2}\left(x+\frac{1}{10}\right)\Rightarrow 2y+\frac{6}{10}=x+\frac{1}{10}$$

$$\Rightarrow 20y+6=10x+1\Rightarrow 2y+\frac{1}{2}=x$$

$$\Rightarrow 2x-4y=1$$

6. The pair of straight lines are

$$18x^2-9xy+y^2=0$$

$$\Rightarrow 18x^2-6xy-3xy+y^2=0$$

$$\Rightarrow 6x(3x-y)-y(3x-y)=0$$

$$\Rightarrow (3x-y)(6x-y)=0$$

Thus, the lines are $y=3x$, $y=6x$ and $y=9$ (Fig. 11.12).

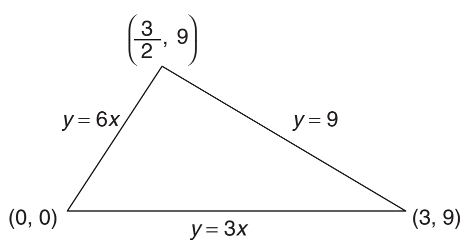


Figure 11.12

Therefore, the area of the triangle is

$$A=\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$A=\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ \frac{3}{2} & 9 & 1 \\ 3 & 9 & 1 \end{vmatrix}$$

$$=\frac{1}{2}\left[\frac{27}{2}-27\right]=\frac{1}{2}\left|-\frac{27}{2}\right|=\frac{27}{4}$$

7.

$$x^2-3xy+2y^2=0$$

$$x^2-2xy-xy+2y^2=0$$

$$\Rightarrow x(x-2y)-y(x-2y)=0$$

$$\Rightarrow (x-2y)(x-y)=0$$

$$\Rightarrow x=y \text{ and } x=2y$$

Here, $m_1=1$ and $m_2=1/2$. Now, the slope of the perpendicular lines are $m_3=-1$ and $m_4=-2$. So, the lines, which are passing through the origin and perpendicular to the pair of straight lines, are

$$y-0=-1(x-0)$$

$$y=-x$$

$$\Rightarrow x+y=0$$

and

$$y-0=-2(x-0)$$

$$2x+y=0$$

$$\Rightarrow (x+y)(2x+y)=0$$

$$2x^2+3xy+y^2=0$$

8. Here, $3x^2-3hxy+4y^2=0$ represents a pair of coincident lines if

$$h^2=ab$$

$$\Rightarrow \left(\frac{3h}{2}\right)^2=3\times 4$$

$$\Rightarrow \frac{9h^2}{4}=3\times 4$$

$$\Rightarrow 9h^2=3\times 16\Rightarrow h^2=\frac{16}{3}\Rightarrow h=\pm\frac{4}{\sqrt{3}}$$

9. We have

$$x^2-2pxy-y^2=0 \quad (1)$$

$$x^2-2qxy-y^2=0 \quad (2)$$

The equation to angle bisectors corresponding to the lines given by Eq. (1) are

$$\frac{x^2-y^2}{2}=\frac{xy}{-p}$$

$$\Rightarrow x^2-y^2+\left(\frac{2}{p}\right)xy=0 \quad (3)$$

Since Eqs. (2) and (3) represent the same pair of lines, on comparing the corresponding coefficients, we have

$$\frac{2}{p}=-2q \text{ or } pq=-1$$

10. The combined equations of OR and OS is

$$5x^2+4y^2-10\left(\frac{ax+3y}{5}\right)^2=0$$

Now,

$$125x^2+100y^2-10(a^2x^2+9y^2+6axy)=0$$

For OR and OS to be orthogonal, we have

$$\text{Coefficient of } x^2+\text{Coefficient of } y^2=0$$

$$(125-10a^2+100-90)=0$$

$$135=10a^2$$

Therefore,

$$a^2 = \frac{27}{2} \text{ or } a = \pm\frac{3\sqrt{3}}{\sqrt{2}}$$

11. Since $x^2 - 4y^2 = 0$ gives $x = 2y$ and $x = -2y$, the slopes of lines are 1/2 and –1/2, that is, $m_1 = 1/2$ and $m_2 = -1/2$.
Similarly, $4x^2 - y^2 = 0$ gives $2x = y$ and $2x = -y$ and hence the slopes of lines are 2 and –2, that is, $m_3 = 2$ and $m_4 = -2$.
Since no pair of opposite sides are parallel, these pairs do not form a square, rectangle or rhombus.
Since $m_1m_4 = -1$ and $m_2m_3 = -1$, the two opposite angles are 90° each and hence, forms a cyclic quadrilateral.

12. We have

$$9x^2 + y^2 - 6xy = (3x - y)(3x - y)$$

Let $(3x - y + r) = 0$ and $(3x - y + r') = 0$ may be the lines. Then, we have

$$(3x - y + r)(3x - y + r') = x^2 + y^2 - 6xy + 42x - 14y + 50$$

$$\Rightarrow r' + r = 14 \quad \text{(on comparing coefficients of } y)$$

and $rr' = 50$

This system has no solution. Hence, the given equation does not represent the pair of straight lines. Now,

$$h^2 = ab$$
$$(3)^2 = 9 \times 1$$

So, the equation represents a parabola.

13. If the second-degree general equation represents a parallel pair of straight lines, then we have

$$\Delta = 0, \qquad h^2 = ab$$

$$\Rightarrow \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$$\Rightarrow a[bc - f^2] - h[hc - gf] + g[hf - bg] = 0$$

$$\Rightarrow abc - af^2 - h^2c + hgf + hgf - bg^2 = 0$$

$$\Rightarrow abc - af^2 - abc + 2hgf - bg^2 = 0$$

$$\Rightarrow 2hgf = af^2 + bg^2$$

$$\Rightarrow 2\sqrt{ab}fg = af^2 + bg^2$$

$$\Rightarrow (\sqrt{a}f - \sqrt{b}g)^2 = 0$$

Therefore,

$$af^2 = bg^2$$

14. The angle between the angle bisectors is always 90°. Thus,

$$\frac{\pi}{2} = \cot^{-1}(0)$$

15. We have

$$12x^2 + 7xy - py^2 - 18x + qy + 6 = 0$$

The lines are perpendicular if

$$a + b = 0$$

$$\Rightarrow 12 - p = 0 \Rightarrow p = 12$$

For a pair of straight lines, we have

$$\Delta = 0$$

$$\Rightarrow \Delta = \begin{vmatrix} 12 & 7/2 & -9 \\ 7/2 & -12 & q/2 \\ -9 & q/2 & 6 \end{vmatrix} = 0$$

$$\Rightarrow 12\left[-72 - \frac{q^2}{4}\right] - \frac{7}{2}\left[21 + \frac{9q}{2}\right] - 9\left[\frac{7q}{4} - 108\right] = 0$$

$$\Rightarrow q = 1$$

16. If the lines are parallel, then $h^2 = ab$; $bg^2 = af^2$.

$$\Rightarrow \frac{a}{h} = \frac{h}{b} = \frac{g}{f}$$

17. We have

$$x^2 + 2\sqrt{2}xy + 2y^2 + 4x + 4\sqrt{2}y + 1 = 0$$

Now,

$$D = \left|2\sqrt{\frac{g^2 - ac}{a(a+b)}}\right|$$

where $a = 1$; $b = 2$; $g = 2$; $c = 1$. So,

$$D = \left|2\sqrt{\frac{4-1}{1(1+2)}}\right|$$

$$\Rightarrow D = \left|2\sqrt{\frac{3}{3}}\right| = 2$$

18. We have $x^2 - y^2 - 2y - 1 = 0$.
Here, $a + b = 0$ and hence, lines are perpendicular to each other.

19. We have $\lambda x^2 + 2xy + \lambda y^2 + 4x + 4y + 3 = 0$ which represents a pair of straight lines if

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

That is,

$$\Delta = \begin{vmatrix} \lambda & 1 & 2 \\ 1 & \lambda & 2 \\ 2 & 2 & 3 \end{vmatrix} = 0$$

$$\Rightarrow \lambda(3\lambda - 4) - 1(3 - 4) + 2(2 - 2\lambda) = 0$$

$$\Rightarrow 3\lambda^2 - 4\lambda + 1 + 4 - 4\lambda = 0$$

$$\Rightarrow 3\lambda^2 - 8\lambda + 5 = 0$$

$$\Rightarrow 3\lambda^2 - 5\lambda - 3\lambda + 5 = 0$$

$$\Rightarrow \lambda(3\lambda - 5) - 1(3\lambda - 5) = 0$$

Therefore, $\lambda = 5/3$ and $\lambda = 1$.

20. We have $ax^2+2hxy+by^2+2gx+2fy+c=0$. Now,

$$\tan\theta=\left|\frac{2\sqrt{h^2-ab}}{a+b}\right|$$

and

$$6x^2-xy-12y^2-8x+29y-14=0$$

Therefore,

$$\tan\theta=\left|\frac{2\sqrt{(1/2)^2+72}}{-6}\right|=\left|\frac{2\sqrt{1+288}}{-6}\right|=\frac{17}{6}$$

21. The pair of straight lines which are perpendicular to the line $ax^2+2hxy+by^2=0$ is $bx^2-2hxy+ay^2=0$.

22. We have

$$8x^2+8xy+2y^2+26x+13y+15=0$$

We know that $h^2=ab$ and hence, they are parallel lines.

Also, $$\Delta=\begin{vmatrix}8 & 4 & 13\\ 4 & 2 & 13/2\\ 13 & 13/2 & 15\end{vmatrix}=0$$

Thus, it represents a pair of parallel straight lines.

23. We have

$$ax^2+2hxy+by^2=0$$
$$\Rightarrow a+2h\left(\frac{y}{x}\right)+b\left(\frac{y}{x}\right)^2=0$$
$$\Rightarrow bm^2-2hm+a=0\ \left\{\frac{y}{x}=m\right.$$

Therefore, $m_1=1$ and $m_2=-1$.
Substituting $m=1$, we get

$$b-2h+a=0.$$

Substituting $m=-1$, we get $b+2h+a=0$. Therefore,

$$a+b=2h$$

and $$a+b=-2h$$

On squaring both these equation, we get

$$(a+b)^2=4h^2$$

24. The distance between two parallel lines. We have

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

Now, $$D=2\sqrt{\frac{g^2-ac}{a(a+b)}}$$

For $8x^2+8xy+2y^2+26x+13y+15=0$, we get

$$D=2\sqrt{\frac{13^2-8\times15}{8(8+2)}}=2\sqrt{\frac{169-120}{8\times10}}=\frac{2\times7}{\sqrt{80}}=\frac{7\sqrt{5}}{10}$$

25. We have the equations $3x^2-8xy-3y^2=0$ and $x+2y=3$

$$\Rightarrow 3x^2-9xy+xy-3y^2=0$$
$$\Rightarrow 3x(x-3y)+y(x-3y)=0$$
$$\Rightarrow (x-3y)(3x+y)=0$$

Now, $AB=BC$. Thus, it is an isosceles triangle (Fig. 11.13).

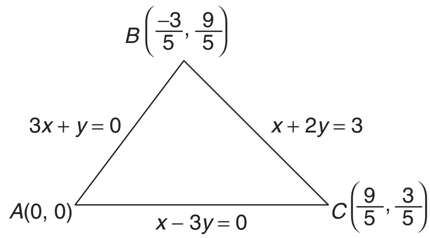


Figure 11.13

26. We have

$$ax^2+2hxy+by^2=0$$
$$\Rightarrow b\left(\frac{y}{x}\right)^2+2h\left(\frac{y}{x}\right)+a=0$$
$$\Rightarrow bm^2+2hm+a=0$$

Hence,

$$m_1+m_2=-\frac{2h}{b};\ m_1m_2=\frac{a}{b}$$

Now, $m_1=m$ and $m_2=\lambda m$.

$$\Rightarrow m+\lambda m=-\frac{2h}{b};\ m\times\lambda m=\frac{a}{b}$$
$$m(1+\lambda)=-\frac{2h}{b};\ m^2\times\lambda=\frac{a}{b} \qquad (1)$$

Therefore,

$$m=-\frac{2h}{b(1+\lambda)}$$

Now, substituting the value of m in Eq. (1), we get

$$\left(\frac{4h^2}{b^2(1+\lambda)^2}\right)\lambda=\frac{a}{b}$$
$$\Rightarrow \frac{(\lambda+1)^2}{\lambda}=\frac{4h^2}{ab}$$

27. We have

$$x^2(\sec^2\theta-\sin^2\theta)-2xy\tan\theta+y^2\sin^2\theta=0$$

Dividing both sides by x^2, we have

$$\Rightarrow\left(\frac{y}{x}\right)^2\sin^2\theta-2\left(\frac{y}{x}\right)\tan\theta+\sec^2\theta-\sin^2\theta=0$$

Now, $m_1+m_2=\dfrac{2\tan\theta}{\sin^2\theta}$, $m_1m_2=\dfrac{\sec^2\theta-\sin^2\theta}{\sin^2\theta}$

$$\Rightarrow (m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1m_2$$

$$= \frac{4\tan^2\theta}{\sin^4\theta} - 4\left(\frac{\sec^2\theta - \sin^2\theta}{\sin^2\theta}\right)$$

$$= \frac{4}{\sin^2\theta\cos^2\theta} - 4\left(\frac{1-\sin^2\theta\cos^2\theta}{\sin^2\theta\cos^2\theta}\right)$$

$$\Rightarrow (m_1 - m_2)^2 = 4$$

Therefore,

$$|m_1 - m_2| = 2$$

28. From the process of homogenisation, we get

$$3x^2 + 8xy + 2y^2 = -5x$$

$$\Rightarrow 1 = \frac{3x^2 + 8xy + 2y^2}{-5x}$$

Substituting in other equation

$$5x^2 + 16xy + 3y^2 + 2gx(1) = 0$$

$$\Rightarrow 5x^2 + 16xy + 3y^2 + 2gx\left(\frac{3x^2 + 8xy + 2y^2}{-5x}\right) = 0$$

$$\Rightarrow -25x^2 - 80xy - 15y^2 + 6gx^2 + 16gxy + 4gy^2 = 0$$

$$\Rightarrow x^2(6g - 25) + y^2(4g - 15) + xy(-80 + 16g) = 0$$

The lines are perpendicular if $a + b = 0$.

$$\Rightarrow 6g - 25 + 4g - 15 = 0$$

$$\Rightarrow 10g = 40$$

$$g = 4$$

29. The equations $2x^2 + 6xy + y^2 = 0$ and $4x^2 + 2hxy + y^2 = 0$ are equally inclined if their bisectors are the same. Hence, the bisectors of the given equation are

$$\frac{x^2 - y^2}{2-1} = \frac{xy}{3}, \quad \frac{x^2 - y^2}{4-1} = \frac{xy}{h}$$

$$\Rightarrow x^2 - y^2 = \frac{xy}{3}, \quad x^2 - y^2 = \frac{3}{h}xy$$

On comparing both equations, we get

$$\frac{1}{3} = \frac{3}{h} \Rightarrow h = 9$$

30. The combined equation of line joining origin to intersection of $x + y + 2 = 0$ with $x^2 + xy + y^2 + x + 3y + 1 = 0$ can be obtained by process of homogenisation.

$$x + y = -2 \Rightarrow \frac{x+y}{-2} = 1$$

That is,

$$x^2 + xy + y^2 + x(1) + 3y(1) + 1 = 0$$

$$\Rightarrow x^2 + xy + y^2 + x\frac{(x+y)}{-2} + 3y\frac{(x+y)}{-2} + \left(\frac{x+y}{-2}\right)^2 = 0$$

$$4x^2 + 4xy + 4y^2 - 2x^2 - 2xy - 6xy - 6y^2 + x^2 + y^2 + 2xy = 0$$

$$3x^2 - 2xy - y^2 = 0$$

The equation of angle bisectors is

$$\frac{x^2 - y^2}{3-(-1)} = \frac{xy}{-1}$$

$$\Rightarrow x^2 - y^2 = -4xy$$

$$\Rightarrow y^2 - x^2 = 4xy$$

31. We have $(3x - 4y)^2 + k(3x - 4y) = 0$

$$\Rightarrow (3x - 4y)(3x - 4y + k) = 0$$

Therefore,

$$D = \left|\frac{k-0}{\sqrt{3^2 + 4^2}}\right| = 3 \Rightarrow k = \pm 15$$

32. The orthocentre of a right-angled triangle is the point of right angle. The lines $2x^2 - 2y^2 + 3xy + 3x + y + 1 = 0$ are perpendicular so that the point of intersection is obtained by

$$\frac{\partial f}{\partial x} = 4x + 3y + 3 = 0 \Rightarrow 4x + 3y + 3 = 0$$

$$\frac{\partial f}{\partial y} = -4y + 3x + 1 = 0 \Rightarrow 3x - 4y + 1 = 0$$

Simplifying further, we get

$$\frac{x}{3+12} = \frac{-y}{4-9} = \frac{1}{-16-9}$$

$$\Rightarrow x = -\frac{15}{25},\ y = \frac{-5}{25} \Rightarrow x = -\frac{3}{5},\ y = -\frac{1}{5}$$

33. We have

$$12x^2 + 7xy - 12y^2 = 0$$

$$\Rightarrow 12x^2 + 16xy - 9xy - 12y^2 = 0$$

$$\Rightarrow 4x[3x + 4y] - 3y[3x + 4y] = 0$$

Therefore,

$$4x - 3y = 0;\ 3x + 4y = 0$$

Similarly, we have

$$12x^2 + 7xy - 12y^2 - x + 7y - 1 = 0$$

$$\Rightarrow (4x - 3y + 1)(3x + 4y - 1) = 0$$

(See Fig. 11.14). These lines form a square.

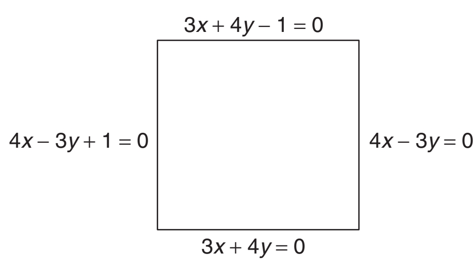


Figure 11.14

34. The point of intersection of lines can be obtained from

$$\frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0$$

Now, we have

$$x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$$

Therefore,

$$\frac{\partial f}{\partial x} = 2x - 5y + 1 = 0$$

$$\frac{\partial f}{\partial y} = -5x + 8y + 2 = 0$$

$$\frac{x}{-10-8} = \frac{-y}{4+5} = \frac{1}{16-25}$$

$$\Rightarrow \frac{21}{-18} = \frac{y}{-9} = \frac{1}{-9}$$

$$\Rightarrow x = 2; y = 1$$

Therefore, $x = 2y$.

35. The angle between the parallel lines remains the same. Thus,

$$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$$

$$\Rightarrow \tan\theta = \left|\frac{2\sqrt{1/4 - (2)(-6)}}{2-6}\right|$$

$$\Rightarrow \tan\theta = \left|\frac{7}{4}\right| = \frac{7}{4} \Rightarrow \theta = \tan^{-1}\left(\frac{7}{4}\right)$$

Practice Exercise 2

1. See Fig. 11.15. Let the lines be $y = m_1x + c_1$ and $y = m_2x + c_2$, since the pair of straight lines are parallel to the x-axis.

Therefore, $m_1 = m_2 = 0$

and the lines will be $y = c_1$ and $y = c_2$.

Given circle is $x^2 + y^2 - 6x - 4y - 12 = 0$, the centre (3, 2) and radius = 5.

Here, the perpendicular drawn from the centre to the lines are CP and CP'.

$$CP = \frac{2 - c_1}{\sqrt{1}} = \pm 5$$

$$\Rightarrow 2 - c_1 = \pm 5$$

$$\Rightarrow c_1 = 7 \text{ and } c_1 = -3$$

Hence, the lines are

$$y - 7 = 0, y + 3 = 0$$

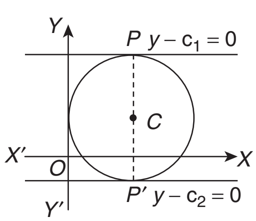


Figure 11.15

That is, $(y-7)(y+3) = 0$

Therefore, the pair of straight lines is $y^2 - 4y - 21 = 0$.

2. Let $ay^4 + bxy^3 + cx^2y^2 + dx^3y + ex^4 = (ax^2 + pxy - ay^2)(x^2 + \cdot qxy + y^2)$

Comparing the co-efficient of similar terms, we get

$$b = aq - p, c = -pq, d = aq + p, e = -a$$

Now,

$$b + d = 2aq, e - a = -2a$$

$$ad + be = 2ap, a + c + e = -pq$$

$$(b+d)(ad+be) = -(e-a)^2(a+c+e)$$

Therefore,

$$(b+d)(ad+eb) + (e-a)^2(a+c+e) = 0$$

3. Let the equations represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

be

$$lx + my + n = 0 \text{ and } l'x + m'y + n' = 0$$

Then the combined equation represented by these lines is given by $(lx + my + n)(l'x + m'y + n') = 0$.

So, it must be similar with the given equation.

On comparing, we get

$$ll' = a,\ mm' = b,\ nn' = c,\ lm' + ml' = 2h,\ ln' + l'n = 2g,\ mn' + nm' = 2f$$

According to the condition, the length of the perpendiculars drawn from the origin to the lines are the same.

$$\text{So, } \frac{n}{\sqrt{l^2 + m^2}} = \frac{n'}{\sqrt{l'^2 + m'^2}} = \frac{(nn')^2}{(l^2 + m^2)(l'^2 + m'^2)}$$

Now on eliminating l, m, l', m' and n, n', we get the required condition $f^4 - g^4 = c(bf^2 - ag^2)$.

4. The separate equations of the lines given by $xy + 2x + 2y + 4 = 0$ are $(x+2)(y+2) = 0$ or $x + 2 = 0$, $y + 2 = 0$. Solving the equations of the sides of the triangle, we obtain the coordinates of the vertices as $A(-2, 0)$, $B(0, -2)$ and $C(-2, -2)$. Clearly, ΔABC is a right-angled triangle with right angle at C. Therefore, the centre of the circum-circle is the midpoint of AB whose coordinates are $(-1, -1)$.

5. Bisector of the angle between the lines $x^2 - 2pxy - y^2 = 0$ is

$$\frac{x^2 - y^2}{xy} = \frac{1-(-1)}{-p}$$

$$\Rightarrow px^2 + 2xy - py^2 = 0$$

But it is represented by

$$x^2 - 2qxy - y^2 = 0$$

Therefore,

$$\frac{p}{1}=\frac{2}{-2q}\Rightarrow pq=-1$$

6. $\alpha=\tan^{-1}\left\{\frac{2\sqrt{\frac{\sin^2\theta}{4}-\sin^2\theta(\cos^2\theta-1)}}{\sin^2\theta+\cos^2\theta-1}\right\}=\tan^{-1}\infty\Rightarrow\alpha=\frac{\pi}{2}$

7. Homogenising the equation of the circle, we get

$$x^2-6xy+y^2=0$$

$$\Rightarrow x=\frac{6y\pm\sqrt{(36-4)y^2}}{2}=\frac{6y\pm4y\sqrt{2}}{2}=3y\pm2\sqrt{2}y$$

Hence, the equations are

$$x=(3+2\sqrt{2})y \text{ and } x=(3-2\sqrt{2})y$$

Also, after rationalizing, these equations become

$$y-(3+2\sqrt{2})x=0 \text{ and } y-(3-2\sqrt{2})x=0$$

8. Solving $y = x$ with each of the given curve we can find their intersection points and hence the lengths of intercepts are as follows:

(i) Solving $y = x$ and $\frac{x^2}{3}+y^2=1$, the points of intersection are $\left(\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}\right)$ and $\left(-\frac{\sqrt{3}}{2},-\frac{\sqrt{3}}{2}\right)$

$$\text{Distance, } D=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{(2x_1)^2+(2y_1)^2}$$

$$=\sqrt{4x_1^2+4y_1^2} \quad (\text{As } x_1=y_1)$$

$$=\sqrt{8x_1^2}=\sqrt{8\times\frac{3}{4}}=\sqrt{6}$$

Therefore, length of intercept is $\sqrt{6}$.

(ii) Solving $y = x$ and $x^2-\frac{y^2}{3}=1$, the points of intersection are $\left(\sqrt{\frac{3}{2}},\sqrt{\frac{3}{2}}\right)$ and $\left(-\sqrt{\frac{3}{2}},-\sqrt{\frac{3}{2}}\right)$

$$\text{Distance, } D=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$=\sqrt{8x^2}=\sqrt{8\times\frac{3}{2}}=\sqrt{12}=2\sqrt{3}$$

Therefore, length of intercept is $2\sqrt{3}$.

(iii) Solving $y = x$ and $x^2+y^2=4$, the points of intersection are $(\sqrt{2},\sqrt{2})$ and $(-\sqrt{2},-\sqrt{2})$

$$\text{Distance, } D=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$=\sqrt{8x^2}=\sqrt{8\times2}=\sqrt{16}=4$$

Therefore, length of intercept is 4.

(iv) Solving $y=x$ and $y^2=4x$, the points of intersection are (0, 0) and (4, 4)

$$\text{Distance, } D=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$=\sqrt{4^2+4^2}=4\sqrt{2}$$

Therefore, length of intercept is $4\sqrt{2}$.

9. See Fig. 11.16. The angle bisectors of the lines given by $x^2-y^2+2y=1$ are $x=0,\ y=1$.

Therefore, the required area $=\frac{1}{2}\times2\times2=2$

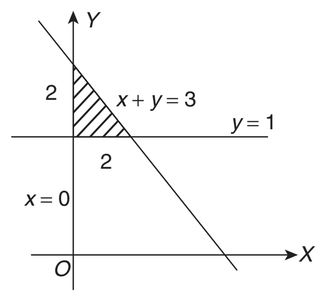


Figure 11.16

10. $abc+2fgh-af^2-bg^2-ch^2=0$

$$\Rightarrow\lambda(2)(3)+2\left(\frac{-7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{-5}{2}\right)-\lambda\left(\frac{-7}{2}\right)^2-2\left(\frac{5}{2}\right)^2-3\left(\frac{-5}{2}\right)^2=0$$

$$\Rightarrow6\lambda+\frac{175}{4}-\frac{49}{4}\lambda-\frac{25}{2}-\frac{75}{4}=0\Rightarrow25\lambda=50\Rightarrow\lambda=2$$

11. To represent a pair of coincident straight lines, $x^2+ky^2+4xy=0$ must be a perfect square. Therefore, $k=4$.

12. From the given equation, we have

$$(2)(3)(\lambda)+2(7)(4)\frac{7}{2}-2(7)^2-3(4)^2-\lambda\left(\frac{7}{2}\right)^2=0$$

$$\Rightarrow6\lambda+196-98-48-\frac{49\lambda}{4}=0\Rightarrow\lambda=8$$

13. The given equation represents a pair of straight lines, if

$$abc+2fgh-af^2-bg^2-ch^2=0$$

$$\Rightarrow-36L+200-64L-75+75=0$$

$$\Rightarrow-100L+200=0$$

$$\Rightarrow L=2$$

14. Here,

$$m_1 + m_2 = -4 \quad (1)$$

and

$$m_1 m_2 = a \quad (2)$$

Given that $m_1 = 3m_2$.
By Eq. (1), we have

$$3m_2 + m_2 = -4 \Rightarrow m_2 = -1$$

Hence, $m_1 = -3$.

Now, by Eq. (2), we have

$$a = 3$$

15. Given that

$$\theta = \tan^{-1}\left(\frac{1}{3}\right) \Rightarrow \tan\theta = \frac{1}{3}$$

Now,

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b} \Rightarrow \frac{1}{3} = \frac{2\sqrt{\left(\frac{-3}{2}\right)^2 - \lambda}}{\lambda + 1}$$

$$\Rightarrow (\lambda+1)^2 = 9(9-4\lambda) \Rightarrow \lambda^2 + 38\lambda - 80 = 0$$

$$\Rightarrow \lambda = \frac{-38 \pm \sqrt{(38)^2 + 320}}{2} \Rightarrow \lambda = \frac{-38 \pm 42}{2} \Rightarrow \lambda = 2$$

12 Circle

12.1 Standard Equation of a Circle

A circle is the locus of a point which moves in a plane such that its distance from a fixed point, called its **centre**, is always equal to a constant distance which is called **radius**. Thus, the equation of a circle with its centre (α, β) and radius a is expressed as

$$(x-\alpha)^2+(y-\beta)^2=a^2$$

If the centre is the origin, then the equation of the circle is given by

$$x^2+y^2=a^2$$

Illustration 12.1 Find the equation of shaded region of Fig. 12.1 in rectangular coordinates.

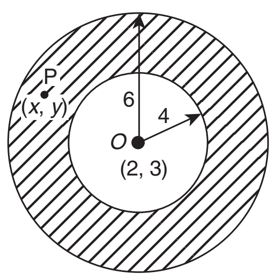


Figure 12.1

Solution: The centre of both circles is (2, 3) and the radii are 4 and 6, respectively. Thus, the equation for the shaded region between two circles is

$$4\le OP\le 6$$
$$\Rightarrow (4)^2\le(x-2)^2+(y-3)^2\le(6)^2$$

12.2 General Equation of a Circle

The general equation of a circle is

$$S\equiv x^2+y^2+2gx+2fy+c=0$$

which has three arbitrary constants. Its centre is ($-g$, $-f$) and the radius is $\sqrt{g^2+f^2-c}$.

The circle is real-point circle or imaginary according as $g^2+f^2-c>0, =0$ or <0. $c=0$ if the circle passes through the origin.

Illustration 12.2 Give the equation of a circle whose ordinate of centre is double of its abscissa and its radius is equal to the sum of ordinate and abscissa.

Solution: Let the centre be ($-g$, $-f$) and hence, the radius be $\sqrt{g^2+f^2-c}$. Now, according to the given condition $f=2g$ and

$$\sqrt{g^2+f^2-c}=3g$$

or $$g^2+f^2-c=9g^2$$

Therefore,

$$c=g^2+4g^2-9g^2=-4g^2$$

Thus, the equation of the given circle is

$$x^2+y^2+2gx+2fy+c=0$$
$$\Rightarrow x^2+y^2+2gx+4gy-4g^2=0$$

or $$(x+g)^2+(y+2g)^2=9g^2$$

12.3 General Equation of a Circle in Second Degree

The equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

shall represent a circle if the coefficient of x^2 and y^2 are equal, that is, $a=b$ and the coefficient of xy is zero, that is, $h=0$.

12.4 Different Forms of Equations of Circle

12.4.1 Parametric Form

A circle with centre (α, β) and radius a is expressed as

$$(x-\alpha)^2+(y-\beta)^2=a^2$$

It can be represented in parametric form as follows:

$$\left.\begin{aligned}x&=\alpha+a\cos\theta\\ y&=\beta+a\sin\theta\end{aligned}\right]\quad \begin{aligned}&\theta \text{ is parameter, such that}\\ &0\le\theta\le 2\pi\end{aligned}$$

A circle with its centre (0, 0) and radius a is expressed as follows:

$$\left.\begin{aligned}x&=a\cos\theta\\ y&=a\sin\theta\end{aligned}\right]\quad \begin{aligned}&\theta \text{ is parameter, such that}\\ &0\le\theta\le 2\pi\end{aligned}$$

Illustration 12.3 Represent the circle $x^2 + y^2 + 4x - 6y - 3 = 0$ in its parametric form.

Solution: We have

$$x^2 + y^2 + 4x - 6y - 3 = 0$$
$$(x + 2)^2 + (y - 3)^2 = 16$$

The centre is (–2, 3) and the radius is $a = 4$. Therefore,

$$x = -2 + 4\cos\theta, \text{ and}$$
$$y = 3 + 4\sin\theta$$

is the parametric form of the given circle.

Illustration 12.4 Parametrically, a circle is $x = -1 + 3\sin\theta$ and $y = -4 + 3\cos\theta$. Give the Cartesian equation of the circle and find its area.

Solution: We have

$$\sin\theta = \frac{x+1}{3}; \cos\theta = \frac{y+4}{3}$$
$$\Rightarrow \sin^2\theta + \cos^2\theta = 1$$

Therefore,

$$\frac{(x+1)^2}{9} + \frac{(y+4)^2}{9} = 1$$
$$\Rightarrow (x+1)^2 + (y+4)^2 = 9$$

which is the Cartesian equation of the circle. The centre is (–1, –4) and its radius is 3. Hence, the area is

$$\pi r^2 = (9\pi) \text{ sq. units}$$

Illustration 12.5 Find the shortest and the longest distance between the following two circles: (a) $x = \cos\theta$ and $y = \sin\theta$ and (b) $x = 3 + 2\cos\theta$; $y = 3 + 2\sin\theta$.

Solution: The shortest distance d between any two circles is $c_1c_2 - r_1 - r_2$ and the longest distance is $c_1c_2 + r_1 + r_2$.

$$c_1 = (0, 0); c_2 = (3, 3), r_1 = 1, r_2 = 2$$

Therefore,

$$c_1c_2 = \sqrt{9+9} = 3\sqrt{2}$$

Thus, the shortest distance between the circles is

$$3\sqrt{2} - 1 - 2 = 3(\sqrt{2} - 1)$$

and the longest distance between the circles is $3(\sqrt{2}+1)$.

Your Turn 1

1. Find the equation of the circle passing through the point of intersection of the lines $x + 3y = 0$ and $2x - 7y = 0$ and whose centre is the point of intersection of the lines $x + y + 1 = 0$ and $x - 2y + 4 = 0$. **Ans.** $x^2 + y^2 + 4x - 2y = 0$
2. Find the equation of the circle whose centre is (1, 2), and passes through the point (4, 6). **Ans.** $x^2 + y^2 - 2x - 4y - 20 = 0$
3. Find the equation of a circle whose radius is 6 and the centre is at the origin. **Ans.** $x^2 + y^2 = 36$
4. Find the equation of a circle which touches the axis of y at a distance of 3 units from the origin and intercepts a distance of 6 units on the axis of x. **Ans.** $x^2 + y^2 \pm 6\sqrt{2}x - 6y + 9 = 0$
5. Find the equation of a circle which touches y-axis at a distance of 2 units from the origin and cuts an intercept of 3 units with the positive direction of x-axis. **Ans.** $x^2 + y^2 \pm 5x - 4y + 4 = 0$

12.4.2 Equation of a Circle in Diametric Form

If two diametrically opposite points on a circle are (x_1, y_1) and (x_2, y_2) (Fig. 12.2), then the centre is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

and the radius is

$$\frac{1}{2}\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

The equation of circle is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

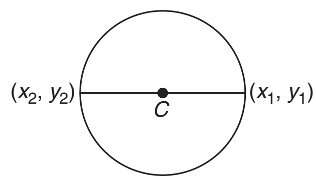


Figure 12.2

Illustration 12.6 A circle with centre at the origin and a point on its periphery is (3, 0). Find the equation of circle in diametric form.

Solution: Since the centre is (0, 0) and point A is (3, 0), the diametrically opposite point B is (–3, 0) (Fig. 12.3). So, the equation of the circle is

$$(x - 3)(x + 3) + (y - 0)(y - 0) = 0$$

or $$(x - 3)(x + 3) + y^2 = 0$$

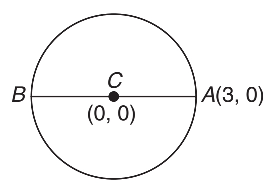


Figure 12.3

12.4.3 Equation of a Circle with Centre (α, β) and Touches x-Axis

As it is obvious from Fig. 12.4 that $r = \beta$, we have

$$(x - \alpha)^2 + (y - \beta)^2 = \beta^2$$

or $$x^2 + y^2 - 2\alpha x - 2\beta y + \alpha^2 = 0$$

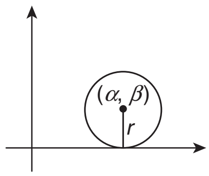


Figure 12.4

12.4.4 Equation of a Circle with Centre (α, β) and Touches y-Axis

Here, $r = \alpha$ (Fig. 12.5). Therefore,

$$(x-\alpha)^2 + (y-\beta)^2 = \alpha^2$$

$$\Rightarrow x^2 + y^2 - 2\alpha x - 2\beta y + \beta^2 = 0$$

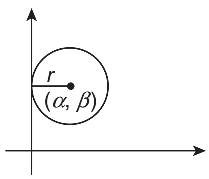


Figure 12.5

12.4.5 Equation of a Circle with Radius a and Touches both Axes

There are four circles (Fig. 12.6) and their centres are ($\pm a$, $\pm a$) and radius a. Therefore,

$$(x \pm a)^2 + (y \pm a)^2 = a^2$$

$$\Rightarrow x^2 + y^2 \pm 2ax \pm 2ay + a^2 = 0$$

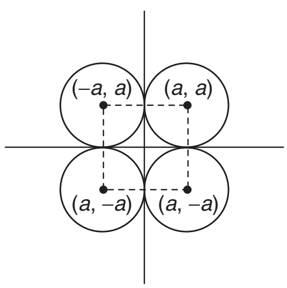


Figure 12.6

Illustration 12.7 Find the equation of a circle in third quadrant of radius 3 touching the y-axis at (0, –4).

Solution: As shown in Fig. 12.7, the required circle will have its centre as (–3, –4) and radius 3.

Hence, the equation of the circle is

$$(x+3)^2 + (y+4)^2 = 9$$

$$\Rightarrow x^2 + y^2 + 6x + 8y + 16 = 0$$

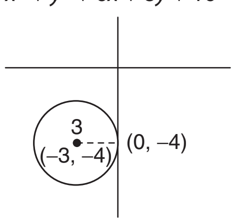


Figure 12.7

12.4.6 Circle Through Three Non-Collinear Points (x_1, y_1), (x_2, y_2) and (x_3, y_3)

Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$. Then all three points must satisfy the circle.

$$\left.\begin{aligned} x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \\ \Rightarrow x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0 \\ x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0 \end{aligned}\right]$$

Simultaneous solution of these three equations gives g, f and c and hence, we can write the equation of the given circle.

Alternate Method: The equation of circle is given by

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

Illustration 12.8 Give the equation of a circle passing through points (3, 0), (0, 3) and origin (0, 0).

Solution:

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ 9 & 3 & 0 & 1 \\ 9 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x^2+y^2 & x & y & 1 \\ 9 & 3 & 0 & 1 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 0 \quad R_3 - R_2$$

$$\Rightarrow \begin{vmatrix} x^2+y^2 & x & y & 1 \\ 9 & 3 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 0 \quad R_2 - R_4$$

$$\Rightarrow \begin{vmatrix} x^2+y^2 & x & y \\ 9 & 3 & 0 \\ 0 & -3 & 3 \end{vmatrix} = 0$$

$$\Rightarrow (x^2+y^2)(9) - x(27) + y(-27) = 0$$

Therefore, the equation of the given circle is

$$x^2 + y^2 - 3x - 3y = 0$$

12.4.7 Intercepts Made by a Circle on Axes

1. The intercept made on x-axis by $S \equiv 0$ is $2\sqrt{g^2 - c}$.
2. The intercept made on y-axis by $S \equiv 0$ is $2\sqrt{f^2 - c}$.
3. If $g^2 = c$ (or respective $f^2 = c$), the circle touches the x-axis (or respective y-axis). If $c = g^2 = f^2$, then the circle touches both the axis.

4. The length of the chord intercepted by $x^2 + y^2 = a^2$ on the line $y = mx + c$ is $2\sqrt{\dfrac{a^2(1+m^2)-c^2}{1+m^2}}$.
5. The intercepts are always positive.

Illustration 12.9 Give the equation of a circle passing through the origin and having intercept on x-axis as double of the intercept on y-axis and radius 4 units.

Solution: Since the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ passes through the origin, we have $c = 0$. Therefore, the x-intercept is $2|g|$ and the y-intercept is $2|f|$. Therefore,

$$2|g| = 2\times 2|f|$$
$$\Rightarrow |g| = 2|f|$$

That is, $g = \pm 2f$ and the radius is

$$4 = \sqrt{4f^2 + f^2 - 0}$$
$$\Rightarrow 16 = 5f^2$$

Therefore,

$$f = \pm\sqrt{\frac{16}{5}} = \frac{+4}{\sqrt{5}}$$

and

$$g = \pm\frac{8}{\sqrt{5}}$$

Hence, the circle is $x^2 + y^2 \pm \dfrac{16}{\sqrt{5}}x \pm \dfrac{8}{\sqrt{5}}y = 0$.

That is, four such circles should exist.

Your Turn 2

1. Find the parametric equations of the circle $x^2 + y^2 - 6x + 4y - 12 = 0$. **Ans.** $x = 3 + 5\cos\theta, y = -2 + 5\sin\theta$
2. Find the Cartesian equations of the curve $x = -2 + 3\cos\theta, y = 3 + 3\sin\theta$. **Ans.** $(x + 2)^2 + (y - 3)^2 = 9$

12.4.7.1 Position of a Point with Respect to a Circle

A point $P(x_1, y_1)$ lies inside, on or outside the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ according as $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ is $< 0, = 0$ or > 0.

12.4.8 Tangent to a Circle

Corresponding to any point $P(x_1, y_1)$, we define the following two expressions:

$$S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

$$T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$$

1. Equation of the tangent at (x_1, y_1) to $S \equiv 0$ is $T = 0$.
2. Equation of the normal at (x_1, y_1) to $S \equiv 0$ is $y(x_1 + g) - x(y_1 + f) + fx_1 - gy_1 = 0$.
3. The line $y = mx + c$ intersects the circle $x^2 + y^2 = a^2$ at
   - **(i)** real and distinct points if $c^2 < a^2(1 + m^2)$
   - **(ii)** real and coincident points if $c^2 = a^2(1 + m^2)$
   - **(iii)** imaginary points if $c^2 > a^2(1 + m^2)$

   Therefore, $y = mx + c$ is a tangent to $x^2 + y^2 = a^2$ if $c^2 = a^2(1+m^2)$.
4. The line $y = mx + a\sqrt{1+m^2}$ is tangent to $x^2 + y^2 = a^2$ at $\left(\dfrac{-am}{\sqrt{1+m^2}}, \dfrac{a}{\sqrt{1+m^2}}\right)$. The line $y = mx - a\sqrt{1+m^2}$ is tangent to $x^2 + y^2 = a^2$ at $\left(\dfrac{am}{\sqrt{1+m^2}}, \dfrac{-a}{\sqrt{1+m^2}}\right)$.
5. The length of the tangent from $P(x_1, y_1)$ to $S \equiv 0$ is equal to $\sqrt{S_1}$, where point P lies outside $S \equiv 0$.
6. The equation of the pair of tangents from $P(x_1, y_1)$ to $S \equiv 0$ is given by $SS_1 = T^2$.

Illustration 12.10 With respect to a circle $x^2 + y^2 + 4x - 6y + 8 = 0$, find the tangent passing through point (4, –2).

Solution: The point (4, –2) is lying outside the circle since

$$16 + 4 + 16 + 12 + 8 > 0$$

That is, $S_1 > 0$. A line passing through (4, –2) is $y + 2 = m(x - 4)$ or $y = mx - 4m - 2$ to be a tangent, the distance of line from the centre (–2, 3) should be equal to the radius $\sqrt{4+9-8} = \sqrt{5}$. Therefore,

$$\frac{-2m - 4m - 2 - 3}{\sqrt{1+m^2}} = \pm\sqrt{5}$$
$$\Rightarrow (6m + 5)^2 = 5 + 5m^2$$
$$\Rightarrow 36m^2 + 25 + 60m = 5 + 5m^2$$
$$\Rightarrow 31m^2 + 60m + 20 = 0$$
$$\Rightarrow m = \frac{-60 \pm \sqrt{3600 - 2480}}{62}$$
$$= \frac{-60 \pm \sqrt{1120}}{62}$$
$$= \frac{-30 \pm \sqrt{280}}{31} = \frac{-30 \pm 2\sqrt{70}}{31}$$

Therefore,

$$y = \left(\frac{-30 \pm 2\sqrt{70}}{31}\right)(x - 4) - 2$$

is the required equations of tangents.

12.4.9 Angle of Intersection of Two Circles

1. **Angle of intersection of two circles $S_1 = 0$ and $S_2 = 0$:** Let

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + C_1$$

and

$$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + C_2$$

If the centre of the two circles S_1 and S_2 be C_1 and C_2, respectively, and the radii be r_1 and r_2, respectively, then the angle of intersection θ of the circles is the angle between their tangents at points of intersection and is given by

$$\cos\theta = \frac{r_1^2 + r_2^2 - (C_1C_2)^2}{2r_1r_2} = \frac{2(g_1g_2 + f_1f_2) - C_1 - C_2}{2\sqrt{g_1^2 + f_1^2 - C_1}\cdot\sqrt{g_2^2 + f_2^2 - C_2}}$$

2. **Orthogonal circles:** The circles $S_1 \equiv 0$ and $S_2 \equiv 0$ are said to intersect orthogonally if $\theta = 90$, that is, if

$$2g_1g_2 + 2f_1f_2 = C_1 + C_2$$

Illustration 12.11 Find the equation of the circle passing through the origin and intersecting the circles orthogonally: $x^2 + y^2 + 2x + 4y + 2 = 0$ and $x^2 + y^2 + 4x + 6y - 3 = 0$.

Solution: Let the circle be $x^2 + y^2 + 2gx + 2fy = 0$. Since the circle intersects the given two circles orthogonally, we get

$$2(g)1 + 2(f)2 = 0 + 2$$

$$\Rightarrow g + 2f = 1 \quad (1)$$

and

$$2(g)2 + 2(f)3 = 0 - 3$$

$$\Rightarrow 4g + 6f + 3 = 0 \quad (2)$$

On solving Eqs. (1) and (2), we get

$$g = -6 \text{ and } f = \frac{7}{2}$$

Therefore, the required circle is $x^2 + y^2 - 12x + 7y = 0$.

12.4.10 Equation of a Circle Through Intersection Points of a Circle and a Line

Let the circle be $S = 0$ and the line be $L = 0$. Now, $S + \lambda L = 0$ is a circle which passes through the points of intersection of both the circle and the line.

1. The equation of any circle that passes through two given points (x_1, y_1) and (x_2, y_2) is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

2. Let $u \equiv ax + by + k = 0$. If $u \equiv 0$ is a tangent to $S \equiv 0$ at the point P, then $S + \lambda u \equiv 0$ is the equation of circles touching $S \equiv 0$ at P.
3. The equation of the circles which touch the line $u \equiv 0$ at (x_1, y_1) is $(x - x_1)^2 + (y - y_1)^2 + \lambda u \equiv 0$.

12.4.11 Equation of a Circle Through Intersection of Two Circles

1. If $S_1 \equiv 0$ and $S_2 \equiv 0$ be two circles, which intersect in real points, then $S_1 + \lambda S_2 = 0$ $(\lambda \neq -1)$ is the equation of the family of circles passing through the common points of $S_1 \equiv 0$ and $S_2 \equiv 0$.
2. If $S_1 \equiv 0$ and $S_2 \equiv 0$ intersect, then $S_1 - S_2 \equiv 0$ is the equation of their common chord.
3. If $S_1 \equiv 0$ and $S_2 \equiv 0$ touch each other, then $S_1 - S_2 \equiv 0$ is the equation of their common tangent at the point of contact.

12.4.12 Common Tangents to Two Circles

Let two circles $S_1 \equiv 0$ and $S_2 \equiv 0$ have centres O_1 and O_2, respectively, and radii r_1 and r_2, respectively.

1. The number of common tangents of $S_1 \equiv 0$ and $S_2 \equiv 0$ is '0' if one circle lies totally inside the other circle (Fig. 12.8), that is, $O_1O_2 < r_1 - r_2$.

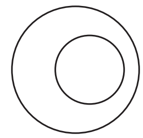


Figure 12.8

2. The number of common tangents of $S_1 \equiv 0$ and $S_2 \equiv 0$ is '1' if the circles touch each other internally (Fig. 12.9), that is, $O_1O_2 = |r_1 - r_2|$.

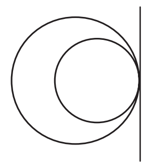


Figure 12.9

3. The number of common tangents of $S_1 \equiv 0$ and $S_2 \equiv 0$ is '2' if the circles intersect at two distinct points (Fig. 12.10), that is, $|r_1 - r_2| < O_1O_2 < r_1 + r_2$.

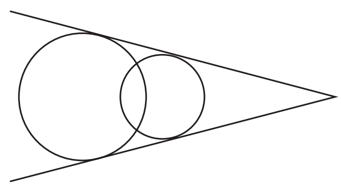


Figure 12.10

4. The number of common tangents of $S_1 \equiv 0$ and $S_2 \equiv 0$ is '3' if the circles touch each other externally (Fig. 12.11), that is, $O_1O_2 = r_1 + r_2$.

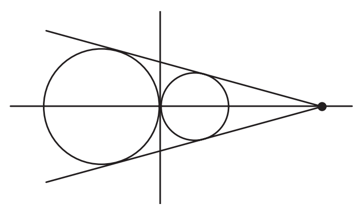


Figure 12.11

5. The number of common tangents of $S_1 \equiv 0$ and $S_2 \equiv 0$ is '4' if the circles lie outside each other, that is, $O_1O_2 > r_1 + r_2$. Two of the tangents are the 'direct common tangents' and the other two of the tangents are called the 'transverse common tangents' (Fig. 12.12).

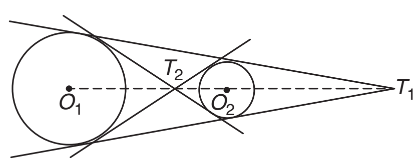


Figure 12.12

If the direct common tangents meet the line O_1O_2 at point T_1, then T_1 divides the segment O_1O_2 externally in the ratio r_1:r_2.

If the transverse common tangents meet the line O_1O_2 at point T_2, then T_2 divides the segment O_1O_2 internally in the ratio $r_1:r_2$.
The coordinates of T_1 and T_2 which are having been found, the corresponding tangents are straight lines through it such that the perpendiculars on them from O_1 are each equal to r_1.

Note: If $S_1 = 0$ and $S_2 = 0$ touch each other, then their point of contact can be obtained by solving $S_1 = 0$ and $S_1 - S_2 = 0$ simultaneously.

12.4.13 Radical Axis, Chord of Contact and Chord with Middle Point

The radical axis of $S_1 \equiv 0$ and $S_2 \equiv 0$ is the locus of a point which moves such that the lengths of the tangents drawn from it to the two circles are equal. The equation of the radical axes is $S_1 - S_2 = 0$. The equation of 'chord of contact' of the tangents drawn from an external point P(x_1, y_1) to $S \equiv 0$ is $T \equiv 0$. The equation of 'chord (of $S \equiv 0$) with middle point' $P(x_1, y_1)$ is $T \equiv S_1$.

Additional Solved Examples

1. Find the equation of the circle circumscribing the triangle formed by the lines $x + y = 6$, $2x + y = 4$ and $x + 2y = 5$.

Solution: We have

$$x + y = 6 \quad (1)$$

$$2x + y = 4 \quad (2)$$

$$x + 2y = 5 \quad (3)$$

The vertices of the triangle ABC formed by Eqs. (1), (2) and (3) are A(–2, 8), B(1, 2), C(7, –1).

Let $x^2 + y^2 + 2gx + 2fy + c = 0$ be the equation of the required circle. Then, A, B and C will lie on this circle, so we get

$$-4g + 16f + c = -68 \quad (4)$$

$$2g + 4f + c = -5 \quad (5)$$

$$14g - 2f + c = -50 \quad (6)$$

From Eqs. (4) and (5), we get

$$2g - 4f = 21 \quad (7)$$

From Eqs. (5) and (6), we get

$$-4g + 2f = 15 \quad (8)$$

On solving Eqs. (7) and (8), we get

$$g = -\frac{17}{2} \text{ and } f = -\frac{19}{2}$$

From Eq. (5), $c = 50$ and hence $x^2 + y^2 - 17x - 19y + 50 = 0$ is the required equation of the circle.

2. Find the equations of circles, which have radius $\sqrt{13}$ and which touch the line $2x - 3y + 1 = 0$ at (1, 1).

Solution: Let one of the circles have centre $C_1(x_1, y_1)$ and let the point P have coordinates (1, 1) (Fig. 12.13). Since C_1P is perpendicular to the line $2x - 3y + 1 = 0$, the equation of C_1P is

$$3x + 2y = 5$$

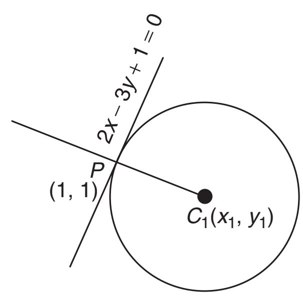


Figure 12.13

Since $C_1(x_1, y_1)$ lies on it, we have

$$3x_1 + 2y_1 = 5 \quad (1)$$

Also,

$$C_1P = \sqrt{13} \Rightarrow \left|\frac{2x_1 - 3y_1 + 1}{\sqrt{13}}\right| = \sqrt{13}$$

Therefore,

$$2x_1 - 3y_1 + 1 = \pm 13$$

Considering positive value, we get

$$2x_1 - 3y_1 = 12 \quad (2)$$

Considering negative value, we get

$$2x_1 - 3y_1 = -14 \quad (3)$$

On solving Eqs. (1) and (2), we get

$$x_1 = 3 \text{ and } y_1 = -2$$

Therefore, the equation of one of the circles is

$$(x - 3)^2 + (y + 2)^2 = 13$$

On solving Eq. (1) and (3), we get

$$x_1 = -1; y_1 = 4$$

So, the equation of the other circle is $(x + 1)^2 + (y - 4)^2 = 13$

3. If the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ cut the coordinate axes in concyclic points, show that $a_1a_2 = b_1b_2$.

Solution: Let the straight line $a_1x + b_1y + c_1 = 0$ cut the coordinate axes at the points A and B respectively (Fig. 12.14). Then, A and B have the coordinates $\left(\frac{-c_1}{a_1}, 0\right)$ and $\left(0, \frac{-c_1}{b_1}\right)$, respectively.

Let the line $a_2x + b_2y + c_2 = 0$ cut the axes at points C and D, respectively. Then C and D have the coordinates $\left(\frac{-c_2}{a_2}, 0\right)$ and $\left(0, \frac{-c_2}{b_2}\right)$, respectively. By geometry, since the points A, B, C and D are concyclic, we have 1(OA) × 1(OC) = 1(OB) × 1(OD). Therefore,

$$\left|\frac{-c_1}{a_1}\right| \cdot \left|\frac{-c_2}{a_2}\right| = \left|\frac{-c_1}{b_1}\right| \cdot \left|\frac{-c_2}{b_2}\right|$$

$$\Rightarrow a_1a_2 = b_1b_2$$

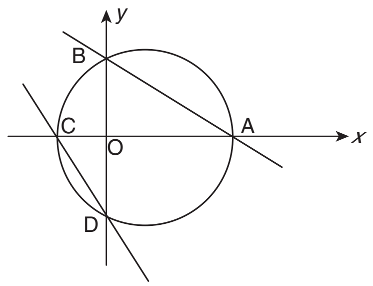


Figure 12.14

4. How many circles can be drawn each touching all the three lines $x + y = 1$, $y = x + 1$, $7x - y = 6$? Find the centre and the radius of one of them.

Solution: The equations of the three lines are

$$x + y = 1 \quad (1)$$

$$y - x = 1 \quad (2)$$

$$7x - y = 6 \quad (3)$$

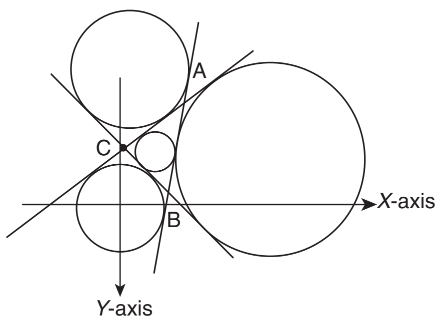


Figure 12.15

The lines expressed in Eqs. (1), (2) and (3), respectively, represent the sides BC, CA and AB of ΔABC which is right angled at C (Fig. 12.15). Hence, there are four circles each touching all three lines. Among these four circles, three are escribed circles and one is the inscribed circle.

We shall find the centre and the radius of the escribed circle opposite to vertex A. Since BC and CA are inclined at angles $3\pi/4$ and $\pi/4$, respectively, to the x-axis. Therefore, the external bisector of $\angle C$ is the y-axis, namely, $x = 0$.

The equations of the bisectors of angle between lines expressed in Eqs. (1) and (3) are

$$\frac{x+y-1}{\sqrt{2}} = \pm\frac{7x-y-6}{\sqrt{50}}$$

$$\Rightarrow 5(x+y-1) = \pm(7x-y-6)$$

$$\Rightarrow 2x - 6y - 1 = 0$$

or $\quad 12x + 4y - 11 = 0$

Consider the bisector whose equation is $2x - 6y - 1 = 0$. Let θ denote the angle between this bisector and the line expressed in Eq. (1). Therefore,

$$\tan\theta = \frac{(-1)-(1/3)}{1+(-1)(1/3)} = -2;$$

$$|\tan\theta| > 1; \quad \theta > 45^\circ$$

and the obtuse angle bisector is

$$2x - 6y - 1 = 0 \quad (4)$$

The lines $x = 0$ and $2x - 6y - 1 = 0$ intersect at $\left(0, -\frac{1}{6}\right)$, the centre of the escribed circle. Now, r is its radius, which is equal to the distance of $\left(0, -\frac{1}{6}\right)$ from the line expressed in Eq. (1) is

$$\frac{|0-(1/6)-1|}{\sqrt{2}} = \frac{7}{6\sqrt{2}}$$

5. Find the equation of the circles which pass through the origin and cut off chords, of length a, from each of the lines $y = x$ and $y = -x$.

Solution: Let us consider

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad (1)$$

be the equation of such circle. We note that $c = 0$ since the circle expressed in Eq. (1) passes through the origin.

Let OP and OQ denote the chords intercepted by the circle expressed in Eq. (1) on $y = x$ and $y = -x$, respectively, where O is the origin. If the coordinates of P are (α, α) then

$$a = OP \Rightarrow \alpha = \pm\frac{a}{\sqrt{2}}$$

The possible coordinates of Q are $(-\alpha, \alpha)$ or $(\alpha, -\alpha)$. Since P and Q lie on Eq. (1), we get $g = 0$ and $f = -\alpha$ or $g = -\alpha$ and $f = 0$. Accordingly, Q is $(-\alpha, \alpha)$ or $(\alpha, -\alpha)$ and where P is (α, α). Therefore, the equation of the required circles are given as follows:

1. When $\alpha = \frac{a}{\sqrt{2}}$, $x^2 + y^2 - \sqrt{2}\,ay = 0$ or $x^2 + y^2 - \sqrt{2}\,ax = 0$
2. When $\alpha = -\frac{a}{\sqrt{2}}$, $x^2 + y^2 + \sqrt{2}ay = 0$ or $x^2 + y^2 + \sqrt{2}ax = 0$

6. A triangle has two of its sides along the coordinate axes, and its third side touches the circle $x^2 + y^2 - 2ax - 2ay + a^2 = 0$. Prove that the locus of the circumcentre of the triangle is $a^2 - 2a(x+y) + 2xy = 0$ where $a > 0$.

Solution: We have

$$x^2 + y^2 - 2ax - 2ay + a^2 = 0 \quad (1)$$

The circle expressed in Eq. (1) has its centre $C(a, a)$ and the radius r is a. Let OAB be the required triangle and let $M(x_1, y_1)$ be any point on the locus. Then, M is the midpoint of the segment AB.

$$A \equiv (2x_1, 0); \; B \equiv (0, 2y_1)$$

The equation of the straight line AB is

$$\frac{x}{2x_1} + \frac{y}{2y_1} = 1$$

$$\Rightarrow xy_1 + yx_1 - 2x_1y_1 = 0$$

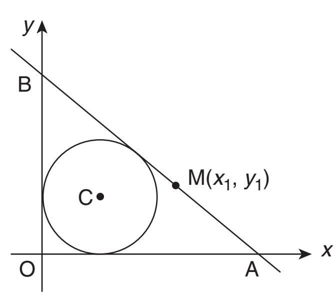


Figure 12.16

Since AB is a tangent to the circle, the length of the perpendicular from the centre C on AB is equal to the radius of the circle (Fig. 12.16). Therefore,

$$\left| \frac{ay_1 + ax_1 - 2x_1y_1}{\sqrt{x_1^2 + y_1^2}} \right| = a$$

$$(ay_1 + ax_1 - 2x_1y_1)^2 = a^2(x_1^2 + y_1^2)$$

$$4x_1^2y_1^2 + 2a^2x_1y_1 - 4ax_1y_1(x_1 + y_1) = 0$$

$$\Rightarrow 2x_1y_1 + a^2 - 2a(x_1 + y_1) = 0 \ \ (\text{since } x_1 \neq 0; y_1 \neq 0)$$

Therefore, the equation of the locus is

$$a^2 - 2a(x + y) + 2xy = 0$$

7. Through a fixed point (h, k), secants are drawn to the circle $x^2 + y^2 = r^2$. Show that the locus of midpoints of the regions of secants intercepted by the circle is $x^2 + y^2 = hx + ky$.

Solution: We have

$$x^2 + y^2 = r^2 \quad (1)$$

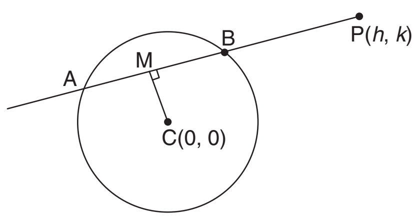


Figure 12.17

The circle expressed in Eq. (1) has centre C(0, 0) and radius r. Let $P \equiv (h, k)$ and let $M \equiv (x_1, y_1)$ be any point on the locus (Fig. 12.17). Then, M is the midpoint of the chord AB which is a part of the secant drawn from $P(h, k)$ to the circle expressed in Eq. (1). Since CM is perpendicular to AB, we have

$$\frac{y_1 - 0}{x_1 - 0} \cdot \frac{y_1 - k}{x_1 - h} = -1 \quad (\text{using, } m_1 \cdot m_2 = -1)$$

Therefore,

$$y_1(y_1 - k) + x_1(x_1 - h) = 0$$

$$x_1^2 + y_1^2 = hx_1 + ky_1$$

Hence, the equation of locus is $x^2 + y^2 = hx + ky$.

8. A variable circle passes through point $A(a, b)$ and touches x-axis. Show that the locus of the other end of the diameter through point A is $(x - a)^2 = 4by$.

Solution: Let equation of the variable circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad (1)$$

The circle [Eq. (1)] has centre $C(-g, -f)$. Let $B(x_1, y_1)$ be any point on the locus. Then B is the other end of the diameter of the circle which is drawn through point A (Fig. 12.18). Therefore,

$$x_1 + a = -2g; y_1 + b = -2f \quad (2)$$

Since $A(a, b)$ lies on the circle [Eq. (1)], we have

$$a^2 + b^2 + 2ga + 2fb + c = 0 \quad (3)$$

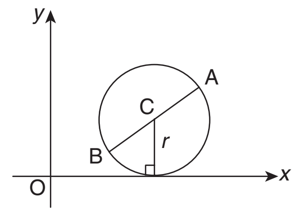


Figure 12.18

Since the circle [Eq. (1)] touches x-axis, the radius of the circle is

|y-coordinate of its centre|

Therefore,

$$|f| = \sqrt{g^2 + f^2 - c}$$

$$\Rightarrow g^2 = c \quad (4)$$

The locus of B is obtained by eliminating g, f and c from Eqs. (2), (3) and (4). From Eqs. (3) and (4), we get

$$a^2 + b^2 + 2ga + 2fb + g^2 = 0$$

From Eq. (2), we get

$$a^2 + b^2 - (x_1 + a)a - (y_1 + b)b + \left(\frac{x_1 + a}{2}\right)^2 = 0$$

Therefore,

$$a^2 + b^2 - ax_1 - a^2 - by_1 - b^2 + \frac{x_1^2 + 2ax_1 + a^2}{4} = 0$$

$$x_1^2 + a^2 - 2ax_1 - 4by_1 = 0$$

$$(x_1 - a)^2 = 4by_1$$

Thus, the equation of the locus is $(x - a)^2 = 4by$.

9. Two circles, each of radius 5 units, touch each other at the point (1, 2). If the equation of their common tangent is $4x + 3y = 10$, find the equation of the circles.

Solution: Let C_1 and C_2 denote the centres of the two circles. The equation of the common tangent at $P(1, 2)$ is

$$4x + 3y = 10 \quad (1)$$

Therefore, the equation of the common normal at (1, 2) is

$$3x - 4y + 5 = 0 \quad (2)$$

Now, the slope of the equation of the common normal at (1, 2) [Eq. (2)] is

$$\tan\theta = \frac{3}{4}$$

$$\cos\theta = \frac{4}{5}; \sin\theta = \frac{3}{5}$$

Also, the equation of the normal in parametric form is

$$x - 1 = r\cos\theta; \ y - 2 = r\sin\theta \quad (3)$$

Since $PC_1 = PC_2 = 5$ units, the coordinates of C_1 and C_2 are obtained by substituting $r = 5$ and $r = -5$ successively in Eq. (3). Hence, $C_1(5, 5)$ and $C_2(-3, -1)$ are the coordinates of the centres. The equations of the two circles, respectively, are

$$(x - 5)^2 + (y - 5)^2 = 25$$

and $$(x + 3)^2 + (y + 1)^2 = 25$$

10. Find the radius of the smallest circle which touches the straight line $3x - y = 6$ at $(1, -3)$ and also touches the line $y = x$.

Solution: Let C and r denote the centre and radius, respectively, of one of the circles which has the lines

$$y = x \quad (1)$$

and $$3x - y = 6 \quad (2)$$

as tangents and Eq. (2) being a tangent touches the circle at point (1, −3). Now, let points A and B denote the points of contact of those given in Eqs. (1) and (2), respectively, with the circle (Fig. 12.19). Lines expressed in Eqs. (1) and (2) intersect at point $P(3, 3)$ and PC bisects the angle $\angle APB = 2\alpha$. So,

$$AP = \sqrt{(3-1)^2 + (3+3)^2}$$
$$= 2\sqrt{10}$$

We know,

$$AP = BP = 2\sqrt{10}$$

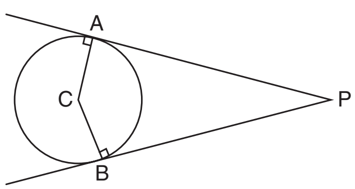


Figure 12.19

Therefore,

$$|\tan 2\alpha| = \left|\frac{3-1}{1+3}\right| = \frac{1}{2}$$

and $$\tan\alpha = \frac{r}{PB} = \frac{r}{2\sqrt{10}}$$

Therefore,

$$\tan 2\alpha = \pm\frac{1}{2}$$

$$\pm 4\tan\alpha = 1 - \tan^2\alpha$$

$$\pm 4\left(\frac{r}{2\sqrt{10}}\right) = 1 - \frac{r^2}{40}$$

$$r^2 \pm 8\sqrt{10}\, r - 40 = 0$$

On considering the positive value, we get

$$r = 10\sqrt{2} - 4\sqrt{10}$$

and on considering the negative value, we get

$$r = 4\sqrt{10} + 10\sqrt{2}$$

Therefore, the radius of the smallest circle is $10\sqrt{2} - 4\sqrt{10}$.

11. A tangent at a point on the circle $x^2 + y^2 = a^2$ intersects a concentric circle C at two points P and Q. The tangents to the circle C at points P and Q meet at a point on the circle $x^2 + y^2 = b^2$. Find the equation to the circle C.

Solution: Let the equation of circle C be

$$x^2 + y^2 = r^2 \quad (1)$$

Let $A(h, k)$ be a point on the circle

$$x^2 + y^2 = a^2 \quad (2)$$

The tangent at point A to the circle expressed in Eq. (2) cuts the circle expressed in Eq. (1) at points P and Q. Let the tangents at points P and Q meet at point $B(b\cos\theta, b\sin\theta)$ on the circle

$$x^2 + y^2 = b^2 \quad (3)$$

The equation of the chord of contact PQ of the tangents drawn from point B to circle C [Eq. (1)] is

$$bx\cos\theta + by\sin\theta = r^2 \quad (4)$$

We know, radius of circle is equal to perpendicular distance of tangent from the centre of circle. Since Eq. (4) is a tangent at point A to the circle [Eq. (2)] is

$$a = \left|\frac{r^2}{\sqrt{b^2\cos^2\theta + b^2\sin^2\theta}}\right|$$

Therefore, $r^2 = ab$ and Eq. (1) becomes $x^2 + y^2 = ab$.

12. Find the equation of the circle which touches the straight lines $x + y = 2$, $x - y = 2$ and also touches the circle $x^2 + y^2 = 1$.

Solution: The centre of the required circle lies on the bisector of the angle between the two lines

$$x + y = 2 \quad (1)$$

and $$x - y = 2 \quad (2)$$

which contains the circle

$$x^2 + y^2 = 1 \quad (3)$$

The equation of the required angle bisector between the lines [Eq. (1) and (2)] is $y = 0$.

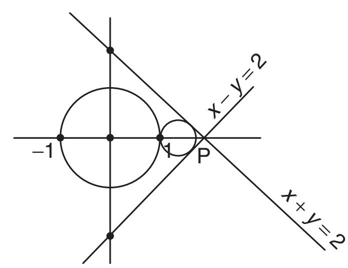


Figure 12.20

Let the centre of the required circle be $(h, 0)$. This circle touches the circle [Eq. (3)] only externally. Therefore, $|h|$ is the distance between the centres of the circles, which is equal to the sum of their radii, that is, $1 + r$, where r is the radius of the required circle. Since the required circle has the line [Eq. (1)] as the tangent, we get

$$r = \left|\frac{h-2}{\sqrt{2}}\right|$$

$$|h| = 1 + \frac{|h-2|}{\sqrt{2}}$$

$$\Rightarrow \sqrt{2}|h| = \sqrt{2} + |h-2|$$

It is obvious (from Fig. 12.20) that $h < 2$. Let $0 \le h < 2$, then

$$\sqrt{2}h = \sqrt{2} + 2 - h$$

That is,

$$h = \sqrt{2};\ r = \sqrt{2} - 1$$

Let $h < 0$. Then

$$-\sqrt{2}h = \sqrt{2} + 2 - h$$

$$(1-\sqrt{2})h = 2 + \sqrt{2}$$

$$h = -\sqrt{2}(\sqrt{2}+1)^2$$

$$= -\sqrt{2}(3+2\sqrt{2})$$

$$= -(4+3\sqrt{2})$$

and $r = |h| - 1 = 3 + 3\sqrt{2}$

Hence, the equation of the required circle is

$$(x-\sqrt{2})^2 + y^2 = (\sqrt{2}-1)^2$$

$$\Rightarrow [x + (4+3\sqrt{2})]^2 + y^2 = (3+3\sqrt{2})^2$$

13. Consider a family of circles passing through the two fixed points $A(3, 7)$ and $B(6, 5)$. Show that the chords in which the circle $x^2 + y^2 - 4x - 6y - 3 = 0$ cuts the members of the family are concurrent at a point. Find the coordinates of this point.

Solution: The equation of the chord AB is

$$2x + 3y = 27 \quad (1)$$

The equation of the family of circles through points A and B is

$$(x-3)(x-6) + (y-7)(y-5) + \lambda(2x+3y-27) = 0$$

That is,

$$x^2 + y^2 - 9x - 12y + 53 + \lambda(2x+3y-27) = 0 \quad (2)$$

$$x^2 + y^2 - 4x - 6y - 3 = 0 \quad (3)$$

The common chords of the circles [Eqs. (2) and (3)] are given by

$$5x + 6y - 56 - 1(2x+3y-27) = 0 \quad [\text{Eq}(3) - \text{Eq}(2)]$$

which is a family of lines through the points of intersection of $5x + 6y - 56 = 0$ and $2x + 3y - 27 = 0$. Therefore, their point of intersection is $\left(2, \dfrac{23}{3}\right)$.

14. Find the equation of the circle through the points of intersection of the circles, and which is intersecting the circle orthogonally.

Solution: The equation of the family of circles through the points of intersection of the circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 4y - 12 = 0$ is given by

$$x^2 + y^2 - 4x - 6y - 12 + \lambda(x^2 + y^2 + 6x + 4y - 12) = 0$$

That is,

$$x^2 + y^2 + \frac{(6\lambda - 4)}{1+\lambda}x + \frac{(4\lambda - 6)}{1+\lambda}y - 12 = 0 \quad (1)$$

Since one of the circles given by Eq. (1) intersects the circle $x^2 + y^2 - 2x = 4$ orthogonally, for that value of λ, we have

$$2\left(\frac{3\lambda - 2}{1+\lambda}\right)(-1) + 0 = -12 - 4$$

$$3\lambda - 2 = 8(1+\lambda)$$

$$5\lambda = -10;\ \lambda = -2$$

The equation of the required circle is

$$x^2 + y^2 + 16x + 14y - 12 = 0$$

Previous Years' Solved JEE Main/AIEEE Questions

1. Consider a family of circles which are passing through the point $(-1, 1)$ and are tangent to x-axis. If (h, k) are the co-ordinates of the centre of the circles, then the set of values of k is given by the interval:

(A) $0 < k < 1/2$ **(B)** $k \ge 1/2$
(C) $-1/2 \le k \le 1/2$ **(D)** $k \le 1/2$

[AIEEE 2007]

Solution: Equation of the circle is $(x-h)^2 + (y-k)^2 = k^2$, which is passing through $(-1, 1)$.

Therefore, $(-1-h)^2 + (1-k)^2 = k^2 \Rightarrow h^2 + 2h - 2k + 2 = 0$

Since h is real, therefore,

$$D \ge 0 \Rightarrow 4 - 4(2-2k) \ge 0 \Rightarrow 2k - 1 \ge 0 \Rightarrow k \ge 1/2$$

Hence, the correct answer is option (B).

2. The point diametrically opposite to the point P (1, 0) on the circle $x^2 + y^2 + 2x + 4y - 3 = 0$ is

(A) (3, –4) **(B)** (–3, 4)
(C) (–3, –4) **(D)** (3, 4)

[AIEEE 2008]

Solution: Centre of the circle $(-g, -f) = (-1, -2)$. Let required point be (x, y). So,

$$\frac{x+1}{2} = -1 \Rightarrow x + 1 = -2 \Rightarrow x = -3$$

and

$$\frac{y+0}{2} = -2 \Rightarrow y = -4$$

Therefore, $(x, y) = (-3, -4)$.

Hence, the correct answer is option (C).

3. If P and Q are the points of intersection of the circles $x^2 + y^2 + 3x + 7y + 2p - 5 = 0$ and $x^2 + y^2 + 2x + 2y - p^2 = 0$, then there is a circle passing through P, Q and (1, 1) for

(A) all values of p
(B) all except one value of p
(C) all except two values of p
(D) exactly one value of p

[AIEEE 2009]

Solution: The circles are

$$S = x^2 + y^2 + 3x + 7y + 2p - 5 = 0;$$

$$S' = x^2 + y^2 + 2x + 2y - p^2 = 0$$

The equation of required circle is

$$S + \lambda S' = 0$$

As it passes through point (1, 1), then we have

$$(1+1+3+7+2p-S)+\lambda\left(1+1+2+2-p^2\right)=0$$

$$\Rightarrow (7+2p)+\lambda\left(6-p^2\right)=0$$

$$\Rightarrow \lambda = \frac{-(7+2p)}{6-p^2}$$

If $p = 1\sqrt{6}$, then λ is not defined.

Therefore, all values of p except $p = \pm\sqrt{6}$.

Hence, the correct answer is option (C).

4. The circle $x^2 + y^2 = 4x + 8y + 5$ intersects the line $3x - 4y = m$ at two distinct points if

(A) $-35 < m < 15$ **(B)** $15 < m < 65$

(C) $35 < m < 85$ **(D)** $-85 < m < -35$

[AIEEE 2010]

Solution: The circle is given by

$$x^2 + y^2 - 4x - 8y - 5 = 0$$

The centre is (2, 4) and radius is $\sqrt{4+16+5} = 5$. If the circle is intersecting line $3x - 4y = m$ at two distinct points, then the length of perpendicular from centre is less than the radius. Therefore,

$$\frac{|3(2)-4(4)-m|}{5} < 5 \Rightarrow \frac{|6-16-m|}{5} < 5 \Rightarrow |10+m| < 25$$

$$\Rightarrow -25 < m+10 < 25 \Rightarrow -35 < m < 15$$

Hence, the correct answer is option (A).

5. The two circles $x^2 + y^2 = ax$ and $x^2 + y^2 = c^2 (c > 0)$ touch each other if

(A) $|a| = c$ **(B)** $a = 2c$

(C) $|a| = 2c$ **(D)** $2|a| = c$

[AIEEE 2011]

Solution: We have,

$$c_1 = \left(\frac{a}{2}, 0\right); c_2 = (0, 0)$$

Therefore,

$$r_1 = \frac{a}{2}; r_2 = c$$

Thus,

$$c_1c_2 = r_1 - r_2 \Rightarrow \frac{a}{2} = c - \frac{a}{2} \Rightarrow c = a$$

Now,

$$c = a, \text{ when } a > 0; c = -a, \text{ when } a < 0$$

Therefore, $c = |a|$.

Hence, the correct answer is option (A).

6. The length of the diameter of the circle which touches the x-axis at the point (1, 0) and passes through the point (2, 3) is

(A) $\frac{10}{3}$ **(B)** $\frac{3}{5}$

(C) $\frac{6}{5}$ **(D)** $\frac{5}{3}$

[AIEEE 2012]

Solution: Let (h, k) be centre of the circle as shown in Fig. 12.21. Therefore,

$$(h-1)^2 + (k-0)^2 = k^2 \quad \Rightarrow h = 1;$$

$$(h-2)^2 + (k-3)^2 = k^2 \quad \Rightarrow k = \frac{5}{3}$$

Therefore, the diameter is

$$2k = \frac{10}{3}$$

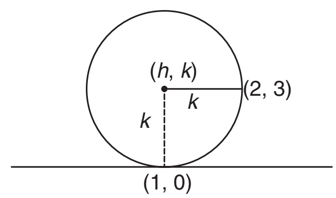


Figure 12.21

Hence, the correct answer is option (A).

7. The circle passing through (1, –2) and touching the axis of x at (3, 0) also passes through the point

(A) (2, –5) **(B)** (5, –2)

(C) (–2, 5) **(D)** (–5, 2)

[JEE MAIN 2013]

Solution: The equation of the circle due to point (3, 0) touching the axis of x is given by

$$(x-3)^2 + (y-0)^2 + \lambda y = 0$$

It is given that the circle passes through point (1, –2). Therefore,

$$(1-3)^2 + (-2)^2 + \lambda(-2) \Rightarrow 4 + 4 - 2\lambda = 0 \Rightarrow \lambda = 4$$

Therefore, the equation of the circle is

$$(x-3)^2 + y^2 + 4y = 0$$

from which it is clear that (5, –2) satisfies the equation of the circle.

Hence, the correct answer is option (B).

8. Let C be the circle with centre at (1, 1) and radius = 1. If T is the circle centred at (0, y), passing through origin and touching the circle C externally, then the radius of T is equal to

(A) $\frac{1}{2}$ **(B)** $\frac{1}{4}$

(C) $\frac{\sqrt{3}}{\sqrt{2}}$ **(D)** $\frac{\sqrt{3}}{2}$

[JEE MAIN 2014 (OFFLINE)]

Solution: See Fig. 12.22.

$$AB = 1 + y$$

$$BC = 1 - y$$

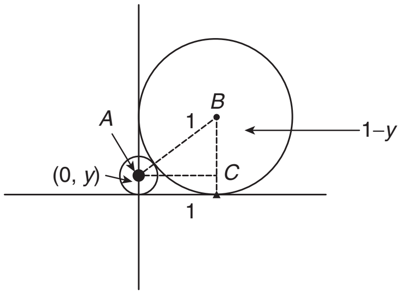


Figure 12.22

Now, by Pythagoras theorem, we have

$$(1 + y)^2 = 1^2 + (1 - y)^2 \Rightarrow 4y = 1$$

Therefore, $y = \dfrac{1}{4}$.

Hence, the correct answer is option (B).

9. If the point (1, 4) lies inside the circle $x^2 + y^2 - 6x - 10y + p = 0$ and the circle does not touch or intersect the coordinate axes, then the set of all possible values of p is the interval:

(A) (0, 25) **(B)** (25, 39)
(C) (9, 25) **(D)** (25, 29)

[JEE MAIN 2014 (ONLINE SET-1)]

Solution: See Fig. 12.23. (1, 4) lies inside the circle

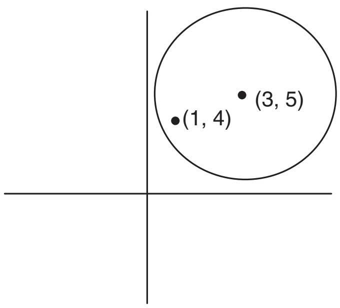


Figure 12.23

Therefore,

$$1 + 16 - 6 - 40 + p < 0$$

$$\Rightarrow p < 29 \quad (1)$$

Also $\quad r < 3$

Therefore,

$$\sqrt{3^2 + 5^2 - p} < 3 \Rightarrow 34 - p < 9$$

$$\Rightarrow 25 < p \text{ or } p > 25 \quad (2)$$

Therefore, from Eqs. (1) and (2), interval is (25, 29).

Hence, the correct answer is option (D).

10. The set of all real values of λ for which exactly two common tangents can be drawn to the circles $x^2 + y^2 - 4x - 4y + 6 = 0$ and $x^2 + y^2 - 10x - 10y + \lambda = 0$ is the interval

(A) (12, 32) **(B)** (18, 42)
(C) (12, 24) **(D)** (18, 48)

[JEE MAIN 2014 (ONLINE SET-2)]

Solution: See Fig. 12.24. For exactly two common tangents,

$$|r_1 - r_2| < AB < r_1 + r_2$$

$$\Rightarrow \left|\sqrt{4+4-6} - \sqrt{25+25-\lambda}\right| < \sqrt{(5-2)^2 + (5-2)^2} < \sqrt{2} + \sqrt{50-\lambda}$$

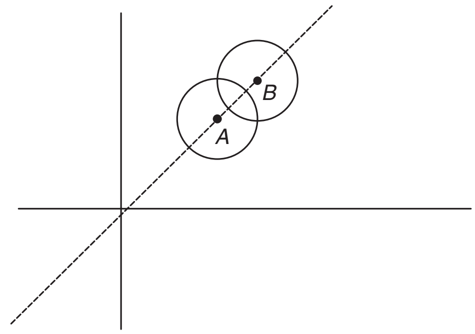


Figure 12.24

$$\Rightarrow \left|\sqrt{2} - \sqrt{50-\lambda}\right| < 3\sqrt{2} < \sqrt{2} + \sqrt{50-\lambda}$$

Now, we take

$$3\sqrt{2} < \sqrt{2} + \sqrt{50-\lambda}$$

$$\Rightarrow 2\sqrt{2} < \sqrt{50-\lambda}$$

$$\Rightarrow 8 < 50 - \lambda \Rightarrow \lambda < 42$$

Also,

$$\left|\sqrt{2} - \sqrt{50-\lambda}\right| < 3\sqrt{2}$$

$$\Rightarrow -\left(\sqrt{2} - \sqrt{50 - \lambda}\right) < 3\sqrt{2}$$

$$\Rightarrow \sqrt{50-\lambda} < 4\sqrt{2} \Rightarrow \lambda > 18$$

Hence, the correct answer is option (B).

11. For the two circles $x^2 + y^2 = 16$ and $x^2 + y^2 - 2y = 0$, there is/are

(A) one pair of common tangents
(B) two pairs of common tangents
(C) three common tangents
(D) no common tangent

[JEE MAIN 2014 (ONLINE SET-3)]

Solution: See Fig. 12.25. Result is obvious from geometry.

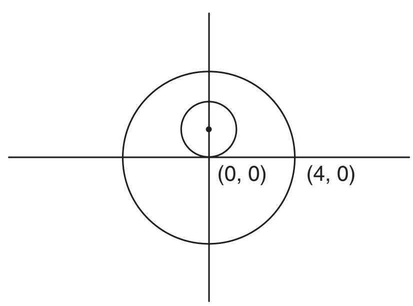


Figure 12.25

Hence, the correct answer is option (D).

12. The equation of the circle described on the chord $3x + y + 5 = 0$ of the circle $x^2 + y^2 = 16$ as diameter is

(1) $x^2 + y^2 + 3x + y - 11 = 0$
(2) $x^2 + y^2 + 3x + y + 1 = 0$
(3) $x^2 + y^2 + 3x + y - 2 = 0$
(4) $x^2 + y^2 + 3x + y - 22 = 0$

[JEE MAIN 2014 (ONLINE SET-4)]

Solution: See Fig. 12.26. Centre of required circle is foot of perpendicular from origin.

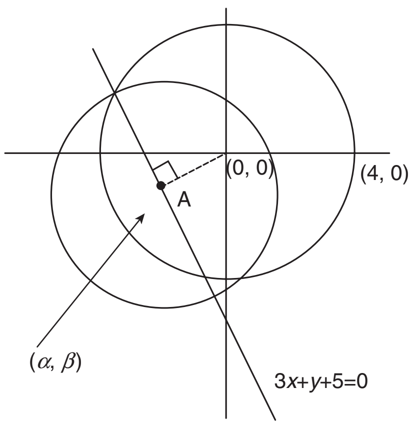


Figure 12.26

Therefore,

$$\frac{\alpha - 0}{3} = \frac{\beta - 0}{1} = \frac{-(3(0)+0+5)}{3^2+1^2} \Rightarrow \frac{\alpha}{3} = \frac{\beta}{1} = \frac{-5}{10}$$

Thus,

$$\alpha = \frac{-3}{2},\ \beta = \frac{-1}{2}$$

Now equation of the circle through $3x + y + 5 = 0$ and $x^2 + y^2 - 16 = 0$ is

$$x^2 + y^2 - 16 + \lambda(3x + y + 5) = 0$$
$$\Rightarrow x^2 + y^2 + 3\lambda x + \lambda y + 5\lambda - 16 = 0$$

Therefore, centre is $\left(\frac{-3\lambda}{2}, \frac{-\lambda}{2}\right)$, but centre is also $\frac{-3}{2}, -\frac{1}{2}$. Thus, $\lambda = 1$. Therefore, required circle is

$$x^2 + y^2 + 3x + y + 5 - 16 = 0$$
$$\Rightarrow x^2 + y^2 + 3x + y - 11 = 0$$

Hence, the correct answer is option (A).

13. The number of common tangents to the circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 18y + 26 = 0$ is

(A) 2 **(B)** 3
(C) 4 **(D)** 1

[JEE MAIN 2015 (OFFLINE)]

Solution:

$$S_1: x^2 + y^2 - 4x - 6y - 12 = 0$$
$$S_2: x^2 + y^2 + 6x + 18y + 26 = 0$$
$$C_1(2, 3);\ r_1 = 5$$
$$C_2(-3, -9);\ r_2 = 8$$

and $$C_1C_2 = \sqrt{(2+3)^2 + (3+9)^2} = 13 = r_1 + r_2$$

The two circles touch each other externally. Hence, three common tangents can be drawn.

Hence, the correct answer is option (B).

14. If $y + 3x = 0$ is the equation of a chord of the circle $x^2 + y^2 - 30x = 0$, then the equation of the circle with this chord as diameter is:

(A) $x^2 + y^2 + 3x + 9y = 0$
(B) $x^2 + y^2 - 3x + 9y = 0$
(C) $x^2 + y^2 - 3x - 9y = 0$
(D) $x^2 + y^2 + 3x - 9y = 0$

[JEE MAIN 2015 (ONLINE SET-1)]

Solution: See Fig. 12.27.

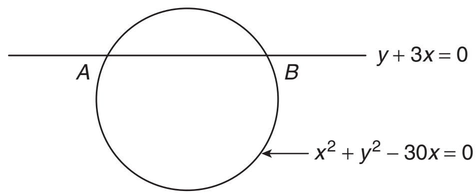


Figure 12.27

At the point of intersection A and B,

$$x^2 + (9x^2) - 30x = 0 \Rightarrow x = 0 \text{ or } x = 3$$

Therefore, $A \equiv (0,0)$ and $B \equiv (3,-9)$

Therefore, equation of circle with AB as diameter is given by

$$(x - 0)(x - 3) + (y - 0)(y + 9) = 0 \text{ (diametric form)}$$
$$\Rightarrow x^2 + y^2 - 3x + 9y = 0$$

Hence, the correct answer is option (B).

15. If the incentre of an equilateral triangle is (1, 1) and the equation of its one side is $3x + 4y + 3 = 0$, then the equation of the circumcircle of this triangle is

(A) $x^2 + y^2 - 2x - 2y - 2 = 0$
(B) $x^2 + y^2 - 2x - 2y - 14 = 0$
(C) $x^2 + y^2 - 2x - 2y + 2 = 0$
(D) $x^2 + y^2 - 2x - 2y - 7 = 0$

[JEE MAIN 2015 (ONLINE SET-2)]

Solution: See Fig. 12.28.

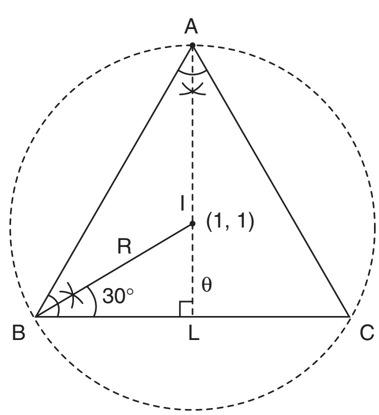


Figure 12.28

$$BC: 3x + 4y + 3 = 0$$

$$\text{In } \Delta IBL,\ IL = \frac{|3+4+3|}{\sqrt{9+16}} = 2 = r$$

Also, $\sin 30° = \frac{IL}{BI} \Rightarrow \frac{1}{2} = \frac{2}{R} \Rightarrow R = 4$

Therefore, equation of circumcircle will be

$$(y-1)^2 + (x-1)^2 = (4)^2$$
$$\Rightarrow x^2 + y^2 - 2x - 2y - 14 = 0$$

Hence, the correct answer is option (B).

16. If a circle passing through the point (–1, 0) touches y-axis at (0, 2), then the length of the chord of the circle along the x-axis is

(A) $\frac{3}{2}$ **(B)** $\frac{5}{2}$

(C) 3 **(D)** 5

[JEE MAIN 2015 (ONLINE SET-2)]

Solution: See Fig. 12.29.

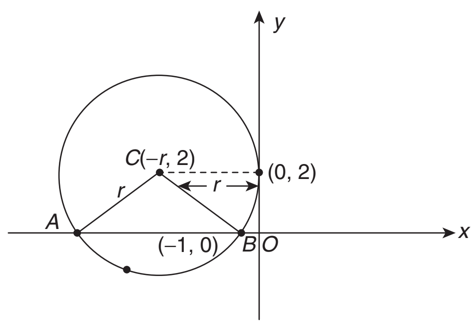


Figure 12.29

Equation of the circle is given by

$$(x + r)^2 + (y - 2)^2 = r^2$$

The above circle passes through (–1, 0)

$$\Rightarrow (r-1)^2 + (-2)^2 = r^2 \Rightarrow r = 5/2$$

So, the equation of the circle is

$$\left(x+\frac{5}{2}\right)^2 + (y-2)^2 = \frac{25}{4}$$

It is intersected by x-axis, where $y = 0$, that is,

$$\left(x+\frac{5}{2}\right) + 4 = \frac{25}{4} \Rightarrow x^2 + 5x + 4 = 0 \Rightarrow (x+1)(x+4) = 0$$
$$\Rightarrow x = -1 \text{ or } x = -4$$

Therefore, $AB = |-1-(-4)| = 3$.

Hence, the correct answer is option (C).

17. If one of the diameters of the circle, given by the equation $x^2 + y^2 - 4x + 6y - 12 = 0$, is a chord of a circle S, whose centre is at (–3, 2), then the radius of S is

(A) 10 **(B)** $5\sqrt{2}$

(C) $5\sqrt{3}$ **(D)** 5

[JEE MAIN 2016 (OFFLINE)]

Solution: See Fig. 12.30. We have

$$x^2 + y^2 - 4x + 6y - 12 = 0$$
$$\Rightarrow (x^2 - 4x + 4) + (y^2 + 6y + 9) - 25 = 0$$
$$\Rightarrow (x-2)^2 + (y+3)^2 = 25$$

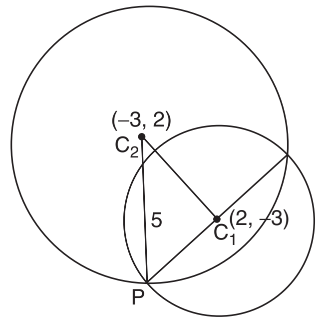


Figure 12.30

Therefore,

$$C_1C_2 = \sqrt{(2+3)^2 + (-3-2)^2} = 5\sqrt{2}$$

Using pythagoras theorem

$$(C_2P)^2 = (C_1C_2)^2 + (C_1P)^2$$
$$= 50 + 25 = 75$$

Therefore, the required radius of the circle S is $C_2P = 5\sqrt{3}$.

Hence, the correct answer is option (C).

18. The centres of those circles which touch the circle, $x^2 + y^2 - 8x - 8y - 4 = 0$, externally and also touch the x-axis, lie on

(A) a parabola

(B) a circle

(C) an ellipse which is not a circle

(D) a hyperbola

[JEE MAIN 2016 (OFFLINE)]

Solution: See Fig. 12.31. We have

$$(x^2 - 8x + 16) + (y^2 - 8y + 16) = 36$$
$$\Rightarrow (x-4)^2 + (y-4)^2 = 36$$

Let the centre of the circle be (α, β) and the radius of the circle be β.

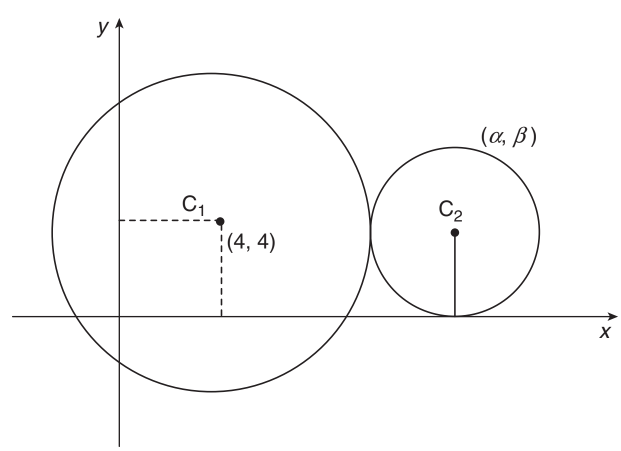


Figure 12.31

Now,

$$C_1C_2 = 4 + \beta$$
$$\Rightarrow \sqrt{(4-\alpha)^2 + (4-\beta)^2} = (4+\beta)$$

$$\Rightarrow (4-\alpha)^2 + (4-\beta)^2 = (4+\beta)^2$$
$$\Rightarrow \alpha^2 + 16 - 8\alpha + \beta^2 + 16 - 8\beta = 16 + \beta^2 + 8\beta$$
$$\Rightarrow (\alpha - 4)^2 = 16\beta$$
$$\Rightarrow (x-4)^2 = 16y$$

which is a parabola.

Hence, the correct answer is option (A).

19. A circle passes through (–2, 4) and touches the y-axis at (0, 2). Which one of the following equations can represent a diameter of this circle?

(A) $2x - 3y + 10 = 0$ **(B)** $3x + 4y - 3 = 0$
(C) $4x + 5y - 6 = 0$ **(D)** $5x + 2y + 4 = 0$

[JEE MAIN 2016 (ONLINE SET-1)]

Solution: The equation of circle is

$$(x-0)^2 + (y-2)^2 + \lambda x = 0 \quad (1)$$

which passes through the points (–2, 4). Therefore, from Eq. (1), we get

$$4 + 4 - 2\lambda = 0 \Rightarrow 2\lambda = 8 \Rightarrow \lambda = 4$$

Put the value of λ in Eq. (1), we get

$$x^2 + (y-2)^2 + 4x + 4 = 4$$
$$\Rightarrow (x+2)^2 + (y-2)^2 = 4$$

Thus, the centre is (–2, 2) and it satisfies the given equation $2x - 3y + 10 = 0$, which represents the diameter of the given circle.

Hence, the correct answer is option (A).

20. Equation of the tangent to the circle, at the point (1, –1), whose centre is the point of intersection of the straight lines $x - y = 1$ and $2x + y = 3$ is

(A) $x + 4y + 3 = 0$ **(B)** $3x - y - 4 = 0$
(C) $x - 3y - 4 = 0$ **(D)** $4x + y - 3 = 0$

[JEE MAIN 2016 (ONLINE SET-2)]

Solution: We have the following two straight lines

$$x - y = 1 \quad (1)$$

$$2x + y = 3 \quad (2)$$

On adding Eqs. (1) and (2), we get

$$3x = 4 \Rightarrow x = \frac{4}{3}$$

and $$y = x - 1 = \frac{4}{3} - 1 = \frac{1}{3}$$

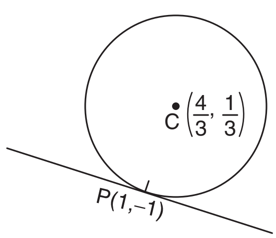


Figure 12.32

Therefore, the centre of the circle is $\left(\frac{4}{3}, \frac{1}{3}\right)$. (See Fig. 12.32.)

The radius of the circle is

$$\sqrt{\left(\frac{4}{3}-1\right)^2 + \left(\frac{1}{3}+1\right)^2} = \sqrt{\frac{1}{9} + \frac{16}{9}} = \frac{\sqrt{17}}{3}$$

The slope of the line CP is

$$\frac{(1/3)+1}{(4/3)-1} = \frac{4/3}{1/3} = 4$$

So, slope of the tangent is –1/4. Therefore,

$$y + 1 = -\frac{1}{4}(x-1)$$
$$\Rightarrow 4y + 4 = -x + 1$$
$$\Rightarrow x + 4y + 3 = 0$$

which is the equation of the tangent to the circle.

Hence, the correct answer is option (A).

Previous Years' Solved JEE Advanced/ IIT-JEE Questions

1. Tangents are drawn from the point (17, 7) to the circle $x^2 + y^2 = 169$ because

Statement-1: The tangents are mutually perpendicular.

Statement-2: The locus of the points from which mutually perpendicular tangents can be drawn to the given circle is $x^2 + y^2 = 338$.

(A) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1

(B) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1

(C) Statement-1 is True, Statement-2 is False

(D) Statement-1 is False, Statement-2 is True

[IIT-JEE 2007]

Solution: The given circle is

$$x^2 + y^2 = 169$$

The equation of its director circle is

$$x^2 + y^2 = 338$$

Since point (17, 7) satisfies the equation of director circle, the tangents, which are drawn from the point (17, 7) are mutually perpendicular and the locus of the perpendicular tangents is the director circle. Hence, both statements are true and Statement-2 explains Statement-1.

Hence, the correct answer is option (A).

2. Match the statements in Column I with the properties in Column II.

Column I	Column II
(A) Two intersecting circles	**(p)** have a common tangent
(B) Two mutually external circles	**(q)** have a common normal
(C) Two circles, one strictly inside the other	**(r)** do not have a common tangent
(D) Two branches of a hyperbola	**(s)** do not have a common normal

[IIT-JEE 2007]

Solution:

When two circles are intersecting, they have a common normal and common tangent.

(A)→ (p), (q)

Two mutually external circles have a common normal and the common tangents.

(B)→ (p), (q)

When one circle lies inside of the other, then they have a common normal but they do not have common tangents.

(C)→ (q), (r)

Two branches of a hyperbola have a common normal but they do not have common tangents.

(D)→ (q), (r)

Hence, the correct matches are (A)→ (p), (q); (B)→ (p), (q); (C)→ (q), (r); (D)→ (q), (r).

Paragraph for Questions 3–5: A circle S of radius 1 is inscribed in an equilateral triangle PQR. The points of contact of S with the sides PQ, QR, RP are D, E, F, respectively. The line PQ is given by the equation $\sqrt{3}x+y-6=0$ and the point D is $\left(\frac{3\sqrt{3}}{2},\frac{3}{2}\right)$. Further, it is given that the origin and the centre C are on the same side of the line PQ.

[IIT-JEE 2008]

3. The equation of the circle is

(A) $(x-2\sqrt{3})^2+(y-1)^2=1$

(B) $(x-2\sqrt{3})^2+\left(y+\frac{1}{2}\right)^2=1$

(C) $(x-\sqrt{3})^2+(y+1)^2=1$

(D) $(x-\sqrt{3})^2+(y-1)^2=1$

Solution: See Fig. 12.33.

We have equation of PQ

$$\sqrt{3}x+y-6=0$$

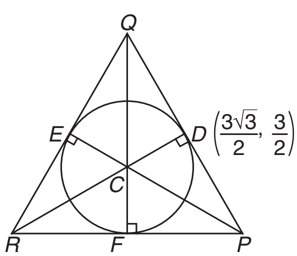


Figure 12.33

So, equation of CD is

$$\frac{x-\frac{3\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}}=\frac{y-\frac{3}{2}}{\frac{1}{2}}=-1$$

$$\Rightarrow c=(\sqrt{3},1)$$

Hence, equation of circle is

$$(x-\sqrt{3})^2+(y-1)^2=1$$

Hence, the correct answer is option (D).

4. Points E and F are given by

(A) $\left(\frac{\sqrt{3}}{2},\frac{3}{2}\right),(\sqrt{3},0)$

(B) $\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right),(\sqrt{3},0)$

(C) $\left(\frac{\sqrt{3}}{2},\frac{3}{2}\right),\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$

(D) $\left(\frac{3}{2},\frac{\sqrt{3}}{2}\right),\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$

Solution: Slope of line $PQ=-\sqrt{3}$

Therefore, PQ makes 120° angle with x-axis.
So, side PR lies along x-axis. Therefore,

$$F=(\sqrt{3},0)$$

Now, equation of CE is

$$\frac{x-\sqrt{3}}{\frac{-\sqrt{3}}{2}}=\frac{y-1}{\frac{1}{2}}=1$$

$$\Rightarrow E=\left(\frac{\sqrt{3}}{2},\frac{3}{2}\right)$$

Hence, the correct answer is option (A).

5. Equations of the side QR and RP are

(A) $y=\frac{2}{\sqrt{3}}x+1,\ y=-\frac{2}{\sqrt{3}}x-1$

(B) $y=\frac{1}{\sqrt{3}}x,\ y=0$

(C) $y=\frac{2}{\sqrt{3}}x+1,\ y=-\frac{\sqrt{3}}{2}x-1$

(D) $y=\sqrt{3}x,\ y=0$

Solution: Equation of PR is x-axis, that is $y=0$ and the equation of side QR is

$$\left(y-\frac{3}{2}\right)=\sqrt{3}\left(x-\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow y=\sqrt{3}x$$

Hence, the correct answer is option (D).

6. Consider

$$L_1: 2x+3y+p-3=0$$
$$L_2: 2x+3y+p+3=0$$

where p is a real number, and C: $x^2+y^2+6x-10y+30=0$.

Statement 1: If line L_1 is a chord of circle C, then line L_2 is not always a diameter of circle C.

Statement 2: If line L_1 is a diameter of circle C, then line L_2 is not a chord of circle C.

(A) Statement 1 is True, Statement 2 is True; Statement 2 is a correct explanation for Statement 1
(B) Statement 1 is True, Statement 2 is True; Statement 2 is NOT a correct explanation for Statement 1
(C) Statement 1 is True, Statement 2 is False
(D) Statement 1 is False, Statement 2 is True

[IIT-JEE 2008]

Solution: We have circle

$$x^2+y^2+6x-10y+30=0$$
$$(x+3)^2+(y-5)^2=4$$

Therefore, centre of the circle is (–3, 5) and radius is 2.

Distance between L_1 and $L_2=\dfrac{6}{\sqrt{13}}<$ radius

So, Statement 2 is false. But, Statement 1 is correct.

Hence, the correct answer is option (C).

7. Tangents drawn from the point $P(1, 8)$ to the circle $x^2+y^2-6x-4y-11=0$ touch the circle at the points A and B. The equation of the circumcircle of the triangle PAB is

(A) $x^2+y^2+4x-6y+19=0$

(B) $x^2+y^2-4x-10y+19=0$

(C) $x^2+y^2-2x+6y-29=0$

(D) $x^2+y^2-6x-4y+19=0$ **[IIT-JEE 2009]**

Solution: The centre of the circle is $C(3, 2)$.

Tangent at any point on the circle makes 90° with radius of circle, so CA and CB are perpendicular to PA and PB, respectively, CP is the diameter of the circumcircle of triangle PAB. Its equation is

$$(x-3)(x-1)+(y-2)(y-8)=0$$

$$\Rightarrow x^2+y^2-4x-10y+19=0.$$

Hence, the correct answer is option (B).

8. The centres of two circles C_1 and C_2 each of unit radius are at a distance of 6 units from each other. Let P be the midpoint of the line segment joining the centres of C_1 and C_2 and C be a circle touching circles C_1 and C_2 externally. If a common tangent to C_1 and C passing through P is also a common tangent to C_2 and C, then the radius of the circle C is ______.

[IIT-JEE 2009]

Solution: See Fig. 12.34.

$$\cos\alpha=\frac{2\sqrt{2}}{3}$$

$$\sin\alpha=\frac{1}{3}$$

$$\tan\alpha=\frac{2\sqrt{2}}{R}$$

$$\Rightarrow R=\frac{2\sqrt{2}}{\tan\alpha}=8\,\text{units}.$$

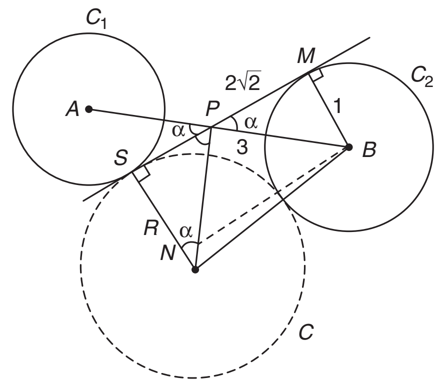


Figure 12.34

Alternate:

$$MS = BN \text{ (Parallel)}$$
$$NO = R-1$$
$$OB = R+1$$
$$OB^2 = NO^2 + NB^2$$
$$\Rightarrow (R+1)^2=(R-1)^2+(4\sqrt{2})^2$$
$$\Rightarrow R=8.$$

Hence, the correct answer is (8).

9. Two parallel chords of a circle of radius 2 are at a distance $\sqrt{3}+1$ apart. If the chords subtend at the centre, angles of $\dfrac{\pi}{k}$ and $\dfrac{2\pi}{k}$, where $k>0$, then the value of $[k]$ is
[**Note:** $[k]$ denotes the largest integer less than or equal to k]

[IIT-JEE 2010]

Solution:

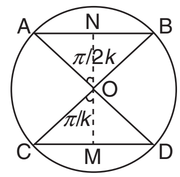


Figure 12.35

$$ON=2\cos\left(\frac{\pi}{2k}\right),$$

$$OM=2\cos\left(\frac{\pi}{k}\right)$$

$$ON+OM=\sqrt{3}+1$$

$$\Rightarrow\ 2\cos\frac{\pi}{2k}+2\cos\frac{\pi}{k}=\sqrt{3}+1$$

$$\cos\frac{\pi}{2k}+\cos\frac{\pi}{k}=\frac{\sqrt{3}+1}{2}$$

Let $\dfrac{\pi}{k}=\theta$. Then

$$\cos\theta+\cos\frac{\theta}{2}=\frac{\sqrt{3}+1}{2}\Rightarrow 2\cos^2\frac{\theta}{2}-1+\cos\frac{\theta}{2}=\frac{\sqrt{3}+1}{2}$$

Let $\cos\dfrac{\theta}{2}=t$,

$$2t^2+t-\frac{\sqrt{3}+3}{2}=0$$

$$t=\frac{-1\pm\sqrt{1+4(3+\sqrt{3})}}{4}=\frac{-1\pm(2\sqrt{3}+1)}{4}$$
$$=\frac{-2-2\sqrt{3}}{4},\ \frac{\sqrt{3}}{2}$$
As $t\in[-1,1]\Rightarrow \cos\frac{\theta}{2}=\frac{\sqrt{3}}{2}$
$$\frac{\theta}{2}=\frac{\pi}{6}\Rightarrow k=3.$$

Hence, the correct answer is (3).

10. The circle passing through the point (–1, 0) and touching the *y*-axis at (0, 2) also passes through the point

(A) $\left(-\frac{3}{2},0\right)$ **(B)** $\left(-\frac{5}{2},2\right)$

(C) $\left(-\frac{3}{2},\frac{5}{2}\right)$ **(D)** $(-4,0)$

[IIT-JEE 2011]

Solution: Circle touching *y*-axis at (0, 2) is $(x-0)^2+(y-2)^2+\lambda x=0$ and it passes through (–1, 0). Therefore,
$$1+4-\lambda=0\Rightarrow\lambda=5$$
Therefore,
$$x^2+y^2+5x-4y+4=0$$
Put $y=0\Rightarrow x=-1,-4$

Therefore, circle passes through (–4, 0).

Hence, the correct answer is option (D).

11. The locus of the mid–point of the chord of contact of tangents drawn from points lying on the straight line $4x-5y=20$ to the circle $x^2+y^2=9$ is

(A) $20(x^2+y^2)-36x+45y=0$

(B) $20(x^2+y^2)+36x-45y=0$

(C) $36(x^2+y^2)-20x+45y=0$

(D) $36(x^2+y^2)+20x-45y=0$ **[IIT-JEE 2012]**

Solution: See Fig. 12.36.

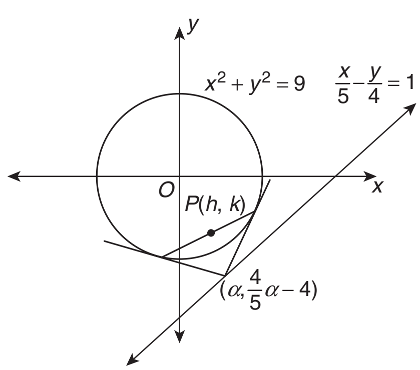


Figure 12.36

Equation of the chord bisected at *P*(*h*, *k*) is
$$hx+ky=h^2+k^2 \quad (1)$$

Let any point on line be $\left(\alpha,\frac{4}{5}\alpha-4\right)$.

Equation of the chord of contact is
$$\alpha x+\left(\frac{4}{5}\alpha-4\right)y=9 \quad (2)$$
Comparing Eqs. (1) and (2), we get
$$\frac{h}{\alpha}=\frac{k}{\frac{4}{5}\alpha-4}=\frac{h^2+k^2}{9}$$
$$\Rightarrow\alpha=\frac{20h}{4h-5k}$$
Now,
$$\frac{h(4h-5k)}{20h}=\frac{h^2+k^2}{9}$$
$$\Rightarrow 20(h^2+k^2)=9(4h-5k)$$
$$\Rightarrow 20(x^2+y^2)-36x+45y=0$$

Hence, the correct answer is option (A).

Paragraph for Questions 12 and 13: A tangent *PT* is drawn to the circle $x^2+y^2=4$ at the point $P(\sqrt{3},1)$. A straight line *L*, perpendicular to *PT* is a tangent to the circle $(x-3)^2+y^2=1$.

[IIT-JEE 2012]

12. A possible equation of *L* is

(A) $x-\sqrt{3}y=1$ **(B)** $x+\sqrt{3}y=1$

(C) $x-\sqrt{3}y=-1$ **(D)** $x+\sqrt{3}y=5$

Solution: Equation of tangent at $P(\sqrt{3},1)$ is
$$\sqrt{3}x+y=4$$
Slope of line perpendicular to above tangent is $\frac{1}{\sqrt{3}}$.

So, equation of tangents with slope $\frac{1}{\sqrt{3}}$ to $(x-3)^2+y^2=1$ will be
$$y=\frac{1}{\sqrt{3}}(x-3)\pm1\sqrt{1+\frac{1}{3}}$$
$$\Rightarrow\sqrt{3}y=x-3\pm(2)$$
$$\Rightarrow\sqrt{3}y=x-1 \text{ or } \sqrt{3}y=x-5$$

Hence, the correct answer is option (A).

13. A common tangent of the two circles is

(A) $x=4$ **(B)** $y=2$

(C) $x+\sqrt{3}y=4$ **(D)** $x+2\sqrt{2}y=6$

Solution: See Fig. 12.37. Point of intersection of direct common tangents is (6, 0).

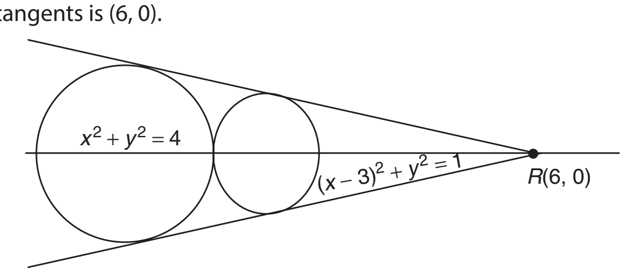


Figure 12.37

So, let the equation of common tangent be

$$y - 0 = m(x - 6)$$

as it touches $x^2 + y^2 = 4$

$$\Rightarrow \left|\frac{0-0+6m}{\sqrt{1+m^2}}\right| = 2$$

$$9m^2 = 1 + m^2$$

$$m = \pm\frac{1}{2\sqrt{2}}$$

So, equations of common tangent are

$$y = \frac{1}{2\sqrt{2}}(x-6),\ y = -\frac{1}{2\sqrt{2}}(x-6) \text{ and } x = 2$$

Hence, the correct answer is option (D).

14. Circle(s) touching x-axis at a distance 3 from the origin and having an intercept of length $2\sqrt{7}$ on y-axis is (are)

(A) $x^2 + y^2 - 6x + 8y + 9 = 0$
(B) $x^2 + y^2 - 6x + 7y + 9 = 0$
(C) $x^2 + y^2 - 6x - 8y + 9 = 0$
(D) $x^2 + y^2 - 6x - 7y + 9 = 0$ **[JEE ADVANCED 2013]**

Solution: From Fig. 12.38, it is clear that the centre is $(3, \alpha)$ and radius is $|\alpha|$.

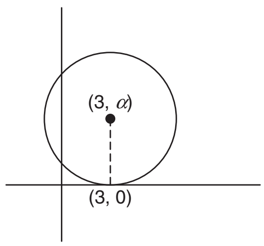


Figure 12.38

Therefore, the equation of the circle is

$$x^2 + y^2 - 6x - 2\alpha y + c = 0$$

Now, the radius is

$$9 + \alpha^2 - c = \alpha^2$$
$$\Rightarrow c = 9$$

The intercept on y-axis is

$$2\sqrt{\alpha^2 - c} = 2\sqrt{7}$$

$$\alpha^2 - 9 = 7$$
$$\Rightarrow \alpha = \pm 4$$

Therefore, the equations are

$$x^2 + y^2 - 6x \pm 8y + 9 = 0$$

Hence, the correct answers are options (A) and (C).

15. A circle S passes through the point (0, 1) and is orthogonal to the circles $(x - 1)^2 + y^2 = 16$ and $x^2 + y^2 = 1$. Then

(A) radius of S is 8
(B) radius of S is 7
(C) centre of S is (–7, 1)
(D) centre of S is (–8, 1)

[JEE ADVANCED 2014]

Solution: Let the required circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad (1)$$

Since Eq. (1) passes through (0, 1), we have

$$0 + 1 + 0 + 2f + c = 0 \Rightarrow 2f + c + 1 = 0 \quad (2)$$

Since $(x - 1)^2 + y^2 = 16$ and $x^2 + y^2 = 1$ are orthogonal to Eq. (1) (or $x^2 + y^2 + 2gx + 2fy + c = 0$), we have

$$2g(-1) + 2f(0) = c - 15 \Rightarrow -2g = c - 15 \quad (3)$$

Also,

$$2g(0) + 2f(0) = c - 1 \text{ (using, } 2g_1g_2 + 2f_1f_2 = c_1 + c_2)$$

$$\Rightarrow c = 1 \quad (4)$$

From Eqs. (2), (3) and (4), we have

$$f = -1,\ g = 7 \text{ and } c = 1$$

Therefore, centre is (–7, 1) and radius is

$$\sqrt{7^2 + 1^2 - 1}$$
$$= \sqrt{49 + \cancel{1} - \cancel{1}}$$
$$= 7$$

Hence, the correct answers are options (B) and (C).

16. Let $n \geq 2$ be an integer. Take n distinct points on a circle and join each pair of points by a line segment. Colour the line segment joining every pair of adjacent points by blue and the rest by red. If the number of red and blue line segments are equal, then the value of n is ______.

[JEE ADVANCED 2014]

Solution: See Fig. 12.39.

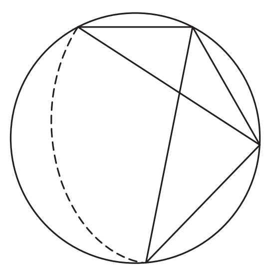


Figure 12.39

Number of blue lines = n = number of sides of polygon so formed.

Number of red lines = ${}^nC_2 - n$

Since by joining n points (not more than 2 on a line) there are ${}^nC_2$ lines formed because for each line two points are required.
Also, red lines come after excluding sides of polygon. Therefore,

$$n = {}^nC_2 - n \text{ or } {}^nC_2 = 2n$$

$$\Rightarrow \frac{\cancel{n}(n-1)}{2} = 2\cancel{n} \text{ or } n - 1 = 4 \qquad (\text{since } n \neq 0)$$

Therefore, $n = 5$.
Hence, the correct answer is (5).

17. Let RS be the diameter of the circle $x^2 + y^2 = 1$, where S is the point (1, 0). Let P be a variable point (other than R and S) on the circle and tangents to the circle at S and P meet at the point Q. The normal to the circle at P intersects a line drawn through Q parallel to RS at point E. Then the locus of E passes through the point(s)

(A) $\left(\frac{1}{3}, \frac{1}{\sqrt{3}}\right)$ **(B)** $\left(\frac{1}{4}, \frac{1}{2}\right)$

(C) $\left(\frac{1}{3}, \frac{-1}{\sqrt{3}}\right)$ **(D)** $\left(\frac{1}{4}, \frac{1}{2}\right)$

[JEE ADVANCED 2016]

Solution: See Fig. 12.40. The given circle is $x^2 + y^2 = 1$.

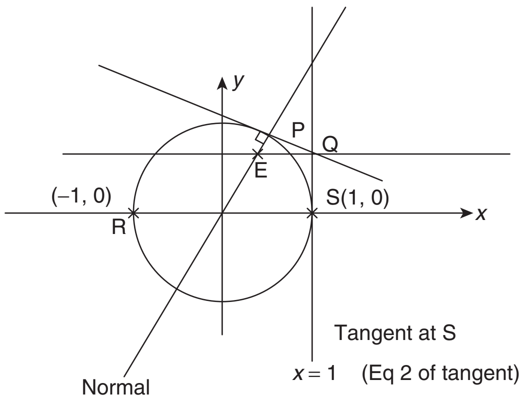


Figure 12.40

The point P on circle is $(\cos\theta, \sin\theta)$.

Equation of tangent is $x\cos\theta + y\sin\theta = 1$ and it meets the tangent at point S.

Now,

$$\cos\theta + y\sin\theta = 1 \Rightarrow y = \frac{1-\cos\theta}{\sin\theta}$$

Therefore,

$$Q = \theta\left(1, \frac{1-\cos\theta}{\sin\theta}\right)$$

Equation of line through point Q parallel to the line RS is

$$y = \frac{1-\cos\theta}{\sin\theta} \quad (1)$$

The normal to the circle at point P is

$$y = x\tan\theta \quad (2)$$

The point of intersection [E(h, k)] of the line [Eq. (1)] and the normal [Eq. (2)] is point is

$$\frac{x\sin\theta}{\cos\theta} = \frac{1-\cos\theta}{\sin\theta} \Rightarrow x\sin^2\theta = \cos\theta - \cos^2\theta$$

$$\Rightarrow x = \frac{\cos\theta - \cos^2\theta}{\sin^2\theta} = \frac{\cos\theta(1-\cos\theta)}{\sin^2\theta} = h$$

Also,

$$y = \frac{1-\cos\theta}{\sin\theta} = k$$

Therefore,

$$\frac{h}{k} = \frac{\cos\theta}{\sin\theta}$$

$$\Rightarrow \tan\theta = \frac{k}{h}$$

$$\Rightarrow \sin\theta = \frac{k}{\sqrt{k^2+h^2}},\ \cos\theta = \frac{h}{\sqrt{h^2+k^2}}$$

That is,

$$k = \frac{1-\cos\theta}{\sin\theta}$$

$$k = \left(1 - \frac{h}{\sqrt{h^2+k^2}}\right)\Big/\frac{k}{\sqrt{h^2+k^2}}$$

$$k = \frac{\sqrt{h^2+k^2} - h}{k}$$

$$k^2 + h = \sqrt{h^2+k^2}$$

$$k^4 + h^2 + 2k^2h = h^2 + k^2$$

$$k^2(k^2 + 2h) = k^2$$

$$k^2 + 2h = 1$$

Therefore, the locus of the point E is

$$y^2 = 1 - 2x$$

which passes through the points $\left(\frac{1}{3}, \pm\frac{1}{\sqrt{3}}\right)$.

Hence, the correct answers are options (A) and (C).

18. Let $a, b \in \mathbb{R}$ and $a^2 + b^2 \neq 0$. Suppose

$$S = \left\{z \in \mathbb{C} : z = \frac{1}{a+ibt},\ t \in \mathbb{R}, t \neq 0\right\}, \text{ where } i = \sqrt{-1}. \text{ If } z = x + iy$$

and $z \in S$, then (x, y) lies on

(A) the circle with radius $\frac{1}{2a}$ and centre $\left(\frac{1}{2a}, 0\right)$ for $a > 0, b \neq 0$.

(B) the circle with radius $-\frac{1}{2a}$ and centre $\left(-\frac{1}{2a}, 0\right)$ for $a < 0, b \neq 0$.

(C) the x-axis for $a \neq 0, b = 0$.

(D) the y-axis for $a = 0, b \neq 0$.

[JEE ADVANCED 2016]

Solution: We can write as

$$z = \frac{1}{a+ibt} = \frac{a-ibt}{(a+ibt)(a-ibt)}$$

$$z = \frac{a-ibt}{a^2+b^2t^2} = \left(\frac{a}{a^2+b^2t^2}\right) + i\left(\frac{-bt}{a^2+b^2t^2}\right) = x + iy$$

$$\Rightarrow x = \frac{a}{a^2+b^2t^2},\ y = \frac{-bt}{a^2+b^2t^2}$$

Therefore,

$$\frac{y}{x} = \frac{-b}{a}t \Rightarrow t = \frac{-ay}{bx}$$

The locus of x is

$$y(a^2 + b^2t^2) = -bt$$

$$y\left(a^2+b^2\frac{a^2y^2}{b^2x^2}\right)=\frac{+bay}{bx}$$

$$y\frac{a^2}{x^2}(x^2+y^2)=\frac{ay}{x}$$

$$x^2+y^2=\frac{x}{a}$$

$$x^2+y^2-\frac{x}{a}=0 \qquad \text{(equation of circle)}$$

Therefore, for $a > 0$ and $b \neq 0$.

Centre of the circle is $\left(\frac{1}{2a},0\right)$, and radius of the circle is $\frac{1}{2a}$.

Hence, option (A) is correct.

For x-axis:

$$y=0=\frac{-bt}{a^2+b^2t^2} \qquad (b=0, a\neq 0)$$

Hence, option (C) is also correct.

For y axis:

$$x=0=\frac{a}{a^2+b^2t^2} \qquad (a=0, b\neq 0)$$

Hence, option (D) is also correct.

Hence, the correct answers are options (A), (C) and (D).

Practice Exercise 1

1. If the straight line $mx - y = 1 + 2x$ intersects the circle $x^2 + y^2 = 1$ at least at one point, then the set of values of m is
   (A) $\left[-\frac{4}{3}, 0\right]$ **(B)** $\left[-\frac{4}{3}, \frac{4}{3}\right]$
   (C) $\left[0, \frac{4}{3}\right]$ **(D)** All of these
2. Circles are drawn having the sides of triangle ABC as their diameters. Radical centre of the circles is the
   (A) circumcentre of triangle ABC
   (B) in-centre of triangle ABC
   (C) orthocentre of triangle ABC
   (D) centroid of triangle ABC
3. The circle described on the line joining the points (0, 1), (a, b) as a diameter cuts the x-axis at the points whose abscissa are roots of the equation
   (A) $x^2 + ax + b = 0$
   (B) $x^2 - ax + b = 0$
   (C) $x^2 + ax - b = 0$
   (D) $x^2 - ax - b = 0$
4. The straight line $y = mx + c$ cuts the circle $x^2 + y^2 = a^2$ at the real points if
   (A) $\sqrt{a^2(1+m^2)} \le |c|$ **(B)** $\sqrt{a^2(1-m^2)} \le |c|$
   (C) $\sqrt{a^2(1+m^2)} \ge |c|$ **(D)** $\sqrt{a^2(1-m^2)} \ge |c|$
5. The centre of a circle passing through the points (0, 0), (1, 0) and touching the circle $x^2 + y^2 = 9$ is
   (A) $\left(\frac{3}{2},\frac{1}{2}\right)$ **(B)** $\left(\frac{1}{2},\frac{3}{2}\right)$
   (C) $\left(\frac{1}{2},\frac{1}{2}\right)$ **(D)** $\left(\frac{1}{2}, -\sqrt{2}\right)$
6. If circles are drawn on the sides of the triangle formed by the lines $x = 0$, $y = 0$ and $x + y = 2$, as diameters, then the radical centre of the three circles is
   (A) (0, 0) **(B)** (1, 1)
   (C) $(\sqrt{2}, 1)$ **(D)** None of these
7. The length of the chord cut off by $y = 2x + 1$ from the circle $x^2 + y^2 = 2$ is
   (A) $\frac{5}{6}$ **(B)** $\frac{6}{5}$
   (C) $\frac{6}{\sqrt{5}}$ **(D)** $\frac{\sqrt{5}}{6}$
8. The coordinates of the middle point of the chord $2x - 5y + 18 = 0$ cut off by the circle $x^2 + y^2 - 6x + 2y - 54 = 0$ is
   (A) (1, 4) **(B)** (–4, 2)
   (C) (4, 1) **(D)** (6, 6)
9. The locus of the point, such that tangents drawn from it to the circle $x^2 + y^2 - 6x - 8y = 0$ are perpendicular to each other, is
   (A) $x^2 + y^2 - 6x - 8y - 25 = 0$
   (B) $x^2 + y^2 + 6x - 8y - 25 = 0$
   (C) $x^2 + y^2 - 6x - 8y + 25 = 0$
   (D) None of these
10. The equation of circle touching the line $2x + 3y + 1 = 0$ at the point (1, –1) and passing through the focus of the parabola $y^2 = 4x$ is
    (A) $3x^2 + 3y^2 - 8x + 3y + 5 = 0$
    (B) $3x^2 + 3y^2 + 8x - 3y + 5 = 0$
    (C) $x^2 + y^2 - 3x + y + 6 = 0$
    (D) None of these
11. The value of k for which two tangents can be drawn from (k, k) to the circle $x^2 + y^2 + 2x + 2y - 16 = 0$ is
    (A) $k \in R^+$ **(B)** $k \in R^-$
    (C) $k \in (-\infty, -4) \cup (2, \infty)$ **(D)** $k \in (0, 1]$
12. The lines $3x - 4y + \lambda = 0$ and $6x - 8y + \mu = 0$ are tangents to the same circle. The radius of the circle is
    (A) $\left|\frac{2\lambda-\mu}{20}\right|$ **(B)** $\left|\frac{2\mu-\lambda}{20}\right|$
    (C) $\left|\frac{2\lambda+\mu}{20}\right|$ **(D)** None of these
13. The locus of the centres of the circles passing through the origin and intersecting the fixed circle $x^2 + y^2 - 5x + 3y - 1 = 0$ orthogonally is
    (A) a straight line of slope $\frac{3}{5}$
    (B) a circle
    (C) a pair of straight line
    (D) None of these
14. If the line $y - mx + m - 1 = 0$ cuts the circle $x^2 + y^2 - 4x - 4y + 4 = 0$ at two real points, then m belongs to
    (A) [1, 1] **(B)** [–2, 2]
    (C) $(-\infty, \infty)$ **(D)** [–4, 4]
15. The equation of the circle with centre on the x-axis and touching the line $3x + 4y - 11 = 0$ at point (1, 2) is

(A) $x^2 + y^2 - x - 4 = 0$
(B) $x^2 + y^2 + 2x - 7 = 0$
(C) $x^2 + y^2 + x - 6 = 0$
(D) None of these

16. Equation of a circle with centre (4, 3) touching the circle $x^2 + y^2 = 1$ is
(A) $x^2 + y^2 - 8x - 6y - 9 = 0$
(B) $x^2 + y^2 - 8x - 6y + 11 = 0$
(C) $x^2 + y^2 - 8x - 6y - 11 = 0$
(D) $x^2 + y^2 - 8x - 6y + 9 = 0$

17. The coordinates of the point on the circle $x^2 + y^2 - 12x - 4y + 30 = 0$ which is farthest from the origin are
(A) (9, 3) (B) (8,5)
(C) (12, 4) (D) None of these

18. The locus of a point from which the length of tangents to the circles $x^2 + y^2 = 4$ and $2(x^2 + y^2) - 10x + 3y - 2 = 0$ are equal is
(A) a straight line inclined at $\frac{\pi}{4}$ with the line joining the centres of the circles
(B) a circle
(C) an ellipse
(D) a straight line perpendicular to the line joining the centres of the circles

19. The centre of the circle inscribed in a square formed by the lines $x^2 - 8x + 12 = 0$ and $y^2 - 14y + 45 = 0$ is
(A) (4, 7) (B) (7, 4)
(C) (9, 4) (D) (4, 9)

20. If from any point on the circle $x^2 + y^2 = a^2$ tangents are drawn to the circle $x^2 + y^2 = a^2\sin^2\alpha$, then the angle between the tangents, is
(A) $\alpha/2$ (B) α
(C) 2α (D) 4α

21. Equation of chord AB of the circle $x^2 + y^2 = 2$ passing through $P(2, 2)$ such that $PB/PA = 3$, is given by
(A) $x = 3y$ (B) $x = y$
(C) $y - 2 = \sqrt{3}(x - 2)$ (D) None of these

22. Four distinct points (2*K*, 3*K*), (1,0), (0,1) and (0,0) lie on a circle when
(A) all values of K are integral (B) $0 < K < 1$
(C) $K < 0$ (D) for two values of K

23. A line is drawn through a fixed point $P(\alpha, \beta)$ to cut the circle $x^2 + y^2 = r^2$ at A and B. Then $PA.PB$ is equal to
(A) $(\alpha + \beta)^2 - r^2$ (B) $\alpha^2 + \beta^2 - r^2$
(C) $(\alpha - \beta)^2 + r^2$ (D) None of these

24. If the tangent to the circle $x^2 + y^2 = 5$ at the point (1, –2) also touches the circle $x^2 + y^2 - 8x + 6y + 20 = 0$, then its point of contact is
(A) (3, 1) (B) (–3, 1)
(C) (3, –1) (D) (–3, –1)

25. The line $4x + 3y - 4 = 0$ divides the circumference of the circle centred at (5, 3) in the ratio 1:2. Then the equation of the circle is
(A) $x^2 + y^2 - 10x - 6y - 66 = 0$
(B) $x^2 + y^2 - 10x - 6y + 100 = 0$
(C) $x^2 + y^2 - 10x - 6y + 66 = 0$
(D) None of these

26. The maximum distance of the point (4, 4) from the circle $x^2 + y^2 - 2x - 15 = 0$ is
(A) 10 (B) 9
(C) 5 (D) None of these

27. If the circle $x^2 + y^2 + 4x + 22y + l = 0$ bisects the circumference of the circle $x^2 + y^2 - 2x + 8y - m = 0$, then $l + m$ is equal to
(A) 60 (B) 50
(C) 40 (D) 56

28. The value(*s*) of m for which the line $y = mx$ lies wholly outside the circle $x^2 + y^2 - 2x - 4y + 1 = 0$ is (are)
(A) $m \in (-4/3, 0)$ (B) $m \in (-4/3, 0)$
(C) $m \in (0, 4/3)$ (D) None of these

29. A, B, C, D are the points of intersection with the coordinate axes of the lines $ax + by = ab$ and $bx + ay = ab$, then
(A) A, B, C, D are concyclic
(B) A, B, C, D forms a parallelogram
(C) A, B, C, D forms a rhombus
(D) None of these

30. The length of the tangent from any point on the circle $15x^2 + 15y^2 - 48x + 64y = 0$ to the two circles $5x^2 + 5y^2 - 24x + 32y + 75 = 0$ and $5x^2 + 5y^2 - 48x + 64y + 300 = 0$ are in the ratio of
(A) 1:2 (B) 2:3
(C) 3:4 (D) None of these

31. If the lines $2x - 3y - 5 = 0$ and $3x - 4y = 7$ are diameters of a circle of area 154 square units, then the equation of the circle is
(A) $x^2 + y^2 + 2x - 2y - 62 = 0$
(B) $x^2 + y^2 + 2x - 2y - 47 = 0$
(C) $x^2 + y^2 - 2x + 2y - 47 = 0$
(D) $x^2 + y^2 - 2x + 2y - 62 = 0$

32. The circles $x^2 + y^2 + x + y = 0$ and $x^2 + y^2 + x - y = 0$ intersect at an angle of
(A) $\pi/6$ (B) $\pi/4$
(C) $\pi/3$ (D) $\pi/2$

33. The centres of a set of circles, each of radius 3, lie on the circle $x^2 + y^2 = 25$. The locus of any point with such circle is
(A) $4 \le x^2 + y^2 \le 64$ (B) $x^2 + y^2 \le 25$
(C) $x^2 + y^2 \le 25$ (D) $3 \le x^2 + y^2 \le 9$

34. The chords of contact of the pair of tangents drawn from each point on the line $2x + y = 4$ to the circle $x^2 + y^2 = 1$ pass through a fixed point
(A) (2, 4) (B) (–1/2, –1/4)
(C) (1/2, 1/4) (D) (–2, –4)

35. Equation of a circle $S(x, y) = 0$, $(S(2, 3) = 16)$ which touches the line $3x + 4y - 7 = 0$ at (1, 1) is given by
(A) $x^2 + y^2 + x + 2y - 5 = 0$
(B) $x^2 + y^2 + 2x + 2y - 6 = 0$
(C) $x^2 + y^2 + 4x - 6y = 0$
(D) None of these

36. If a circle $S(x, y) = 0$ touches a line $x + y = 5$ at a point (2, 3) and $S(1, 2) = 0$, then radius of such circle
(A) 2 units (B) 4 units
(C) 1/2 units (D) $1/\sqrt{2}$ units

37. The number of common tangents that can be drawn to the circle $x^2 + y^2 - 4x - 6y - 3 = 0$ and $x^2 + y^2 + 2x + 2y + 1 = 0$ is

(A) 1 **(B)** 2
(C) 3 **(D)** 4

38. A circle S of radius 'a' is the director circle of another circle S_1. S_1 is the director circle of circle S_2 and so on. If the sum of the radii of all these circles is 2, then the value of a is

(A) $2+\sqrt{2}$ **(B)** $2-\sqrt{2}$
(C) $2-\frac{1}{\sqrt{2}}$ **(D)** $2+\frac{1}{\sqrt{2}}$

39. If the distance of the chord of contact of a circle from any point on its director circle is α, then the radius of the circle is

(A) 2α **(B)** $\sqrt{2\alpha}$
(C) $\sqrt{2}\alpha$ **(D)** $2\sqrt{\alpha}$

40. Let AB be a chord of the circle $x^2 + y^2 = r^2$ subtending a right angle at the centre. Then the centroid of the triangle PAB as P moves on the circle is

(A) a parabola **(B)** an ellipse
(C) a circle **(D)** a pair of straight lines

41. Find the equation of the circle whose centre is (1, –2) and radius is 4.

(A) $x^2 + y^2 - 2x - 4y + 11 = 0$
(B) $x^2 + y^2 - 2x + 4y - 11 = 0$
(C) $x^2 + y^2 + 2x - 4y + 11 = 0$
(D) $x^2 - 2x + 4y + 11 = 0$

42. Find the equation of the circle which passes through the point of intersection of the lines $3x - 2y - 1 = 0$ and $4x + y - 27 = 0$ and whose centre is (2, – 3).

(A) 10 **(B)** $\sqrt{97}$
(C) $\sqrt{109}$ **(D)** None of these

43. Find the centre and radius of the circle whose equation is $x^2 + y^2 - 4x + 6y + 12 = 0$.

(A) (2, –3), 1 **(B)** (2, –3), 3
(C) (3, – 2), 1 **(D)** (3, –2), 3

44. Find the equation of the circle, the coordinates of the end points of whose diameter are (–1, 2) and (4, –3).

(A) $x^2 + y^2 + 3x + y - 10 = 0$
(B) $x^2 + y^2 - 3x - y - 10 = 0$
(C) $x^2 + y^2 - 3x + y + 10 = 0$
(D) $x^2 + y^2 - 3x + y - 10 = 0$

45. Find the equation to the circle touching the y-axis at a distance –3 from the origin and intercepting a length 8 on the x-axis.

(A) $x^2 + y^2 \pm 14x + 6y + 9 = 0$
(B) $x^2 + y^2 \pm 6x + 10y + 10 = 0$
(C) $x^2 + y^2 \pm 10x + 6y + 9 = 0$
(D) None of these

46. Find the parametric equations of the circle $x^2 + y^2 - 4x - 2y + 1 = 0$.

(A) $x = 2 + 2\cos\theta$; $y = 1 + 2\sin\theta$
(B) $x = -2 + 2\cos\theta$; $y = -1 + 2\sin\theta$
(C) $x = 4 + 2\cos\theta$; $y = 2 + 2\sin\theta$
(D) None of these

47. Find the centre and the radius of the circle $x = a + c\cdot\cos\theta$, $y = b + c \cdot \sin\theta$.

(A) $(a, c), b$ **(B)** $(b, a), c$
(C) $(a, b), c$ **(D)** None of these

48. Discuss the position of the points (1, 2) and (6, 0) with respect to the circle $x^2 + y^2 - 4x + 2y - 11 = 0$.

(A) (1, 2) inside; (6, 0) inside
(B) (1, 2) outside; (6, 0) inside
(C) (1, 2) outside; (6, 0) outside
(D) (1, 2) inside; (6, 0) outside

49. For what value of c will the line $y = 2x + c$ be a tangent to the circle $x^2 + y^2 = 5$?

(A) ± 5 **(B)** ± 4
(C) ± 3 **(D)** ± 6

50. Find the equation of the tangent to the circle $x^2 + y^2 - 30x + 6y + 109 = 0$ at (4, –1).

(A) $11x + 2y + 46 = 0$ **(B)** $11x - 2y + 46 = 0$
(C) $5x - 3y + 4 = 0$ **(D)** $5x + 3y + 4 = 0$

51. Find the equation of tangents to the circle $x^2 + y^2 - 6x + 4y - 12 = 0$ which are parallel to the line $4x + 3y + 5 = 0$.

(A) $4x + 2y - 11 = 0$ & $4x + 2y + 7 = 0$
(B) $4x + 3y + 19 = 0$ & $4x + 3y - 31 = 0$
(C) $4x - 3y + 11 = 0$ & $4x - 3y - 7 = 0$
(D) None of these

52. Find the equation of the normal to the circle $x^2 + y^2 - 5x + 2y - 48 = 0$ at the point (5, 6).

(A) $14x - 5y - 40 = 0$ **(B)** $14x + 5y + 40 = 0$
(C) $14x + 5y - 40 = 0$ **(D)** $14x - 5y + 40 = 0$

53. Find the equation of the pair of tangents drawn to the circle $x^2 + y^2 - 2x + 4y = 0$ from the point (0, 1).

(A) $x - 2y - 2 = 0$ & $2x - y + 1 = 0$
(B) $x - 2y - 2 = 0$ & $2x + y + 1 = 0$
(C) $x + 2y + 2 = 0$ & $2x - y - 1 = 0$
(D) $x - 2y + 2 = 0$ & $2x + y - 1 = 0$

54. Find the length of the tangent drawn from the point (5, 1) to the circle $x^2 + y^2 + 6x - 4y - 3 = 0$.

(A) 3 **(B)** 5
(C) 7 **(D)** 9

55. Find the equation of director circle of the circle $(x - 2)^2 + (y + 1)^2 = 2$.

(A) $x^2 + y^2 - 4x - 2y - 1 = 0$
(B) $x^2 + y^2 + 4x - 2y - 1 = 0$
(C) $x^2 + y^2 - 4x + 2y + 1 = 0$
(D) $x^2 + y^2 - 4x - 2y - 1 = 0$

56. Find the equation of the chord of contact of the tangents drawn from (1, 2) to the circle $x^2 + y^2 - 2x + 4y + 7 = 0$.

(A) $2y + 5 = 0$ **(B)** $2x + 5 = 0$
(C) $3x - 7 = 0$ **(D)** $3y - 7 = 0$

57. The tangents are drawn to the circle $x^2 + y^2 = 12$ at the points where it is touched by the circle $x^2 + y^2 - 5x + 3y - 2 = 0$. Find the point of intersection of these tangents.

(A) $\left(-6, \frac{18}{5}\right)$ **(B)** $\left(6, \frac{18}{5}\right)$
(C) $\left(6, -\frac{18}{5}\right)$ **(D)** $\left(-6, -\frac{18}{5}\right)$

58. Find the equation of the chord of the circle $x^2 + y^2 + 6x + 8y - 11 = 0$, whose middle point is $(1, -1)$.

(A) $4x + 3y + 12 = 0; 4x + 3y + 1 = 0$
(B) $4x - 3y + 12 = 0; 4x - 3y + 1 = 0$
(C) $4x - 3y - 12 = 0; 4x - 3y - 1 = 0$
(D) $4x + 3y - 12 = 0; 4x + 3y - 1 = 0$

Practice Exercise 2

Single/Multiple Correct Choice Type Questions

1. P is a point on the circle $x^2 + y^2 = 9$ and Q is a point on the line $7x + y + 3 = 0$. If the line $x - y + 1 = 0$ is perpendicular bisector of PQ, then co-ordinates of P may be

(A) $(3, 0)$ **(B)** $\left(\frac{72}{25}, \frac{-21}{25}\right)$
(C) $(0, 3)$ **(D)** $\left(-\frac{72}{25}, \frac{21}{25}\right)$

2. A variable circle touches the angle bisector of the pair of lines $\lambda x^2 + \lambda y^2 + \mu xy = 0$ (where $\lambda, \mu \in R$).

(A) Then locus of centre of circle is a rectangular hyperbola.
(B) Then locus of centre of circle is a pair of straight lines.
(C) If radius of such a circle is 4 units. Then 4 such circles are possible
(D) Then centre of all such circles lies on x-axis.

3. If the circle $x^2 + y^2 - 2x - 2y + 1 = 0$ is inscribed in a triangle whose two sides are axes and one side has negative slope cutting intercepts a and b on x and y axis, then

(A) $\frac{1}{a} + \frac{1}{b} - 1 = -\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$ **(B)** $\frac{1}{a} + \frac{1}{b} < 1$
(C) $\frac{1}{a} + \frac{1}{b} > 1$ **(D)** $\frac{1}{a} + \frac{1}{b} - 1 = \sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$

4. Consider the circle $x^2 + y^2 - 10x - 6y + 30 = 0$. Let O be the centre of the circle and tangent at $A(7, 3)$ and $B(5, 1)$ meet at C. If $S = 0$ represents family of circles passing through A and B, then

(A) area of quadrilateral $OACB = 4$.
(B) the radical axis for the family of circles $S = 0$ is $x + y = 10$.
(C) the smallest possible circle of the family $S = 0$ is $x^2 + y^2 - 12x - 4y + 38 = 0$.
(D) the coordinates of point C are $(7, 1)$.

5. Let x, y be real variable satisfying the $x^2 + y^2 + 8x - 10y - 40 = 0$. If $a = \max\{(x + 2)^2 + (y - 3)^2\}$ and $b = \min\{(x + 2)^2 + (y - 3)^2\}$, then

(A) $a + b = 18$ **(B)** $a + b = 4\sqrt{2}$
(C) $a - b = 4$ **(D)** $a \cdot b = 73$

6. Coordinates of the centre of a circle, whose radius is 2 units and which touches the line pair $x^2 - y^2 - 2x + 1 = 0$ are

(A) $(4, 0)$ **(B)** $(1+2\sqrt{2}, 0)$
(C) $(4, 1)$ **(D)** $(1, 2\sqrt{2})$

7. Point M moved on the circle $(x - 4)^2 + (y - 8)^2 = 20$. Then it broke away from it and moving along a tangent to the circle, cuts the x-axis at the point $(-2, 0)$. The coordinates of a point on the circle at which the moving point broke away is

(A) $\left(-\frac{3}{5}, \frac{46}{5}\right)$ **(B)** $\left(-\frac{2}{5}, \frac{44}{5}\right)$
(C) $(6, 4)$ **(D)** $(3, 5)$

8. If the area of the quadrilateral formed by the tangents from the origin to the circle $x^2 + y^2 + 6x - 10y + c = 0$ and the radii corresponding to the points of contact is 15, then values of c is/are

(A) 9 **(B)** 4
(C) 5 **(D)** 25

Comprehension Type Questions

Paragraph for Questions 9–11: A system of circles is said to be coaxial when every pair of the circles has the same radical axis. It follows from this definition that

1. The centres of all circles of a coaxial system lie on one straight line, which is perpendicular to the common radical axis.
2. Circles passing through two fixed points form a coaxial system for which the line joining the fixed points is the common radical axis.
3. The equation to a coaxial system, of which two members are $S_1 = 0$ and $S_2 = 0$, is $S_1 + \lambda S_2 = 0$, λ is parameter. If we choose the line of centres as x-axis and the common radical axis as y-axis, then the simplest form of equation of coaxial circles is

$$x^2 + y^2 + 2gx + c = 0 \quad (1)$$

where c is fixed and g is arbitrary.

If $g = \pm\sqrt{c}$, then the radius $\sqrt{g^2 - c}$ vanishes and the circles become point circles. The points $(\pm\sqrt{c}', 0)$ are called the limiting points of the system of coaxial circles given by Eq. (1).

9. The equation of the circle which belongs to the coaxial system of circles for which the limiting points are $(1, -1)$, $(2, 0)$ and which passes through the origin is

(A) $x^2 + y^2 - 4x = 0$ **(B)** $x^2 + y^2 + 4x = 0$
(C) $x^2 + y^2 - 4y = 0$ **(D)** $x^2 + y^2 + 4y = 0$

10. If origin be a limiting point of a coaxial system one of whose member is $x^2 + y^2 - 2\alpha x - 2\beta y + c = 0$, then the other limiting point is

(A) $\left(\frac{c\alpha}{\alpha^2+\beta^2}, -\frac{c\beta}{\alpha^2+\beta^2}\right)$ **(B)** $\left(\frac{c\alpha}{\alpha^2+\beta^2}, \frac{c\beta}{\alpha^2+\beta^2}\right)$
(C) $\left(\frac{\alpha\beta}{\alpha^2+\beta^2}, \frac{c\alpha}{\alpha^2+\beta^2}\right)$ **(D)** $\left(-\frac{c\beta}{\alpha^2+\beta^2}, \frac{c\alpha}{\alpha^2+\beta^2}\right)$

11. The equation of the radical axis of the system of coaxial circles $x^2 + y^2 + 2ax + 2by + c + 2\lambda(ax - by + 1) = 0$ is

(A) $ax - by + 1 = 0$ **(B)** $bx + ay - 1 = 0$
(C) $2(ax + by) + 1 = 0$ **(D)** $2(bx - ay) + 1 = 0$

Paragraph for Questions 12–14: Let α-chord of a circle be that chord of the circle which subtends an angle α at the centre.

12. If $x + y = 1$ is α-chord of $x^2 + y^2 = 1$, then α is equal to

(A) $\frac{\pi}{4}$ **(B)** $\frac{\pi}{2}$

(C) $\frac{\pi}{6}$ **(D)** $x + y = 1$ is not a chord

13. If slope of a $\frac{\pi}{3}$ – chord of $x^2 + y^2 = 4$ is 1, then its equation is

(A) $x - y + \sqrt{6} = 0$ **(B)** $x - y = 2\sqrt{3}$

(C) $x - y = \sqrt{3}$ **(D)** $x - y + \sqrt{3} = 0$

14. Distance of $\frac{2\pi}{3}$– chord of $x^2 + y^2 + 2x + 4y + 1 = 0$ from the centre is

(A) 1 **(B)** 2

(C) $\sqrt{2}$ **(D)** $\frac{1}{\sqrt{2}}$

Paragraph for Questions 15–17: See Fig. 12.41. Two variable chords AB and BC of a circle $x^2 + y^2 = a^2$ are such that $AB = BC = a$, and M and N are the midpoints of AB and BC, respectively, such that line joining MN intersect the circle at P and Q where P is closer to AB and O is the centre of the circle

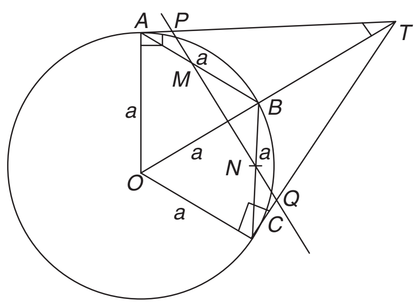


Figure 12.41

15. $\angle OAB$ is

(A) 30° **(B)** 60°

(C) 45° **(D)** 15°

16. Angle between tangents at A and C is

(A) 90° **(B)** 120°

(C) 60° **(D)** 150°

17. Locus of point of intersection of tangents at A and C is

(A) $x^2 + y^2 = a^2$ **(B)** $x^2 + y^2 = 2a^2$

(C) $x^2 + y^2 = 4a^2$ **(D)** $x^2 + y^2 = 8a^2$

Paragraph for Questions 18–20: P is a variable point on the line $L = 0$. Tangents are drawn to the circle $x^2 + y^2 = 4$ from P to touch it at Q and R. The parallelogram $PQSR$ is completed.

18. If $L \equiv 2x + y - 6 = 0$, then the locus of circumcentre of ΔPQR is

(A) $2x - y = 4$ **(B)** $2x + y = 3$

(C) $x - 2y = 4$ **(D)** $x + 2y = 3$

19. If $P \equiv (6, 8)$, then the area of ΔQRS is

(A) $\frac{(6)^{3/2}}{25}$ sq. units **(B)** $\frac{(24)^{3/2}}{25}$ sq. units

(C) $\frac{48\sqrt{6}}{25}$ sq. units **(D)** $\frac{196\sqrt{6}}{25}$ sq. units

20. If $P \equiv (3, 4)$, then coordinate of S is

(A) $\left(-\frac{46}{25}, -\frac{63}{25}\right)$ **(B)** $\left(-\frac{51}{25}, -\frac{68}{25}\right)$

(C) $\left(-\frac{46}{25}, -\frac{68}{25}\right)$ **(D)** $\left(-\frac{68}{25}, -\frac{51}{25}\right)$

Matrix Match Type Questions

21. Match the following:

List I	List II
(A) If $ax + by - 5 = 0$ is the equation of the chord of the circle $(x - 3)^2 + (y - 4)^2 = 4$, which passes through (2, 3) and at the greatest distance from the centre of the circle, then $\|a + b\|$ is equal to	**(p)** 6
(B) Let O be the origin and P be a variable point on the circle $x^2 + y^2 + 2x + 2y = 0$. If the locus of midpoint of OP is $x^2 + y^2 + 2gx + 2fy = 0$, then the value of $(g + f)$ is equal to	**(q)** 3
(C) The x-coordinates of the centre of the smallest circle which cuts the circles $x^2 + y^2 - 2x - 4y - 4 = 0$ and $x^2 + y^2 - 10x + 12y + 52 = 0$ orthogonally is	**(r)** 2
(D) If θ be the angle between two tangents which are drawn to the circle $x^2 + y^2 - 6\sqrt{3}x - 6y + 27 = 0$ from the origin, then $2\sqrt{3}\tan\theta$ equals	**(s)** 1
	(t) 4

22. Match the following:

List I	List II
(A) The length of the common chord of two circles of radii 3 and 4 units which intersect orthogonally is $\frac{k}{5}$, then k equals	**(p)** 1
(B) The circumference of the circle $x^2 + y^2 + 4x + 12y + p = 0$ is bisected by the circle $x^2 + y^2 - 2x + 8y - q = 0$, then $p + q$ is equal to	**(q)** 24
(C) Number of distinct chords of the circle $2x(x - \sqrt{2}) + y(2y - 1) = 0$ passing through the point $\left(\sqrt{2}, \frac{1}{2}\right)$ and are bisected by x-axis is	**(r)** 32

List I	List II
(D) One of the diameters of the circle circumscribing the rectangle $ABCD$ is $4y = x + 7$. If A and B are the points $(-3, 4)$ and $(5, 4)$ respectively, then the area of the rectangle is equal to	**(s)** 2
	(t) 36

Integer Type Questions

23. A circle touches the hypotenuse of a right-angled triangle at its middle point and passes through the middle point of shorter side. If 3 and 4 units be the length of the sides and 'r' be the radius of the circle, then find the value of '$3r$'.

24. A circle with centre in the first quadrant is tangent to $y = x + 10$, $y = x - 6$ and the y-axis. Let (h, k) be the centre of the circle. If the value of $(h + k) = a + b\sqrt{a}$, where $(a, b \in Q)$, find the value of $(a + b)$.

25. S is a circle having centre at $(0, \alpha)$ and radius $b(b < a)$. A variable circle centred at $(\alpha, 0)$ and touching circle S, meets the x-axis at M and N. A point $P \equiv (0, \pm\lambda\sqrt{a^2 - b^2})$ on the Y-axis, such that $\angle MPN$ is a constant for any choice of α, then find λ.

26. If C_1: $x^2 + y^2 = (3 + 2\sqrt{2})^2$ be a circle and PA and PB are pair of tangents on C_1, where P is any point on the director circle of C_1, then find the radius of smallest circle which touches C_1 externally and also the two tangents PA and PB.

Answer Key

Practice Exercise 1

1. (D) **2.** (C) **3.** (B) **4.** (C) **5.** (D) **6.** (A)
7. (C) **8.** (A) **9.** (A) **10.** (A) **11.** (C) **12.** (A)
13. (D) **14.** (C) **15.** (C) **16.** (D) **17.** (A) **18.** (D)
19. (A) **20.** (C) **21.** (B) **22.** (D) **23.** (B) **24.** (C)
25. (A) **26.** (B) **27.** (B) **28.** (A) **29.** (A) **30.** (A)
31. (C) **32.** (D) **33.** (A) **34.** (C) **35.** (A) **36.** (D)
37. (C) **38.** (B) **39.** (C) **40.** (C) **41.** (B) **42.** (C)
43. (A) **44.** (D) **45.** (C) **46.** (A) **47.** (C) **48.** (D)
49. (A) **50.** (B) **51.** (B) **52.** (A) **53.** (D) **54.** (C)
55. (C) **56.** (A) **57.** (C) **58.** (D)

Practice Exercise 2

1. (A), (D) **2.** (B), (C) **3.** (A), (B) **4.** (A), (C), (D) **5.** (A), (C), (D) **6.** (B), (D)
7. (B), (C) **8.** (A), (D) **9.** (D) **10.** (B) **11.** (A) **12.** (B)
13. (A) **14.** (A) **15.** (B) **16.** (C) **17.** (C) **18.** (B)
19. (D) **20.** (B) **21.** (A) $\to$ (r), (B) $\to$ (s), (C) $\to$ (q), (D) $\to$ (p) **22.** (A) $\to$ (q), (B) $\to$ (t), (C) $\to$ (p), (D) $\to$ (r)
23. 5 **24.** 10 **25.** 1 **26.** 1

Solutions

Practice Exercise 1

1. $x^2 + (mx - 2x - 1)^2 = 1$

$$\Rightarrow x^2 + (m-2)^2 x^2 - 2(m-2)x = 0$$
$$\Rightarrow [1 + (m-2)^2]x^2 - 2(m-2)x = 0$$

Disc ≥ 0 which is true, $\forall\, m$

2.

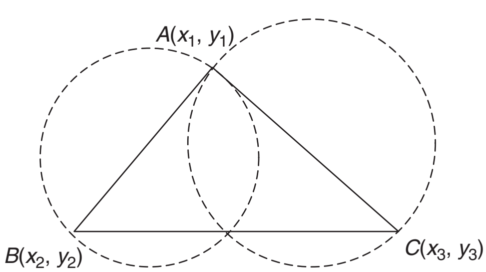


Figure 12.42

See Fig. 12.42. Equation of circle through AB as diameter is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0 \quad (1)$$

Equation of circle through AC as diameter is

$$(x - x_1)(x - x_3) + (y - y_1)(y - y_3) = 0 \quad (2)$$

Radical axis of circles (1) and (2) is

$$(x - x_1)(x_3 - x_2) + (y - y_1)(y_3 - y_2) = 0$$

Slope of radical axis is $m_1 = -\dfrac{(x_3 - x_2)}{y_3 - y_2}$

Slope of AC is $m_2 = \dfrac{y_3 - y_2}{x_3 - x_2}$

Therefore,

$$m_1 m_2 = -1$$

So, radical axis is altitude through A. Hence, intersection of altitudes is orthocentre.

3. The equation of the circle described on the line joining (0, 1), (a, b) as diameter is

$$(x-0)(x-a)+(y-1)(y-b)=0$$
$$\Rightarrow x^2+y^2-ax-y(1+b)+b=0$$

This meets the x-axis at $y = 0$. Therefore, the abscissa of the points where the circle meets the x-axis are roots of the equation $x^2 - ax + b = 0$.

4. If the straight line $y = mx + c$ cuts the circle $x^2 + y^2 = a^2$ in the real points, then the equation $x^2 + (mx + c)^2 = a^2$, that is, $x^2(1 + m^2) + 2mcx + c^2 - a^2 = 0$ has real roots. Hence,

$$4m^2c^2 - 4(1+m^2)(c^2-a^2) \ge 0$$
$$\Rightarrow -c^2 + a^2(1+m^2) \ge 0$$
$$\Rightarrow a^2(1+m^2) \ge c^2$$
$$\Rightarrow \sqrt{a^2(1+m^2)} \ge |c|$$

5. Let the circle be given as $x^2 + y^2 + 2gx + 2fy + c = 0$. This passes through (0, 0) and (1, 0).
Therefore,

$$c = 0, \text{ and}$$
$$1 + 2g = 0 \Rightarrow g = -\frac{1}{2}$$

It is given that the above circle touches $x^2 + y^2 = 9$. The centre of this circle (0, 0) lies on the above given circle. From this, it follows that the given circle touches the circle, $x^2 + y^2 = 9$ internally.
Thus, the diameter of the required circle must be equal to the radius of the circle $x^2 + y^2 = 9$. Hence, we have

$$2\sqrt{g^2+f^2} = 3 \Rightarrow f = \pm\sqrt{2}$$

Hence, centres of the required circle are $\left(\frac{1}{2}, \pm\sqrt{2}\right)$.

6. If the circles are drawn on the sides of the triangle formed by $x = 0$, $y = 0$, and $x + y = 2$ as diameter, then radical centre will lie on orthocenter (Figure 12.43).

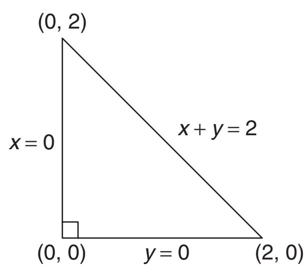


Figure 12.43

The orthocentre of right-angled triangle lies on point of right angle. So, radical centre is (0, 0).

7.

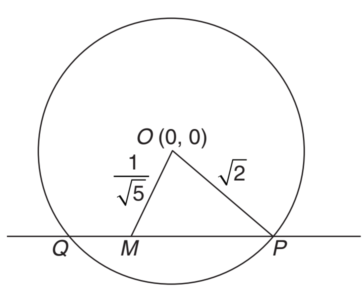


Figure 12.44

We have

OM = length of the ⊥ from (0, 0) on $y = 2x + 1 = \frac{1}{\sqrt{5}}$

and OP = radius of the given circle $= \sqrt{2}$

$$PQ = 2\,PM = 2\sqrt{OP^2 - OM^2} = 2\sqrt{2-\frac{1}{5}} = \frac{6}{\sqrt{5}}$$

8. The mid-point is the intersection of the chord and the perpendicular line on it from the centre (3, −1). The equation of the perpendicular line is $5x + 2y - 13 = 0$. Therefore, the required point is (1, 4).

9. For director circle, centre remains same and radius is $\sqrt{2}\,r$. So, equation of director circle is

$$x^2 + y^2 - 6x - 8y - 25 = 0$$

10. Equation of the circle is

$$\lambda(2x+3y+1) + (x-1)^2 + (y+1)^2 = 0$$

It passes through (1, 0) $\Rightarrow \lambda = \frac{-1}{3}$

Therefore, equation of circle is $3x^2 + 3y^2 - 8x + 3y + 5 = 0$

11. Two tangents can be drawn if point lies outside the circle, or $s_1 > 0$. Hence,

$$2k^2 + 4k - 16 = 2(k^2 + 2k - 8) > 0$$
$$\Rightarrow k \in (-\infty, -4) \cup (2, \infty)$$

12. Diameter = distance between the parallel lines

$$= \left|\frac{\lambda}{5} - \frac{\mu}{10}\right| = \left|\frac{2\lambda-\mu}{10}\right| \Rightarrow \text{Radius} = \left|\frac{2\lambda-\mu}{20}\right|$$

13. Let the centre is (h, k) passing through (0, 0). So, the equation of circles is

$$x^2 + y^2 - 2hx - 2ky = 0$$

and $$x^2 + y^2 - 5x + 3y - 1 = 0$$

cut each other orthogonally. Therefore,

$$2g_1g_2 + 2f_1f_2 = c_1 + c_2$$
$$\Rightarrow 2(-h)\left(\frac{-5}{2}\right) + 2(-k)\left(\frac{3}{2}\right) = -1$$
$$\Rightarrow 5h - 3k = -1$$

Therefore, straight line is $5x - 3y = -1$.

14. Equation of line is $y - 1 = m(x - 1) \Rightarrow$ line passes through (1, 1) since (1, 1) lies inside the circle, m can take any real value.

15. Equation of circle is

$$(x-1)^2 + (y-2)^2 + \lambda(3x+4y-11) = 0$$
$$x^2 + y^2 + x(3\lambda - 2) + y(4\lambda - 4) + (5 - 11\lambda) = 0$$
$$\text{Centre} \equiv \left(\frac{2-3\lambda}{2},\ 2-2\lambda\right)$$

Centre lies on the x-axis $\Rightarrow \lambda = 1$
Therefore, equation of the circle is $x^2 + y^2 + x - 6 = 0$.

16. Let the circle be $(x-4)^2 + (y-3)^2 = r^2$. The point (4, 3) lies outside $x^2 + y^2 = 1$. So, they touch each other externally.

Therefore, $r + 1 = \sqrt{(4-0)^2 + (3-0)^2} \Rightarrow r = 4$

Hence, the circle is

$$(x-4)^2+(y-3)^2=4^2$$
$$\Rightarrow x^2+y^2-8x-6y+9=0$$

17. See Fig. 12.45. $r=\sqrt{36+4-30}=\sqrt{10}$

$$OC=\sqrt{6^2+2^2}=\sqrt{40}$$
$$OC=2r \Rightarrow OC=2PC$$

Now,

$$OC{:}PC=2{:}1$$
$$\Rightarrow P\equiv(9,3)$$

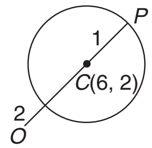


Figure 12.45

18. Locus of points from which length of tangents are equal is radical axis.

Equation of radical is $S_1-S_2=0$

$$\Rightarrow x^2+y^2-4-\left[x^2+y^2-5x+\frac{3}{2}y-1\right]=0$$

$$\Rightarrow -4+5x-\frac{3}{2}y+1=0$$

$$\Rightarrow 10x-3y-6=0$$

Therefore, the locus of a point is a straight line perpendicular to the line joining the centres of the circles.

19. See Fig. 12.46. Centre is the mid-point of $AC\equiv(4,7)$.

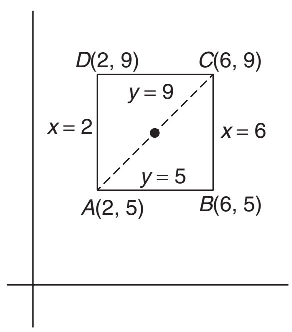


Figure 12.46

20. See Fig. 12.47. The two given circles are concentric.

From figure, $OP=a$, $OA=a\sin\alpha$

If 2θ is the angle between the tangents,

$$\sin\theta=\frac{a\sin\alpha}{a}\Rightarrow\theta=\alpha$$

Hence, the required angle between PA and $PB=2\alpha$.

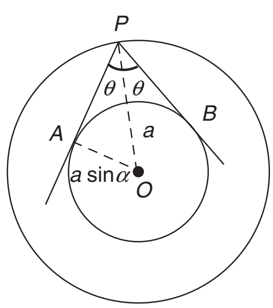


Figure 12.47

21. Any line passing through (2, 2) will be of the form

$$\frac{y-2}{\sin\theta}=\frac{x-2}{\cos\theta}=r$$

When this line cuts the circle $x^2+y^2=2$,

$$(r\cos\theta+2)^2+(r\sin\theta+2)^2=2$$

$$\Rightarrow r^2+4(\sin\theta+\cos\theta)r+6=0$$

$$\frac{PB}{PA}=\frac{r_2}{r_1}$$

Now if $r_1=\alpha, r_2=3\alpha$, then,

$4\alpha=-4(\sin\theta+\cos\theta)$, $3\alpha^2=6\Rightarrow\sin2\theta=1\Rightarrow\theta=\pi/4$

So, required chord will be $y-2=1(x-2)\Rightarrow y=x$.

Alternative solution:

$$PA\cdot PB=PT^2=2^2+2^2-2=6 \quad (1)$$

$$\frac{PB}{PA}=3 \quad (2)$$

From Eqs. (1) and (2), we have $PA=\sqrt{2}$, $PB=3\sqrt{2}$

$$\Rightarrow AB=2\sqrt{2}$$

Now, diameter of the circle is $2\sqrt{2}$ (as radius is $\sqrt{2}$).

Hence, the line passes through the centre is $y=x$.

22. The equation of the circle passing through the point (1, 0), (0, 1) and (0, 0) is $x^2+y^2-x-y=0$, which passes through $(2K, 3K)$, if

$$4K^2+9K^2-2K-3K=0\Rightarrow K=0, K=\frac{5}{13}$$

23. The equation of any line through $P(\alpha,\beta)$ is $\frac{x-\alpha}{\cos\theta}=\frac{y-\beta}{\sin\theta}=k$ (say). Any point on this line is $(\alpha+k\cos\theta, \beta+k\sin\theta)$. This point lies on the given circle if

$$(\alpha+k\cos\theta)^2+(\beta+k\sin\theta)^2=r^2$$

$$\Rightarrow k^2+2k(\alpha\cos\theta+\beta\sin\theta)+\alpha^2+\beta^2-r^2=0 \quad (1)$$

This, being quadratic in k, gives two values of k and hence, the distance of two points A and B on the circle from the point P.

Let $PA=k_1$, $PB=k_2$, where k_1, k_2 are the roots of (I). Then,

$$PA\cdot PB=k_1k_2=\alpha^2+\beta^2-r^2$$

24. Equation of tangent to $x^2+y^2=5$ at $(1,-2)$ is

$$x-2y=5 \quad (1)$$

Also, equation of tangent to $x^2+y^2-8x+6y+20=0$ at (α,β) is

$$(\alpha-4)x+(\beta+3)y=4\alpha-3\beta-20 \quad (2)$$

Since Eqs. (1) and (2) represent the same line, comparing Eqs. (1) and (2), we get

$$\frac{\alpha-4}{1}=\frac{\beta+3}{-2}=\frac{4\alpha-3\beta-20}{5}$$

On simplifying, we get

$$\alpha=3, \beta=-1$$

25. Since $4x+3y-4=0$ is dividing the circumference in the ratio 1:2. Therefore, angle subtended at the centre $=2\pi/3$. Also, the perpendicular distance from the centre to the given line is 5.

So, radius = 10
Therefore, equation of the circle is $x^2 + y^2 - 10x - 6y - 66 = 0$.

26. See Fig. 12.48.
Maximum distance of P from the circle = $PC + CB$

$$= \sqrt{(4-1)^2 + (4-0)^2} + 4 = 9$$

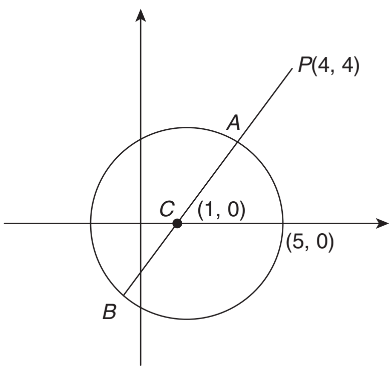


Figure 12.48

27. The common chord of the given circles can be obtained by $S_1 - S_2 = 0$

Hence,

$$6x + 14y + l + m = 0 \qquad (1)$$

Since, $x^2 + y^2 + 4x + 22y + l = 0$ bisects the circumference of the circle $x^2 + y^2 - 2x + 8y - m = 0$, Eq. (1) passes through the centre of the second circle, that is, $(1, -4)$.
Therefore, $6 - 56 + l + m = 0 \Rightarrow l + m = 50$

28. Points of intersection of the line and circle are given by

$$x^2(1 + m^2) - 2x(1 + 2m) + 1 = 0$$

The given line will lie outside the circle if, $D < 0$

$$\Rightarrow 4(1 + 2m)^2 - 4(1 + m^2) < 0$$

$$\Rightarrow 3m^2 + 4m < 0$$

$$\Rightarrow 3m\left(m + \frac{4}{3}\right) < 0 \Rightarrow m \in \left(-\frac{4}{3},\ 0\right)$$

29. See Fig. 12.49. Points of intersection are

$$A(b, 0), B(0, a), C(a, 0), D(0, b)$$

$$\Rightarrow OA \cdot OC = ab = OD \cdot OB$$

Therefore, A, B, C, D are concyclic.

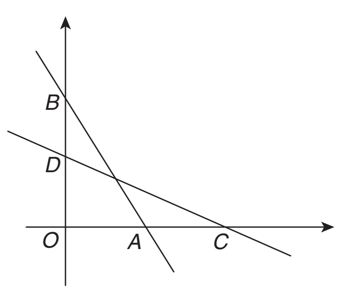


Figure 12.49

30. Let (h, k) be the point on

$$15x^2 + 15y^2 - 48x + 64y = 0 \qquad (1)$$

From which tangents are drawn to the two circles, then

$$\frac{\sqrt{5h^2 + 5k^2 - 24h + 32k + 75}}{\sqrt{5h^2 + 5k^2 - 48h + 64k + 300}} = \frac{1}{r}$$

$\Rightarrow r^2(5h^2 + 5k^2 - 24h + 32k + 75) = 5h^2 + 5k^2 - 48h + 64k + 300$
$\Rightarrow 5(r^2 - 1)(h^2 + k^2) - 24h(r^2 - 2) + 32k(r^2 - 2) + 75(r^2 - 4) = 0$
But, $15h^2 + 15k^2 - 48h + 64k = 0$, since (h, k) lies on Eq. (1).
Both these are simultaneously true only if $r = 2$.

Therefore, required ratio = $\frac{1}{2}$

31. Centre of the circle is given by solving

$$2x - 3y - 5 = 0$$

$$3x - 4y - 7 = 0$$

Therefore,

$$\frac{x}{21-20} = \frac{y}{-15+14} = \frac{1}{-8+9}$$

$$\Rightarrow x = 1, y = -1$$

Therefore, centre is $(1, -1)$.

$$\pi r^2 = 154$$

$$\Rightarrow r^2 = \frac{154 \times 7}{22} = 49 \Rightarrow r = 7$$

Therefore, circle is $(x - 1)^2 + (y + 1)^2 = 49$, that is,

$$x^2 + y^2 - 2x + 2y - 47 = 0$$

32. The angle of intersection of two circles is given by $\cos\theta = \frac{r_1^2 + r_2^2 - (C_1C_2)^2}{2r_1r_2}$ where r_1, r_2 are radii of two circles and C_1C_2 is the distance between their centres. Here,

$$r_1 = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{1}{2}} = r_2 \text{ and } C_1C_2 = 1$$

Hence,

$$\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$$

33. See Fig. 12.50. The centres are $(5\cos\theta, 5\sin\theta)$. So, equation of such circle is

$$(x - 5\cos\theta)^2 + (y - 5\sin\theta)^2 = 9$$

Let (h, k) be any point within such circle. Then, (h, k) lie in shaded area. Therefore, $2 \le$ distance of (h, k) from $0 \le 8$.
Hence, $4 \le x^2 + y^2 \le 64$.

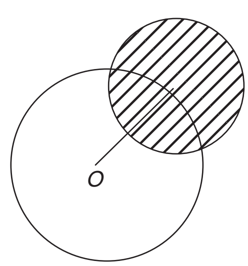


Figure 12.50

34. The chord of contact of tangents from (α, β) is

$$\alpha x + \beta y = 1 \qquad (1)$$

Also, (α, β) lies on $2x + y = 4$, so $2\alpha + \beta = 4 \Rightarrow \frac{\alpha}{2} + \frac{\beta}{4} = 1$

Hence, Eq. (1) passes through $\left(\frac{1}{2}, \frac{1}{4}\right)$.

35. Any circle which touches $3x + 4y - 7 = 0$ at (1, 1) will be of the form

$$S(x, y) \equiv (x - 1)^2 + (y - 1)^2 + \lambda(3x + 4y - 7) = 0$$

Since, $S(2, 3) = 16 \Rightarrow \lambda = 1$

So, required circle will be

$$x^2 + y^2 + x + 2y - 5 = 0$$

36. Desired equation of the circle is $(x-2)^2 + (y-3)^2 + \lambda(x + y - 5) = 0$

Since, $S(1, 2) = 0$

$$\Rightarrow 1 + 1 + \lambda(1 + 2 - 5) = 0 \Rightarrow \lambda = 1$$

$$x^2 - 4x + 4 + y^2 - 6y + 9 + x + y - 5 = 0$$

$$\Rightarrow x^2 + y^2 - 3x - 5y + 8 = 0$$

$$\left(x^2 - \frac{3}{2}\right)^2 + \left(y - \frac{5}{2}\right)^2 = -8 + \frac{25}{4} + \frac{9}{4} = \frac{2}{4} = \frac{1}{2}$$

37. The two circles are

$x^2 + y^2 - 4x - 6y - 3 = 0$ and $x^2 + y^2 + 2x + 2y + 1 = 0$

Centre: $C_1 \equiv (2, 3)$, $C_2 \equiv (-1, -1)$; radii: $r_1 = 4$, $r_2 = 1$

We have $C_1 C_2 = 5 = r_1 + r_2$, therefore there are 3 common tangents to the given circles.

38. $a + \frac{a}{\sqrt{2}} + \frac{a}{2} + \frac{a}{2\sqrt{2}} + \cdots = 2$

$$a\left[\frac{1}{1 - \frac{1}{\sqrt{2}}}\right] = 2 \Rightarrow a = 2 - \sqrt{2}$$

39.

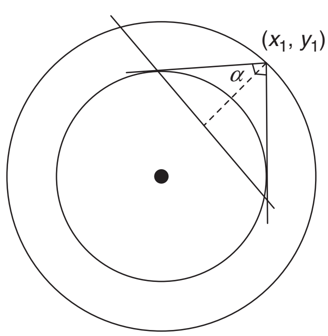


Figure 12.51

(See Fig. 12.51). If equation of circle is $x^2 + y^2 = r^2$ and equation of director circle is $x^2 + y^2 = 2r^2$. Let (x_1, y_1) is a point on director circle

$$x^2 + y^2 = 2r^2.$$

So,

$$x_1^2 + y_1^2 = 2r^2$$

Now equation of chord of contact is

$$xx_1 + yy_1 - 2r^2 = 0$$

Distance from (x_1, y_1) is $\frac{x_1^2 + y_2^2 - r^2}{\sqrt{x_1^2 + y_1^2}} = 9$

$$\Rightarrow \frac{2r^2 - r^2}{\sqrt{2r^2}} = \alpha \Rightarrow \frac{r}{\sqrt{2}} = \alpha \Rightarrow r = \sqrt{2}\,\alpha$$

40. See Fig. 12.52. Let the point P be $(r \cos \theta, r \sin \theta)$ and the centroid be (α, β). Therefore,

$$\alpha = \frac{r + r\cos\theta}{3}, \beta = \frac{r + r\sin\theta}{3}$$

Locus is $\left(x - \frac{r}{3}\right)^2 + \left(y - \frac{r}{3}\right)^2 = \left(\frac{r}{3}\right)^2$, which is a circle.

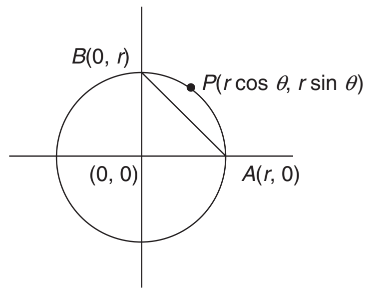


Figure 12.52

41. The equation of the circle is

$$(x - 1)^2 + [y - (-2)]^2 = 4^2$$

$$\Rightarrow (x - 1)^2 + (y + 2)^2 = 16$$

$$\Rightarrow x^2 + y^2 - 2x + 4y - 11 = 0$$

42. Let P be the point of intersection of the lines AB and LM whose equations, respectively, are

$$3x - 2y - 1 = 0 \qquad (1)$$

and

$$4x + y - 27 = 0 \qquad (2)$$

Solving Eqs. (1) and (2), we get $x = 5$, $y = 7$.

Hence, the coordinates of point P are (5, 7).

Let C(2, –3) be the centre of the circle. Since the circle passes through point P, we have

CP = radius

$$\Rightarrow \sqrt{(5-2)^2 + (7+3)^2} = \text{radius}$$

Therefore, the radius is $\sqrt{109}$.

43. On comparing with the general equation $x^2 + y^2 + 2gx + 2fy + c = 0$, we have

$$2g = -4 \Rightarrow g = -2$$

$$2f = 6 \Rightarrow f = 3$$

and

$$c = 12$$

Therefore, the centre is $(-g, -f)$, that is, (2, –3) and the radius is

$$\sqrt{g^2 + f^2 - c} = \sqrt{(-2)^2 + (3)^2 - 12} = 1$$

44. We know that the equation of the circle described on the line segment joining (x_1, y_1) and (x_2, y_2) as a diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$. Here, $x_1 = -1, x_2 = 4, y_1 = 2$ and $y_2 = -3$. Hence, the equation of the required circle is

$$(x + 1)(x - 4) + (y - 2)(y + 3) = 0$$
$$\Rightarrow x^2 + y^2 - 3x + y - 10 = 0$$

45. Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$. Since it touches y-axis at (0, –3) and (0, –3) lies on the circle. Therefore,

$$c = f^2 \quad (1)$$
$$9 - 6f + c = 0 \quad (2)$$

From Eqs. (1) and (2), we get

$$9 - 6f + f^2 = 0$$
$$\Rightarrow (f - 3)^2 = 0$$
$$\Rightarrow f = 3$$

Substituting $f = 3$ in Eq. (1), we get $c = 9$. It is given that the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ intercepts length 8 on x-axis. Therefore,

$$2\sqrt{g^2 - c} = 8$$
$$\Rightarrow 2\sqrt{g^2 - 9} = 8$$
$$\Rightarrow g^2 - 9 = 16$$
$$\Rightarrow g = \pm 5$$

Hence, the required circle is $x^2 + y^2 \pm 10x + 6y + 9 = 0$.

46. We have

$$x^2 + y^2 - 4x - 2y + 1 = 0$$
$$\Rightarrow (x^2 - 4x) + (y^2 - 2y) = -1$$
$$\Rightarrow (x - 2)^2 + (y - 1)^2 = 2^2$$

Thus, the parametric equations of this circle are

$$x = 2 + 2\cos\theta, y = 1 + 2\sin\theta$$

47. We have

$$x = a + c\cdot\cos\theta; y = b + c\cdot\sin\theta$$
$$\Rightarrow \cos\theta = \frac{x-a}{c},\ \sin\theta = \frac{y-b}{c}$$
$$\Rightarrow \left(\frac{x-a}{c}\right)^2 + \left(\frac{y-b}{c}\right)^2 = \cos^2\theta + \sin^2\theta = 1$$
$$\Rightarrow (x - a)^2 + (y - b)^2 = c^2$$

Hence, it is a circle with its centre at (a, b) and radius c.

48. We have

$$x^2 + y^2 - 4x + 2y - 11 = 0 \text{ or } S = 0$$

where

$$S = x^2 + y^2 - 4x + 2y - 11$$

For the point (1, 2), we have

$$S_1 = 1^2 + 2^2 - 4 \times 1 + 2 \times 2 - 11 < 0$$

For the point (6, 0), we have

$$S_2 = 6^2 + 0^2 - 4 \times 6 + 2 \times 0 - 11 > 0$$

Hence, the point (1, 2) lies inside the circle and the point (6, 0) lies outside the circle.

49. We have

$$y = 2x + c \text{ or } 2x - y + c = 0 \quad (1)$$

and
$$x^2 + y^2 = 5 \quad (2)$$

If the line expressed in Eq. (1) touches the circle expressed in Eq. (2), then the length of the perpendicular from the centre (0, 0) is equal to the radius of circle [Eq. (2)].

$$\left|\frac{2\times 0 - 0 + c}{\sqrt{2^2 + (-1)^2}}\right| = \sqrt{5} \Rightarrow \left|\frac{c}{\sqrt{5}}\right| = \sqrt{5}$$
$$\Rightarrow \frac{c}{\sqrt{5}} = \pm\sqrt{5} \Rightarrow c = \pm 5$$

Hence, the line expressed in Eq. (1) touches the circle expressed in Eq. (2) for $c = \pm 5$.

50. The equation of the tangent is

$$4x + (-y) - 30\left(\frac{x+4}{2}\right) + 6\left(\frac{y+(-1)}{2}\right) + 109 = 0$$
$$\Rightarrow 4x - y - 15x - 60 + 3y - 3 + 109 = 0$$
$$\Rightarrow -11x + 2y + 46 = 0$$
$$\Rightarrow 11x - 2y - 46 = 0$$

Hence, the required equation of the tangent is $11x - 2y - 46 = 0$.

51. It is given that the circle is

$$x^2 + y^2 - 6x + 4y - 12 = 0 \quad (1)$$

and the given line is

$$4x + 3y + 5 = 0 \quad (2)$$

The centre of circle [Eq. (1)] is (3, –2) and its radius is 5. The equation of any line, $4x + 3y + k = 0$, which is parallel to the line expressed in Eq. (2) is tangent to circle expressed in Eq. (1), then

$$\frac{|4(3) + 3(-2) + k|}{\sqrt{4^2 + 3^2}} = 5$$
$$\Rightarrow |6 + k| = 25$$
$$\Rightarrow 6 + k = \pm 25$$

Therefore, $k = 19, -31$. Hence, the equation of the required tangents are $4x + 3y + 19 = 0$ and $4x + 3y - 31 = 0$.

52. The equation of the tangent to the circle $x^2 + y^2 - 5x + 2y - 48 = 0$ at (5, 6) is

$$5x + 6y - 5\left(\frac{x+5}{2}\right) + 2\left(\frac{x+6}{2}\right) - 48 = 0$$
$$\Rightarrow 10x + 12y - 5x - 25 + 2y + 12 - 96 = 0$$
$$\Rightarrow 5x + 14y - 109 = 0$$

Therefore, the slope of the tangent is –5/14 and hence, the slope of the normal is 14/5. Thus, the equation of the normal at (5, 6) is

$$y - 6 = \left(\frac{14}{5}\right)(x - 5)$$
$$\Rightarrow 14x - 5y - 40 = 0$$

53. It is given that the circle is

$$S = x^2 + y^2 - 2x + 4y = 0 \quad (1)$$

Let $P \equiv (0, 1)$. For the point P,

$$S_1 = 0^2 + 1^2 - 1(0) + 4(1) = 5$$

Hence, the point P lies outside the circle and

$$T \equiv x(0) + y(1) - (x + 0) + 2(y + 1)$$

$$\Rightarrow T \equiv -x + 3y + 2$$

Now, the equation of pair of tangents from P(0, 1) to the circle [Eq. (1)] is

$$SS_1 = T^2$$

$$\Rightarrow 5(x^2 + y^2 - 2x + 4y) = (-x + 3y + 2)^2$$
$$\Rightarrow 5x^2 + 5y^2 - 10x + 20y = x^2 + 9y^2 + 4 - 6xy - 4x + 12y$$
$$\Rightarrow 4x^2 - 4y^2 - 6x + 8y + 6xy - 4 = 0$$
$$\Rightarrow 2x^2 - 2y^2 + 3xy - 3x + 4y - 2 = 0 \quad (2)$$

Note: Separate equation of pair of tangents: From Eq. (2),

$$2x^2 + 3(y - 1)x - 2(2y^2 - 4y + 2) = 0$$

Therefore,

$$x = \frac{3(y-1) \pm \sqrt{9(y-1)^2 + 8(2y^2 - 4y + 2)}}{4}$$

$$\Rightarrow 4x - 3y + 3 = \pm\sqrt{25y^2 - 50y + 25} = \pm 5(y-1)$$

Therefore, the separate equations of tangents are

$$x - 2y + 2 = 0$$

and $$2x + y - 1 = 0$$

54. The given circle is

$$x^2 + y^2 + 6x - 4y - 3 = 0 \quad (1)$$

The given point is (5, 1). Let P = (5, 1). Now, the length of the tangent from P(5, 1) to circle [Eq. (1)] is

$$\sqrt{5^2 + 1^2 + 6.5 - 4.1 - 3} = 7$$

55. The centre and radius of given circle are (2, –1) and $\sqrt{2}$, respectively. The centre and radius of the director circle is (2, –1) and $\sqrt{2} \times \sqrt{2} = 2$, respectively. Therefore, the equation of the director circle is

$$(x - 2)^2 + (y + 1)^2 = 4$$
$$\Rightarrow x^2 + y^2 - 4x + 2y + 1 = 0$$

56. The given circle is

$$x^2 + y^2 - 2x + 4y + 7 = 0 \quad (1)$$

Let P = (1, 2). For the point P(1, 2), we have

$$x^2 + y^2 - 2x + 4y + 7 = 1 + 4 - 2 + 8 + 7 = 18 > 0$$

Hence, the point P lies outside the circle. For the point P(1, 2), we have

$$T = x(1) + y(2) - (x + 1) + 2(y + 2) + 7$$

$$T = 4y + 10$$

Now, the equation of the chord of contact of point P(1, 2), w.r.t. circle [Eq. (1)] is

$$4y + 10 = 0$$
$$\Rightarrow 2y + 5 = 0$$

57. See Fig. 12.53. The given circles are

$$S_1 \equiv x^2 + y^2 - 12 = 0 \quad (1)$$

and $$S_2 = x^2 + y^2 - 5x + 3y - 2 = 0 \quad (2)$$

Now, the equation of the common chord of circles [Eqs. (1) and (2)] is

$$S_1 - S_2 = 0$$

That is,

$$5x - 3y - 10 = 0 \quad (3)$$

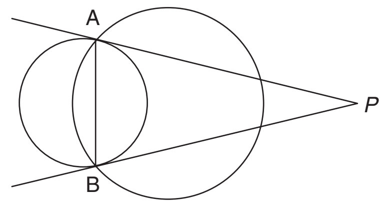


Figure 12.53

Let this line meet circle [Eq. (1) or Eq. (2)] at points A and B (Fig. 12.51). Let the tangents to circle [Eq. (1)] at points A and B meet at $P(\alpha, \beta)$. Then AB is the chord of contact of the tangents to the circle [Eq. (1)] from point P. Therefore, the equation of AB is

$$x\alpha + y\beta - 12 = 0 \quad (4)$$

Now, lines [Eqs. (3) and (4)] are same. So, the equations [Eqs. (3) and (4)] are identical. Therefore,

$$\frac{\alpha}{5} = \frac{\beta}{-3} = \frac{-12}{-10}$$

$$\alpha = 6, \beta = -\frac{18}{5}$$

Hence,

$$P = \left(6, -\frac{18}{5}\right)$$

58. The equation of the given circle is

$$S \equiv x^2 + y^2 + 6x + 8y - 11 = 0$$

Let $L \equiv (1, -1)$. For the point L(1, –1), we have

$$S_1 = 1^2 + (-1)^2 + 6(1) + 8(-1) - 11 = -11$$

and $$T \equiv x(1) + y(-1) + 3(x + 1) + 4(y - 1) - 11$$

$$T \equiv 4x + 3y - 12$$

Now, the equation of the chord of circle [Eq. (1)] whose middle point is L(1, –1) is

$$T = S_1$$
$$\Rightarrow 4x + 3y - 12 = -11$$
$$\Rightarrow 4x + 3y - 1 = 0$$

Practice Exercise 2

1. Let $P(3\cos\alpha, 3\sin\alpha)$

Let $Q(x_1, -7x_1 - 3)$

$x - y + 1 = 0$ is perpendicular bisector of PQ, then midpoint of PQ lie on

$$x - y + 1 = 0$$

$$24x_1 + 9\cos\alpha - 9\sin\alpha + 15 = 0 \quad (1)$$

Also,

$$1 \times \frac{3\sin\alpha + 7x_1 + 3}{3\cos\alpha - x_1} = -1 \quad (2)$$

Solving Eqs. (1) and (2), we get $\cos\alpha = 1$ and $\cos\alpha = \frac{-24}{25}$

Therefore, P is (3, 0) or $\left(\frac{-72}{25}, \frac{21}{25}\right)$.

2. Equation of angle bisector is $x^2 - y^2 = 0$.

Let the centre of the circle be (h, k). Then

$$|h - k| = |h + k| \Rightarrow x = 0 \text{ or } y = 0$$

So, option (B) is correct.

And when radius is 4 units, only 4 such circles are possible.

So, option (C) is correct.

3. See Fig. 12.54.

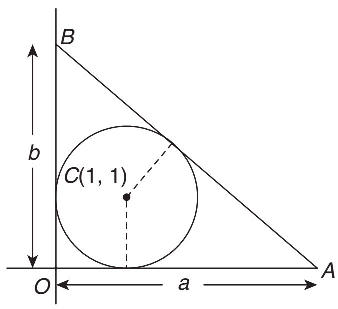


Figure 12.54

It is clear that $a > 2$, $b > 2$. Therefore,

$$\frac{1}{a} < \frac{1}{2}, \frac{1}{b} < \frac{1}{2} \Rightarrow \frac{1}{a} + \frac{1}{b} < 1$$

Now, equation of the line AB is $\frac{x}{a} + \frac{y}{b} = 1$.

Since perpendicular from (1, 1) = radius

$$\Rightarrow \left| \frac{\frac{1}{a} + \frac{1}{b} - 1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} \right| = 1$$

But $\frac{1}{a} + \frac{1}{b} - 1 < 0$. So,

$$\frac{1}{a} + \frac{1}{b} - 1 = -\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$$

4. Coordinates of O are (5, 3) and radius = 2

Equation of tangent at $A(7, 3)$ is

$$7x + 3y - 5(x + 7) - 3(y + 3) + 30 = 0$$
$$\Rightarrow 2x - 14 = 0$$
$$\Rightarrow x = 7$$

Equation of tangent at $B(5, 1)$ is

$$5x + y - 5(x + 5) - 3(y + 1) + 30 = 0$$
$$\Rightarrow -2y + 2 = 0$$
$$\Rightarrow y = 1$$

Therefore, coordinate of C are (7, 1).

Therefore, area of $OACB = 4$

Equation of AB is $x - y = 4$ (radical axis)

Equation of the smallest circle is

$$(x - 7)(x - 5) + (y - 3)(y - 1) = 0$$
$$\Rightarrow x^2 + y^2 - 12x - 4y + 38 = 0$$

5. See Fig. 12.55.

$$x^2 + y^2 + 8x - 10y - 40 = 0$$

Centre of the circle is (–4, 5) and its radius is 9.

Distance of the centre (–4, 5) from the point (–2, 3) is

$$\sqrt{4+4} = 2\sqrt{2}$$

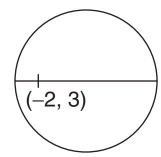


Figure 12.55

Thus,

$$a = 2\sqrt{2} + 9 \text{ and } b = -2\sqrt{2} + 9$$

Therefore,

$$a + b = 18$$
$$a - b = 4\sqrt{2}$$
$$a \cdot b = 81 - 8 = 73$$

6. See Fig. 12.56. Line pair is $(x - 1)^2 - y^2 = 0$.

That is, $x + y - 1 = 0$, $x - y - 1 = 0$. Let the centre be $(\alpha, 0)$. Then its distance from $x + y - 1 = 0$ is

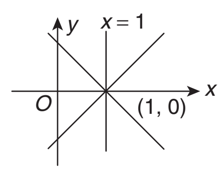


Figure 12.56

$$\left| \frac{\alpha - 1}{\sqrt{2}} \right| = 2 \text{ (radius)}$$
$$\Rightarrow \alpha = 1 \pm 2\sqrt{2}$$

Therefore, the centre may be $(1 + 2\sqrt{2}, 0)$, $(1 - 2\sqrt{2}, 0)$.

Now, let the centre be $(1, \beta)$. Then

$$\left| \frac{1 + \beta - 1}{\sqrt{2}} \right| = 2$$
$$\Rightarrow \beta = \pm 2\sqrt{2}$$

Therefore, the centre may be $(1, 2\sqrt{2})$, $(1, -2\sqrt{2})$.

7. $x^2 + y^2 - 8x - 16y + 60 = 0 \quad (1)$

Equation of chord of contact from (– 2, 0) is

$$-2x - 4(x - 2) - 8y + 60 = 0$$
$$3x + 4y - 34 = 0 \quad (2)$$

From Eqs. (1) and (2)

$$x^2 + \left(\frac{34 - 3x}{4}\right)^2 - 8x - 16\left(\frac{34 - 3x}{4}\right) + 60 = 0$$

$16x^2 + 1156 - 204x + 9x^2 - 128x - 2176 + 192x + 960 = 0$

$5x^2 - 28x - 12 = 0 \Rightarrow (x-6)(5x+2) = 0$

$x = 6, -\dfrac{2}{5}$

Therefore, points are (6, 4), $\left(-\dfrac{2}{5}, \dfrac{44}{5}\right)$.

8. Area of the quadrilateral $= \sqrt{c} \times \sqrt{9+25-c} = 15$
Therefore, $c = 9, 25$

9. $(x-1)^2 + (y+1)^2 + \lambda((x-2)^2 + y^2) = 0$ (1)
Put (0, 0) in Eq. (1) to get λ. Therefore

$$\lambda = -\frac{1}{2}$$

The equation of the circle is $x^2 + y^2 + 4y = 0$.

10. Family of circle $x^2 + y^2 - 2\alpha x - 2\beta y + c + \lambda(x^2 + y^2) = 0$

$$x^2 + y^2 - \frac{2\alpha}{\lambda+1}x - \frac{2\beta}{\lambda+1}y + \frac{c}{\lambda+1} = 0$$

Radius of this circle = 0. So,

$$\left(\frac{\alpha}{\lambda+1}\right)^2 + \left(\frac{\beta}{\lambda+1}\right)^2 - \frac{c}{\lambda+1} = 0 \Rightarrow \alpha^2 + \beta^2 - c(\lambda+1) = 0$$

$$\Rightarrow \lambda + 1 = \frac{\alpha^2 + \beta^2}{c}$$

The other limiting point is

$$\left(\frac{\alpha}{\lambda+1}, \frac{\beta}{\lambda+1}\right) = \left(\frac{c\alpha}{\alpha^2+\beta^2}, \frac{c\beta}{\alpha^2+\beta^2}\right)$$

11. $S + \lambda L = 0$

$L = 0$ represents the radical axis. Therefore,

$$ax - by + 1 = 0$$

12. Distance of chord from the origin is $\dfrac{1}{\sqrt{2}}$.

Angle between length of perpendicular from origin and radius is $\dfrac{\pi}{4}$.
Hence, angle made by chord at the centre of circle is $\dfrac{\pi}{2}$.

13. See Fig. 12.57.

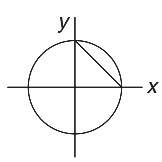


Figure 12.57

Slope of chord = 1

Since the chord is $\dfrac{\pi}{3}$– chord. Therefore, distance from the origin is $\sqrt{3}$.

Let the equation of the chord be $x - y + k = 0$. Then

$$\left|\frac{k}{\sqrt{2}}\right| = \sqrt{3}$$

$$\Rightarrow k = \pm\sqrt{6}$$

14. Radius of the circle = 2
Distance from the origin $= 2\cos\dfrac{\pi}{3} = 1$

15. See Fig. 12.58.

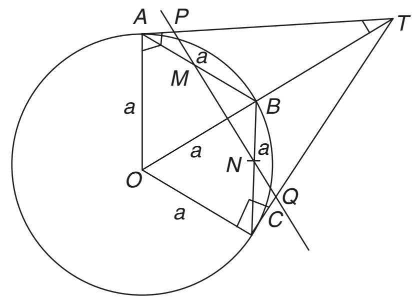


Figure 12.58

Since ΔOAB is an equilateral triangle. Therefore, $\angle OAB = 60°$.

16. Let T be the point of intersection of the tangents.
Since $\angle AOC = 120°$
Therefore, angle between tangents is 60°.

17. Locus of point of intersection of tangents at A and C is a circle whose centre is $O(0, 0)$ and radius is $OT = \sqrt{a^2 + a^2\cot^2 30} = 2a$
So, locus is $x^2 + y^2 = 4a^2$.

18. See Fig. 12.59. Since, $PQ = PR$. So, parallelogram $PQRS$ is a rhombus.
Therefore,
midpoint of QR = midpoint of PS and $QR \perp PS$
Therefore, S is the mirror image of P with respect to QR.

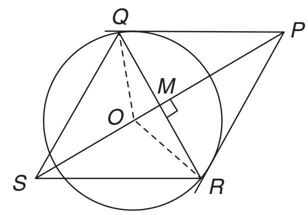


Figure 12.59

Since, $L \equiv 2x + y = 6$

Let $P \equiv (k, 6-2k)$

Since, $\angle PQO = \angle PRO = \dfrac{\pi}{2}$

Therefore, OP is the diameter of the circumcircle PQR, then the centre is $\left(\dfrac{k}{2}, 3-k\right)$.

So,

$$x = \frac{k}{2} \Rightarrow k = 2x$$

Now,

$$y = 3 - k$$

Hence, $2x + y = 3$.

19. As $P(6, 8)$, equation of QR is

$$6x + 8y = 4$$
$$\Rightarrow 3x + 4y - 2 = 0$$

Therefore,

$$PM = \frac{48}{5} \text{ and } PQ = \sqrt{96}$$

$$QM = \sqrt{96 - \frac{(48)^2}{25}} = \sqrt{\frac{96}{25}}$$

So, $$QR = 2\sqrt{\frac{96}{25}}$$

Therefore, area of $\Delta PQR = \frac{1}{2} \cdot PM \cdot QR = \frac{196\sqrt{6}}{25}$

Since $PQRS$ is a rhombus, we have

$$\text{area of } \Delta QRS = \text{area of } \Delta PQR = \frac{196\sqrt{6}}{25} \text{ sq. units}$$

20. Since $P \equiv (3, 4)$

Therefore, equation of QR is $3x + 4y = 4$. (1)

Let $S \equiv (x_1, y_1)$

Since S is the mirror image of P with respect to Eq. (1), then

$$\frac{x_1 - 3}{3} = \frac{y_1 - 4}{4} = \frac{-2(3 \times 3 + 4 \times 4 - 4)}{3^2 + 4^2} = -\frac{42}{25}$$

Therefore, $x_1 = -\frac{51}{25}, y_1 = -\frac{68}{25}$

$$\Rightarrow S\left(-\frac{51}{25}, -\frac{68}{25}\right)$$

21. (A) Since (2, 3) lies on $ax + by - 5 = 0$

Therefore, $2a + 3b - 5 = 0$

Since line is at greatest distance from the centre

$$\Rightarrow \left(\frac{4-3}{3-2}\right)\left(-\frac{a}{b}\right) = -1$$

$$\Rightarrow a = b$$

Therefore, $a = 1, b = 1$

Hence, $|a + b| = 2$

(B) Let P be the point (α, β), then $\alpha^2 + \beta^2 + 2\alpha + 2\beta = 0$

Midpoint of OP is $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$

Therefore, locus of $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$ is

$$4x^2 + 4y^2 + 4x + 4y = 0$$

$$\Rightarrow x^2 + y^2 + x + y = 0$$

Therefore, $2g = 1, 2f = 1$

$$\Rightarrow g + f = 1$$

(C) Centres of the circles are (1, 2), (5, – 6)

Equation of C_1C_2 is $y - 2 = -\frac{8}{4}(x - 1)$

$\Rightarrow 2x + y - 4 = 0$

Equation of radical axis is $8x - 16y - 56 = 0$

$\Rightarrow x - 2y - 7 = 0$

Points of intersection are (3, – 2)

(D) $x^2 + y^2 - 6\sqrt{3}x - 6y + 27 = 0$

Equation of the pair of tangents is given by

$$(-3\sqrt{3}x - 3y + 27)^2 = 27(x^2 + y^2 - 6\sqrt{3}x - 6y + 27)$$

$$27x^2 + 9y^2 + 27^2 + 18\sqrt{3}xy - 6 \times 27\sqrt{3}x - 6 \times 27y$$

$$= 27x^2 + 27y^2 - 6 \times 27\sqrt{3}x - 6 \times 27y + 27^2$$

$$18y^2 - 18\sqrt{3}xy = 0$$

$$y(y - \sqrt{3}x) = 0$$

Therefore, the tangents are

$$y = 0 \; y = \sqrt{3}x$$

So, angle between the tangents is $\frac{\pi}{3}$.

$$\Rightarrow 2\sqrt{3}\tan\theta = 2\sqrt{3} \times \sqrt{3} = 6$$

22. See Fig. 12.60.

(A) Let length of common chord be $2a$, then

$$\sqrt{9 - a^2} + \sqrt{16 - a^2} = 5$$

$$\sqrt{16 - a^2} = 5 - \sqrt{9 - a^2}$$

$$16 - a^2 = 25 + 9 - a^2 - 10\sqrt{9 - a^2}$$

$$10\sqrt{9 - a^2} = 18$$

$$\Rightarrow 100(9 - a^2) = 324$$

$$\Rightarrow 100a^2 = 576$$

$$\Rightarrow a = \sqrt{\frac{576}{100}} = \frac{24}{10}$$

Therefore,

$$2a = \frac{24}{5} = \frac{k}{5} \Rightarrow k = 24$$

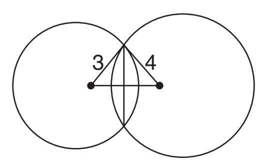


Figure 12.60

(B) Equation of common chord is $6x + 4y + p + q = 0$

Common chord pass through the centre (–2, – 6) of circle

$$x^2 + y^2 + 4x + 12y + p = 0$$

$$\Rightarrow p + q = 36$$

(C) Equation of the circle is $2x^2 + 2y^2 - 2\sqrt{2}x - y = 0$.

Let $(\alpha, 0)$ be midpoint of a chord. Then equation of the chord is

$$2\alpha x - \sqrt{2}(x + \alpha) - \frac{1}{2}(y + 0) = 2\alpha^2 - 2\sqrt{2}\alpha$$

Since it passes through the point $\left(\sqrt{2}, \frac{1}{2}\right)$

Therefore,

$$2\sqrt{2}\alpha - \sqrt{2}(\sqrt{2} + \alpha) - \frac{1}{4} = 2\alpha^2 - 2\sqrt{2}\alpha$$

$$\Rightarrow 8\alpha^2 - 12\sqrt{2}\alpha + 9 = 0$$

$$\Rightarrow (2\sqrt{2}\alpha - 3)^2 = 0$$

$$\Rightarrow \alpha = \frac{3}{2\sqrt{2}}, \frac{3}{2\sqrt{2}}$$

Therefore, the number of chords is 1.

(D) Midpoint of $AB = (1, 4)$

Therefore, equation of perpendicular bisector of AB is $x = 1$.
A diameter of the circle is $4y = x + 7$.
Therefore, centre of the circle is (1, 2)
So, sides of the rectangle are 8 and 4.
Thus, area of the rectangle is 32 sq. units.

23. See Fig. 12.61.

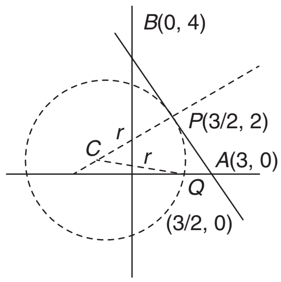


Figure 12.61

Let the centre be $C(h, k)$
Since,

$$CP \perp AB$$

$$\Rightarrow \frac{2-k}{\frac{3}{2}-h} = \frac{3}{4}$$

$$\Rightarrow 6h - 8k = -7 \qquad (1)$$

As, $CP = CQ$

$$\Rightarrow \left(h-\frac{3}{2}\right)^2 + (k-2)^2 = \left(h-\frac{3}{2}\right)^2 + k^2$$

$$\Rightarrow k = 1$$

Putting in Eq. (1), we get

$$6h = 1$$

$$\Rightarrow h = \frac{1}{6}$$

$$\text{So, radius } (r) = CQ = \sqrt{\left(\frac{1}{6}-\frac{3}{2}\right)^2 + 1}$$

$$= \sqrt{\left(\frac{1-9}{6}\right)^2 + 1} = \frac{5}{3}$$

$$\Rightarrow 3r = 5$$

24. See Fig. 12.62.

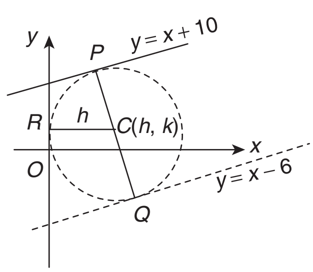


Figure 12.62

As,

$$CP = CR$$

$$\Rightarrow \frac{|h-k+10|}{\sqrt{2}} = h$$

$$h - k + 10 = h\sqrt{2} \qquad (1)$$

Since, $CP = CQ$. Therefore,

$$h - k + 10 = -h + k + 6$$

$$h - k = -2 \qquad (2)$$

Putting in Eq. (1), we get

$$h = 4\sqrt{2}$$

Therefore,

$$k = h + 2$$

$$\Rightarrow k = 2 + 4\sqrt{2}$$

So,

$$h + k = 2 + 8\sqrt{2}$$

$$\Rightarrow a = 2, b = 8$$

Hence, $a + b = 10$.

25. Let the radius of variable circle be r.
Therefore, from Fig. 12.63, we have

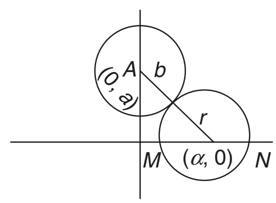


Figure 12.63

$$\sqrt{\alpha^2 + a^2} = b + r \qquad (1)$$

Consider a point $P(0, k)$ on the y-axis
$M(\alpha - r, 0)$ and $N(\alpha + r, 0)$

Now,

$$\text{slope of } MP = \frac{-k}{\alpha - r}$$

$$\text{slope of } NP = \frac{-k}{\alpha + r}$$

If $\angle MPN = \theta$, then

$$\tan\theta = \left| \frac{\frac{-k}{\alpha-r} - \frac{-k}{\alpha+r}}{1 + \frac{k^2}{\alpha^2 - r^2}} \right| = \left| \frac{2kr}{\alpha^2 - r^2 + k^2} \right|$$

According to the given condition, θ is a constant for any choice α

$$\frac{2kr}{\alpha^2 - r^2 + k^2} = \text{constant}$$

$$\Rightarrow \frac{r}{\alpha^2 - r^2 + k^2} = \text{constant}$$

$$\Rightarrow \frac{\sqrt{\alpha^2 + a^2} - b}{\alpha^2 - (\sqrt{\alpha^2 + a^2} - b)^2 + k^2} 0 = \text{constant} \quad \text{(from Eq. (1))}$$

$$\Rightarrow \frac{\sqrt{\alpha^2+a^2}-b}{2b\sqrt{\alpha^2+a^2}-a^2-b^2+k^2} = \text{constant}$$

$$\frac{\sqrt{\alpha^2+a^2}-b}{\sqrt{\alpha^2+a^2}-\lambda} = \text{constant}\left\{\text{putting}\ \frac{a^2+b^2-k^2}{2b} = \lambda\right\}$$

which is possible only if $\lambda = b$

$$\Rightarrow \frac{a^2+b^2-k^2}{2b} = b \Rightarrow k = \pm\sqrt{a^2-b^2}$$

Therefore,

$$P \equiv (0, \pm\sqrt{a^2-b^2})$$

$$\Rightarrow \lambda = 1$$

26. See Fig. 12.64.

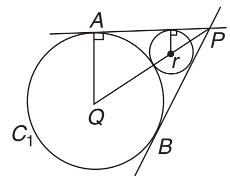


Figure 12.64

$$AQ = 3 + 2\sqrt{2}$$

$$PQ = 3\sqrt{2} + 4$$

Let 'r' be required radius

$$3\sqrt{2} + 4 = 3 + 2\sqrt{2} + r + r\sqrt{2}$$

$$\sqrt{2} + 1 = r(1 + \sqrt{2})$$

$$\Rightarrow r = 1$$

Solved JEE 2017 Questions

JEE Main 2017

1. The radius of a circle, having minimum area, which touches the curve $y = 4 - x^2$ and the lines, $y = |x|$ is

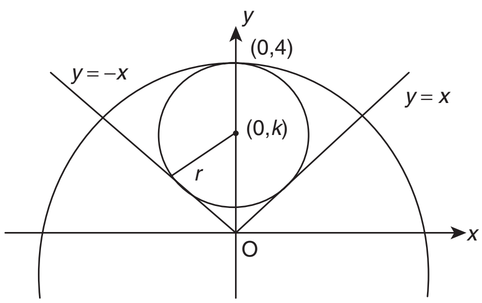
(A) $2(\sqrt{2}-1)$ **(B)** $4(\sqrt{2}-1)$

(C) $4(\sqrt{2}+1)$ **(D)** $2(\sqrt{2}+1)$

(OFFLINE)

Solution: From the following figure, which depicts the given situation, the circle touches the line.

Also, from the graph, the radius is obtained as $4 - k$.



Perpendicular distance from the centre = Radius of the circle

$$4 - k = \left|\frac{0-k}{\sqrt{2}}\right|$$

$$16 + k^2 - 8k = \frac{k^2}{2}$$

$$k^2 - 16k + 32 = 0$$

The solution of this quadratic equation is

$$k = \frac{16 \pm \sqrt{256 - 4(32)}}{2} = \frac{16 \pm 8\sqrt{2}}{2}$$

That is, $k = 8 \pm 4\sqrt{2}$.
Now, we consider $k = 8 - 4\sqrt{2}$ (k should be $0 < k < 4$).
Therefore, the radius of the circle is

$$4 - k = 4 - (8 - 4\sqrt{2}) = 4(\sqrt{2}-1)$$

Hence, the correct answer is option (B).

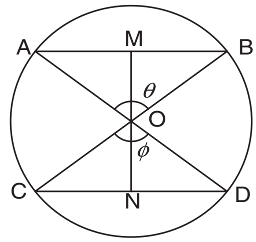
2. If two parallel chords of a circle, having diameter 4 units, lie on the opposite sides of the centre and subtend angles $\cos^{-1}\left(\frac{1}{7}\right)$ and $\sec^{-1}(7)$ at the centre, respectively, then the distance between these chords, is

(A) $\frac{16}{7}$ **(B)** $\frac{8}{\sqrt{7}}$

(C) $\frac{8}{7}$ **(D)** $\frac{4}{\sqrt{7}}$

(ONLINE)

Solution: The given geometrical situation is shown in the following figure:



Here, we have $\theta = \cos^{-1}\left(\frac{1}{7}\right)$ and $\phi = \sec^{-1}(7)$

The radius is 2 units; now, we have to find the distance between two chords (i.e. the distance between the points M and N).
Now, from ΔAMO, we have $\angle AMO = 90°$ and $\angle AOM = \theta/2$. Therefore,

$$\frac{OM}{AO} = \cos\left(\frac{\theta}{2}\right)$$

$$\Rightarrow \frac{OM}{AO} = \cos\left[\frac{1}{2}\cos^{-1}\left(\frac{1}{7}\right)\right]$$

Now, AO = radius = 2 units. Therefore,

$$\frac{OM}{2} = \cos\left[\frac{1}{2}\cos^{-1}\left(\frac{1}{7}\right)\right] \quad (1)$$

Evaluating RHS of Eq. (1),

$$\cos\left(\frac{1}{2}\cos^{-1}\left(\frac{1}{7}\right)\right)$$

Put,

$$\cos^{-1}\left(\frac{1}{7}\right) = \theta \quad (2)$$

$$\Rightarrow \frac{1}{7} = \cos\theta$$

Adding 1 on both sides of Eq. (3), we get

$$1 + \cos\theta = 1 + \frac{1}{7} = \frac{8}{7}$$

Using $\cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1$, we get

$$\cos\theta + 1 = 2\cos^2\left(\frac{\theta}{2}\right)$$

$$\Rightarrow 2\cos^2\left(\frac{\theta}{2}\right) = \frac{8}{7}$$

$$\Rightarrow \cos^2\left(\frac{\theta}{2}\right) = \frac{4}{7}$$

$$\Rightarrow \cos\left(\frac{\theta}{2}\right) = \frac{2}{\sqrt{7}} \quad (3)$$

Substituting Eq. (2) in Eq. (1), we get

$$\frac{OM}{2} = \cos\left(\frac{\theta}{2}\right)$$

Substituting Eq. (3), we get

$$\frac{OM}{2} = \frac{2}{\sqrt{7}} \Rightarrow OM = \frac{4}{\sqrt{7}} \quad (4)$$

Similarly from ΔCNO, we have $\angle CNO = 90°$ and $\angle CON = \frac{\phi}{2}$. Therefore,

$$\frac{ON}{OC} = \cos\left(\frac{\phi}{2}\right)$$

where OC = radius = 2 units. Now,

$$\phi = \sec^{-1}(7)$$

$$\Rightarrow \frac{ON}{2} = \cos\left(\frac{1}{2}\sec^{-1}7\right) \quad (5)$$

Evaluating RHS of Eq. (5):

$$\cos\left(\frac{1}{2}\sec^{-1}7\right)$$

Substituting

$$\sec^{-1}(7) = \phi \quad (6)$$

$$\Rightarrow \sec\phi = 7 \Rightarrow \cos\phi = \frac{1}{7} \left(\text{since } \sec\phi = \frac{1}{\cos\phi}\right)$$

Adding 1 on both sides, we get

$$1 + \cos\phi = \frac{8}{7} \Rightarrow 2\cos^2\left(\frac{\phi}{2}\right) = \frac{8}{7} \left(\text{since } \cos\phi = 2\cos^2\frac{\phi}{2} + 1\right)$$

$$\cos^2\left(\frac{\phi}{2}\right) = \frac{4}{7} \Rightarrow \cos\left(\frac{\phi}{2}\right) = \frac{2}{\sqrt{7}} \quad (7)$$

Substituting Eq. (6) in Eq. (5), we get

$$\frac{ON}{2} = \cos\left(\frac{1}{2}\phi\right)$$

Substituting $\cos\frac{\phi}{2} = \frac{2}{\sqrt{7}}$ [from Eq. (7)], we get

$$\frac{ON}{2} = \frac{2}{\sqrt{7}} \Rightarrow ON = \frac{4}{\sqrt{7}} \quad (8)$$

Now,

$$MN = MO + ON$$

Substituting the value of MO and ON from Eq. (4) and (8), we get

$$MN = \frac{4}{\sqrt{7}} + \frac{4}{\sqrt{7}} \Rightarrow MN = \frac{8}{\sqrt{7}}$$

Hence, the correct answer is option (B).

3. If a point P has coordinates (0, –2) and Q is any point on the circle, $x^2 + y^2 - 5x - y + 5 = 0$, then the maximum value of $(PQ)^2$ is

(A) $\frac{25+\sqrt{6}}{2}$ **(B)** $8+5\sqrt{3}$

(C) $14+5\sqrt{3}$ **(D)** $\frac{47+10\sqrt{6}}{2}$

(ONLINE)

Solution: We have

$$x^2 + y^2 - 5x - y + 5 = 0$$

Factorizing the equation, adding and subtracting by $\frac{25}{4}$ and $\frac{1}{4}$, respectively, we get

$$\left(x^2 - 5x + \frac{25}{4}\right) - \frac{25}{4} + \left(y^2 - y + \frac{1}{4}\right) - \frac{1}{4} + 5 = 0$$

$$\Rightarrow \left(x - \frac{5}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 - \frac{25}{4} - \frac{1}{4} + 5 = 0$$

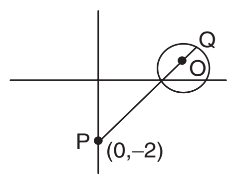
$$\Rightarrow \left(x - \frac{5}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 - \frac{26}{4} + 5 = 0$$

$$\Rightarrow \left(x - \frac{5}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 - \frac{6}{4} = 0$$

$$\Rightarrow \left(x - \frac{5}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{3}{2}$$

$$\Rightarrow \left(x - \frac{5}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \left(\sqrt{\frac{3}{2}}\right)^2$$

Therefore, the centre of the circle is $\left(\frac{5}{2}, \frac{1}{2}\right)$ and the radius is $\sqrt{\frac{3}{2}}$.



Let the coordinates of O be (x_1, y_1). Then the maximum distance between the points PQ is

$$PQ = \sqrt{(x_1 - 0)^2 + (y_1 + 2)^2} + \text{radius}$$

$$= \sqrt{\left(\frac{5}{2} - 0\right)^2 + \left(\frac{1}{2} + 2\right)^2} + \sqrt{\frac{3}{2}}$$

$$= \sqrt{\left(\frac{5}{2}\right)^2 + \left(\frac{5}{2}\right)^2} + \sqrt{\frac{3}{2}}$$

$$= \sqrt{2\left(\frac{5}{2}\right)^2} + \sqrt{\frac{3}{2}}$$

$$= \sqrt{2}\frac{5}{2} + \sqrt{\frac{3}{2}}$$

$$= \frac{5}{\sqrt{2}} + \frac{\sqrt{3}}{\sqrt{2}} = \frac{5+\sqrt{3}}{\sqrt{2}}$$

Therefore,

$$(PQ)^2 = \left(\frac{5+\sqrt{3}}{\sqrt{2}}\right)^2 = \frac{(5+\sqrt{3})^2}{(\sqrt{2})^2}$$

Using $(a+b)^2 = a^2 + b^2 + 2ab$, we get the maximum value of $(PQ)^2$ as

$$(PQ)^2 = \frac{25+3+(2\times5\sqrt{3})}{2} = \frac{28+10\sqrt{3}}{2} = 14+5\sqrt{3}$$

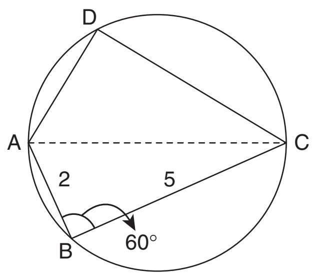
Hence, the correct answer is option (C).

4. The two adjacent sides of a cyclic quadrilateral are 2 and 5 and the angle between them is 60°. If the area of the quadrilateral is $4\sqrt{3}$, then the perimeter of the quadrilateral is

(A) 12 **(B)** 12.5
(C) 13 **(D)** 13.2

(ONLINE)

Solution: The given geometrical situation is depicted in the following figure:



The area of the quadrilateral ABCD is

$$A(\Delta ABC) + A(\Delta ADC) = 4\sqrt{3}$$

- **In ΔABC:** $\text{Area} = \frac{1}{2}\times2\times5\times\sin60° = \frac{5\sqrt{3}}{2}$.

Now, the area of ΔADC is expressed as

$$4\sqrt{2} - \frac{5}{2}\sqrt{3} = \frac{3}{2}\sqrt{3} \quad (1)$$

- **In ΔADC:**

$$\text{Area} = \frac{1}{2}\times AD\times CD\times\sin120° \Rightarrow \frac{3}{2}\sqrt{x} = \frac{1}{x}\times c\times d\times\frac{\sqrt{3}}{2}.$$

Therefore,

$$cd = 6 \quad (2)$$

Now, for ΔABC, we have

$$\cos B = \frac{2^2+5^2-(AC)^2}{2\times2\times5} \quad \text{(cosine rule)}$$

$$\frac{1}{2} = \frac{4+25-(AC)^2}{20}$$

$$\Rightarrow (AC)^2 = 19 \quad (3)$$

Now, for ΔADC, we have

$$\cos120° = \frac{c^2+d^2-(AC)^2}{2cd}$$

$$\Rightarrow -\frac{1}{2} = \frac{c^2+d^2-19}{2\times6} \quad \text{[from Eqs. (2) and (3)]}$$

$$\Rightarrow c^2+d^2 = 13 \quad (4)$$

Now,

$$[\text{Eq. (4)}] + [\text{Eq. } 2\times(2)] = c^2+d^2+2cd$$

$$13+2\times6 = 13+12 = 25$$

$$(c+d)^2 = 25$$

The perimeter of the quadrilateral ABCD is

$$2+5+c+d = 7+5 = 12$$

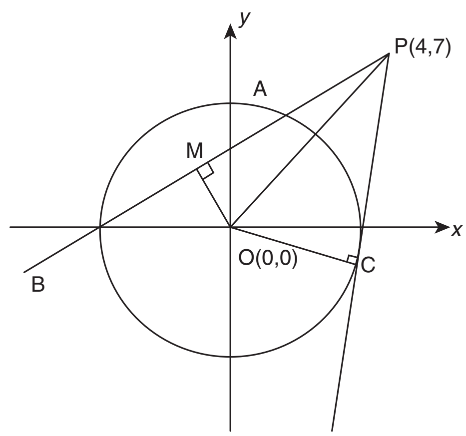
Hence, the correct answer is option (A).

5. A line drawn through the point P(4, 7) cuts the circle $x^2+y^2 = 9$ at the points A and B. Then PA·PB is equal to

(A) 56 **(B)** 74
(C) 65 **(D)** 53

(ONLINE)

Solution: The given geometrical situation is depicted in the following figure:



Now, $PA\cdot PB = (PC)^2$, where C is the point of contact of a tangent to the circle from point P.

If the centre of the circle is O, ΔPOC forms a right-angled triangle, where we have

$$(PO)^2 = (OC)^2 + (PC)$$

$$\Rightarrow (4-0)^2 + (7-0)^2 = 9 + (PC)^2$$

$$\Rightarrow (PC)^2 = 16+49-9 = 56$$

Therefore, $PA\times PB = 56$

Hence, the correct answer is option (A).

JEE Advanced 2017

1. For how many values of p, the circle $x^2 + y^2 + 2x + 4y - p = 0$ and the coordinate axes have exactly three common points?

Solution: We consider the following three cases:

- **Case 1:** The circle passes through the origin, that is, $p = 0$; now, the equation of circle becomes

$$x^2+y^2+2x+4y = 0$$

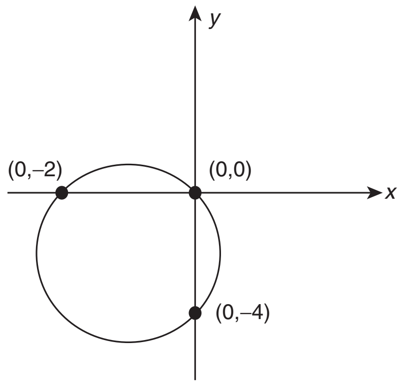
Now, $x = 0$. Therefore,

$$y^2+4y = 0$$

$$\Rightarrow y(y+4) = 0$$

Therefore, $y = 0$ and $y = -4$.
Thus, $y = 0$ gives

$$x^2 + 2x = 0$$
$$\Rightarrow x(x + 2) = 0$$
$$\Rightarrow x = 0, -2$$



- **Case 2:** When the circle touches y-axis, then the circle intersects *x*-axis at two distinct points.

Substituting $y = 0$ in the equation of circle, we get

$$x^2 + 2x - p = 0$$

Now, from $g^2 - c > 0$ and, we get

$$1^2 - (-p) > 0 \text{ and } 2^2 - (-p) = 0$$
$$1 + p > 0 \text{ and } 4 + p = 0$$
$$p > -1 \text{ and } p = -4$$

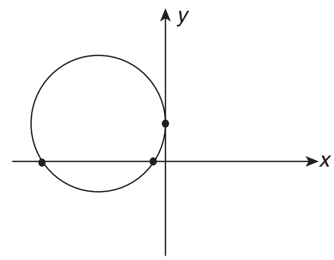
which is a contradiction. Also, for $p = -4$, we get

$$x^2 - 2x + 4 = 0$$

Therefore,

$$x = \frac{2 \pm \sqrt{4 - 4 \times 4}}{2} = \frac{2 \pm 2\sqrt{3}i}{2} = 1 \pm \sqrt{3}i$$

which are imaginary roots. Therefore, this Case 2 is not possible.

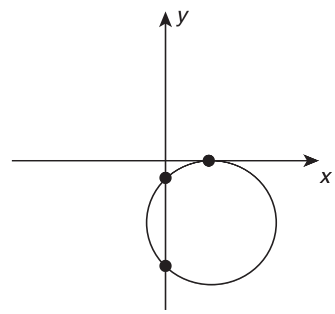


- **Case 3:** When the circle touches *x*-axis, the circle intersects *y*-axis at two distinct points. Substituting $x = 0$, we get

$$y^2 + 4y - p = 0$$

Now, from $g^2 - c = 0$ and $f^2 - c > 0$, we have

$$1^2 - (-p) = 0 \text{ and } 2^2 = (-p) > 0$$
$$p = -1 \text{ and } 4 + p > 0 \Rightarrow p > -4$$



Therefore, the equation of circle becomes

$$y^2 + 4y + 1 = 0$$
$$\Rightarrow y = \frac{-4 \pm \sqrt{16 - 4}}{2} = \frac{-4 \pm \sqrt{12}}{2}$$
$$y = \frac{-4 \pm \sqrt{4 \times 3}}{2} = \frac{-4 \pm 2\sqrt{3}}{2} = -2 \pm \sqrt{3}$$

which are real values. Therefore, Case 3 is possible.
Thus, for two values of p ($p = 0$ and $p = -1$), the circle and the coordinate axes have exactly three points in common.
Hence, the correct answer is (2).

13 Parabola

13.1 Understanding Conic Section

A 'conic section' or a 'conic' is the locus of a point which moves in a plane so that its distance from a fixed point is in a constant ratio to its perpendicular distance from a fixed straight line. Following are the important definitions related to a conic section:

1. A fixed point is called the 'focus'.
2. A fixed line is called the 'directrix'.
3. The constant ratio is called the 'eccentricity' which is denoted by *e*.
4. The line passing through the focus which is perpendicular to the directrix is called an 'axis'.
5. The point of intersection of conic with axis is called a 'vertex'.

For example, if a fixed point is S(α, β) (Fig. 13.1) and the directrix is $lx + my + n = 0$, then

$$PS = \sqrt{(x-\alpha)^2 + (y-\beta)^2}$$
$$PM = \frac{lx + my + n}{\sqrt{l^2 + m^2}}$$
$$\frac{PS}{PM} = e$$
$$\Rightarrow (l^2 + m^2)[(x-\alpha)^2 + (y-\beta)^2] = e^2(lx + my + n)^2$$

which is of the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

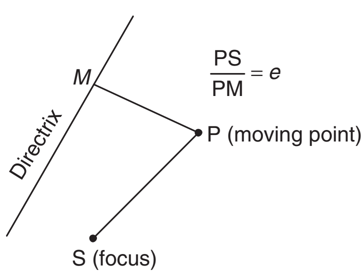


Figure 13.1

13.1.1 Section of Right Circular Cone by a Different Plane

1. Section of a right circular cone by a plane passing through its vertex is pair of straight lines passing through the vertex (Fig. 13.2).

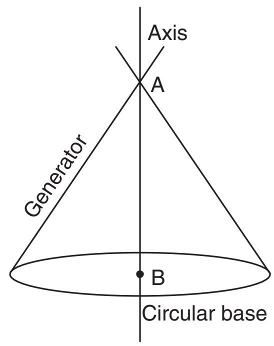


Figure 13.2

2. Section of right circular cone by a plane parallel to its base is a circle (Fig. 13.3).

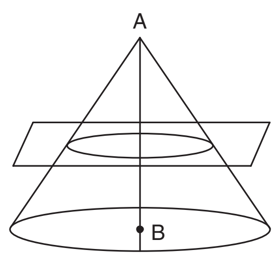


Figure 13.3

3. Section of a right circular cone by a plane parallel to a generator of the cone is a parabola.

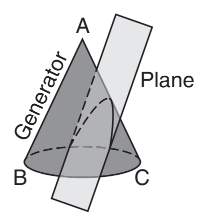


Figure 13.4

4. Section of a right circular cone by a plane neither parallel to any generator of the cone which is not perpendicular or parallel

to the axis of the cone is an ellipse or hyperbola as shown in Fig. 13.5.

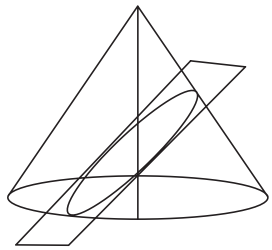


Figure 13.5

13.1.2 Distinguishing Various Conics

The nature of the conic depends on the position of the focus S w.r.t. the directrix and it also depends on the value of the eccentricity e.

1. When the focus lies on the directrix, we have

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Here, the conic represents a pair of straight lines if

(i) $h^2 > ab$ which denotes that the lines are real and distinct that are intersecting in S.
(ii) $h^2 = ab$ which denotes that the lines are coincident.
(iii) $h^2 < ab$ which denotes that the lines are imaginary.

2. When the focus does not lie on the directrix, we have the categories listed in Table 13.1.

Table 13.1 Categorisation of different conics

Parabola	Ellipse	Hyperbola	Rectangular Hyperbola
$e = 1, \Delta \neq 0$	$0 < e < 1, \Delta \neq 0$	$e > 1, \Delta \neq 0$	$e = \sqrt{2}, \Delta \neq 0$
$h^2 = ab$	$h^2 < ab$	$h^2 > ab$	$h^2 > ab$, $a + b = 0$

Illustration 13.1 Which type of conic do the following equations represent? (a) $25(x^2 + y^2 - 2x + 1) = (4x - 3y + 1)^2$ and (b) $13x^2 - 18xy + 37y^2 + 2x + 14y - 2 = 0$

Solution:

(a) The given equation is

$$25(x^2 + y^2 - 2x + 1) = (4x - 3y + 1)^2$$

Now, rewriting this equation, we get

$$25[(x-1)^2 + (y-0)^2] = 25\left[\frac{(4x-3y+1)}{\sqrt{(4)^2+(3)^2}}\right]^2$$

$$\Rightarrow \sqrt{(x-1)^2+(y-0)^2} = \frac{|4x-3y+1|}{\sqrt{4^2+3^2}}$$

Here, $e = 1$ and $S(1, 0)$ does not lie on $4x - 3y + 1 = 0$. Hence, it represents a parabola.

(b) The given equation is

$$13x^2 - 18xy + 37y^2 + 2x + 14y - 2 = 0$$

On comparing with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 13 & -9 & 1 \\ -9 & 37 & 7 \\ 1 & 7 & -2 \end{vmatrix} = -1600 \neq 0$$

Also,

$$h^2 - ab = (-9)^2 - 13 \times 37 = -400 < 0$$

Hence, the given equation represents an ellipse.

Illustration 13.2 Find the equation of the conic whose focus is (–1, 1) and eccentricity is 1/2. The equation of the directrix is $x - y + 3 = 0$.

Solution: See Fig. 13.6.

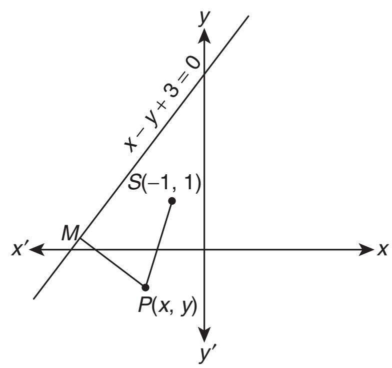


Figure 13.6

By definition, we have

$$SP = e(PM)$$

$$\Rightarrow SP^2 = e^2(PM^2)$$

$$\Rightarrow (x+1)^2 + (y-1)^2 = \frac{1}{4}\left[\frac{x-y+3}{\sqrt{2}}\right]^2$$

On simplifying further, we get

$$7x^2 + 7y^2 + 2xy + 10x - 10y + 7 = 0$$

Your Turn 1

1. Find the locus of a point which moves such that its distance from the point (0, –1) is twice its distance from the line $3x + 4y + 1 = 0$.

Ans. $11x^2 + 39y^2 + 96xy + 24x - 18y - 21 = 0$

2. Which type of conic is represented by the equation $\sqrt{ax} + \sqrt{by} = 1$? **Ans.** Parabola

3. If the equation of the conic $2x^2 + xy + 3y^2 - 3x + 5y + \lambda = 0$ represent a single point, then find the value of λ. **Ans.** 4

4. For what value of λ, the equation of the conic $2xy + 4x - 6y + \lambda = 0$ represents two intersecting straight lines. **Ans.** $\lambda = -12$

5. If the equation $ax^2 + 4xy + y^2 + ax + 3y + 2 = 0$ represents a parabola, then find the value of a. **Ans.** 4

13.2 Parabola: Definition and Its Terminologies

A parabola is a locus of a point which moves in a plane such that its distance from a fixed point is equal to its perpendicular distance from a fixed line.

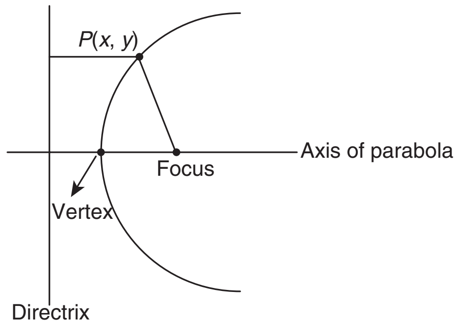


Figure 13.7

See Fig. 13.7. We know that the fixed point is called focus and the fixed line is called directrix. Now, the different terminologies of parabola are discussed as follows:

1. **Axis of parabola:** The line through the focus which is perpendicular to the directrix.
2. **Vertex:** The mid-point of the point of intersection of the axis with the directrix and the focus (Fig. 13.7).
3. **Double-ordinate:** A straight line which is drawn perpendicular to the axis and terminated at both ends of the curve (parabola) is called a double-ordinate.
4. **Latus rectum:** The double-ordinate passing through the focus is called a latus rectum.
5. **Centre:** The point which bisects every chord of the conic passing through it is called the centre (of the parabola).
6. Now, if we consider the axis of parabola as x-axis and its origin as the vertex, focus as point $S(a, 0)$. So, the directrix is $x + a = 0$ (Fig. 13.8).
7. Let, $P(x, y)$ be any general point on the parabola. Thus,

$$PS = PM$$
$$\Rightarrow (x - a)^2 + y^2 = (x + a)^2$$
$$\Rightarrow y^2 = 4ax$$

which is the standard equation of a parabola.

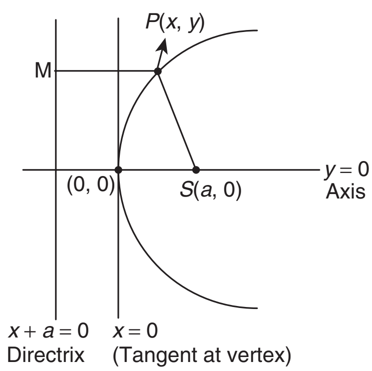


Figure 13.8

8. **Focal chord:** Any chord passing through the focus is called the focal chord.
   - **(a)** The line RT passes through the focus $S(a, 0)$ and hence it can be termed as focal chord (Fig. 13.9).

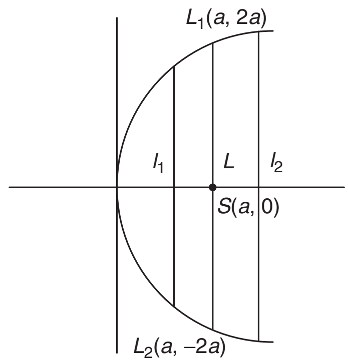


Figure 13.9

   - **(b)** Here, the lines l_1 and l_2 are the double-ordinates and the line L is a double-ordinate passing through the focus $(a, 0)$ and is termed as 'latus rectum'. The end points of latus rectum are $L_1(a, 2a)$ and $L_2(a, -2a)$. Also, the length of latus rectum is $4a$.
   - **(c)** Two parabolas are said to be equal if the length of their latus rectum is equal.
9. **Focal distance:** The distance of any point P on the parabola from its focus is called 'focal distance' of the point P.
   - **(a)** The focal distance of any point on the parabola is equal to sum of one-fourth of the length of the latus rectum and its distance from the tangent at the vertex.
   - **(b)** The focal distance of $P(x, y)$ is a point on the parabola $y^2 = 4ax$ is equal to $|x + a|$.
   - **(c)** $PS = \sqrt{(x-a)^2 + (y-0)^2} = \sqrt{(x+a)^2} = |x + a|$.

Illustration 13.3 Find the equation of the parabola whose focus is at (– 1, 2) and the directrix is the straight line $x - 2y + 3 = 0$.

Solution: See Fig. 13.10. By definition, we have

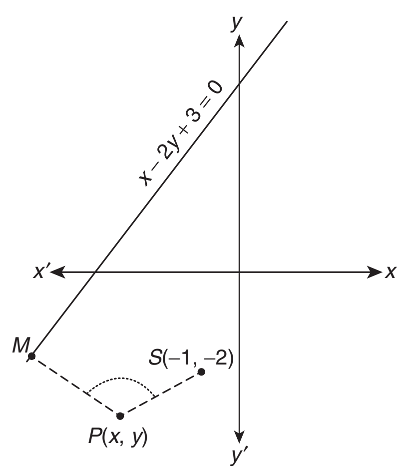


Figure 13.10

$$SP = PM$$

$$\Rightarrow SP^2 = PM^2$$

$$\Rightarrow (x+1)^2 + (y+2)^2 = \left(\frac{|x-2y+3|}{\sqrt{(1)^2+(-2)^2}}\right)^2$$

On simplifying further, we get

$$4x^2 + y^2 + 4xy + 4x + 32y + 16 = 0$$

13.2.1 Forms of Standard Parabola

1. See Fig. 13.11. If we have $y^2 = 4ax$, then
- **(a)** the vertex is (0, 0).
- **(b)** the focus is $(a, 0)$.
- **(c)** the equation of the directrix is $x = -a$.
- **(d)** the equation of axis is $y = 0$.
- **(e)** the equation of the tangent at vertex is $x = 0$.

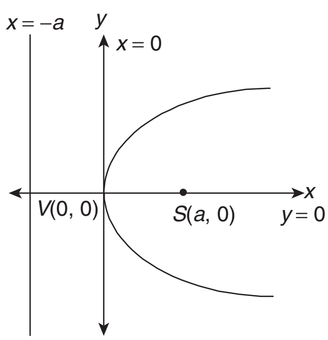


Figure 13.11

2. See Fig. 13.12. If we have $y^2 = -4ax$, then
- **(a)** the vertex is (0, 0).
- **(b)** the focus $(-a, 0)$.
- **(c)** the equation of the directrix is $x = a$.
- **(d)** the equation of axis is $y = 0$.
- **(e)** the equation of the tangent at the vertex is $x = 0$.

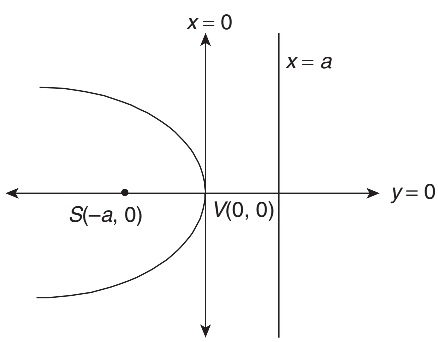


Figure 13.12

3. See Fig. 13.13. If we have $x^2 = 4ay$, then
- **(a)** the vertex is (0, 0).
- **(b)** the focus is $(0, a)$.
- **(c)** the equation of the directrix is $y = -a$.
- **(d)** the equation of the axis is $x = 0$.
- **(e)** the equation of tangent at the vertex is $y = 0$.

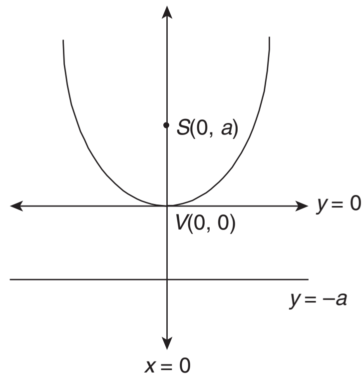


Figure 13.13

4. See Fig. 13.14. If we have $x^2 = -4ay$, then
- **(a)** the vertex is (0, 0).
- **(b)** the focus is $(0, -a)$.
- **(c)** the equation of the directrix is $y = a$.
- **(d)** the equation of the axis is $x = 0$.
- **(e)** the equation of the tangent at the vertex is $y = 0$.

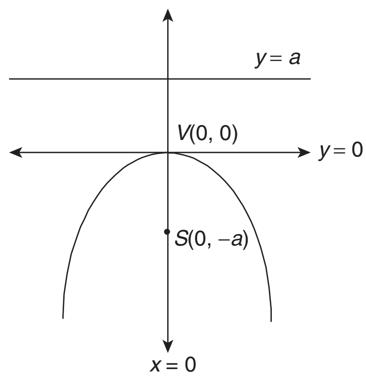


Figure 13.14

If the vertex is not the origin and the axis and directrix are parallel to the coordinate axis, then the equation of parabola with vertex at (h, k) can be obtained by using the transformation of axis. The different forms are listed in Table 13.2.

Table 13.2 Forms of parabola with horizontal or vertical axis and vertex at (h, k)

Form	Vertex	Focus	Equation of Directrix	Equation of Axis	Tangent at Vertex
$(y-k)^2 = 4a(x-h)$	(h, k)	$(h+a, k)$	$x = h - a$	$y = k$	$x = h$
$(y-k)^2 = -4a(x-h)$	(h, k)	$(h-a, k)$	$x = h + a$	$y = k$	$x = h$
$(x-h)^2 = 4a(y-k)$	(h, k)	$(h, k+a)$	$y = k - a$	$x = h$	$y = k$
$(x-h)^2 = -4a(y-k)$	(h, k)	$(h, k-a)$	$y = k + a$	$x = h$	$y = k$

Illustration 13.4 For the equation $y^2 = 8x$, find the (a) vertex, (b) directrix, (c) axis of the parabola, (d) tangent at the vertex, (e) end points of the latus rectum and (f) length of the latus rectum.

Solution: See Fig. 13.15.

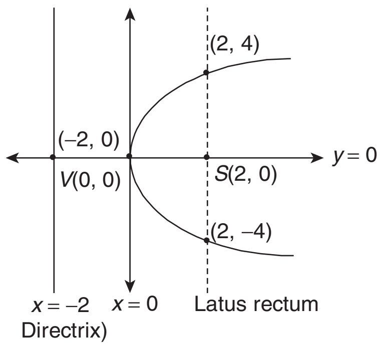


Figure 13.15

We have $y^2 = 8x$. Here, $a = 2$.

(a) The vertex is (0, 0).
(b) The directrix is $x = -a \Rightarrow x = -2$.
(c) The axis of parabola is $y = 0$.
(d) The tangent at the vertex is $x = 0$.
(e) The end points of the latus rectum are (a, 2a) and (a, –2a), that is, (2, 4) and (2, –4).
(f) The length of the latus rectum is $4a$, that is, 8 unit.

Illustration 13.5 Find the vertex, focus, equations of directrix, axis and tangent at the vertex of the parabola $x^2 + 4x + 6y - 8 = 0$.

Solution: The equation can be rewritten as

$$(x+2)^2 = -6y + 8 + 4$$

$$\Rightarrow (x+2)^2 = -6[y-2]$$

Thus, the vertex is (–2, 2) and $a = 3/2$; the focus is (–2, 1/2); the equation of directrix is $y = 7/2$; the equation of the axis is $x = -2$ and the equation of the tangent at the vertex is $y = 2$.

13.2.2 General Equation of Parabola

Let S(h, k) be the focus and line $lx + my + n = 0$ the equation of the directrix ZM of a parabola. Let (x, y) be the coordinates of any point P on the parabola. Then

PS = Perpendicular distance of P from ZM

$$(x-h)^2 + (y-k)^2 = \frac{(lx+my+n)^2}{l^2+m^2}$$

$$\Rightarrow (mx - ly)^2 + 2gx + 2fy + d = 0$$

which is the general equation of a parabola. It is clear that second-degree terms in the equation of a parabola form a perfect square. However, the converse is also true, that is, if an equation of the second-degree terms form a perfect square, then the equation represents a parabola unless it represents two parallel straight lines.

The general equation of second degree, for example, $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a parabola if $\Delta \neq 0$ and $h^2 = ab$, where

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Illustration 13.6 Find the equation of the parabola whose focus is (4, –3) and its vertex is (4, –1).

Solution:

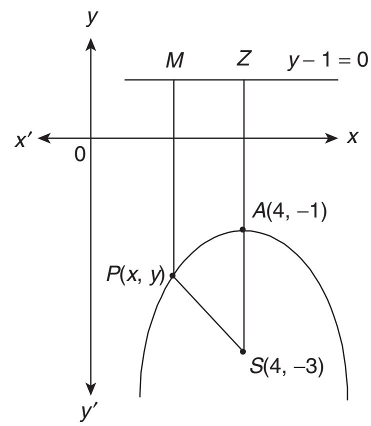


Figure 13.16

Let, $A(4, -1)$ be the vertex and $S(4, -3)$ be the focus (Fig. 13.16), The slope of AS is

$$\frac{-3+1}{4-4} = \infty$$

which is parallel to y-axis. Therefore, the directrix is parallel to x-axis. Let $Z(x_1, y_1)$ be any point on the directrix. Then A is the mid-point of SZ is obtained as follows:

$$4 = \frac{x_1+4}{2} \Rightarrow x_1 = 4$$

and $$-1 = \frac{y_1-3}{2} \Rightarrow y_1 = 1$$

Therefore,

$$Z \equiv (4, 1)$$

Thus, the equation of directrix is

$$y - 1 = 0$$

Now, using the definition, $SP = PM$, we get

$$(SP)^2 = (PM)^2$$

$$\Rightarrow (x-4)^2 + (y+3)^2 = \left(\frac{|y-1|}{\sqrt{(1)^2}}\right)^2$$

On simplifying further, we get

$$x^2 - 8x + 8y + 24 = 0$$

Illustration 13.7 The focal distance of a point on the parabola $y^2 = 8x$ is 8. Find the point.

Solution: On comparing $y^2 = 8x$ with $y^2 = 4ax$, we get $a = 2$. The equation of the directrix is

$$x + 2 = 0$$

Let $P(x_1, y_1)$ be a point on the parabola $y^2 = 8x$. Then

$$(y_1)^2 = 8x_1$$

Here, $SP = 8$; therefore,

$$PM = 8 \text{ (Fig. 13.17)}$$
$$\Rightarrow x_1 + 2 = 8 \Rightarrow x_1 = 6$$

That is,

$$(y_1)^2 = 8 \times 6$$
$$\Rightarrow y_1 = \pm 4\sqrt{3}$$

Hence, the required points are $(6, 4\sqrt{3})$ and $(6, -4\sqrt{3})$.

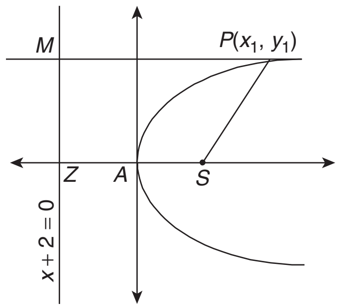


Figure 13.17

Your Turn 2

1. Find (a) the focus and (b) the directrix of the parabola $y^2 = -16x$.
Ans. (a) (–4, 0); (b) $x = 4$
2. If the equation of the directrix of a parabola is $2x + 3 = 0$, the axis is $2y - 3 = 0$ and the length of the latus rectum is 6 unit, find the focus. **Ans.** $\left(\frac{3}{2}, \frac{3}{2}\right)$ or $\left(\frac{-9}{2}, \frac{3}{2}\right)$
3. Find the equation of (a) the axis and (b) the length of the latus rectum of the parabola $x^2 - 2x + 4y = 0$.
Ans. (a) $x = 1$; (b) 4 units
4. Find all the points on the parabola $y^2 = 8x$ whose focal distance is 4 unit. **Ans.** (2, 4) and (2, – 4)
5. Prove that the equation $y^2 + 2ax + 2by + c = 0$ represents a parabola whose axis is parallel to x-axis. Find the vertex and the equation of the double-ordinate through the focus?
Ans. $\left(\frac{b^2 - c}{2a}, -b\right)$, $2ax = b^2 - a^2 - c$
6. Find the equation of the parabola whose vertex is at (2, 1) and the directrix is $x = y - 1$.
Ans. $x^2 + y^2 + 2xy - 14x + 2y + 17 = 0$

13.2.3 Parametric Equation

Let $y^2 = 4ax$ be a parabola then for any real value of t, $x = at^2$ and $y = 2at$ satisfy the equation of parabola, that is, $(at^2, 2at)$ lie on the parabola. Now, $x = at^2$, $y = 2at$ is known as parametric representation of parabola and, in short, we denote $P(at^2, 2at)$ as $P(t)$ where t is a parameter.

1. **Focal Chord:** A focal chord is any chord to the parabola, $y^2 = 4ax$, which passes through the focus. Let $y^2 = 4ax$ be the equation of a parabola and $(at^2, 2at)$ a point P on it (Fig. 13.18). Suppose the coordinates of the other extremity Q of the focal chord through point P are $(at_1^2, 2at_1)$. Then PS and SQ where S is the focus $(a, 0)$ have the same slopes. Therefore,

$$\frac{2at - 0}{at^2 - a} = \frac{2at_1 - 0}{at_1^2 - a}$$

$$\Rightarrow tt_1^2 - t = t_1t^2 - t_1 \Rightarrow (t, t+1)(t_1 - t) = 0$$

Hence, $t_1 = -1/t$, that is, point Q is $\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$.

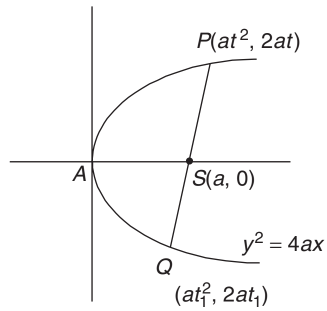


Figure 13.18

2. **Parametric Equation of Other Forms of Parabola**
   (a) $y^2 = -4ax$; $(-at^2, 2at)$
   (b) $x^2 = 4ay$; $(2at, at^2)$
   (c) $x^2 = -4ay$; $(2at, -at^2)$

Illustration 13.8 Show that the locus of a point that divides a chord of slope 2 of the parabola $y^2 = 4x$ internally in the ratio 1:2 is a parabola. Find the vertex of the parabola.

Solution: Let the two points on the given parabola be $(t_1^2, 2t_1)$ and $(t_2^2, 2t_2)$. The slope of the line joining these points is

$$2 = \frac{2t_2 - 2t_1}{t_2^2 - t_1^2} = \frac{1}{t_1 + t_2}$$

$$\Rightarrow t_1 + t_2 = 1$$

Hence, the two points are $(t_1^2, 2t_1)$ and $[(1 - t_1)^2, 2(1 - t_1)]$.Let (h, k) be the point which divides these points in the ratio 1:2. Then

$$h = \frac{(1 - t_1)^2 + 2t_1^2}{3} = \frac{1 - 2t_1 + 3t_1^2}{3} \quad (1)$$

and

$$k = \frac{2(1 - t_1) + 4t_1}{3} = \frac{2 + 2t_1}{3} \quad (2)$$

On eliminating t_1 from Eqs. (1) and (2), we get

$$4h = 9k^2 - 16 + 8$$

Hence, the locus is

$$\left(y - \frac{8}{9}\right)^2 = \frac{4}{9}\left(x - \frac{2}{9}\right)$$

which is a parabola. Hence, the vertex is $\left(\frac{2}{9}, \frac{8}{9}\right)$.

Illustration 13.9 Prove that the circle with the focal distance of a point on a parabola $y^2 = 4ax$ as diameter touches the tangent at the vertex.

Solution: Let $(at^2, 2at)$ be a point on the parabola $y^2 = 4ax$. The equation of the circle with the focal distance of the point as diameter is

$$x^2 + y^2 - (at^2 + a)x - 2aty + a^2t^2 = 0$$

When this circle cuts the tangents at the vertex, that is, at $x = 0$, we get

$$y^2 - 2aty + a^2t^2 = 0$$
$$\Rightarrow (y - at)^2 = 0$$

Thus, the circle cuts the line at two coinciding points. Hence, it touches the tangent at the vertex.

13.2.4 Position of a Point w.r.t. Parabola

Let $y^2 = 4ax$ be a parabola and P(h, k) be a general point (Fig. 13.19).

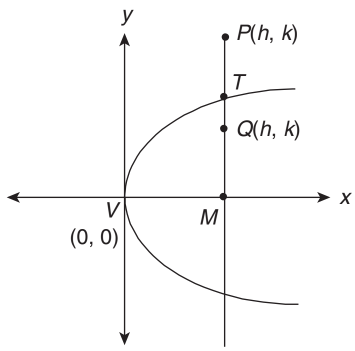


Figure 13.19

If $k^2 = 4ah$, that is, $k^2 - 4ah = 0$, then the point lies on the parabola. If P(h, k) lie outside the parabola, then

$$PM > TM$$
$$\Rightarrow PM^2 > TM^2$$
$$\Rightarrow k^2 > 4ah$$

If Q(h, k) lie inside the parabola, then

$$QM < TM$$
$$\Rightarrow QM^2 < TM^2$$
$$\Rightarrow k^2 < 4ah$$

Hence, if

$k^2 - 4ah < 0$ (point lies inside parabola)
$k^2 - 4ah = 0$ (point lies on the parabola)
$k^2 - 4ah > 0$ (point lies outside parabola)

Illustration 13.10 Find whether the point (2, 6) lie on, inside or outside parabola $2x - y^2 + 2y + 3 = 0$.

Solution: Rewriting the equation of parabola so that the coefficient of y^2 is positive, we get

$$y^2 - 2y - 2x - 3 = 0$$

On substituting the point (2, 6) for x and y, we get

$$(6)^2 - 2 \times (6) - 2(2) - 3 = 36 - 12 - 4 - 3$$
$$= 36 - 19$$
$$= 17 > 0$$

Therefore, the point lies outside the parabola.

Illustration 13.11 For what values of α, the point P(α, α) lies inside or lies on or lies outside the parabola $(y - 2)^2 = 4(x - 3)$.

Solution: The given equation can be written as

$$y^2 - 4y - 4x + 16 = 0$$

Point P(α, α) lies inside parabola if

$$\alpha^2 - 8\alpha + 16 < 0$$
$$\Rightarrow (\alpha - 4)^2 < 0$$

which implies that no such α exists. Point P(α, α) lies on the parabola if

$$(\alpha - 4)^2 = 0 \Rightarrow \alpha = 4$$

Point P(α, α) lies outside parabola if

$$(\alpha - 4)^2 > 0 \Rightarrow \alpha \in R - \{4\}$$

13.2.5 Intersection of Straight Line with Parabola

Consider a parabola $y^2 = 4ax$ and the straight line $y = mx + c$. On eliminating y from both of these equations, we get

$$m^2x^2 + 2(mc - 2a)x + c^2 = 0 \quad (13.1)$$

Now, the discriminant of Eq. (13.1) is

$$D = 4(mc - 2a)^2 - 4m^2c^2 = 16(a^2 - amc)$$

Now, the following three cases arise:

1. If $a < mc$, the line $y = mx + c$ is an imaginary chord to the parabola.
2. If $a = mc$, the line is a tangent to the parabola.
3. If $a > mc$, the line is the chord of the parabola.

13.2.6 Length of Chord

The abscissa of the points common to the straight line $y = mx + c$ and the parabola $y^2 = 4ax$ are given by the equation

$$m^2x^2 + (2mc - 4a)x + c^2 = 0$$

If (x_1, y_1) and (x_2, y_2) are the points of intersection, then

$$(x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2$$
$$= \frac{4(mc-2a)^2}{m^4} - \frac{4c^2}{m^2} = \frac{16a(a-mc)}{m^4}$$

and $\quad y_1 - y_2 = m(x_1 - x_2)$

Hence, the required length is

$$\sqrt{(x_1-x_2)^2+(y_1-y_2)^2} = \sqrt{1+m^2}\,|(x_1-x_2)|$$
$$= \frac{4}{m^2}\sqrt{1+m^2}\,\sqrt{a(a-mc)}$$

Also, for the equation

$$m^2x^2 + 2(mc - 2a)x + c^2 = 0$$

If m is too small, the line is almost parallel to axis of the parabola. One of the roots of the equation is much large. Hence, in this case, the line cuts the parabola at one point at infinity. Hence, it is concluded that the line cuts the parabola at one point only.

Illustration 13.12 Prove that for any non-zero, real m, the line $y = mx + (3/m)$ is a real chord to the parabola $y^2 = 16x$. Find the value of m if the length of the chord is 3.

Solution: It is given that $a = 4$, $c = 3/m$. Here, $a > mc$ for any m and hence, the line is a real chord. Now, the length of the chord is

$$\frac{4}{m^2}\sqrt{1+m^2}\sqrt{a(a-mc)} = \frac{4}{m^2}\sqrt{1+m^2}\sqrt{4(4-3)}$$

$$= \frac{8}{m^2}\sqrt{1+m^2} = 3 \qquad \text{(given)}$$

$$\Rightarrow 64(1+m^2) = 9m^4$$

$$\Rightarrow 9m^4 - 64m^2 - 64 = 0$$

$$\Rightarrow m^2 = 8 \Rightarrow m = \pm 2\sqrt{2}$$

Illustration 13.13 Prove that the line $\frac{x}{l}+\frac{y}{m}=1$ touches the parabola $y^2 = 4a(x+b)$ if $m^2(l+b) + al^2 = 0$.

Solution: The given line and the parabola are

$$\frac{x}{l}+\frac{y}{m}=1 \qquad (1)$$

and

$$y^2 = 4a(x+b) \qquad (2)$$

On substituting the value of x from Eq. (1), we get

$$x = l\left(l-\frac{y}{m}\right)$$

On substituting this value of x in Eq. (2), we get

$$y^2 = 4a\left[l\left(l-\frac{y}{m}\right)+b\right]$$

$$\Rightarrow y^2 + \frac{4al}{m}y - 4a(l+b) = 0 \qquad (3)$$

Since the line [Eq. (1)] touches the parabola [Eq. (2)], then the roots of Eq. (3) are equal.

$$D = 0$$

$$\Rightarrow \left(\frac{4al}{m}\right)^2 - 4\times 1\times[-4a(l+b)] = 0$$

$$\Rightarrow \frac{al^2}{m^2} + (l+b) = 0$$

$$\Rightarrow al^2 + m^2(l+b) = 0$$

$$\Rightarrow m^2(l+b) + al^2 = 0$$

Your Turn 3

1. Find if the point $\left(2, \frac{10}{3}\right)$ lie inside or outside the parabola $2x - x^2 + 3y - 2 = 0$. **Ans.** Inside the parabola
2. Find whether the line $2x - y = 0$ is a tangent, a real chord or an imaginary chord to the parabola $y^2 - 2y + 4x = 0$ **Ans.** Tangent
3. Find the length of the side of an equilateral triangle inscribed in the parabola $y^2 = 4ax$ so that one angular point is at the vertex. **Ans.** $8a\sqrt{3}$
4. If the point $(at^2, 2at)$ be the extremity of a focal chord of parabola $y^2 = 4ax$, then show that the length of the focal chord is $a\left(t+\frac{1}{t}\right)^2$.
5. Find the position of the point $(-2, 2)$ w.r.t. the parabola $y^2 - 4y + 9x + 13 = 0$. **Ans.** Inside the parabola
6. Show that the line $x\cos\alpha + y\sin\alpha = p$ touches the parabola $y^2 = 4ax$, if $p\cos\alpha + a\sin^2\alpha = 0$.

13.2.7 Tangent to Parabola

13.2.7.1 Tangent at Point (x_1, y_1) on Parabola

Let the equation of the parabola be $y^2 = 4ax$. Hence, dy/dx at $P(x_1, y_1)$ is $2a/y$. Then the equation of tangent at P is

$$y - y_1 = \frac{2a}{y_1}(x - x_1)$$

$$\Rightarrow y_1 y = 2a(x - x_1) + y_1^2$$

That is, $yy_1 = 2a(x + x_1)$.

13.2.7.2 Tangent in Terms of Slope (*m*)

Let the equation of tangent to parabola $y^2 = 4ax$ be $y = mx + c$. The abscissa of the points of intersection of tangent and parabola is $(mx + c)^2 = 4ax$. Therefore,

$$m^2x^2 + (2mc - 4a)x + c^2 = 0$$

However, the condition that the tangent should touch the parabola is that it should meet the parabola in coincident points. So,

$$(mc - 2a)^2 - m^2c^2 = 0$$

$$\Rightarrow (-2a)(mc - 2a + mc) = 0$$

$$\Rightarrow c = \frac{a}{m}$$

Hence, the tangent is

$$y = mx + \frac{a}{m}$$

Now, the equation $(mx + c)^2 = 4ax$ becomes

$$\left(mx - \frac{a}{m}\right)^2 = 0$$

$$\Rightarrow x = \frac{a}{m^2} \text{ and } y^2 = 4ax$$

$$\Rightarrow y = \frac{2a}{m}$$

Thus, the point of contact of the tangent $y = mx + \frac{a}{m}$ is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

13.2.7.3 Tangent in Terms of *t*

If t is the parameter in the parabola $y^2 = 4ax$ such that $x = at^2$, $y = 2at$, then the equation of tangent at t is

$$ty = x + at^2$$

At $P(t_1)$, the equation of tangent is

$$t_1 y = x + at_1^2$$

and at Q(t_2), the equation of tangent is

$$t_2 y = x + at_2^2$$

The point of intersection of the tangents at t_1 and t_2 is $[at_1t_2, a(t_1 + t_2)]$.

13.2.7.4 Tangent from an External Point

Let P(h, k) be an external point, then the combined equation of the tangents (pair of tangents) from point P to the parabola is given by

$$SS_1 = T^2$$
$$\Rightarrow [yk - 2a(x + h)]^2 = (y^2 - 4ax)(k^2 - 4ah)$$

13.2.8 Locus of Point of Intersection of Perpendicular Tangents Drawn to a Parabola

Consider a parabola, $y^2 = 4ax$. Now, the equation of tangent in slope form is

$$y = mx + \frac{a}{m}$$
$$\Rightarrow my = m^2x + a \quad (13.2)$$

Let the point of intersection be P(h, k). Then

$$mk = m^2h + a$$
$$\Rightarrow m^2h - mk + a = 0 \quad (13.3)$$

Since the tangents are perpendicular and m_1, m_2 are the roots of Eq. (13.3), we have

$$m_1m_2 = -1$$
$$\Rightarrow \frac{a}{h} = -1$$
$$\Rightarrow h + a = 0$$

Hence, the required locus is $x + a = 0$ which is directrix of the parabola.

Illustration 13.14 If the line $2x + 3y = 1$ touches the parabola $y^2 = 4ax$ at point P, find the focal distance of point P.

Solution: Let point P be (at^2, $2at$). Then the tangent at point P is

$$ty = x + at^2$$

On comparing this with given line $2x + 3y = 1$, we get

$$\frac{t}{3} = \frac{-1}{2} = at^2 \Rightarrow t = \frac{-3}{2}, a = \frac{-2}{9}$$

Thus, the focal distance of point P is

$$|a|(1+t^2) = \frac{2}{9}\left(1+\frac{9}{4}\right) = \frac{13}{18}$$

Illustration 13.15 Find the equation of the common tangents to the parabola $y^2 = 4ax$ and $x^2 = 4by$.

Solution: The equation of any tangent in terms of slope (m) to the parabola $y^2 = 4ax$ is

$$y = mx + \frac{a}{m} \quad (1)$$

If this line is also tangent to the parabola $x^2 = 4ay$, then the tangent [Eq. (1)] meets $x^2 = 4by$ in two coincident points. Substituting the value of y from Eq. (1) in $x^2 = 4by$, we get

$$x^2 = 4b\left(mx + \frac{a}{m}\right)$$
$$\Rightarrow x^2 - 4bmx - \frac{4ab}{m} = 0$$

Now, the roots are equal if $b^2 = 4ac$. So,

$$(-4bm)^2 = 4(1)\left(\frac{-4ab}{m}\right)$$
$$\Rightarrow 16b^2m^3 + 16ab = 0 \quad (m \neq 0)$$
$$\Rightarrow m^3 = \frac{-a}{b} \Rightarrow x = -\left(\frac{a}{b}\right)^{1/3}$$

Substituting value of m in Eq. (1), we get

$$y = \frac{-a^{1/3}}{b^{1/3}}x - \frac{ab^{1/3}}{a^{1/3}}$$
$$\Rightarrow a^{1/3}x + b^{1/3}y + a^{2/3}b^{2/3} = 0$$

Illustration 13.16 The tangents to the parabola $y^2 = 4ax$ at P(at_1^2, $2at_1$) and Q(at_2^2, $2at_2$) intersect at point R. Find the area of ΔPQR.

Solution: The equations of tangents at P(at_1^2, $2at_1$) and Q(at_2^2, $2at_2$) (Fig. 13.20) are

$$t_1y = x + at_1^2 \quad (1)$$
$$t_2y = x + at_2^2 \quad (2)$$

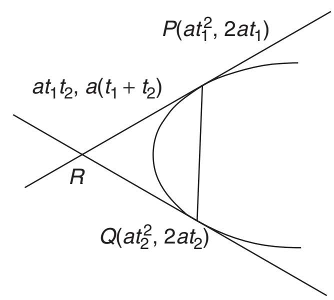


Figure 13.20

The point of intersection of the tangents [Eqs. (1) and (2)] is

$$R \equiv [at_1t_2, a(t_1 + t_2)]$$

Therefore, the area of ΔPQR is

$$\frac{1}{2}\begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ at_1t_2 & a(t_1+t_2) & 1 \end{vmatrix} = \frac{1}{2}a^2|(t_1 - t_2)^3|$$

13.2.9 Chords of Parabola

13.2.9.1 Chord of Contact

From an external point $P(h, k)$ (Fig. 13.21), let PT_1 and PT_2 be the two tangents that are drawn to the parabola $y^2 = 4ax$. Then the line joining the points of contact T_1 and T_2 is known as chord of contact and its equation is given by $T = 0$.

$ky - 2a(x+h) = 0$

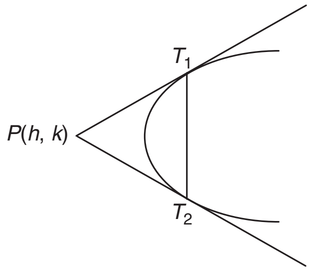


Figure 13.21

13.2.9.2 Chord of Parabola with Mid-Point

Consider the parabola $y^2 = 4ax$ and let P(h, k) be an internal point of the parabola. Then the equation of the chord with P(h, k) as mid-point is given by T = S_1, $ky - 2a(x + h) = k^2 - 4ah$.

Illustration 13.17 Show that the length of the chord of contact of tangents drawn from (x_1, y_1) to $y^2 = 4ax$ is

$$\frac{1}{a}[\sqrt{(y_1^2 - 4ax_1)(y_1^2 + 4a^2)}]$$

Solution: Let Q(at_1^2, $2at_1$) and R(at_2^2, $2at_2$) be the points of contact of tangents drawn from P(x_1, y_1) to the parabola $y^2 = 4ax$. Then the coordinates of point P, the point of intersection of tangents to the parabola at R and Q, is

$$[at_1t_2, a(t_1 + t_2)]$$

$$\Rightarrow x_1 = at_1t_2 \quad (1)$$

$$y_1 = a(t_1 + t_2) \quad (2)$$

$$\Rightarrow (t_1 - t_2)^2 = (t_1 + t_2)^2 - 4t_1t_2 = \frac{y_1^2 - 4ax_1}{a^2} \quad (3)$$

$$\Rightarrow QR = \sqrt{(at_1^2 - at_2^2)^2 + (2at_1 - 2at_2)^2}$$

On using Eqs. (2) and (3), we get

$$QR = \frac{\sqrt{(y_1^2 - 4ax_1)(y_1^2 + 4a)}}{a}$$

Illustration 13.18 Find the locus of the mid-points of the chord of contact of orthogonal tangents to the parabola $y^2 = 4a$.

Solution: The equation of the chord of contact of tangents from (h, k) is

$$yk = 2a(x + h) \quad (1)$$

and the equation of the chord of the parabola whose mid-point is (α, β) is

$$y\beta - 2a(x + \alpha) = \beta^2 - 4a\alpha \quad (2)$$

Since Eqs. (1) and (2) are the same, we get

$$\frac{k}{\beta} = \frac{2a}{2a} = \frac{2ah}{2a\alpha + \beta^2 - 4a\alpha}$$

$$\Rightarrow k = \beta \text{ and } \frac{2ah}{\beta^2 - 2a\alpha} = 1$$

$$\Rightarrow h = \frac{\beta^2 - 2a\alpha}{2a} \text{ and } k = \beta$$

Since the tangents from (h, k) are at right angles, the point (h, k) lies on the directrix. So,

$$h + a = 0$$

$$\Rightarrow \frac{\beta^2 - 2a\alpha}{2a} + a = 0$$

$$\Rightarrow \beta^2 - 2a\alpha + 2a^2 = 0$$

Thus, the locus is

$$y^2 - 2ax + 2a^2 = 0$$

or

$$y^2 = 2a(x - a)$$

Illustration 13.19 Find the equation of chord of the parabola $y^2 = 12x$ which is bisected at the point (5, – 7).

Solution: Here, $(x_1, y_1) \equiv (5, -7)$. The required equation is

$$T = S_1$$

That is,

$$y_1y - 12\left(\frac{x + x_1}{2}\right) = y_1^2 - 12x_1$$

$$\Rightarrow -7y - 12\left(\frac{x + 5}{2}\right) = (-7)^2 - 12(5)$$

$$\Rightarrow 6x + 7y + 19 = 0$$

Your Turn 4

1. The tangents are drawn from any point on the line $x + 4a = 0$ to the parabola $y^2 = 4ax$. Prove that their chord of contact subtends a right angle at the vertex.
2. Prove that the area of the triangle formed by the tangents to the parabola $y^2 = 4ax$ from the point (x_1, y_1) and the chord of contact is $(1/2a)(y_1^2 - 4ax_1)^{3/2}$ sq. unit.
3. Two tangents to a parabola $y^2 = 4ax$ meet at an angle of 45°. Prove that the locus of their point of intersection is the curve $y^2 - 4ax = (x + a)^2$.
4. If a tangent to the parabola $y^2 = 4ax$ meets the axis of the parabola in T and the tangent at the vertex A in Y and the rectangle TAYG is completed, show that the locus of G is $y^2 + ax = 0$.
5. Find the angle between the tangents drawn from point (4, 0) to the parabola $y^2 + 4x = 0$.

   Ans. $2\tan^{-1}\left(\frac{1}{2}\right)$
6. Find the locus of the orthocentre of the triangle formed by any three tangents on the parabola $y^2 - 4y + 4x + 8 = 0$.

   Ans. y-axis

13.2.10 Normal to Parabola

13.2.10.1 Normal at Point (x_1, y_1) on Parabola

The slope of tangent at (x_1, y_1) to the parabola $y^2 = 4ax$ is $2a/y_1$. Thus, the slope of the normal is $-y_1/2a$. Hence, the equation of normal is

$$y - y_1 = \frac{-y_1}{2a}(x - x_1)$$

13.2.10.2 Normal at Point *t*

Let P(at^2, $2at$) be a point on the parabola $y^2 = 4ax$. Then the equation of normal at point P is

$$y - 2at = \frac{-2at}{2a}(x - at^2)$$

$$\Rightarrow y + tx = 2at + at^3$$

Now, let this normal pass through the point (h, k). Then

$$k + th = 2at + at^3$$

$$\Rightarrow at^3 + (2a - h)t - k = 0 \quad (13.4)$$

If t_1, t_2 and t_3 be the roots of Eq. (13.4), then

$$t_1 + t_2 + t_3 = 0 \quad (13.5)$$

$$t_1t_2 + t_2t_3 + t_1t_3 = \frac{2a - h}{a} \quad (13.6)$$

$$t_1t_2t_3 = \frac{k}{a} \quad (13.7)$$

Hence, we can conclude that the three normals can be drawn from a given point (h, k) to the parabola. Now, multiplying Eq. (13.5) by $2a$, Eq. (13.6) by $4a^2$ and Eq. (13.7) by $8a^3$, we get

$$y_1 + y_2 + y_3 = 0$$

$$y_1y_2 + y_1y_3 + y_2y_3 = 4a(2a - h)$$

$$y_1y_2y_3 = 8a^2k$$

where y_1, y_2 and y_3 are the ordinates of the feet of the normals drawn from the point (h, k) to the parabola $y^2 = 4ax$. If normals at three points on parabola $y^2 = 4ax$ are concurrent, then the sum of the ordinates of the feet of normals is zero.

13.2.10.3 Normal in Terms of Slope *m*

In the equation $y + tx = 2at + at^3$, we can substitute the slope of normal $-t = m$ and we get

$$y - mx + 2am + am^3 = 0$$

and the point from where the normal is drawn as (am^2, $-2am$). If the normal passes through (h, k), then

$$am^3 + (3a - h)m + k = 0 \quad (13.8)$$

The roots of Eq. (13.8) give the slopes of the normal which are drawn from the point (h, k).

$$m_1 + m_2 + m_3 = 0$$

$$m_1m_2 + m_1m_3 + m_2m_3 = \frac{2a - h}{a}$$

$$m_1m_2m_3 = \frac{-k}{a}$$

If the three normals to the parabola $y^2 = 4ax$ are concurrent, then the sum of the slopes of normals is zero.

13.2.10.4 Results for Conormal Points

(i) The algebraic sum of the slopes of three concurrent normals is zero.

(ii) The algebraic sum of the ordinates of the feet of the three normals drawn to a parabola from a given point is zero.

(iii) If three normals are drawn to any parabola $y^2 = 4ax$ from point (h, k) be real, then $h > 2a$.

(iv) If three normals drawn to any point $y^2 = 4ax$ from a given point (h, k)be real and distinct, then $27ak^2 < 4(h - 2a)^3$.

(v) The centroid of the triangle formed by the feet of the three normals lies on the axis of the parabola.

Illustration 13.20 Show that the normal at a point (at_1^2, $2at_1$) on the parabola $y^2 = 4ax$ cuts the curve again at the point whose parameter t_2 is $-t_1 - (2/t_1)$.

Solution: The equation of the normal at point (at_1^2, $2at_1$) is

$$y - 2at_1 = -t_1(x - at_1^2)$$

If it meets the parabola once again at point (at_2^2, $2at_2$), then

$$2at_2 - 2at_1 = -t_1(at_2^2 - at_1^2)$$

$$\Rightarrow t_1(t_2 + t_1) = -2$$

$$\Rightarrow t_2 = -t_1 - \frac{2}{t_1}$$

Illustration 13.21 If P and Q are the points with parameters t_1 and t_2 on the parabola $y^2 = 4ax$. If the normals at points P and Q meet on the parabola at point R(t_3), show that $t_1t_2 = 2$ and $t_3 = -t_1 - t_2$.

Solution: Let the normals at points P and Q meet at R(x_1, y_1) where $x_1 = at_3^2$ and $y_1 = 2at_3$. The normal to the parabola at t is

$$y + tx = 2at + at^3$$

If it passes through R(x_1, y_1), then

$$y_1 + tx_1 = 2at + at^3$$

or

$$at^3 + (2a - x_1)t - y_1 = 0 \quad (1)$$

If t_1, t_2 and t_3 are the roots of Eq. (1), we get

$$t_1t_2t_3 = -\left(\frac{-y_1}{a}\right) = \frac{y_1}{a} = \frac{2at_3}{a} = 2t_3$$

Therefore, $t_1t_2 = 2$. Also,

$$t_1 + t_2 + t_3 = 0 \Rightarrow t_3 = -t_1 - t_2$$

Illustration 13.22 Find the locus of a point P which moves such that the two of the three normals drawn from it to the parabola $y^2 = 4ax$ are mutually perpendicular.

Solution: Let point P be (h, k) and the normal to parabola be $y = mx - 2am - am^3$. If this normal passes through P(h, k), then

$$am^3 + (2a - h)m + k = 0 \quad (1)$$

If m_1, m_2 and m_3 be the roots of Eq. (1), then

$$m_1 + m_2 + m_3 = 0 \quad (2)$$

$$m_1m_2 + m_1m_3 + m_2m_3 = \frac{2a - h}{a} \quad (3)$$

$$m_1m_2m_3 = \frac{-k}{a} \quad (4)$$

Here, we get $m_1m_2 = -1$. Thus,

$$m_3 = \frac{k}{a} \Rightarrow m_1 + m_2 = \frac{-k}{a}$$

Substituting the value in Eq. (3), we get

$$\frac{-1-k^2}{a^2} = \frac{2a-h}{a} \Rightarrow k^2 = a(h-3a)$$

Therefore, the locus of point P is

$$y^2 = a(x-3a)$$

Illlustration 13.23 Prove that the locus of the intersection of the normals at the ends of a system of parallel chords of a parabola is a straight line which is normal to the curve.

Solution: Let the system of parallel chords be

$$y = mx + c \quad (1)$$

where m is a constant. Also, let the parabola be

$$y^2 = 4ax \quad (2)$$

The ordinates of the point of intersection of the chords [Eq. (1)] and the parabola [Eq. (2)] are the roots of the equation

$$y^2 = 4a\left(\frac{y-c}{m}\right) \quad (3)$$

If y_1 and y_2 are the roots, then

$$y_1 + y_2 = \frac{4a}{m} \quad (4)$$

which is a constant. Let two normals be drawn at points with ordinates y_1 and y_2. Let the ordinate of the point in which the third normal is drawn be y_3. Then

$$y_1 + y_2 + y_3 = 0$$

From Eq. (4), we get

$$y_3 = \frac{-4a}{m}$$

Now, since $4a/m$ is a constant, the third point on the parabola is fixed. Hence, the normals at the extremities of a system of parallel chords of a parabola intersect upon a fixed normal.

Your Turn 5

1. Prove that the locus of the mid-points of the normal chords of the parabola $y^2 = 4ax$ is $\frac{y^2}{2a} + \frac{4a^3}{y^2} = x - 2a$.
2. For what values of a, the tangents that are drawn to the parabola $y^2 = 4ax$ from a point, which is not located on y-axis, are normal to the parabola $x^2 = 4y$. **Ans.** $a < -2\sqrt{2}$ or $a > 2\sqrt{2}$
3. Find the equation of the normal to the parabola $y^2 = 4x$ which is (a) parallel to the line $y = 2x - 5$ and (b) perpendicular to the line $x + 3y + 1 = 0$. **Ans.** (a) $y = 2x - 12$; (b) $y = 3(x - 11)$
4. Three normals to $y^2 = 4x$ pass through the points (13, 12). Show that one of the normals is given by $y = x - 3$. Also find equation of the other normal. **Ans.** $4x + y = 72$; $3x - y = 33$
5. If a chord joining the points t_1 and t_2 of the parabola $y^2 = 4ax$ is normal at the point t_2. Find the value of $t_2(t_1 + t_2)$. **Ans.** – 2
6. Prove that the centroid of ΔABC where A, B and C are the points on the parabola $y^2 = 4ax$ whose normals are concurrent and lie on the axis of the parabola.
7. Find the set of points on the axis of the parabola $x^2 + 8y = 0$ from where three distinct normals can be drawn to the parabola. **Ans.** [(0, h), $h < -4$]

13.2.11 Reflection Property of Parabola

The tangent (PT) and the normal (PN) of the parabola $y^2 = 4ax$ at point P are the internal and external bisectors of $\angle$SPM and BP is parallel to the axis of the parabola (Fig. 13.22). Also, $\angle BPN = \angle SPN$.

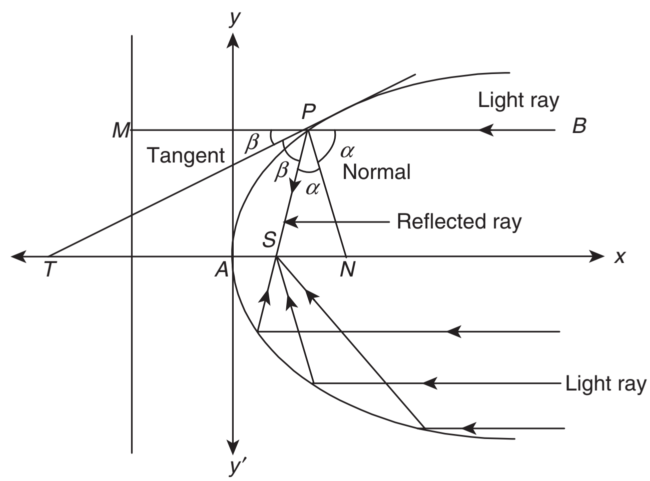


Figure 13.22

Illustration 13.24 A ray of light is coming along the line $y = b$ from the positive direction of x-axis and strikes a concave mirror whose intersection with the xy plane is a parabola $y^2 = 4ax$. Find the equation of the reflected ray and show that it passes through the focus of the parabola. Both a and b are positive.

Solution: See Fig. 13.23. The given parabola is

$$y^2 = 4ax$$

The equation of tangent at P $\left(\frac{b^2}{4a}, b\right)$ is

$$yb = 2a\left(x + \frac{b^2}{4a}\right)$$

The slope of tangent is $2a/b$. Hence, the slope of the normal is

$$\frac{-b}{2a} = \tan(180° - \alpha)$$

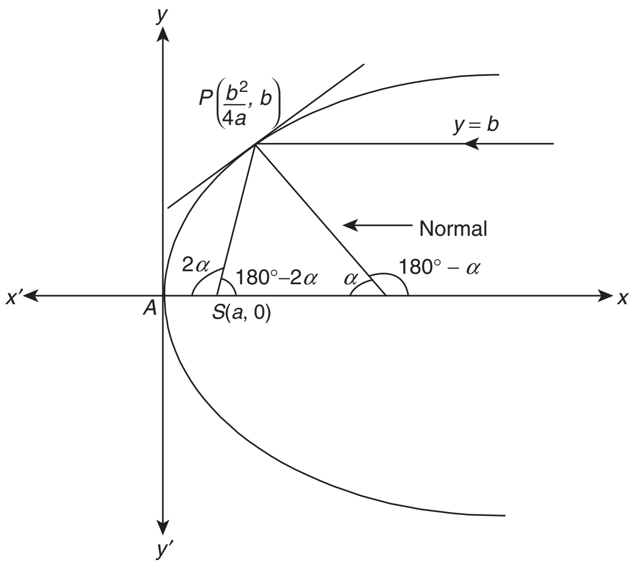


Figure 13.23

Therefore,

$$\tan\alpha = \frac{b}{2a}$$

The slope of reflected ray is

$$\tan(180° - 2\alpha) = -\tan 2\alpha$$

$$= -\left[\frac{2\tan\alpha}{1-\tan^2\alpha}\right]$$

$$= -\left[\frac{2(b/2a)}{1-(b^2/4a^2)}\right]$$

$$= \frac{-4ab}{(4a^2-b^2)}$$

Hence, the equation of the reflected ray is

$$y - b = \frac{4ab}{(4a^2-b^2)}\left(x - \frac{b^2}{4a}\right)$$

$$\Rightarrow (y-b)(4a^2-b^2) = -(4ax-b^2)b$$

which obviously passes through $S(a, 0)$.

13.2.12 Important Properties of Parabola

1. The section of a tangent to a parabola cut off between the directrix and the curve subtends a right angle at the focus.
2. The tangents at the extremities of a focal chord intersect at right angles on the directrix and hence circle on any focal chord as diameter touches the directrix (Fig. 13.24). Also, a circle on any focal radii of a point P(at^2, $2at$) as diameter touches the tangent at the vertex and intercepts a chord of length $a\sqrt{1+t^2}$ on a normal at the point P.

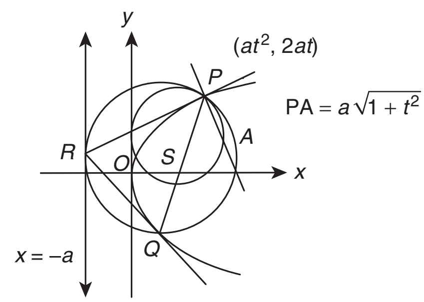


Figure 13.24

3. Any tangent to a parabola and the perpendicular on it from the focus meet on the tangent at the vertex (Fig. 13.25).

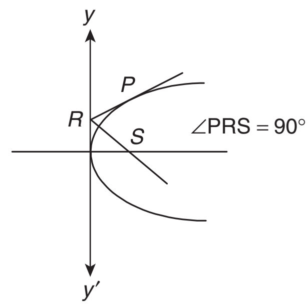


Figure 13.25

4. If the tangents at points P and Q meet at point T (Fig. 13.26), then
   (a) TP and TQ subtend equal angles at the focus S.
   (b) $ST^2 = SP \cdot SQ$.
   (c) ΔSPT and STQ are similar.

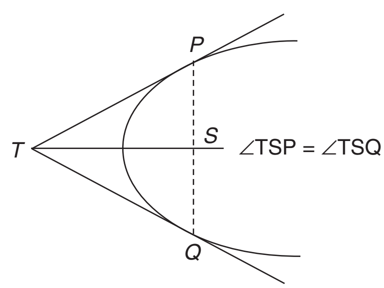


Figure 13.26

5. A semi-latus rectum of the parabola $y^2 = 4ax$ is the harmonic mean between the segments of any focal chord of the parabola (Fig. 13.27).

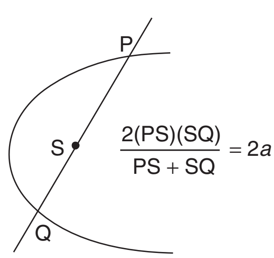


Figure 13.27

6. The area of the triangle formed by the three points on a parabola is twice the area of the triangle formed by the tangents at these points.
7. The condition for three real and distinct normals to be drawn from a point P(h, k) is $h > 2a$ and $k^2 < \frac{4}{27a}(h-2a)^3$.
8. The centroid of the triangle formed by three co-normal points lies on the x-axis.
9. The length of sub-tangent at any point P(x, y) on the parabola $y^2 = 4ax$ equals twice the abscissa of the point P (Fig. 13.28). Note that subtangent is bisected at the vertex.

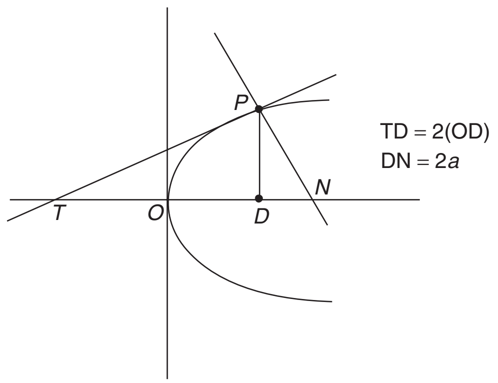


Figure 13.28

10. The length of subnormal is constant for all points on the parabola and is equal to the semi-latus rectum.
11. The tangent at any point P on a parabola bisects the angle between the focal chord through P and the perpendicular from P on the directrix.

In Fig. 13.29, $\angle$MPT = $\angle$TPS

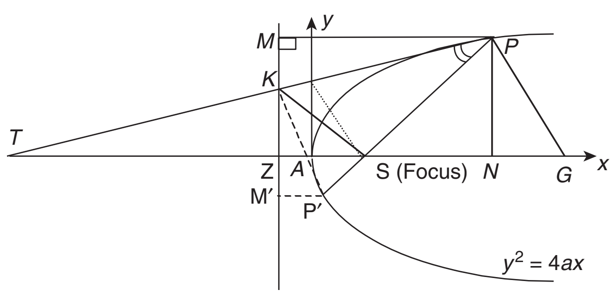


Figure 13.29

Similarly, the normal at any point on a parabola bisects the angle between the focal chord and the line parallel to the axis through that point.

Illustration 13.25 If the tangents at two points (1, 2) and (3, 6) on a parabola intersect at the point (– 1, 1), then the slope of the directrix of the parabola is

(A) $\sqrt{2}$ **(B)** -2
(C) -1 **(D)** None of these

Solution: Midpoint of the points, say P and Q, the parabola is R (2, 4). If the tangent at P and Q intersect at T, then axis of parabola is parallel to TR. So, the slope of axis $= \frac{4-1}{2+1} = 1$.

Therefore, slope of directrix = – 1.

Hence, the correct answer is option (C).

Illustration 13.26 If the normals drawn at the end points of a variable chord PQ of the parabola $y^2 = 4ax$ intersect at parabola, then the locus of the point of intersection of the tangent drawn at the points P and Q is

(A) $x + a = 0$ **(B)** $x - 2a = 0$
(C) $y^2 - 4x + 6 = 0$ **(D)** None of these

Solution: Let the points P and Q be $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$, respectively. Since the normals at P and Q intersects at parabola, $t_1t_2 = 2$.

Let (h, k) be the point of intersection of the tangents at P and Q. Then

$$at_1t_2 = h \text{ and } a(t_1 + t_2) = k$$
$$\Rightarrow h = 2a$$
$$\Rightarrow x - 2a = 0$$

Hence, the correct answer is option (B).

Illustration 13.27 For the parabola $y^2 = 4x$, let P be the point of concurrency of three normals and S be the focus. If α_1 be the sum of the angles made by three normals from the positive direction of x-axis and α_2 be the angle made by PS with the positive direction of x-axis then $\frac{\alpha_1 - \alpha_2}{\pi}$ can be equal to

(A) 1 **(B)** 2
(C) $\frac{1}{2}$ **(D)** $\frac{3}{2}$

Solution: Let the point P be (h, k). Then

$$m^3 + (2 - h)m + k = 0$$

$$\Rightarrow \tan\alpha_1 = \frac{k}{h-1} \text{ and } \tan\alpha_2 = \frac{k}{h-1}$$

$$\Rightarrow \alpha_1 - \alpha_2 = n\pi \Rightarrow \frac{\alpha_1 - \alpha_2}{\pi} \text{ is an integer.}$$

Hence, the correct answers are options (A) and (B).

Illustration 13.28 Tangents are drawn from (– 2, 0) to $y^2 = 8x$, radius of circle(s) that would touch these tangents and the corresponding chord of contact, can be equal to

(A) $4(\sqrt{2}+1)$ **(B)** $4(\sqrt{2}-1)$
(C) $8\sqrt{2}$ **(D)** None of these

Solution: See Fig. 13.30.

Point 'P' clearly lies on the directrix of $y^2 = 8x$.

Thus, slope of PA and PB are 1 and – 1, respectively.

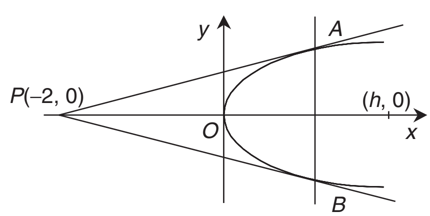


Figure 13.30

Equation of PA: $y = x + 2$

Equation of PB: $y = -x - 2$

Equation of AB : $x = 2$

Let the centre of the circle be (h, 0) and radius be 'r'. Then

$$\frac{|h+2|}{\sqrt{2}} = \frac{|h-2|}{1} = r$$

$$\Rightarrow h^2 + 4 + 4h = 2(h^2 + 4 - 4h) \Rightarrow h^2 - 12h + 4 = 0$$

$$h = \frac{12 \pm 8\sqrt{2}}{2} = 6 \pm 4\sqrt{2} \Rightarrow |h-2| = 4(\sqrt{2}-1), 4(\sqrt{2}+1)$$

Hence, the correct answers are options (A) and (B).

Illustration 13.29 Slope of tangent to $x^2 = 4y$ from (–1, –1) can be

(A) $\frac{-1+\sqrt{5}}{2}$ **(B)** $\frac{-1-\sqrt{5}}{2}$

(C) $\frac{1-\sqrt{5}}{2}$ **(D)** $\frac{1+\sqrt{5}}{2}$

Solution:

$$y' = x/2 = m$$

$$\Rightarrow x = 2m \Rightarrow y = m^2$$

So, equation of tangent is $y - m^2 = m(x - 2m)$ which passes through (– 1, – 1). Therefore,

$$-1 - m^2 = m(-1 - 2m) \Rightarrow m^2 + m - 1 = 0 \Rightarrow m = \frac{-1 \pm \sqrt{5}}{2}$$

Hence, the correct answers are options (A) and (B).

Illustration 13.30 Tangent is drawn at any point (x_1, y_1) other than vertex on the parabola $y^2 = 4ax$. If tangents are drawn from any point on this tangent to the circle $x^2 + y^2 = a^2$ such that all the chords of contact pass through a fixed point (x_2, y_2), then

(A) x_1, a, x_2 are in GP **(B)** $\frac{y_1}{2}, a, y_2$ are in GP

(C) $-4, \frac{y_1}{y_2}, \frac{x_1}{x_2}$ are in GP **(D)** $x_1x_2 + y_1y_2 = a^2$

Solution: Let $(x_1, y_1) \equiv (at^2, 2at)$. Then tangent at this point is

$$ty = x + at^2$$

Any point on this tangent is $\left(h, \frac{h+at^2}{t}\right)$.

Chord of contact of this point with respect to the circle $x^2 + y^2 = a^2$ is

$$hx + \left(\frac{h+at^2}{t}\right)y = a^2 \text{ or } \left(aty - a^2\right) + h\left(x + \frac{y}{t}\right) = 0$$

which is a family of straight lines passing through point of intersection of $ty - a = 0$ and $x + \frac{y}{t} = 0$.

So, the fixed point is $\left(-\frac{a}{t^2}, \frac{a}{t}\right)$. Therefore,

$$x_2 = -\frac{a}{t^2}, y_2 = \frac{a}{t}$$

Clearly, $x_1x_2 = -a^2$, $y_1y_2 = 2a^2$

$$\frac{y_1}{2}, a, y_2 \text{ are in GP}$$

Now,

$$x_1x_2 + y_1y_2 = a^2$$

Also, $\frac{x_1}{x_2} = -t^4$, $\frac{y_1}{y_2} = 2t^2 \Rightarrow 4\frac{x_1}{x_2} + \left(\frac{y_1}{y_2}\right)^2 = 0$

$$-4, \frac{y_1}{y_2}, \frac{x_1}{x_2} \text{ are in GP.}$$

Hence, the correct answers are options (B), (C) and (D).

Illustration 13.31 The integral values of α such that $\left(\alpha, \frac{24}{\alpha^2+4}\right)$ lie in the interior of the larger segment of the circle $x^2 + y^2 = \frac{99}{2}$ cut off by the parabola $y^2 + 8\sqrt{2}x = 0$ (where $\alpha \in [-10, 10]$) can be

(A) 2 **(B)** 3

(C) 4 **(D)** 8

Solution: Intersection of parabola with $\frac{24}{x^2+14} = \sqrt{-8\sqrt{2}x}$

$$x = -\sqrt{2}$$

Also

$$\frac{24}{x^2+4} + x^2 = \frac{99}{2}$$

$$\Rightarrow x = \sqrt{44}$$

$$-\sqrt{2} < \alpha < \sqrt{44}$$

Therefore, the integral values of α are 8.

Hence, the correct answer is option (D).

Additional Solved Examples

1. A point moves on the parabola $y^2 = 4ax$ and its distance from the focus is minimum for what value of x?

   (A) –1 **(B)** 0

   (C) 1 **(D)** a

Solution: For $y^2 = 4ax$, the focus is S(a, 0). The distance of any point P on a parabola from the focus is minimum only if the point is located at the vertex (0, 0).

Hence, the correct answer is option (B).

2. From point (– 1, 2), tangent lines are drawn to the parabola $y^2 = 4x$, then the equation of chord of contact is

   (A) $y = x + 1$ **(B)** $y = x - 1$

   (C) $y + x = 1$ **(D)** None of these

Solution: For the parabola $y^2 = 4ax$ and the external point (x_1, y_1) equation of chord of contact is

$$y_1y = 2a(x + x_1)$$

Therefore, for the parabola $y^2 = 4x$ and point (–1, 2), we get

$$2y = 2(x - 1)$$

$$\Rightarrow y = x - 1$$

Hence, the correct answer is option (B).

3. The normals to the parabola $y^2 = 4ax$ from the point ($5a$, $2a$) are

   (A) $y = -x - 3a$ **(B)** $y = -2x + 12a$

   (C) $y = -3x + 33a$ **(D)** $y = x + 3a$

Solution: We have

$$y^2 = 4ax \quad (1)$$

$$\Rightarrow \frac{dy}{dx} = \frac{2a}{y} = \frac{2a}{2at} = \frac{1}{t} \quad \text{[at P}(at^2, 2at)]$$

The equation of normal at point P is

$$(y-2at)\frac{dy}{dx}+(x-at^2)=0$$

$$\Rightarrow (y-2at)\times\frac{1}{t}+(x-at^2)=0$$

$$\Rightarrow y-2at+tx-at^3=0$$

$$\Rightarrow y=-tx+2at+at^3 \qquad (2)$$

This equation of normal [Eq. (2)] passes through $(5a, 2a)$. Therefore,

$$2a=-t(5a)+2at+at^3$$

$$\Rightarrow 2a=-3at+at^3$$

$$\Rightarrow t^3-3t-2=0$$

$$\Rightarrow (t+1)^2(t-2)=0$$

$$\Rightarrow t=-1, 2$$

When $t = 2$, we get

$$2x+y=12a$$

and when $t = -1$, we get

$$x-y=3a$$

Hence, the correct answer is option (B).

4. If the normal to $y^2 = 12x$ at (3, 6) meets the parabola once again at the point (27, –18), then the circle on the normal chord as diameter is

(A) $x^2 + y^2 + 30x + 12y - 27 = 0$
(B) $x^2 + y^2 + 30x + 12y + 27 = 0$
(C) $x^2 + y^2 - 30x - 12y - 27 = 0$
(D) $x^2 + y^2 - 30x + 12y - 27 = 0$

Solution: Here, there are two end points of the diameter, that is, (3, 6) and (27, –18). Therefore, the equation of circle is

$$(x-3)(x-27)+(y-6)(y+18)=0$$

$$\Rightarrow x^2+y^2-30x+12y-27=0$$

Hence, the correct answer is option (D).

5. AB is a chord of the parabola $y^2 = 4ax$ with vertex A. The line BC is drawn perpendicular to AB meeting the axis at point C. The projection of BC on the axis of the parabola is

(A) a **(B)** 2
(C) $4a$ **(D)** $8a$

Solution: We have

$$\tan\theta=\frac{y}{x}$$

The projection of BC (Fig. 13.31) on the axis gives

$$LC=\frac{y}{\tan(90^\circ-\theta)}=\frac{y\tan\theta}{1}=\frac{y^2}{x}=4a$$

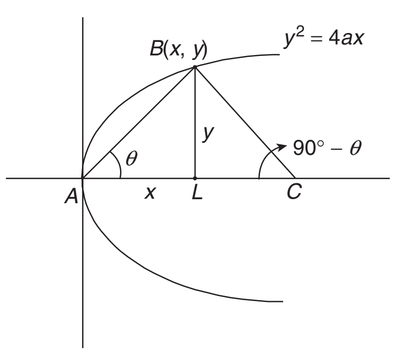


Figure 13.31

Hence, the correct answer is option (C).

6. A circle drawn on any focal chord of the parabola $y^2 = 4ax$ as diameter cuts the parabola at two points t and t' (other than the extremity of focal chord), then

(A) $tt' = -1$ **(B)** $tt' = 2$
(C) $tt' = 3$ **(D)** None of these

Solution: The equation of the circle (where t_1 and t_2 are extremities of the focal chord) is

$$(x-at_1^2)(x-at_2^2)+(y-2at_1)(y-2at_2)=0$$

$$\Rightarrow x^2+y^2-xa(t_1^2+t_2^2)-2ay(t_1+t_2)-3a^2=0$$

Let $(at^2, 2at)$ be any point on it. Then

$$a^2t^4+4a^2t^2-a^2t^2(t_1^2+t_2^2)-2a(2at)(t_1+t_2)-3a^2=0$$

which has the roots t_1, t_2, t_3 and t_4.

$$t_1t_2t_3t_4=-3$$

$$\Rightarrow t_3t_4=3 \qquad (\because t_1t_2=-1)$$

$$\Rightarrow tt'=3$$

Hence, the correct answer is option (C).

7. Two parabolas exist with the same axis. The focus of each parabola is exterior to the other and the latus rectum of the parabolas is $4a$ and $4b$, respectively. The locus of the middle points of the intercepts between the parabolas made on the lines parallel to the common axis is a

(A) straight line if $a = b$ **(B)** circle if $a \neq b$
(C) parabola for all a, b **(D)** ellipse if $b > a$

Solution: The equation of the two parabolas can be written as follows:

$$y^2=4a(x-k)$$

and

$$y^2=-4b(x+k)$$

A line parallel to the common axis is $y = h$. Then

$$A\equiv\left(\frac{h^2}{4a}+k, h\right) \text{ and } B\equiv\left(\frac{-k-h^2}{4b}, h\right)$$

If $P \equiv (\alpha, \beta)$, then

$$\alpha=\frac{1}{2}\left(\frac{h^2}{4a}+k-k-\frac{h^2}{4b}\right)$$

Now, $\beta = h$. Therefore,

$$2\alpha=\frac{h^2}{4}\left(\frac{1}{a}-\frac{1}{b}\right)$$

The locus of P is

$$2x=\frac{y^2}{2}\left(\frac{b-a}{ab}\right)$$

Hence, the correct answer is option (A).

8. If three distinct normals can be drawn to the parabola $y^2 - 2y = 4x - a$ from the point $(2a, 0)$, then the range of values of a is

(A) No real values possible **(B)** $(2, \infty)$
(C) $(-\infty, 2)$ **(D)** None of these

Solution: The given parabola is

$$(y-1)^2=4(x-2)$$

The equation of any normal to the given parabola is

$$y - 1 = m(x - 2) - m - m^3$$

$$\Rightarrow y = mx - 4m - m^3 + 1$$

Since this passes through $(2a, 0)$, the equation

$$m^3 + 2m(2 - a) - 1 = 0$$

has three distinct and real values of m. Also $3m^2 + 2(2 - a) = 0$ has all real and distinct root, which implies that $a > 2$.

Hence, the correct answer is option (B).

9. A tangent and a normal is drawn at a point $P \equiv (16, 16)$ of the parabola $y^2 = 16x$ which cut the axis of the parabola at the points A and B, respectively. If the centre of the circle through P, A and B is C, then the angle between PC and the x-axis is

(A) $\tan^{-1}(1/2)$ **(B)** $\tan^{-1}2$
(C) $\tan^{-1}(3/4)$ **(D)** $\tan^{-1}(4/3)$

Solution: We know that centre of circle coincides with the focus of parabola (Fig. 13.32).

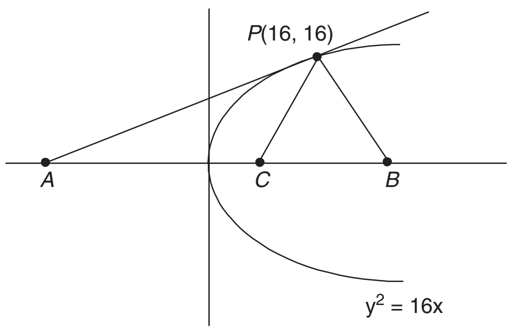


Figure 13.32

$$C \equiv (4, 0)$$

Therefore, $\tan\alpha$ is the slope of $PC = 16/12$. Thus,

$$\alpha = \tan^{-1}\left(\frac{4}{3}\right)$$

Hence, the correct answer is option (D).

10. Equation of a normal to the curve, $y = x^2 - 6x + 6$, which is perpendicular to the straight line joining the origin to the vertex of the parabola is

(A) $4x - 4y - 11 = 0$ **(B)** $4x - 4y + 1 = 0$
(C) $4x - 4y - 21 = 0$ **(D)** $4x - 4y + 21 = 0$

Solution: We have

$$y = (x - 3)^2 - 3$$

$$\Rightarrow y + 3 = (x - 3)^2$$

Hence, the vertex is $(3, -3)$. The slope of the line joining the origin $(0, 0)$ and the vertex $(3, -3)$ is -1. Hence, the slope of the normal is 1.

$$\frac{dy}{dx} = 2x - 6 \text{ and } -\frac{dx}{dy} = 1$$

$$\Rightarrow -1 = \frac{1}{2x-6}$$

$$\Rightarrow 2x - 6 = -1$$

$$\Rightarrow x = \frac{5}{2} \text{ and } y = \left(\frac{5}{2}\right)^2 - 6\left(\frac{5}{2}\right) + 6 = -\frac{11}{4}$$

Hence, the normal is

$$y + \frac{11}{4} = 1\left(x - \frac{5}{2}\right)$$

$$\Rightarrow 4x - 4y - 21 = 0$$

Hence, the correct answer is option (C).

11. If two different tangents of $y^2 = 4x$ are the normals to $x^2 = 4by$, then

(A) $|b| > 1/2\sqrt{2}$ **(B)** $|b| < 1/2\sqrt{2}$
(C) $|b| > 1/\sqrt{2}$ **(D)** $|b| < 1/\sqrt{2}$

Solution: The tangent of $y^2 = 4x$ in terms of m is

$$y = mx + \frac{1}{m}$$

and the normal to $x^2 = 4by$ in terms of m is

$$y = mx + 2b + \frac{b}{m^2}$$

If these lines are the same lines, then

$$\frac{1}{m} = 2b + \frac{b}{m^2} \Rightarrow 2bm^2 - m + b = 0$$

For two different tangents, we get

$$1 - 8b^2 > 0 \Rightarrow |b| < \frac{1}{2\sqrt{2}}$$

Hence, the correct answer is option (B).

12. The two parabolas $y^2 = 4x$ and $x^2 = 4y$ intersect at point P, whose abscissa is not zero, such that

(A) the tangents touch the curve at P and makes complementary angles with x-axis
(B) they both touch each other at P
(C) they cut at right angles at P
(D) None of these

Solution: We have

$$y^2 = 4x \quad (1)$$

$$x^2 = 4y \quad (2)$$

and $x \neq 0$. From Eqs. (1) and (2), we get

$$4x = y^2 = \left(\frac{x^2}{4}\right)^2$$

$$\Rightarrow 64x = x^4$$

$$\Rightarrow x^3 = 64$$

$$\Rightarrow x = 4 \quad (\because x \neq 0)$$

Substituting this value in Eq. (2), we get

$$(4)^2 = 4y \Rightarrow y = 4$$

Therefore, the point of intersection P is (4, 4). On differentiating Eq. (1), we get

$$2y\frac{dy}{dx} = 4 \Rightarrow m_1 = \frac{dy}{dx} = \frac{2}{y} = \frac{2}{4} = \frac{1}{2} = \tan\theta_1$$

On differentiating Eq. (2), we get

$$4\frac{dy}{dx} = 2x \Rightarrow m_2 = \frac{dy}{dx} = \frac{4}{2} = 2 = \tan\theta_2$$

Now, obviously $m_1 m_2 \neq -1$. Also,

$$\tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2}$$

$$= \frac{m_1 + m_2}{1 - m_1 m_2} = \frac{2 + (1/2)}{1 - 2 \times (1/2)} = \infty$$

Therefore,

$$\theta_1 + \theta_2 = \frac{\pi}{2}$$

where m_1 and m_2 are the slopes of the tangents to $y^2 = 4x$ and $x^2 = 4y$ at point P.

Hence, the correct answer is option (A).

13. If a focal chord with positive slope m of the parabola $y^2 = 16x$ touches the circle $x^2 + y^2 - 12x + 34 = 0$, then m is

(A) $\sqrt{3}$ **(B)** 1
(C) 2 **(D)** 1/2

Solution: Any tangent to circle is given by

$$y = m(x - 6) \pm \sqrt{2}\sqrt{1 + m^2}$$

If it is a focal chord to parabola, then

$$0 = m(4 - 6) \pm \sqrt{2}\sqrt{1 + m^2}$$
$$\Rightarrow m = \pm 1$$

and hence $m = 1$.

Hence, the correct answer is option (B).

14. If $2y = x + 24$ is a tangent to a parabola $y^2 = 24x$, then its distance from the parallel normal is

(A) $15\sqrt{5}/2$ **(B)** 1
(C) $15\sqrt{5}$ **(D)** $3\sqrt{5}$

Solution: Rewriting the equation of the tangent as

$$y = \frac{1}{2}x + \frac{6}{1/2}$$

Thus, it is a tangent with slope 1/2. Now, the normal with the same slope is

$$y - \left(\frac{1}{2}\right)x + \frac{27}{4} = 0$$

The distance between these two parallel lines is $15\sqrt{5}/2$.

Hence, the correct answer is option (A).

15. The coordinates of the point on the parabola $y^2 = 8x$, which is at minimum distance from the circle $x^2 + (y + 6)^2 = 1$ are

(A) (+2, – 4) **(B)** (18, –12)
(C) (2, 4) **(D)** None of these

Solution: The equation of normal to the parabola $y^2 = 8x$ at $Q(2m^2, -4m)$ is

$$y = mx - 4m - 2m^3 \quad (1)$$

Now, for the minimum distance, the equation of normal [Eq. (1)] is common for both. Thus, the normal [Eq. (1)] must pass through the centre C(0, –1).

$$m^3 + 2m - 3 = 0$$
$$\Rightarrow (m - 1)(m^2 + m + 3) = 0$$
$$\Rightarrow m = 1$$

Thus, the required point is (2, – 4).

Hence, the correct answer is option (A).

16. The angle between the tangents to the parabola $y^2 = 4ax$ at the points where it intersects with the line $x - y - a = 0$ is

(A) $\pi/3$ **(B)** $\pi/4$
(C) $\pi/6$ **(D)** $\pi/2$

Solution: The coordinates of the focus of the parabola $y^2 = 4ax$ are $(a, 0)$. The line $x - y - a = 0$ passes through this point. Therefore, it is a focal chord of the parabola. Hence, the tangents intersect at right angle.

Hence, the correct answer is option (D).

17. A water jet from a fountain reaches its maximum height of 4 m at a distance 0.5 m from the vertical passing through the point O of water outlet. The height of the jet above the horizontal OX at a distance of 0.75 m from the point O is

(A) 5 m **(B)** 6 m
(C) 3 m **(D)** 7 m

Solution: The path of the water jet is nothing but a parabola (Fig. 13.33). Let its equation be $y = ax^2 + bx + c$. It should pass through (0, 0), (0.5, 4) and (1, 0) which implies that $c = 0$, $a = -16$ and $b = 16$, Therefore,

$$y = -16x^2 + 16x$$

If $x = 0.75$, we get $y = 3$.

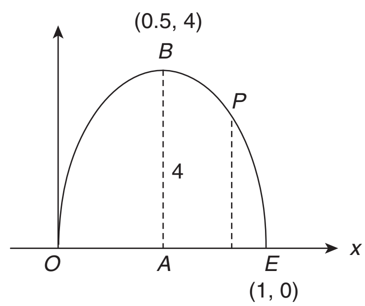


Figure 13.33

Hence, the correct answer is option (C).

18. Two tangents are drawn from the point (– 2, – 1) to the parabola $y^2 = 4x$. If α is the angle between them, then $\tan\alpha$ is equal to

(A) 3 **(B)** 1/3
(C) 2 **(D)** 1/2

Solution: We have

$$y^2 = 4x \quad (1)$$

The equation of tangent from an external point (– 2, – 1) to $y^2 = 4x$ is given by

$$T^2 = SS_1$$
$$\Rightarrow (2x + y - 4)^2 = (y^2 - 4x) = 9$$

Here,

$$T: -y - 2(x - 2) = -(2x + y - 4)$$
$$S_1: 1 + 8 = 9$$

On simplifying further, we get

$$4x^2 - 8y^2 + 20x - 8y + 4xy = 0$$

Now,

$$\tan\alpha = \left|\frac{2\sqrt{4 + 32}}{4 - 8}\right| = 3$$

$$\left[\tan\alpha = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|\right]$$

Hence, the correct answer is option (A).

19. If b and c are the lengths of the segments of any focal chord of a parabola $y^2 = 4ax$, then the length of the semi-latus rectum is

(A) $bc/b+c$ (B) $\sqrt{bc}$
(C) $b+c/a$ (D) $2bc/b+c$

Solution: A semi-latus rectum is harmonic mean of the segments of any focal chord. Therefore, the semi-latus rectum is $2bc/b+c$.

Hence, the correct answer is option (D).

20. The equation of the tangent to the parabola $y^2 = 9x$ which goes through the point (4, 10) is

(A) $x + 4y + 1 = 0$ (B) $9x + 4y + 4 = 0$
(C) $x - 4y - 36 = 0$ (D) $9x - 4y + 4 = 0$

Solution: We have

$$y^2 = 9x \quad (1)$$

Here, $a = 9/4$. Now, the tangent to Eq. (1) is given by

$$y = mx + \frac{9}{4m} \quad (2)$$

Since this tangent [Eq. (2)] passes through (4, 10), we get

$$10 = 4m + \frac{9}{4m}$$

$$\Rightarrow 16m^2 - 40m + 9 = 0$$

Therefore,

$$m = \frac{1}{4}, \frac{9}{4}$$

Substituting the values of m in Eq. (2), we get the following two cases:

Case 1. If $m = \frac{1}{4}, y = \frac{x}{4} + 9 \Rightarrow x - 4y + 36 = 0.$

Case 2. If $m = \frac{9}{4}, y = \frac{9x}{4} + 1 \Rightarrow 9x - 4y + 4 = 0.$

Hence, the correct answer is option (D).

21. Find the centre and the radius of the smaller of the two circles that touch the parabola $75y^2 = 64(5x - 3)$ at $\left(\frac{6}{5}, \frac{8}{5}\right)$ and the x-axis.

Solution: See Fig. 13.34. The equation of the tangent to the given parabola at $P\left(\frac{6}{5}, \frac{8}{5}\right)$ is

$$75y\left(\frac{8}{5}\right) = 160\left(x + \frac{6}{5}\right) - 192$$

$$\Rightarrow 4x - 3y = 0$$

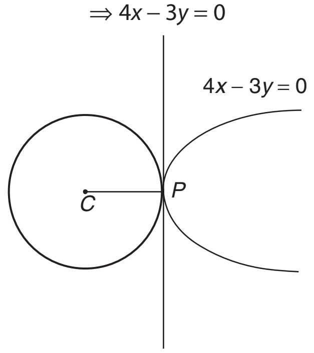


Figure 13.34

The equation of line through point $P\left(\frac{6}{5}, \frac{8}{5}\right)$, and which is perpendicular to the tangent is

$$3x + 4y - \left[3\left(\frac{6}{5}\right) + 4\left(\frac{8}{5}\right)\right] = 0$$

$$\Rightarrow 3x + 4y + 10 = 0$$

Therefore, the slope is

$$\frac{-3}{4} = \tan\theta$$

Thus,

$$\cos\theta = -\frac{4}{5}, \quad \sin\theta = \frac{3}{5}$$

II Quadrant: The coordinates of the centre of the required circle are

$$\left(\frac{6}{5} + a\cos\theta, \ \frac{8}{5} + a\sin\theta\right)$$

That is,

$$\left(\frac{6-4a}{5}, \frac{8+3a}{5}\right)$$

where the radius is $|a|$. Since the circle also touches x-axis, we have

$$|y\text{-coordinate of the centre}| = |a|$$

$$\Rightarrow \left|\frac{8+3a}{5}\right| = |a|$$

$$\Rightarrow \frac{8+3a}{5} = \pm a$$

$$\Rightarrow a = 4, -1$$

When $a = 4$, the centre is (–2, 4) and radius is 4 and when $a = -1$ the centre is (2, 1) and radius is 1.

Hence, the radius of the smaller circle is 1 and its centre is (2, 1).

22. A line AB makes intercepts of length a and b on the coordinate axes. Find the equation of the parabola passing through points A and B and the origin if AB is the shortest focal chord of the parabola.

Solution: See Fig. 13.35. AB is the shortest focal chord of the parabola, that is, its latus rectum. Therefore, the focus of the parabola is the mid-point of AB $\equiv \left(\frac{a}{2}, \frac{b}{2}\right)$.

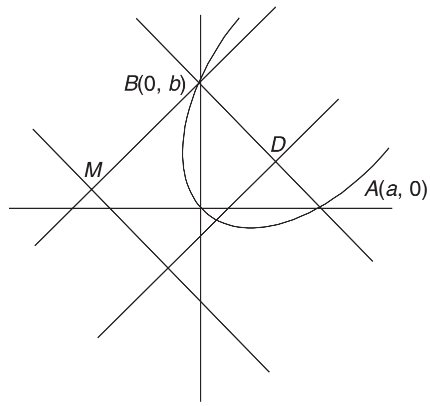


Figure 13.35

Now, the equation of AB is

$$\frac{x}{a}+\frac{y}{b}=1$$

The equation of the directrix is

$$\frac{x}{a}+\frac{y}{b}=\lambda$$

By the definition of parabola, BD = BM. So,

$$\frac{\sqrt{a^2+b^2}}{2}=\left|\frac{0\cdot a+b/b\ -\lambda}{\sqrt{(1/a^2)+(1/b^2)}}\right|$$

$$\Rightarrow 2(1-\lambda)ab=\pm(a^2+b^2)$$

$$\Rightarrow \lambda=\frac{-(a-b)^2}{2ab} \text{ and } \frac{(a+b)^2}{2ab}$$

Therefore, the directrices are

$$\frac{x}{a}+\frac{y}{b}=\frac{-(a-b)^2}{2ab} \text{ and } \frac{x}{a}+\frac{y}{b}=\frac{(a+b)^2}{2ab}$$

Thus, two parabolas are possible whose equations are given by

$$\left(x-\frac{a}{2}\right)^2+\left(y-\frac{b}{2}\right)^2=\frac{\{(x/a)+(y/b)+[(a-b)^2/2ab]\}^2}{(1/a^2)+(1/b^2)}$$

$$\left(x-\frac{a}{2}\right)^2+\left(y-\frac{b}{2}\right)^2=\frac{\{(x/a)+(y/b)-[(a+b)^2/2ab]\}^2}{(1/a^2)+(1/b^2)}$$

23. Prove that the equation of parabola whose focus is (0, 0) and the tangent at the vertex is $x-y+1=0$, is $x^2+y^2+2xy-4x+4y-4=0$.

Solution: Let the focus be S(0, 0) and A be the vertex of the parabola. Consider any point Z such that AS = AZ. The given tangent at vertex is $x-y+1=0$. Since the directrix is parallel to the tangent at the vertex, the equation of directrix is

$$x-y+\lambda=0$$

where λ is a constant. This implies that A is the mid-point of SZ.

$$SZ = 2SA$$

$$\Rightarrow \frac{|0-0+\lambda|}{\sqrt{(1)^2+(-1)^2}}=2\times\frac{|0-0+1|}{\sqrt{(1)^2+(-1)^2}}$$

$$\Rightarrow \lambda\pm 2$$

Therefore, $\lambda = 2$ since λ is positive due to the directrix lies in II quadrant in this case. Thus, the equation of the directrix is

$$x-y+2=0$$

Consider P(x, y) be any point on the parabola such as PM$\perp$ZM.

$$SP = PM$$

$$\Rightarrow SP^2 = PM^2$$

$$\Rightarrow (x-0)^2+(y-0)^2=\frac{|(x-y+2)|^2}{(\sqrt{2})^2}$$

On further simplification, we get

$$x^2+y^2+2xy-4x+4y-4=0$$

24. Prove that the normal chord to a parabola at the point whose ordinate is equal to the abscissa subtends a right angle at the focus.

Solution: Let the equation of the parabola be

$$y^2 = 4ax$$

Let PQ be a normal chord of the parabola at P(at^2, $2at$). Since, the ordinate and abscissa of P are equal, we have

$$at^2 = 2at \Rightarrow t = 2 \quad [\because t = 0 \text{ is not possible}]$$

Now, we have to prove that the $\angle$PSQ = 90° where S is the focus (a, 0). If the coordinates of Q are ($at_1{}^2$, $2at_1$), then

$$t_1=-t-\frac{2}{t}=-3$$

Since S(a, 0) is the focus, the slopes of SP and SQ are

$$\frac{2at-0}{at^2-a}=\frac{2t}{t^2-1} \text{ and } \frac{2at_1-0}{at_1^2-a}=\frac{2t_1}{t_1^2-1}$$

where $t = 2$ and $t_1 = -3$. Hence, their product is

$$\frac{2t}{t^2-1}\times\frac{2t_1}{t_1^2-1}=\frac{4}{2^2-1}\times\frac{-6}{(-3)^2-1}=-1$$

Therefore, SP and SQ are perpendicular and $\angle$PSQ = 90°.

25. Prove that the area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.

Solution: Let the three points on the parabola be ($at_1{}^2$, $2at_1$), ($at_2{}^2$, $2at_2$) and ($at_3{}^2$, $2at_3$). The area of the triangle formed by these points is given by

$$\left|\frac{1}{2}\left[at_1^2\,(2at_2-2at_3)+at_2^2(2at_3-2at_1)+at_3^2(2at_1-2at_2)\right]\right|$$

$$=\left|-a^2(t_2-t_3)(t_3-t_1)(t_1-t_2)\right|$$

$$=|a^2(t_2-t_3)(t_3-t_1)(t_1-t_2)|$$

Now, as discussed in this chapter, we know that the intersection of these tangents at these points is

$$\{at_2t_3,\ a(t_2+t_3)\},\ \{at_3t_1,\ a(t_3+t_1)\} \text{ and } \{at_1t_2,\ a(t_1+t_2)\}$$

The area of triangle formed by these three points

$$\frac{1}{2}\left|\left[at_1t_3(at_3-at_2)+at_1t_3(at_1-at_3)+at_1t_2(at_2-at_1)\right]\right|$$

$$=\frac{1}{2}|\,a^2(t_2-t_3)(t_3-t_1)(t_1-t_2)$$

Hence, the area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.

26. Find the shortest distance between the circle $x^2+y^2-24y+128=0$ and the parabola $y^2=4x$.

Solution: We know that the shortest distance between two curves occurs along common normal. Any normal to parabola is of the form $y+tx-2t-t^3=0$. If it is common normal of circle and parabola, it passes through the centre of the circle. So,

$$12 = 2t + t^3$$

$$\Rightarrow (t-2)(t^2+2t+6)=0$$

$$\Rightarrow t = 2$$

Thus, the point on the parabola is (4, 4) and the distance of this point to the centre of the circle is $\sqrt{80}$. Hence, the shortest distance is $\sqrt{80}-4=4(\sqrt{5}-1)$.

27. A variable chord PQ of the parabola $y = x^2$ subtends a right angle at the vertex. Find the locus of points of intersection of the normals at points P and Q.

Solution: See Fig. 13.36. The vertex V of the parabola is (0, 0) and any point on $y = x^2$ has coordinates (t, t^2). Let us consider $P \equiv (t_1, t_1^2)$, $Q \equiv (t_2, t_2^2)$ and PVQ = 90°.

We have m of VP as

$$\frac{t_1^2-0}{t_1-0}=t_1$$

and m of VQ as

$$\frac{t_2^2-0}{t_2-0}=t_2$$

Also, VP⊥VQ. Therefore,

$$t_1t_2=-1 \quad (1)$$

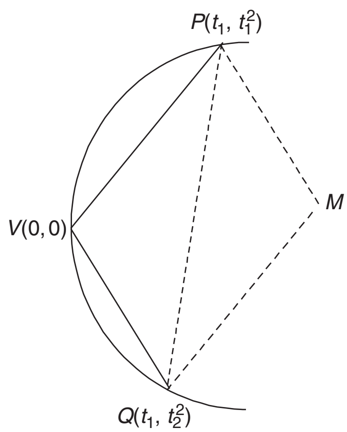


Figure 13.36

The equation of the normal to a curve at point $P(t_1, t_1^2)$ is

$$y-t_1^2=\frac{-1}{2t_1}(x-t_1)$$

This is due to the slope of normal at (t_1, t_1^2) being $-1/2t_1$.

$$2t_1y+x=2t_1^3+t_1 \quad (2)$$

Similarly, the equation of normal at point $Q(t_2, t_2^2)$ is

$$2t_2y+x=2t_2^3+t_2 \quad (3)$$

On eliminating t_1 and t_2 from Eqs. (1), (2) and (3), we get the locus of M. On subtracting Eq. (3) from Eq. (2), we get

$$2y(t_1-t_2)=2(t_1^3-t_2^3)+(t_1-t_2)$$

$$\Rightarrow 2y=2(t_1^2+t_1t_2+t_2^2)+1 \quad (4)$$

Now, Eq. (2) × t_2 – Eq. (3) × t_1, we get

$$(t_2-t_1)x=(2t_1^3+t_1)t_2-(2t_2^3+t_2)t_1$$

$$=2t_1t_2(t_1^2-t_2^2)$$

or

$$x=-2t_1t_2(t_1+t_2) \quad (5)$$

From Eqs. (1) and (5), we get

$$x=2(t_1+t_2)$$

or

$$t_1+t_2=\left(\frac{1}{2}\right)x \quad (6)$$

From Eq. (4), we get

$$2y=2[(t_1+t_2)^2-t_1t_2]+1$$

$$=2\left\{\left(\frac{1}{2}x\right)^2+1\right\}+1 \quad \text{[Using Eqs. (1) and (6)]}$$

Therefore, the equation of the locus is

$$2y=\frac{x^2}{2}+3$$

$$\Rightarrow x^2=2(2y-3)$$

which is a parabola.

28. Find the locus of the poles of the tangents to the parabola $y^2 = 4ax$ w.r.t. the parabola $y^2 = 4bx$.

Solution: Any tangent to the parabola $y^2 = 4ax$ is

$$ty=x+at^2 \quad (1)$$

Let (α, β) be the pole of Eq. (1) w.r.t. the parabola $y^2 = 4bx$. Then Eq. (1) is the polar of (α, β) w.r.t. $y^2 = 4bx$. Thus, $ty = x + at^2$ and $y\beta = 2b(x + \alpha)$ are identical. On comparing these, we get

$$\frac{t}{\beta}=\frac{1}{2b}=\frac{at^2}{2b\alpha}$$

Thus,

$$t=\frac{\beta}{2b},\ t^2=\frac{\alpha}{a}$$

Therefore,

$$\left(\frac{\beta}{2B}\right)^2=\frac{\alpha}{a}$$

or

$$a\beta^2=4b^2\alpha$$

Hence, the equation of the locus of the poles is

$$ay^2=4b^2x$$

29. Show that the tangent and the normal at a point P on the parabola $y^2 = 4ax$ are the bisectors of the angle between the focal radius SP and the perpendicular from P on the directrix.

Solution: See Fig. 13.37. Let $P \equiv (at^2, 2at)$ and $S \equiv (a, 0)$. The equation of SP is

$$y-0=\frac{2at-0}{at^2-a}(x-a)$$

$$\Rightarrow 2tx+(1-t^2)y+(-2at)=0 \quad (1)$$

The equation of PM is

$$y-2at=0 \quad (2)$$

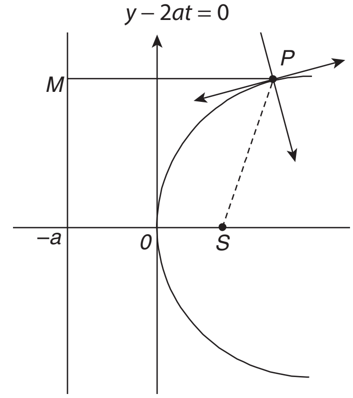


Figure 13.37

The angle bisectors of Eqs. (1) and (2) are

$$\frac{y-2at}{\sqrt{0+1}}=\pm\frac{\left[2tx+(1-t^2)y-2at\right]}{\sqrt{4t^2+(1-y^2)^2}}$$

$$\Rightarrow ty = x + at^2 \text{ and } tx + y = 2at + at^3$$

Hence, the tangent and normal at point P are the bisectors of SP and PM.

Previous Years' Solved JEE Main/AIEEE Questions

1. The equation of a tangent to the parabola $y^2 = 8x$ is $y = x + 2$. The point on this line from which the other tangent to the parabola is perpendicular to the given tangent is

(A) (–1, 1) **(B)** (0, 2)
(C) (2, 4) **(D)** (–2, 0)

[AIEEE 2007]

Solution: Now since locus of point of intersection of the perpendicular tangents is the directrix. Therefore,

$$x\text{-coordinate of point} = -2$$
$$\Rightarrow y = -2 + 2 = 0$$

Therefore, point is (–2, 0).

Hence, the correct answer is option (D).

2. Given: A circle, $2x^2 + 2y^2 = 5$ and a parabola, $y^2 = 4\sqrt{5}x$.

Statement-I: An equation of a common tangent to these curves is $y = x + \sqrt{5}$.

Statement-II: If the line, $y = mx + \frac{\sqrt{5}}{m}(m \neq 0)$, is their common tangent, then m satisfies $m^4 - 3m^2 + 2 = 0$.

(A) Statement – I is True; Statement – II is true; Statement – II is not a correct explanation for Statement – I.
(B) Statement – I is True; Statement – II is False.
(C) Statement – I is False; Statement– II is True.
(D) Statement – I is True; Statement – II is True; Statement – II is a correct explanation for Statement – I.

[JEE MAIN 2013]

Solution: Let us consider that the common tangent to the parabola be

$$y + mx + \frac{\sqrt{5}}{m} \qquad (m \neq 0)$$

Its distance from the centre of the circle, (0, 0) must be equal to the radius of the circle, $\sqrt{\frac{5}{2}}$. Therefore,

$$\left|\frac{\sqrt{5}}{m}\right| = \frac{\sqrt{5}}{\sqrt{2}}\sqrt{1+m^2} \Rightarrow (1+m^2)m^2 = 2 \Rightarrow m^4 + m^2 - 2 = 0$$

Hence,

$$(m^2-1)(m^2+2) = 0 \Rightarrow m = \pm 1$$

Therefore, the common tangents are obtained as $y = x + \sqrt{5}$ and $y = -x - \sqrt{5}$. Both statements are correct as the condition $m = \pm 1$ satisfies Statement-II.

Hence, the correct answer is option (A).

3. The slope of the line touching both the parabolas $y^2 = 4x$ and $x^2 = -32y$ is

(A) $\frac{1}{8}$ **(B)** $\frac{2}{3}$
(C) $\frac{1}{2}$ **(D)** $\frac{3}{2}$

[JEE MAIN 2014 (OFFLINE)]

Solution: See Fig. 13.38. Let a parametric point on $y^2 = 4x$ be $(t^2, 2t)$.

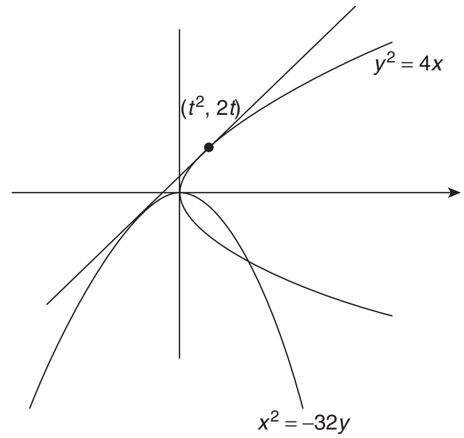


Figure 13.38

Therefore, equation of tangent is

$$yt = x + t^2 \qquad (1)$$

or $$x = yt - t^2$$

Now solving, $x = yt - t^2$ and $x^2 = -32y$ for points of intersection, we have

$$(yt - t^2)^2 = -32y$$

or $$y^2t^2 + t^4 - 2yt^3 = -32y$$

or $$y^2t^2 + y(32 - 2t^3) + t^4 = 0$$

For line to be tangent, discriminant = 0. Therefore,

$$(32 - 3t^3)^2 - 4t^6 = 0 \text{ or } 4(16 - t^3)^2 - 4(t^3)^2 = 0$$

$$\Rightarrow (16 - \cancel{t^3} + \cancel{t^3})(16 - t^3 - t^3) = 0 \Rightarrow d^3 = 8 \Rightarrow t = 2$$

Now slope $\frac{1}{t} = \frac{1}{2}$ (from 1)

Hence, the correct answer is option (C).

4. Let L_1 be the length of the common chord of the curves $x^2 + y^2 = 9$ and $y^2 = 8x$, and L_2 be the length of the latus rectum of $y^2 = 8x$. Then

(A) $L_1 > L_2$ **(B)** $L_1 = L_2$
(C) $L_1 < L_2$ **(D)** $\frac{L_1}{L_2} = \sqrt{2}$

[JEE MAIN 2014 (ONLINE SET-2)]

Solution: See Fig. 13.39.

$$x^2 + y^2 = 9; y^2 = 8x$$

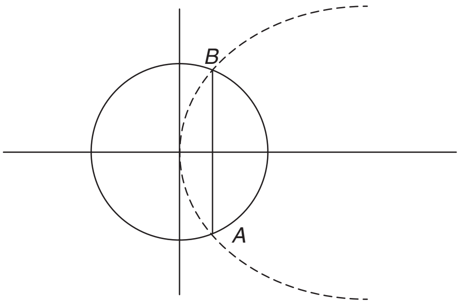


Figure 13.39

For L_1, solving $x^2 + 8x - 9 = 0$. Therefore,

$$x = \frac{-8 \pm \sqrt{64+36}}{2} = \frac{8 \pm 10}{2} = 1, -9$$

Thus, $y^2 = 8$.

Since, – 9 does not satisfy parabola.

Therefore, $y = \pm 2\sqrt{2}$ and $A(1, -2\sqrt{2})$ and $B(1, 2\sqrt{2})$.

So,

$$L_1 = 4\sqrt{2}, L_2 = 4a = 8; \frac{L_1}{L_2} = \frac{4\sqrt{2}}{8} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow L_2 = \sqrt{2}L_1 \Rightarrow L_2 > L_1$$

Hence, the correct answer is option (C).

5. Two tangents are drawn from a point (–2, –1) to the curve, $y^2 = 4x$. If α is the angle between them, then $|\tan \alpha|$ is equal to

(A) $\frac{1}{3}$ **(B)** $\frac{1}{\sqrt{3}}$

(C) $\sqrt{3}$ **(D)** 3

[JEE MAIN 2014 (ONLINE SET-3)]

Solution: See Fig. 13.40.

Joint equation of pair of lines is $SS_1 = T^2$, that is,

$$(y^2 - 4x)(y_1^2 - 4x_1) = (yy_1 - 2(x + x_1))^2$$

Therefore,

$$(y^2 - 4x)(1 - 4(-2)) = \{y(-1) - 2(x - 2)\}^2$$
$$\Rightarrow 9(y^2 - 4x) = \{-y - 2x + 4\}^2$$
$$\Rightarrow x^2 - 2y^2 + xy + 5x - 2y + 4 = 0$$

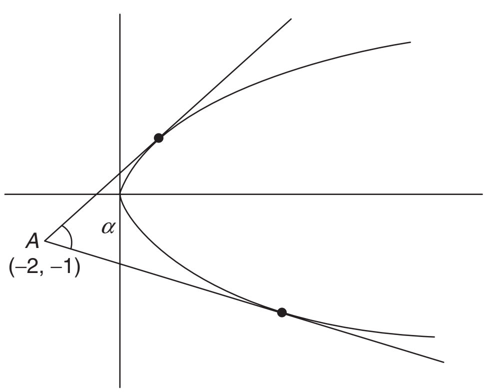


Figure 13.40

Now, angle between this pair is given by $\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$.

Therefore,

$$\tan\alpha = \left|\frac{2\sqrt{\frac{3}{4}+2}}{1-2}\right| = \left|\frac{3}{-1}\right| = 3 \Rightarrow |\tan\alpha| = 3$$

Hence, the correct answer is option (D).

6. A chord is drawn through the focus of the parabola $y^2 = 6x$ such that its distance from the vertex of this parabola is $\frac{\sqrt{5}}{2}$, then its slope can be

(A) $\frac{\sqrt{5}}{2}$ **(B)** $\frac{\sqrt{3}}{2}$

(C) $\frac{2}{\sqrt{5}}$ **(D)** $\frac{2}{\sqrt{3}}$

[JEE MAIN 2014 (ONLINE SET-4)]

Solution: See Fig. 13.41.

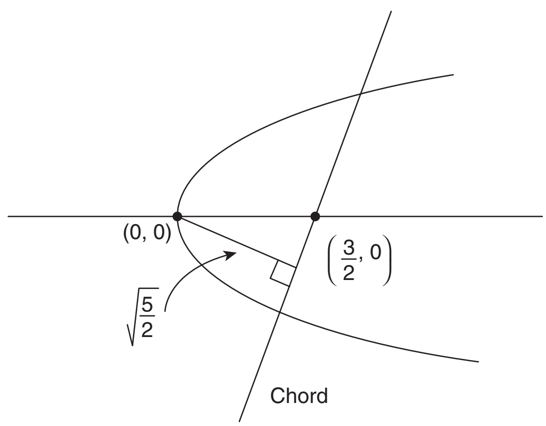


Figure 13.41

Let chord be $y - 0 = m\left(x - \frac{3}{2}\right)$ or $mx - y - \frac{3m}{2} = 0$. Then

$$\frac{\sqrt{5}}{2} = \frac{\left|m(0) - 0 - \frac{3m}{2}\right|}{\sqrt{1+m^2}} \Rightarrow 5(1+m) = \frac{9m^2}{4} \times 4$$
$$\Rightarrow 5 + 5m^2 = 9m^2 \Rightarrow 4m^2 = 5$$

Therefore,

$$m^2 = \frac{5}{4} \Rightarrow m = \pm\frac{\sqrt{5}}{2}$$

Hence, the correct answer is option (A).

7. Let O be the vertex and Q be any point on the parabola, $x^2 = 8y$. If the point P divides the line segment OQ internally in the ratio 1:3, then the locus of P is

(A) $y^2 = x$ **(B)** $y^2 = 2x$

(C) $x^2 = 2y$ **(D)** $x^2 = y$

[JEE MAIN 2015 (OFFLINE)]

Solution: See Fig. 13.42.

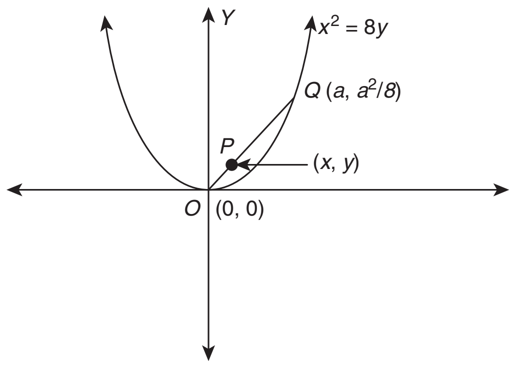


Figure 13.42

Since P divides OQ internally in the ratio 1:3. Therefore

$$x=\frac{a}{4}, y=\frac{\frac{a^2}{8}}{4}=\frac{a^2}{32}$$

$$\Rightarrow y=\frac{1}{2}x^2$$

Hence, the correct answer is option (C).

8. Let PQ be a double ordinate of the parabola, $y^2 = -4x$, where P lies in the second quadrant. If R divides PQ in the ratio 2 : 1, then the locus of R is

(A) $9y^2 = 4x$ **(B)** $9y^2 = -4x$
(C) $3y^2 = 2x$ **(D)** $3y^2 = -2x$

[JEE MAIN 2015 (ONLINE SET-2)]

Solution: See Fig. 13.43. Now

$$PR:RQ = 2:1$$

$$\Rightarrow x=\frac{-2t^2-t^2}{3}, y=\frac{-4t+2t}{3}$$

$$\Rightarrow x=-t^2, y=\frac{-2t}{3}\Rightarrow x=-\left(\frac{-3y}{2}\right)^2=\frac{-9y^2}{4}$$

$$\Rightarrow 9y^2=-4x$$

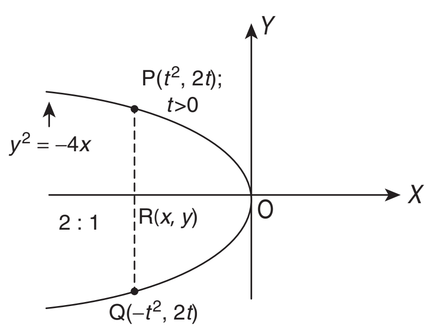


Figure 13.43

Hence, the correct answer is option (B).

9. Let P be the point on the parabola, $y^2 = 8x$ which is at a minimum distance from the centre C of the circle, $x^2 + (y + 6)^2 = 1$. Then, the equation of the circle, passing through C and having its centre at P is

(A) $x^2 + y^2 - 4x + 9y + 18 = 0$
(B) $x^2 + y^2 - 4x + 8y + 12 = 0$
(C) $x^2 + y^2 - x + 4y - 12 = 0$
(D) $x^2+y^2-\frac{x}{4}+2y-24=0$

[JEE MAIN 2016 (OFFLINE)]

Solution: See Fig. 13.44. The parabola is

$$y^2 = 8x$$

Therefore,

$$4a = 8 \Rightarrow a = 2$$

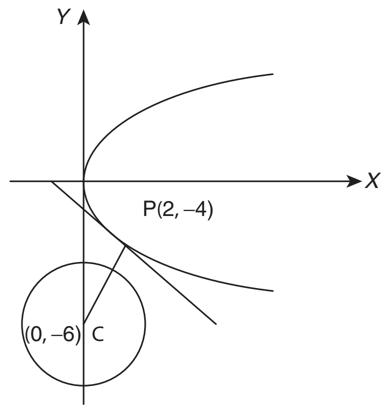


Figure 13.44

The normal of the parabola that meets at points $P(am^2, -2am)$ i.e. $P(2m^2, -4m)$ given by

$$y = mx - 4m - 2m^3$$

passes through the centre of the circle (0, –6).

$$-6 = -4m - 2m^3$$

$$2m^3 + 4m - 6 = 0$$

$$m^3 + 2m - 3 = 0$$

Therefore,

$$m^2(m - 1) + m(m - 1) + 3(m - 1) = 0$$

$$\Rightarrow m = 1 \text{ and } m^2 + m + 3 \neq 0$$

Hence, the point on the parabola is P(2, –4). Therefore,

$$CP = \text{Radius} = \sqrt{(0-2)^2+(-6+4)^2} = \sqrt{4+4}=2\sqrt{2}$$

The equation of circle is

$$(x - 2)^2 + (y + 4)^2 = 8$$

$$\Rightarrow x^2 + y^2 - 4x + 8y + 12 = 0$$

Hence, the correct answer is option (B).

10. P and Q are two distinct points on the parabola, $y^2 = 4x$, with parameters t and t_1, respectively. If the normal at P passes through Q, then the minimum value of t_1^2 is

(A) 8 **(B)** 4
(C) 6 **(D)** 2

[JEE MAIN 2016 (ONLINE SET-2)]

Solution: Let P(t) and Q(t_1) be the points, on the given parabola $y^2 = 4x$, which are given by

$$P(t) \equiv (t^2, 2t), Q(t_1) \equiv (t_1^2, 2t_1)$$

The normal at P(t) $\Rightarrow y + tx = 2t + t^3$ passes through Q(t_1).

$$2t_1+tt_1^2=2t+t^3$$

$$2(t_1-t)=-t\left(t_1^2-t^2\right)$$

$$2=-t(t_1+t)$$

$$\Rightarrow\frac{-2}{t}-t=t_1$$

$$t_1=-t-\frac{2}{t}$$

Therefore,

$$t_1^2=\left(t^2+\frac{4}{t^2}\right)+4$$

$$\frac{t^2+\left(4/t^2\right)}{2}\geq\sqrt{t^2\times\frac{4}{t^2}}$$

$$t^2+\frac{4}{t^2}\geq4$$

$$t_1^2\geq8$$

Hence, the correct answer is option (A).

11. Let C be a curve given by $y(x)=1+\sqrt{4x-3}, x>\frac{3}{4}$. If P is a point on C, such that the tangent at P has slope $\frac{2}{3}$ then a point through which the normal at P passes, is

(A) (1, 7) **(B)** (3, –4)
(C) (4, –3) **(D)** (2, 3)

[JEE MAIN 2016 (ONLINE SET-2)]

Solution: We have

$$y=1+\sqrt{4x-3} \qquad \left(x>\frac{3}{4}\right)$$

Let us consider P as a point (x_1, y_1). Now,

$$\frac{dy}{dx}=\frac{4}{2\sqrt{4x-3}}=\frac{2}{\sqrt{4x-3}}$$

$$\left.\frac{dy}{dx}\right|_{\text{point}P}=\frac{2}{\sqrt{4x_1-3}}=\frac{2}{3}$$

Now,

$$\sqrt{4x_1-3}=3$$

$$\Rightarrow 4x_1-3=9$$

$$\Rightarrow x_1=3$$

and

$$y_1=1+\sqrt{12-3}=4$$

Therefore, the point is obtained as P(3, 4).
Hence, the equation of the normal at point P is

$$y-4=-\frac{3}{2}(x-3)$$

$$\Rightarrow 2y-8=-3x+9$$

Therefore, the point (1, 7) lies on the normal $3x + 2y = 17$.
Hence, the correct answer is option (A).

Previous Years' Solved JEE Advanced/ IIT-JEE Questions

Paragraph for Questions 1 – 3: Consider the circle $x^2 + y^2 = 9$ and the parabola $y^2 = 8x$. They intersect at P and Q in the first and the fourth quadrants, respectively. Tangents to the circle at P and Q intersect the x-axis at R and tangents to the parabola at P and Q intersect the x-axis at S.

[IIT-JEE 2007]

1. The ratio of the areas of the triangles PQS and PQR is

(A) $1:\sqrt{2}$ **(B)** 1:2
(C) 1:4 **(D)** 1:8

Solution: See Fig. 13.45.

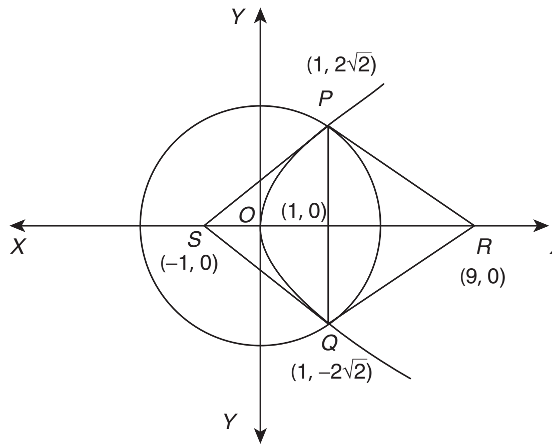


Figure 13.45

We have the circle

$$x^2+y^2=9$$

and parabola

$$y^2=8x$$

Points P and Q are the point of intersection of the circle and the parabola. Therefore, we get

$$P(1, 2\sqrt{2})$$

$$Q(1, -2\sqrt{2})$$

The equation of PQ is $x=1$. Now, let $R = (\alpha, 0)$ and PQ is the chord of contact of point R about the circle. Therefore, the line

$$\alpha(x)+0(\beta)=9$$

is similar to PQ. This implies that

$$\alpha=9$$

$$R=(9,0)$$

Now, let $S(\beta, 0)$ and PQ is chord of contact of S about the parabola. Therefore, the line

$$y(0)=4(x+\beta)$$

is similar to PQ. This implies that

$$\beta=-1$$

$$S(-1, 0)$$

Now,

$$\text{Area of }\Delta PQR=\frac{1}{2}\times4\sqrt{2}\times8=16\sqrt{2}$$

$$\text{Area of }\Delta PQS=\frac{1}{2}\times4\sqrt{2}\times2=4\sqrt{2}$$

The ratio of the area of ΔPQS and ΔPQR is 1:4.
Hence, the correct answer is option (C).

2. The radius of the circumcircle of the triangle PRS is

(A) 5 **(B)** $3\sqrt{3}$
(C) $3\sqrt{2}$ **(D)** $2\sqrt{3}$

Solution: The circumcircle of ΔPRS is

$$(x+1)(x-9)+y^2+\lambda y=0$$

It will pass through the point $(1, 2\sqrt{2})$. Then

$$\lambda=2\sqrt{2}$$

The equation of circumcircle is

$$x^2+y^2-8x+2\sqrt{2y}-9=0$$

Hence, its radius is $3\sqrt{3}$.

Hence, the correct answer is option (B).

3. The radius of the incircle of the triangle PQR is
   (A) 4 **(B)** 3
   (C) $\frac{8}{3}$ **(D)** 2

Solution: The radius of in-circle is

$$r=\frac{\Delta}{s}$$

Since $\Delta=16\sqrt{2}$, we get

$$s=\frac{6\sqrt{2}+6\sqrt{2}+4\sqrt{2}}{2}=8\sqrt{2}$$

$$r=\frac{16\sqrt{2}}{8\sqrt{2}}=2$$

Hence, the correct answer is option (D).

4. Consider the two curves $C_1: y^2 = 4x$, $C_2 : x^2 + y^2 - 6x + 1 = 0$. Then,
   (A) C_1 and C_2 touch each other only at one point
   (B) C_1 and C_2 touch each other exactly at two points
   (C) C_1 and C_2 intersect (but do not touch) at exactly two points
   (D) C_1 and C_2 neither intersect nor touch each other.

[IIT-JEE 2008]

Solution: See Fig. 13.46.

$$c_1: y^2=4x \qquad (1)$$

$$c_2: x^2+y^2-6x+1=0 \qquad (2)$$

Put $y^2 = 4x$ in Eq. (2)

$$x^2+4x-6x+1=0$$

$$x^2-2x+1=0$$

$$x=1$$

That is, points of intersection of circle and parabola (1, 2) and (1, –2).

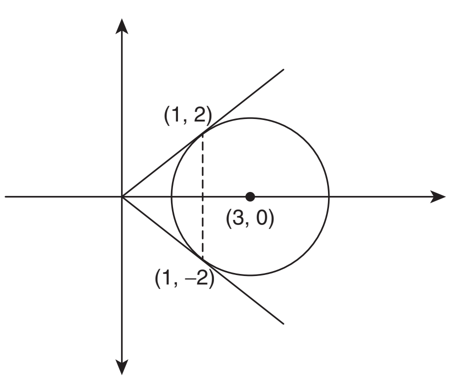


Figure 13.46

Clearly, c_1 and c_2 touch each other exactly at two points.

Hence, the correct answer is option (B).

5. The tangent PT and the normal PN to the parabola $y^2 = 4ax$ at a point P on it meet its axis at points T and N, respectively. The locus of the centroid of the triangle PTN is a parabola whose
   (A) vertex is $\left(\frac{2a}{3},0\right)$ **(B)** directrix is $x = 0$
   (C) latus rectum is $\frac{2a}{3}$ **(D)** focus is $(a, 0)$

[IIT-JEE 2009]

Solution:

$$G = (h, k)$$

$$\Rightarrow h=\frac{2a+at^2}{3}, k=\frac{2at}{3}$$

$$\Rightarrow \left(\frac{3h-2a}{a}\right)=\frac{9k^2}{4a^2}$$

Therefore, required parabola is

$$\frac{9y^2}{4a^2}=\frac{(3x-2a)}{a}=\frac{3}{a}\left(x-\frac{2a}{3}\right)$$

$$\Rightarrow y^2=\frac{4a}{3}\left(x-\frac{2a}{3}\right)$$

Vertex $\equiv\left(\frac{2a}{3}, 0\right)$, Focus $\equiv(a,0)$

Hence, the correct answers are options (A) and (D).

6. Let A and B be two distinct points on the parabola $y^2 = 4x$. If the axis of the parabola touches a circle of radius r having AB as its diameter, then the slope of the line joining A and B can be
   (A) $-\frac{1}{r}$ **(B)** $\frac{1}{r}$
   (C) $\frac{2}{r}$ **(D)** $-\frac{2}{r}$

[IIT-JEE 2010]

Solution: Let $A(t_1^2, 2t_1)$ and $B(t_2^2, 2t_2)$. Then centre of circle drawn with AB as diameter is $\left(\frac{t_1^2+t_2^2}{2}, t_1+t_2\right)$.

As circle touches x-axis, therefore

$$r=|t_1+t_2| \Rightarrow t_1+t_2=\pm r$$

Also

$$\text{Slope of } AB=\frac{2(t_2-t_1)}{t_2^2-t_1^2}=\frac{2}{t_2+t_1}=\frac{\pm 2}{r}$$

Hence, the correct answers are options (C) and (D).

7. Consider the parabola $y^2 = 8x$. Let Δ_1 be the area of the triangle formed by the end points of its latus rectum and the point $P\left(\frac{1}{2},2\right)$ on the parabola, and Δ_2 be the area of the triangle formed by drawing tangents at P and at the end points of the latus rectum. Then $\frac{\Delta_1}{\Delta_2}$ is _____.

[IIT-JEE 2011]

Solution: See Fig. 13.47.

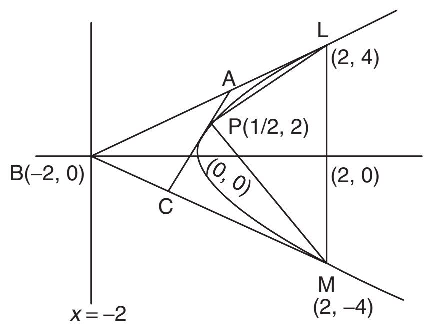


Figure 13.47

Δ_1= Area of $\Delta PLM = \frac{1}{2}\times 8\times\frac{3}{2} = 6$

Equation of $AC: y = 2x + 1$; Equation of $AB: y = x + 2$;

Equation of $BC: -y = x + 2$.

Solving above equations, we get A(1, 3), B(–2, 0) and C(–1, –1).

Therefore

$$\Delta_2 = \frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ -2 & 0 & 1 \\ -1 & -1 & 1 \end{vmatrix} = 3$$

$$\Rightarrow \frac{\Delta_1}{\Delta_2} = 2$$

Hence, the correct answer is (2).

8. Let (*x*, *y*) be any point on the parabola $y^2 = 4x$. Let *P* be the point that divides the line segment from (0, 0) to (*x*, *y*) in the ratio 1 : 3. Then the locus of *P* is

(A) $x^2 = y$ **(B)** $y^2 = 2x$

(C) $y^2 = x$ **(D)** $x^2 = 2y$

[IIT-JEE 2011]

Solution: See Fig. 13.48.

Let *P*(*h*, *k*) divides *OQ* in the ratio 1 : 3. Then *Q* (4*h*, 4*k*).

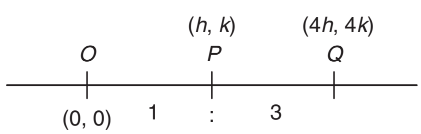


Figure 13.48

$$y^2 = 4x$$

and *Q* will lie on it

$$(4k)^2 = 4\times 4h$$

$$\Rightarrow k^2 = h$$

$$\Rightarrow y^2 = x \text{(replacing } h \text{ by } x \text{ and } k \text{ by } y\text{)}$$

Hence, the correct answer is option (C).

9. Let *L* be a normal to the parabola $y^2 = 4x$. If *L* passes through the point (9, 6), then *L* is given by

(A) $y - x + 3 = 0$ **(B)** $y + 3x - 33 = 0$

(C) $y + x - 15 = 0$ **(D)** $y - 2x + 12 = 0$

[IIT-JEE 2011]

Solution:

$$y^2 = 4x$$

Equation of normal is $y = mx - 2m - m^3$.

It passes through (9, 6), so

$$m^3 - 7m + 6 = 0$$

$$\Rightarrow m = 1, 2, -3$$

$$\Rightarrow y - x + 3 = 0, y + 3x - 33 = 0, y - 2x + 12 = 0$$

Hence, the correct answers are options (A), (B) and (D).

10. Let *S* be the focus of the parabola $y^2 = 8x$ and let *PQ* be the common chord of the circle $x^2 + y^2 - 2x - 4y = 0$ and the given parabola. The area of the triangle *PQS* is ______.

[IIT-JEE 2012]

Solution: The parabola is $x = 2t^2$, $y = 4t$.

Solving it with the circle we get

$$4t^2 + 16t^2 - 4t^2 - 16t = 0$$

$$\Rightarrow t^4 + 3t^2 - 4t = 0 \Rightarrow t = 0, 1$$

So, the points *P* and *Q* are (0, 0) and (2, 4) which are also diametrically opposite points on the circle. The focus is $S = (2, 0)$.

The area of $\Delta PQS = \frac{1}{2}\times 2\times 4 = 4$.

Hence, the correct answer is (4).

Paragraph for Questions 11 and 12: Let *PQ* be a focal chord of the parabola $y^2 = 4ax$. The tangents to the parabola at *P* and *Q* meet at a point lying on the line $y = 2x + a$, $a > 0$.

11. Length of chord *PQ* is

(A) 7*a* **(B)** 5*a*

(C) 2*a* **(D)** 3*a*

[JEE ADVANCED 2013]

Solution: See Fig. 13.49. We have $a(t_1 + t_2) = a$. Therefore, $t_1 + t_2 = 1$.

Also $t_1 t_2 = -1$

Therefore,

$$PQ = a(t_1 - t_2)^2 \quad (1)$$

$$= a\left[(t_1 + t_2)^2 - 4t_1 t_2\right]$$

From Eq. (1), we get

$$PQ = a[1 + 4] = 5a$$

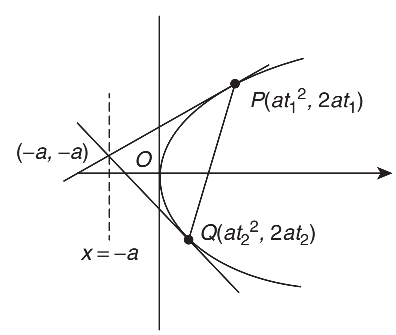


Figure 13.49

Hence, the correct answer is option (B).

12. If chord PQ subtends an angle θ at the vertex of $y^2 = 4ax$, then $\tan\theta = ?$

(A) $\frac{2}{3}\sqrt{7}$ **(B)** $\frac{-2}{3}\sqrt{7}$

(C) $\frac{2}{3}\sqrt{5}$ **(D)** $\frac{-2}{3}\sqrt{5}$

[JEE ADVANCED 2013]

Solution: See Fig. 13.50. We have $t_1 + t_2 = 1$.

$$t_1t_2 = -1$$

$$t_1 = \frac{1 \pm \sqrt{5}}{2},\ t > 0$$

Therefore,

$$t_1 = \frac{1 \pm \sqrt{5}}{2}$$

and

$$\tan\theta = \frac{\frac{2}{t_1} - \frac{2}{t_2}}{1 + \frac{4}{t_1t_2}} = \frac{2(t_2 - t_1)}{t_1t_2 + 4} = \frac{2(1-(1+\sqrt{5}))}{3} = -\frac{2\sqrt{5}}{3}$$

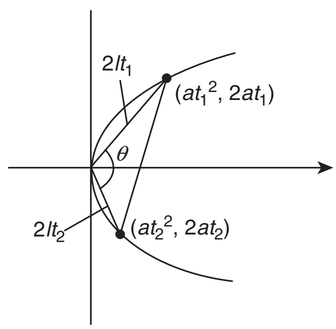


Figure 13.50

Hence, the correct answer is option (D).

13. A line L: $y = mx + 3$ meets y-axis at $E(0, 3)$ and the arc of the parabola $y^2 = 16x$, $0 \le y \le 6$ at the point $F(x_0, y_0)$. The tangent to the parabola at $F(x_0, y_0)$ intersects the y-axis at $G(0, y_1)$. The slope m of the line L is chosen such that the area of the triangle EFG has a local maximum. Match List I with List II and select the correct answer using the code given below the list:

List I	List II
P. $m =$	**1.** $\frac{1}{2}$
Q. Maximum area of Δ EFG is	**2.** 4
R. $y_0 =$	**3.** 2
S. $y_1 =$	**4.** 1

Codes:

	P	Q	R	S
(A)	4	1	2	3
(B)	3	4	1	2
(C)	1	3	2	4
(D)	1	3	4	2

[JEE ADVANCED 2013]

Solution: See Fig. 13.51.

The parabola is $y^2 = 16x$ and line is $y = mx + 3$. Solving, we get

$$(mx+3)^2 = 16x$$

$$\Rightarrow m^2x^2 + (6m-16)x + 9 = 0 \quad (1)$$

Also, the tangent at F is

$$yy_0 = 8(x + x_0)$$

Thus,

$$y_1 = \frac{8x_0}{y_0} \quad (2)$$

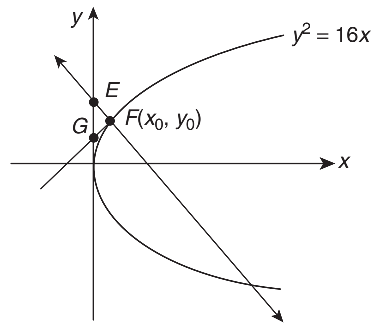


Figure 13.51

Now, area of ΔEFG

$$\Delta = \frac{1}{2}(3 - y_1)x_0$$

$$= \frac{1}{2}\left(3 - \frac{8x_0}{y_0}\right)x_0$$

$$= \frac{1}{2}\left(3x_0 - \frac{8x_0^2}{y_0}\right)$$

$$= \frac{1}{2}\left(3x_0 - \frac{8x_0^2}{4\sqrt{x_0}}\right)$$

For Δ to be maximum, we have

$$\frac{d\Delta}{dx_0} = 0$$

$$\Rightarrow 3 - 2 \times \frac{3}{2}\sqrt{x_0} = 0$$

$$\Rightarrow x_0 = 1$$

Thus, $y_0 = 4$. Therefore,

$$y_1 = \frac{8x_0}{y_0} = 2$$

From $y_0 = mx_0 + 3 \Rightarrow m = 1$.

Hence, the correct answer is option (A).

14. The common tangents to the circle $x^2 + y^2 = 2$ and the parabola $y^2 = 8x$ touch the circle at the points P, Q and the parabola at the points R, S. Then the area of the quadrilateral $PQRS$ is

(A) 3 **(B)** 6
(C) 9 **(D)** 15

[JEE ADVANCED 2014]

Solution: See Fig. 13.52.

We know that equation of tangent to parabola with slope m is

$$y = mx + \frac{2}{m} \quad (1)$$

Since, Eq. (1) is also a tangent to the circle, so length of perpendicular from the centre is equal to the radius. Therefore,

$$\left| \frac{m(0) + \frac{2}{m} - 0}{\sqrt{m^2 + (-1)^2}} \right| = \sqrt{2}$$

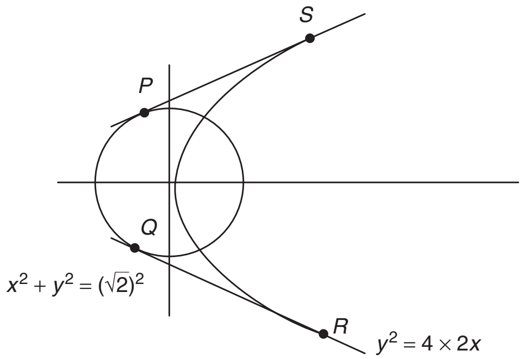


Figure 13.52

$$\Rightarrow \frac{4}{m^2} = 2(m^2 + 1) \text{ or}$$

$$\Rightarrow 4 = 2m^2 (m^2 + 1) \text{ or } 2 = m^4 + m^2$$

$$\Rightarrow (m^2)^2 + m^2 - 2 = 0 \Rightarrow m^2 = \frac{-1 \pm \sqrt{1+8}}{2}$$

$$\Rightarrow m^2 = \frac{-1 \pm 3}{2} = 1, -2 \qquad (\because m^2 \neq -2 \therefore m^2 = 1)$$

$$\Rightarrow m = \pm 1$$

Therefore, tangents are $y = x + \frac{2}{1}$ and $y = -x - \frac{2}{1}$.

Points of contact $\left(\frac{2}{1^2}, \frac{4}{1}\right)$ and $\left(\frac{2}{(-1^2)}, \frac{4}{-1}\right)$. (Parabola)

That is, (2, 4) and (2, 4) (S and R)

$$\left(\frac{a}{m^2}, \frac{2a}{m}\right) \text{ for } y = mx + \frac{a}{m}$$

For point of contact of circle:

Solving $y = x + 2$ and $x^2 + y^2 = 2$

Therefore,

$$x^2 + (x + 2)^2 \Rightarrow x^2 + x^2 4 + 4x - 2 = 0$$

$$\Rightarrow 2x^2 + 4x + 2 = 0 \Rightarrow x^2 + 2x + 1 = 0$$

$$\Rightarrow (x + 1)^2 = 0 \Rightarrow x = -1 \therefore y = 1$$

Therefore, (–1, 1) and (–1, –1), by symmetry are P and Q.

Now, area of the trapezium $PQRS = \frac{1}{2}(PQ + RS) \times$ Distance

$$= \frac{1}{2}\{(2+8)(1+2)\} = \frac{1}{2} \times 10^5 \times 3 = 15$$

Hence, the correct answer is option (D).

Paragraph for Questions 15 and 16: Let a, r, s, t be nonzero real numbers, Let $P(at^2, 2at)$, Q, $R(ar^2, 2ar)$ and $S(as^2, 2as)$ be distinct points on the parabola $y^2 = 4ax$. Suppose that PQ is the focal chord and lines QR and PK are parallel, where K is the point $(2a, 0)$.

[JEE ADVANCED 2014]

15. The value r is

(A) $-\frac{1}{t}$ **(B)** $\frac{t^2+1}{t}$

(C) $\frac{1}{t}$ **(D)** $\frac{t^2-1}{t}$

Solution: See Fig. 13.53. Since $QR \parallel PK$. Therefore,

slope of QR = slope of PK

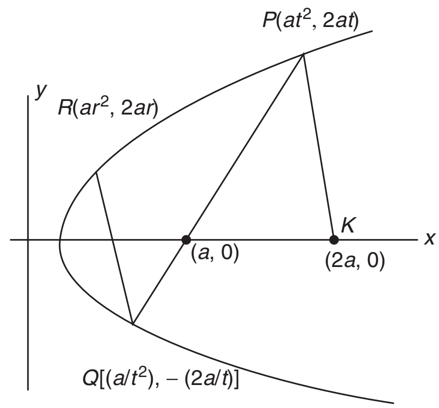


Figure 13.53

$$\Rightarrow \frac{2ar + \frac{2a}{t}}{2r^2 - \frac{a}{t^2}} = \frac{2at}{at^2 - 2a}$$

$$\Rightarrow \frac{r + \frac{1}{t}}{r^2 - \frac{1}{t^2}} = \frac{t}{t^2 - 2}$$

$$\Rightarrow \frac{\left(r + \frac{1}{t}\right)}{\left(r + \frac{1}{t}\right)\left(r - \frac{1}{t}\right)} = \frac{t}{t^2 - 2}$$

$$\Rightarrow tr - 1 = t^2 - 2$$

$$\Rightarrow r = \frac{t^2 - 1}{t}$$

Since PQ is a focal chord, therefore

$$t_1 t = -1$$

$$\Rightarrow t_1 = \frac{-1}{t}$$

Therefore, $(Q(at_1^2, 2at_1)) = Q\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$.

Hence, the correct answer is option (D).

16. If $st = 1$. then the tangent at P and the normal at S to the parabola meet at a point whose ordinate is

(A) $\frac{(t^2+1)^2}{2t^3}$ **(B)** $\frac{a(t^2+1)^2}{2t^3}$

(C) $\frac{a(t^2+1)^2}{t^3}$ **(D)** $\frac{a(t^2+2)^2}{t^3}$

Solution: As

$$st = 1 \Rightarrow s = \frac{1}{t}$$

Therefore,

$$S\left(a\left(\frac{1}{t}\right)^2, \frac{2a}{t}\right) = S\left(\frac{a}{t^2}, \frac{2a}{t}\right)$$

Now equation of tangent of P is

$$yt = x + at^2 \quad (1)$$

Equation of normal at S is

$$y = -sx + 2as + as^3$$
$$= -\frac{x}{t} + \frac{2a}{t} + \frac{a}{t^3} \quad (2)$$

From Eq. (1), $x = yt - at^2$

From Eq. (2), $x = \left(\frac{2a}{t} + \frac{a}{t^3} - y\right)t$

Therefore,

$$yt - at^2 = 2a + \frac{a}{t^2} - yt$$

$$\Rightarrow 2yt = at^2 + \frac{a}{t^2} + 2a = a\left(t^2 + \frac{1}{t^2} + 2\right)$$

$$= a\left(t + \frac{1}{t}\right)^2$$

$$\Rightarrow y = \frac{a\left(t - \frac{1}{t}\right)^2}{2t} = \frac{a(t^2+1)^2}{2t^3}$$

Hence, the correct answer is option (B).

17. Let the curve C be the mirror image of the parabola $y^2 = 4x$ with respect to the line $x + y + 4 = 0$. If A and B are the points of intersection of C with the line $y = -5$, then the distance between A and B is ______.

[JEE ADVANCED 2015]

Solution: Let $P(x_0, y_0)$ be any point on parabola $y^2 = 4x$ and $P'(x_0', y_0')$ be image of P on line $x + y + 4 = 0$. Then

$$\frac{x_0' - x_0}{1} - \frac{y_0' - y_0}{1} = \frac{-2(x_0 + y_0 + 4)}{1+1}$$

$$\Rightarrow x_0' = x_0 - x_0 - y_0 - 4$$

$$\Rightarrow x_0' = 4 - y_0$$

and $\quad y_0' = y_0 - x_0 - y_0 - 4 \Rightarrow y_0' = -4 - x_0$

But $x_0 = y_0^2/4$, so

$$y_0' = 4 - \frac{y_0^2}{4}$$

At the point of intersection of C and the line, $y = -5$, we will have

$$y_0' = -5$$

$$\Rightarrow y_0' = -5 = -4 - \frac{y_0^2}{4}$$

$$\Rightarrow \frac{y_0^2}{4} = 1$$

$$\Rightarrow y_0 = \pm 2$$

$$\Rightarrow x_0' = -4 - y_0 = -4 - (\pm 2)$$

$$A \equiv (-6, -6) \text{ and } B \equiv (-2, -5)$$

$$\Rightarrow |AB| = 4$$

Hence, the correct answer is (4).

18. If the normals of the parabola $y^2 = 4x$ drawn at the end points of its latus rectum are tangents to the circle $(x - 3)^2 + (y + 2)^2 = r^2$, then the value of r^2 is ______.

[JEE ADVANCED 2015]

Solution: Parabola: $y^2 = 4x$

Circle: $(x - 3)^2 + (y + 2)^2 = r^2$

End points of latus rectum are $A(1, 2)$; $B(1, -2)$

Equations of normal at A and B to parabola $y^2 = 4x$ are given by

$$(y-2) = \frac{-1}{1}(x-1) \text{ and } (y+2) = \frac{-1}{-1}(x-1)$$

That is, $\quad x + y - 3 = 0$ and $x - y - 3 = 0$

Above two lines are also tangents to the circle

Therefore, their perpendicular distances from centre $(3, -2) = 4$ (radius)

$$\frac{|3-2-3|}{\sqrt{2}} = r$$

$$\Rightarrow r = \sqrt{2}$$

$$\Rightarrow r^2 = 2$$

Hence, the correct answer is (2).

19. Let P and Q be distinct points on the parabola $y^2 = 2x$ such that a circle with PQ as diameter passes through the vertex O of the parabola. If P lies in the first quadrant and the area of the triangle ΔOPQ is $3\sqrt{2}$, then which of the following is (are) the coordinates of P?

(A) $(4, 2\sqrt{2})$ (B) $(9, 3\sqrt{2})$

(C) $\left(\frac{1}{4}, \frac{1}{\sqrt{2}}\right)$ (D) $(1, \sqrt{2})$

[JEE ADVANCED 2015]

Solution: See Fig. 13.54. Equation of parabola is

$$y^2 = 2x \quad (1)$$

Area of $\Delta OPQ = 3\sqrt{2}$, where $P \equiv \left(\frac{1}{2}t_1^2, t_1\right)$ and $Q \equiv \left(\frac{1}{2}t_2^2, t_2\right)$

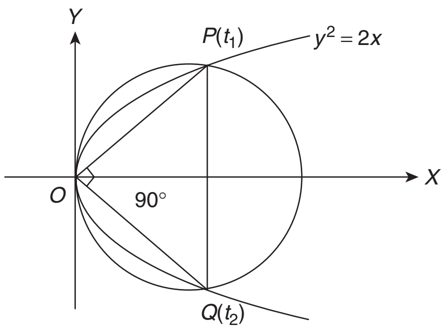


Figure 13.54

PQ is the diameter and circle passes through origin. Therefore,

$$\angle POQ = 90^\circ$$

$$\Rightarrow OP \perp OQ \Rightarrow \left(\frac{t_1}{\frac{1}{2}t_1^2}\right)\left(\frac{t_2}{\frac{1}{2}t_2^2}\right) = -1$$

$$\Rightarrow \frac{4}{t_1 t_2} = -1 \Rightarrow t_1 t_2 = -4$$

Also

$$\text{Area of } \Delta OPQ = 3\sqrt{2}$$

$$\Rightarrow \frac{1}{2}(OP)(OQ) = 3\sqrt{2}$$

$$\Rightarrow \frac{1}{2}\left|\left(\sqrt{t_1^2 + \frac{1}{4}t_1^4}\,\sqrt{t_2^2 + \frac{1}{4}t_2^4}\right)\right| = 3\sqrt{2}$$

$$\Rightarrow \frac{1}{2}\frac{\sqrt{(4+t_1^2)(4+t_2^2)}\,|t_1 t_2|}{4} = 3\sqrt{2}$$

$$\Rightarrow 4t_1^2 + 4t_2^2 + (t_1 t_2)^2 + 16 = 72$$

$$\Rightarrow 4t_1^2 + 4t_2^2 + 32 = 72$$

$$\Rightarrow t_1^2 + t_2^2 - 10 = 0$$

$$\Rightarrow t_1^2 + \left(\frac{16}{t_1^2}\right) - 10 = 0$$

$$\Rightarrow t_1^4 - 10t_1^2 + 16 = 0$$

$$\Rightarrow (t_1^2 - 2)(t_1^2 - 8) = 0$$

$$\Rightarrow t_1^2 = 2 \text{ or } t_1^2 = 8$$

$$\Rightarrow t_2 = \sqrt{2} \text{ or } t_2 = 2\sqrt{2} \text{ as } t_1 > 0$$

Therefore, coordinate of P are $(1, \sqrt{2})$ or $(4, 2\sqrt{2})$.

Hence, the correct answers are options (A) and (D).

20. The circle C_1: $x^2 + y^2 = 3$, with centre at O, intersects the parabola $x^2 = 2y$ at the point P in the first quadrant. Let the tangent to the circle C_1 at P touches other two circles C_2 and C_3 at R_2 and R_3, respectively. Suppose C_2 and C_3 have equal radii $2\sqrt{3}$ and centres Q_2 and Q_3, respectively. If Q_2 and Q_3 lie on the y-axis, then

(A) $Q_2Q_3 = 12$

(B) $R_2R_3 = 4\sqrt{6}$

(C) area of the triangle OR_2R_3 is $6\sqrt{2}$

(D) area of the triangle PQ_2Q_3 is $4\sqrt{2}$

[JEE ADVANCED 2016]

Solution: See Fig. 13.55. The equation of circle C_1 is $x^2 + y^2 = 3$. The parabola is $x^2 = 2y$.

Point P is obtained as follows:

$$y^2 + 2y - 3 = 0$$

$$y^2 + 3y - y - 3 = 0$$

$$y(y - 3) - (y + 3) = 0$$

$$(y + 3)(y - 1) = 0$$

Therefore, $y = -3$ and $y = 1$. Thus, point P is $(\sqrt{2}, 1)$.

Now, the tangent at point P is $\sqrt{2}\,x + y = 3$.

Point A is $\left(\frac{3}{\sqrt{2}}, 0\right)$.

Now, $\angle OBA = 90^\circ - \alpha; \sin\alpha = \sqrt{\frac{2}{3}}; \cos\alpha = \frac{1}{\sqrt{3}}$

and $Q_2R_2 = Q_3R_3 = 2\sqrt{3}$

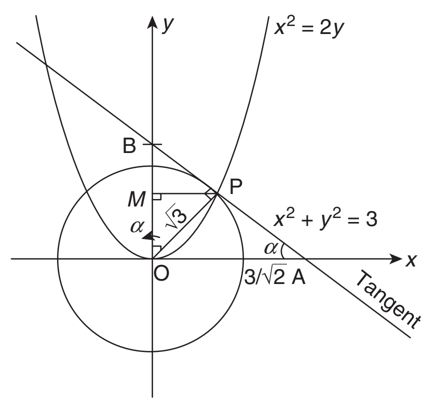


Figure 13.55

From ΔQ_2BR_2, we get

$$\sin(90^\circ - \alpha) = \frac{Q_2R_2}{Q_2B} = \frac{2\sqrt{3}}{Q_2B}$$

$$Q_2B = \frac{2\sqrt{3}}{\cos\alpha} = \frac{2\sqrt{3}}{1/\sqrt{3}} = 6$$

From ΔQ_2BR_2, we also get

$$\cos(90^\circ - \alpha) = \frac{BR_2}{Q_2B}$$

That is, $BR_2 = 6(\sin\alpha) = 6\sqrt{\frac{2}{3}}$

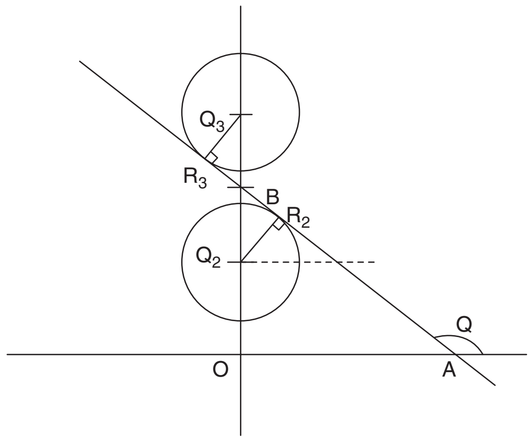


Figure 13.56

From Figure 13.56, we see that ΔQ_2BR_2 and ΔQ_3BR_3 are similar triangle. Therefore,

$$Q_2B = Q_3B = 6$$

and $$BR_3 = BR_2 = 6\sqrt{\frac{2}{3}}$$

Therefore,

$$Q_2Q_3 = 6 + 6 = 12$$

Hence, option (A) is correct.

Now, $$R_2R_3 = 6\sqrt{\frac{2}{3}} + 6\sqrt{\frac{2}{3}} = 12\sqrt{\frac{2}{3}} = \sqrt{\frac{2}{3}\times 144} = \sqrt{2\times 48} = 4\sqrt{6}$$

Hence, option (B) is correct.

Now, the area of OR_2R_3 is

$$\frac{1}{2}\times R_2R_3\times OP = \frac{1}{2}\times 4\sqrt{6}\times\sqrt{3} = 2\sqrt{18} = 6\sqrt{2}$$

Now, from ΔOPM, we have

$$\sin\alpha = \frac{PM}{OP}$$

$$\sqrt{\frac{2}{3}} = \frac{PM}{\sqrt{3}} \Rightarrow PM = \sqrt{2}$$

The area of ΔPQ_2Q_3 is

$$\frac{1}{2}Q_2Q_3(PM) = \frac{1}{2}\times 2\sqrt{2} = 6\sqrt{2}$$

Hence, option (C) is correct.

Hence, the correct answers are options (A), (B) and (C).

21. Let P be the point on the parabola $y^2 = 4x$, which is at the shortest distance from the centre S of the circle $x^2 + y^2 - 4x - 16y + 64 = 0$. Let Q be the point on the circle dividing the line segment SP internally. Then

(A) $SP = 2\sqrt{5}$

(B) $SQ:QP = (\sqrt{5}+1):2$

(C) the x-intercept of the normal to the parabola at P is 6

(D) the slope of the tangent to the circle at Q is $\frac{1}{2}$

[JEE ADVANCED 2016]

Solution: See Fig. 13.57. The given parabola is

$$y^2 = 4x$$

The point on the parabola is $p(am^2, -2am)$. Therefore,

$$4a = 4 \Rightarrow a = 1$$

Hence, the point P becomes $P(m_1^2 - 2m)$.

The equation of normal to parabola at point $P(m_1^2 - 2m)$ is

$$y = mx - 2m - m^3$$

For the shortest distance from circle $x^2 + y^2 - 4x - 16y + 64 = 0$, this normal is also normal to the given circle; therefore, it passes through the centre of the circle (2, 8).

$$8 = 2m - 2m - m^3$$

$$m^3 = -8 \Rightarrow m = -2$$

Therefore, point P is (4, 4).

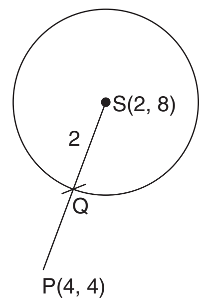


Figure 13.57

The radius of the circle is

$$\sqrt{4+64-64} = 2$$

Therefore, the segment SP is given by

$$SP = \sqrt{(4-2)^2 + (4-9)^2} = \sqrt{20} = 2\sqrt{5}$$

Hence, option (A) is correct.

Now, $$\frac{SQ}{QP} = \frac{2}{SP-SQ} = \frac{2}{2\sqrt{5}-2} = \frac{1}{\sqrt{5}-1}\times\frac{\sqrt{5}+1}{\sqrt{5}+1} = \frac{\sqrt{5}+1}{4}$$

The normal of the parabola at point P is given by

$$y = -2x + 4 + 8 \Rightarrow 2x + y = 12$$

Therefore, the x intercept is 6.

Hence, option (C) is correct.

Therefore, the slope of the tangent to the circle at point Q is $+\frac{1}{2}$.

Hence, option (D) is correct.

Hence, the correct answers are options (A), (C) and (D).

Practice Exercise 1

1. If a double-ordinate of the parabola $y^2 = 4ax$ be of length $8a$, then the angle between the lines joining the vertex of the parabola to the ends of this double-ordinate is

(A) 30° **(B)** 60°

(C) 90° **(D)** 120°

2. PQ is a double-ordinate of the parabola $y^2 = 4ax$. The locus of the points of trisection of PQ is

(A) $9y^2 = 4ax$ **(B)** $9x^2 = 4ay$

(C) $9y^2 + 4ax = 0$ **(D)** $9x^2 + 4ay = 0$

3. If the vertex of a parabola be at origin and directrix be $x + 5 = 0$, then its latus rectum is

(A) 5 (B) 10
(C) 20 (D) 40

4. The latus rectum of a parabola, whose directrix is $x+y-2=0$ and focus is (3, – 4), is
(A) $-3\sqrt{2}$ (B) $3\sqrt{2}$
(C) $-3/\sqrt{2}$ (D) $3/\sqrt{2}$

5. The equation of the lines joining the vertex of the parabola $y^2=6x$ to the points on it, whose abscissa is 24, is
(A) $y\pm 2x=0$ (B) $2y\pm x=0$
(C) $x\pm 2y=0$ (D) $2x\pm y=0$

6. The points on the parabola $y^2=36x$, whose ordinate is three times the abscissa, are
(A) (0, 0), (4, 12) (B) (1, 3), (4, 12)
(C) (4, 12) (D) None of these

7. The points on the parabola $y^2=12x$, whose focal distance is 4, are
(A) $(2,\sqrt{3}),(2,-\sqrt{3})$ (B) $(1,2\sqrt{3}),(1,-2\sqrt{3})$
(C) (1, 2) (D) None of these

8. The focal distance of a point on the parabola $y^2=16x$, whose ordinate is twice the abscissa, is
(A) 6 (B) 8
(C) 10 (D) 12

9. The coordinates of the extremities of the latus rectum of the parabola $5y^2=4x$ are
(A) $(1/5,2/5),(-1/5,2/5)$ (B) $(1/5,2/5),(1/5,-2/5)$
(C) $(1/5,4/5),(1/5,-4/5)$ (D) None of these

10. A parabola passing through the point (–4, –2) has its vertex at the origin and y-axis as its axis. The latus rectum of the parabola is
(A) 6 (B) 8
(C) 10 (D) 12

11. The focus of the parabola $x^2=-16y$ is
(A) (4, 0) (B) (0, 4)
(C) (–4, 0) (D) (0, –4)

12. If (2, 0) is the vertex and y-axis the directrix of a parabola, then its focus is
(A) (2, 0) (B) (–2, 0)
(C) (4, 0) (D) (–4, 0)

13. If the parabola $y^2=4ax$ passes through (–3, 2), then length of its latus rectum is
(A) 2/3 (B) 1/3
(C) 4/3 (D) 4

14. The ends of latus rectum of parabola $x^2+8y=0$ are
(A) (–4, –2) and (4, 2) (B) (4, –2) and (–4, 2)
(C) (–4, –2) and (4, –2) (D) (4, 2) and (–4, 2)

15. The end points of latus rectum of the parabola $x^2=4ay$ are
(A) $(a,2a),(2a,-a)$ (B) $(-a,2a),(2a,a)$
(C) $(a,-2a),(2a,a)$ (D) $(-2a,a),(2a,a)$

16. The equation of the parabola with its vertex at the origin, axis on the y-axis and passing through the point (6, –3) is
(A) $y^2=12x+6$ (B) $x^2=12y$
(C) $x^2=-12y$ (D) $y^2=-12x+6$

17. Focus and directrix of the parabola $x^2=-8ay$ are
(A) $(0,-2a)$ and $y=2a$ (B) $(0,2a)$ and $y=-2a$
(C) $(2a,0)$ and $x=-2a$ (D) $(-2a,0)$ and $x=2a$

18. The equation of the parabola with focus (3, 0) and the directrix $x+3=0$ is
(A) $y^2=3x$ (B) $y^2=2x$
(C) $y^2=12x$ (D) $y^2=6x$

19. Locus of the poles of focal chords of a parabola is
(A) The tangent at the vertex
(B) The axis
(C) A focal chord
(D) The directrix

20. The parabola $y^2=x$ is symmetric about
(A) x-axis (B) y-axis
(C) Both x-axis and y-axis (D) The line $y=x$

21. The point on the parabola $y^2=18x$, for which the ordinate is three times the abscissa, is
(A) (6, 2) (B) (–2, –6)
(C) (3, 18) (D) (2, 6)

22. The equation of latus rectum of a parabola is $x+y=8$ and the equation of the tangent at the vertex is $x+y=12$, then length of the latus rectum is
(A) $4\sqrt{2}$ (B) $2\sqrt{2}$
(C) 8 (D) $8\sqrt{2}$

23. The vertex of the parabola $y^2+2y+x=0$ lies in which quadrant?
(A) First (B) Second
(C) Third (D) Fourth

24. The equation $x^2-2xy+y^2+3x+2=0$ represents
(A) A parabola (B) An ellipse
(C) A hyperbola (D) A circle

25. $x-2=t^2$, $y=2t$ are the parametric equations of the parabola
(A) $y^2=4x$ (B) $y^2=-4x$
(C) $x^2=-4y$ (D) $y^2=4(x-2)$

26. The equation of the latus rectum of the parabola $x^2+4x+2y=0$ is
(A) $2y+3=0$ (B) $3y=2$
(C) $2y=3$ (D) $3y+2=0$

27. The vertex of the parabola $9x^2-6x+36y+9=0$ is
(A) $(1/3,-2/9)$ (B) $(-1/3,-1/2)$
(C) $(-1/3,1/2)$ (D) $(1/3,1/2)$

28. The equation of the parabola whose axis is vertical and passes through the points (0, 0), (3, 0) and (–1, 4) is
(A) $x^2-3x-y=0$ (B) $x^2+3x+y=0$
(C) $x^2-4x+2y=0$ (D) $x^2-4x-2y=0$

29. The equation of the parabola whose vertex is (–1, –2), axis is vertical, and which passes through the point (3, 6), is
(A) $x^2+2x-2y-3=0$ (B) $2x^2=3y$
(C) $x^2-2x-y+3=0$ (D) None of these

30. The axis of the parabola $x^2-4x-3y+10=0$ is
(A) $y+2=0$ (B) $x+2=0$
(C) $y-2=0$ (D) $x-2=0$

31. The equation of the parabola whose directrix is $y=2x-9$ and focus (–8, –2) is

(A) $x^2+4y^2+4xy+16x+2y+259=0$
(B) $x^2+4y^2+4xy+116x+2y+259=0$
(C) $x^2+y^2+4xy+116x+2y+259=0$
(D) None of these

32. The equation of the parabola with (–3, 0) as focus, and $x+5=0$ as directrix, is
(A) $x^2=4(y+4)$ **(B)** $x^2=4(y-4)$
(C) $y^2=4(x+4)$ **(D)** $y^2=4(x-4)$

33. The equation of the parabola whose vertex and focus lies on the *x*-axis at distance *a* and *a′* from the origin is
(A) $y^2=4(a'-a)(x-a)$
(B) $y^2=4(a'-a)(x+a)$
(C) $y^2=4(a'+a)(x-a)$
(D) $y^2=4(a'+a)(x+a)$

34. The focus of the parabola $y^2=4y-4x$ is
(A) (0, 2) **(B)** (1, 2)
(C) (2, 0) **(D)** (2, 1)

35. The vertex of the parabola $x^2+4x+2y-7=0$ is
(A) (–2, 11/2) **(B)** (–2, 2)
(C) (–2, 11) **(D)** (2, 11)

36. If the axis of a parabola is horizontal and it passes through the points (0, 0), (0, –1) and (6, 1), then its equation is
(A) $y^2+3y-x-4=0$ **(B)** $y^2-3y+x-4=0$
(C) $y^2-3y-x-4=0$ **(D)** None of these

37. The equation of the latus rectum of the parabola represented by equation $y^2+2Ax+2By+C=0$ is
(A) $x=\dfrac{B^2+A^2-C}{2A}$ **(B)** $x=\dfrac{B^2-A^2+C}{2A}$
(C) $x=\dfrac{B^2-A^2-C}{2A}$ **(D)** $x=\dfrac{A^2-B^2-C}{2A}$

38. The parametric equation of the curve $y^2=8x$ are
(A) $x=t^2, y=2t$ **(B)** $x=2t^2, y=4t$
(C) $x=2t, y=4t^2$ **(D)** None of these

39. The equations $x=\dfrac{t}{4}, y=\dfrac{t^2}{4}$ represents
(A) A circle **(B)** A parabola
(C) An ellipse **(D)** A hyperbola

40. The equation of parabola whose vertex and focus are (0, 4) and (0, 2), respectively, is
(A) $y^2-8x=32$ **(B)** $y^2+8x=32$
(C) $x^2+8y=32$ **(D)** $x^2-8y=32$

41. The curve $16x^2+8xy+y^2-74x-78y+212=0$ represents
(A) Parabola **(B)** Hyperbola
(C) Ellipse **(D)** None of these

42. The length of the latus rectum of the parabola $9x^2-6x+36y+19=0$ is
(A) 36 **(B)** 9
(C) 6 **(D)** 4

43. The axis of the parabola $9y^2-16x-12y-57=0$ is
(A) $3y=2$ **(B)** $x+3y=3$
(C) $2x=3$ **(D)** $y=3$

44. The vertex of a parabola is the point (*a*, *b*) and latus rectum is of length *l*. If the axis of the parabola is along the positive direction of *y*-axis, then its equation is
(A) $(x+a)^2=\dfrac{l}{2}(2y-2b)$
(B) $(x-a)^2=\dfrac{l}{2}(2y-2b)$
(C) $(x+a)^2=\dfrac{l}{4}(2y-2b)$
(D) $(x-a)^2=\dfrac{l}{8}(2y-2b)$

45. If the vertex of the parabola $y=x^2-8x+c$ lies on *x*-axis, then the value of *c* is
(A) –16 **(B)** –4
(C) 4 **(D)** 16

46. The points of intersection of the curves whose parametric equations are $x=t^2+1, y=2t$ and $x=2s, y=\dfrac{2}{s}$ is given by
(A) (1, – 3) **(B)** (2, 2)
(C) (–2, 4) **(D)** (1, 2)

47. The latus rectum of the parabola $y^2=5x+4y+1$ is
(A) 5/4 **(B)** 10
(C) 5 **(D)** 5/2

48. The equation of the locus of a point which moves so as to be at equal distances from the point (*a*, 0) and the *y*-axis is
(A) $y^2-2ax+a^2=0$ **(B)** $y^2+2ax+a^2=0$
(C) $x^2-2ay+a^2=0$ **(D)** $x^2+2ay+a^2=0$

49. The focus of the parabola $x^2=2x+2y$ is
(A) $\left(\dfrac{3}{2},\dfrac{-1}{2}\right)$ **(B)** $\left(1,\dfrac{-1}{2}\right)$
(C) (1, 0) **(D)** (0, 1)

50. The latus rectum of the parabola $y^2-4y-2x-8=0$ is
(A) 2 **(B)** 4
(C) 8 **(D)** 1

51. The equation of the parabola with focus (*a, b*) and directrix $\dfrac{x}{a}+\dfrac{y}{b}=1$ is given by
(A) $(ax-by)^2-2a^3x-2b^3y+a^4+a^2b^2+b^4=0$
(B) $(ax+by)^2-2a^3x-2b^3y-a^4+a^2b^2-b^4=0$
(C) $(ax-by)^2+a^4+b^4-2a^3x=0$
(D) $(ax-by)^2-2a^3x=0$

52. The length of latus rectum of the parabola $4y^2+2x-20y+17=0$ is
(A) 3 **(B)** 6
(C) 1/2 **(D)** 9

53. Eccentricity of the parabola $x^2-4x-4y+4=0$ is
(A) $e=0$ **(B)** $e=1$
(C) $e>4$ **(D)** $e=4$

54. The vertex of the parabola $3x-2y^2-4y+7=0$ is
(A) (3, 1) **(B)** (–3, –1)
(C) (–3, 1) **(D)** None of these

55. The focus of the parabola $4y^2-6x-4y=5$ is
(A) (–8/5, 2) **(B)** (–5/8, 1/2)
(C) (1/2, 5/8) **(D)** (5/8, –1/2)

56. The vertex of the parabola $x^2+8x+12y+4=0$ is
(A) (–4, 1) **(B)** (4, –1)
(C) (–4, –1) **(D)** (4, 1)

57. The focus of the parabola $(y-2)^2=20(x+3)$ is
(A) (3, –2) **(B)** (2, –3)
(C) (2, 2) **(D)** (3, 3)

58. The length of the latus rectum of the parabola $x^2-4x-8y+12=0$ is
(A) 4 **(B)** 6
(C) 8 **(D)** 10

59. The focus of the parabola $y=2x^2+x$ is
(A) (0, 0) **(B)** $\left(\frac{1}{2},\frac{1}{4}\right)$
(C) $\left(-\frac{1}{4},0\right)$ **(D)** $\left(-\frac{1}{4},\frac{1}{8}\right)$

60. The focus of the parabola $y^2-x-2y+2=0$ is
(A) (1/4, 0) **(B)** (1, 2)
(C) (3/4, 1) **(D)** (5/4, 1)

61. The vertex of the parabola $(y-2)^2=16(x-1)$ is
(A) (2, 1) **(B)** (1, –2)
(C) (–1, 2) **(D)** (1, 2)

62. The equation of the parabola with its vertex at (1, 1) and focus (3, 1) is
(A) $(x-1)^2=8(y-1)$ **(B)** $(y-1)^2=8(x-3)$
(C) $(y-1)^2=8(x-1)$ **(D)** $(x-3)^2=8(y-1)$

63. The equation of parabola, whose focus is (5, 3) and directrix is $3x-4y+1=0$, is
(A) $(4x+3y)^2-256x-142y+849=0$
(B) $(4x-3y)^2-256x-142y+849=0$
(C) $(3x+4y)^2-142x-256y+849=0$
(D) $(3x-4y)^2-256x-142y+849=0$

64. Which of the following points lie on the parabola $x^2=4ay$?
(A) $x=at^2, y=2at$ **(B)** $x=2at, y=at$
(C) $x=2at^2, y=at$ **(D)** $x=2at, y=at^2$

65. The equation of the parabola whose vertex is at (2, –1) and focus at (2, –3) is
(A) $x^2+4x-8y-12=0$ **(B)** $x^2-4x+8y+12=0$
(C) $x^2+8y=12$ **(D)** $x^2-4x+12=0$

66. The directrix of the parabola $x^2-4x-8y+12=0$ is
(A) $x=1$ **(B)** $y=0$
(C) $x=-1$ **(D)** $y=-1$

67. The equation of the parabola with focus (0, 0) and directrix $x+y=4$ is
(A) $x^2+y^2-2xy+8x+8y-16=0$
(B) $x^2+y^2-2xy+8x+8y=0$
(C) $x^2+y^2+8x+8y-16=0$
(D) $x^2-y^2+8x+8y-16=0$

68. If (0, 6) and (0, 3) are, respectively, the vertex and focus of a parabola, then its equation is
(A) $x^2+12y=72$ **(B)** $x^2-12y=72$
(C) $y^2-12x=72$ **(D)** $y^2+12x=72$

69. The equation of the directrix of the parabola $x^2+8y-2x=7$ is
(A) $y=3$ **(B)** $y=-3$
(C) $y=2$ **(D)** $y=0$

70. The equation of axis of the parabola $2x^2+5y-3x+4=0$ is
(A) $x=\frac{3}{4}$ **(B)** $y=\frac{3}{4}$
(C) $x=-\frac{1}{2}$ **(D)** $x-3y=5$

71. If $x^2+6x+20y-51=0$, then the axis of parabola is
(A) $x+3=0$ **(B)** $x-3=0$
(C) $x=1$ **(D)** $x+1=0$

72. The equation of the tangent to the parabola $y=x^2-x$ at the point where $x=1$, is
(A) $y=-x-1$ **(B)** $y=-x+1$
(C) $y=x+1$ **(D)** $y=x-1$

73. The point of intersection of the latus rectum and the axis of the parabola $y^2+4x+2y-8=0$
(A) (5/4, –1) **(B)** (9/4, –1)
(C) (7/2, 5/2) **(D)** None of these

74. The point of contact of the tangent $18x-6y+1=0$ to the parabola $y^2=2x$ is
(A) $\left(\frac{-1}{18},\frac{-1}{3}\right)$ **(B)** $\left(\frac{-1}{18},\frac{1}{3}\right)$
(C) $\left(\frac{1}{18},\frac{-1}{3}\right)$ **(D)** $\left(\frac{1}{18},\frac{1}{3}\right)$

75. The equation of the common tangent of the parabolas $x^2=108y$ and $y^2=32x$, is
(A) $2x+3y=36$ **(B)** $2x+3y+36=0$
(C) $3x+2y=36$ **(D)** $3x+2y+36=0$

76. The line $lx+my+n=0$ touches the parabola $y^2=4ax$ if
(A) $mn=al^2$ **(B)** $lm=an^2$
(C) $ln=am^2$ **(D)** $mn=al$

77. The line $x\cos\alpha+y\sin\alpha=p$ touches the parabola $y^2=4a(x+a)$ if
(A) $p\cos\alpha+a=0$ **(B)** $p\cos\alpha-a=0$
(C) $a\cos\alpha+p=0$ **(D)** $a\cos\alpha-p=0$

78. The equation of a tangent to the parabola $y^2=4ax$ which is making an angle θ with x-axis is
(A) $y=x\cot\theta+a\tan\theta$
(B) $x=y\tan\theta+a\cot\theta$
(C) $y=x\tan\theta+a\cot\theta$
(D) None of these

79. The equation of the tangent to the parabola $y^2 = 4x + 5$ parallel to the line $y = 2x + 7$ is

(A) $2x - y - 3 = 0$ **(B)** $2x - y + 3 = 0$
(C) $2x + y + 3 = 0$ **(D)** None of these

80. The point of the contact of the tangent to the parabola $y^2 = 4ax$, which makes an angle of 60° with x-axis, is

(A) $\left(\frac{a}{3}, \frac{2a}{\sqrt{3}}\right)$ **(B)** $\left(\frac{2a}{\sqrt{3}}, \frac{a}{3}\right)$
(C) $\left(\frac{a}{\sqrt{3}}, \frac{2a}{3}\right)$ **(D)** None of these

81. The straight line $y = 2x + \lambda$ does not meet the parabola $y^2 = 2x$ if

(A) $\lambda < \frac{1}{4}$ **(B)** $\lambda > \frac{1}{4}$
(C) $\lambda = 4$ **(D)** $\lambda = 1$

82. The equation of the tangent at a point $P(t)$ where t is any parameter to the parabola $y^2 = 4ax$, is

(A) $yt = x + at^2$ **(B)** $y = xt + at^2$
(C) $y = xt + \frac{a}{t}$ **(D)** $y = tx$

83. The line $y = 2x + c$ is a tangent to the parabola $y^2 = 16x$, if c equals

(A) -2 **(B)** -1
(C) 0 **(D)** 2

84. The line $y = mx + 1$ is a tangent to the parabola $y^2 = 4x$, if

(A) $m = 1$ **(B)** $m = 2$
(C) $m = 4$ **(D)** $m = 3$

85. The angle of intersection between the curves $y^2 = 4x$ and $x^2 = 32y$ at point (16, 8), is

(A) $\tan^{-1}\left(\frac{3}{5}\right)$ **(B)** $\tan^{-1}\left(\frac{4}{5}\right)$
(C) π **(D)** $\frac{\pi}{2}$

86. The locus of a foot of perpendicular drawn to the tangent of parabola $y^2 = 4ax$ from focus, is

(A) $x = 0$ **(B)** $y = 0$
(C) $y^2 = 2a(x + a)$ **(D)** $x^2 + y^2(x + a) = 0$

87. If the straight line $x + y = 1$ touches the parabola $y^2 - y + x = 0$, then the coordinates of the point of contact are

(A) (1, 1) **(B)** $\left(\frac{1}{2}, \frac{1}{2}\right)$
(C) (0, 1) **(D)** (1, 0)

88. If the line $y = mx + c$ is a tangent to the parabola $y^2 = 4a(x + a)$ then $ma + \frac{a}{m}$ is equal to

(A) c **(B)** $2c$
(C) $-c$ **(D)** $3c$

89. A tangent to the parabola $y^2 = 8x$ makes an angle of 45° with the straight line $y = 3x + 5$, then the equation of tangent is

(A) $2x + y - 1 = 0$ **(B)** $x + 2y - 1 = 0$
(C) $2x + y + 1 = 0$ **(D)** None of these

90. The angle between the tangents drawn at the end points of the latus rectum of parabola $y^2 = 4ax$ is

(A) $\pi/3$ **(B)** $2\pi/3$
(C) $\pi/4$ **(D)** $\pi/2$

91. The line $y = mx + c$ touches the parabola $x^2 = 4ay$, if

(A) $c = -am$ **(B)** $c = -a/m$
(C) $c = -am^2$ **(D)** $c = a/m^2$

92. The locus of the point of intersection of the perpendicular tangents to the parabola $x^2 = 4ay$ is

(A) Axis of the parabola
(B) Directrix of the parabola
(C) Focal chord of the parabola
(D) Tangent at vertex to the parabola

93. The angle between the tangents drawn from the origin to the parabola $y^2 = 4a(x - a)$ is

(A) 90° **(B)** 30°
(C) $\tan^{-1}(1/2)$ **(D)** 45°

94. If the line $x = my + k$ touches the parabola $x^2 = 4ay$, then what is the value of k?

(A) a/m **(B)** am
(C) am^2 **(D)** $-am^2$

95. If y_1 and y_2 are the ordinates of two points P and Q on the parabola and y_3 is the ordinate of the point of intersection of tangents at P and Q, then

(A) y_1, y_2, y_3 are in AP.
(B) y_1, y_3, y_2 are in AP.
(C) y_1, y_2, y_3 are in GP.
(D) y_1, y_3, y_2 are in GP.

96. The two parabolas $y^2 = 4x$ and $x^2 = 4y$ intersect at a point P, whose abscissa is not zero, such that

(A) They both touch each other at P
(B) They cut at right angles at P
(C) The tangents to each curve at P make complementary angles with x-axis
(D) None of these

97. The line $y = 2x + c$ is tangent to the parabola $y^2 = 4x$, then what is the value of c?

(A) $-1/2$ **(B)** $1/2$
(C) $1/3$ **(D)** 4

98. The condition for which the straight line $y = mx + c$ touches the parabola $y^2 = 4ax$ is

(A) $a = c$ **(B)** $\frac{a}{c} = m$
(C) $m = a^2c$ **(D)** $m = ac^2$

99. If the parabola $y^2 = 4ax$ passes through the point (1, −2), then the tangent at this point is

(A) $x + y - 1 = 0$ **(B)** $x - y - 1 = 0$
(C) $x + y + 1 = 0$ **(D)** $x - y + 1 = 0$

100. The equation of the tangent to the parabola $y^2 = 16x$, which is perpendicular to the line $y = 3x + 7$ is

(A) $y - 3x + 4 = 0$ **(B)** $3y - x + 36 = 0$
(C) $3y + x - 36 = 0$ **(D)** $3y + x + 36 = 0$

101. The equation of the tangent to the parabola $y^2 = 4ax$ at point $(a/t^2, 2a/t)$ is

(A) $ty = xt^2 + a$ (B) $ty = x + at^2$
(C) $y = tx + at^2$ (D) $y = tx + (a/t^2)$

102. The equation of common tangent to the circle $x^2 + y^2 = 2$ and parabola $y^2 = 8x$ is
(A) $y = x + 1$ (B) $y = x + 2$
(C) $y = x - 2$ (D) $y = -x + 2$

103. If the line $lx + my + n = 0$ is a tangent to the parabola $y^2 = 4ax$, then the locus of its point of contact is
(A) A straight line (B) A circle
(C) A parabola (D) Two straight lines

104. The line $x - y + 2 = 0$ touches the parabola $y^2 = 8x$ at the point
(A) $(2, -4)$ (B) $(1, 2\sqrt{2})$
(C) $(4, -4\sqrt{2})$ (D) $(2, 4)$

105. The tangent to the parabola $y^2 = 4ax$ at the point $(a, 2a)$ makes with x-axis an angle equal to
(A) $\pi/3$ (B) $\pi/4$
(C) $\pi/2$ (D) $\pi/6$

106. If $lx + my + n = 0$ is tangent to the parabola $x^2 = y$, then the condition of tangency is
(A) $l^2 = 2mn$ (B) $l = 4m^2n^2$
(C) $m^2 = 4ln$ (D) $l^2 = 4mn$

107. The equation of the tangent to the parabola $y^2 = 9x$ which goes through the point (4, 10), is
(A) $x + 4y + 1 = 0$ (B) $9x + 4y + 4 = 0$
(C) $x - 4y + 36 = 0$ (D) $9x - 4y + 4 = 0$

108. Two perpendicular tangents to $y^2 = 4ax$ always intersect on the line if
(A) $x = a$ (B) $x + a = 0$
(C) $x + 2a = 0$ (D) $x + 4a = 0$

109. The equation of the common tangent touching the circle $(x-3)^2 + y^2 = 9$ and the parabola $y^2 = 4x$ above the x-axis is
(A) $\sqrt{3}y = 3x + 1$ (B) $\sqrt{3}y = -(x + 3)$
(C) $\sqrt{3}y = x + 3$ (D) $\sqrt{3}y = -(3x + 1)$

110. The point at which the line $y = mx + c$ touches the parabola $y^2 = 4ax$ is
(A) $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ (B) $\left(\frac{a}{m^2}, \frac{-2a}{m}\right)$
(C) $\left(-\frac{a}{m^2}, \frac{2a}{m}\right)$ (D) $\left(-\frac{a}{m^2}, -\frac{2a}{m}\right)$

111. The tangent drawn at any point P to the parabola $y^2 = 4ax$ meets the directrix at the point K, then the angle which KP subtends at its focus is
(A) 30° (B) 45°
(C) 60° (D) 90°

112. The point of intersection of the parabola at the points t_1 and t_2 is
(A) $[at_1t_2, a(t_1 + t_2)]$ (B) $[2at_1t_2, a(t_1 + t_2)]$
(C) $[2at_1t_2, 2a(t_1 + t_2)]$ (D) None of these

113. The angle of intersection between the curves $x^2 = 4(y+1)$ and $x^2 = -4(y+1)$ is
(A) $\pi/6$ (B) $\pi/4$
(C) 0 (D) $\pi/2$

114. The angle between the two curves $y^2 = 4(x+1)$ and $x^2 = 4(y+1)$ is
(A) 0° (B) 90°
(C) 60° (D) 30°

115. If the tangent to the parabola $y^2 = ax$ makes an angle of 45° with x-axis, then the point of contact is
(A) $\left(\frac{a}{2}, \frac{a}{2}\right)$ (B) $\left(\frac{a}{4}, \frac{a}{4}\right)$
(C) $\left(\frac{a}{2}, \frac{a}{4}\right)$ (D) $\left(\frac{a}{4}, \frac{a}{2}\right)$

116. The tangents at the extremities of any focal chord of a parabola intersect
(A) At right angles
(B) On the directrix
(C) On the tangents at vertex
(D) None of these

117. The point of intersection of tangents at the ends of the latus rectum of the parabola $y^2 = 4x$ is equal to
(A) (1, 0) (B) (–1, 0)
(C) (0, 1) (D) (0, –1)

118. The angle between the tangents drawn from the points (1, 4) to the parabola $y^2 = 4x$ is
(A) $\pi/2$ (B) $\pi/3$
(C) $\pi/4$ (D) $\pi/6$

119. The locus of the middle points of the chords of the parabola $y^2 = 4ax$ which passes through the origin
(A) $y^2 = ax$ (B) $y^2 = 2ax$
(C) $y^2 = 4ax$ (D) $x^2 = 4ay$

120. The point on the parabola $y^2 = 8x$ at which the normal is parallel to the line $x - 2y + 5 = 0$ is
(A) $(-1/2, 2)$ (B) $(1/2, -2)$
(C) $(2, -1/2)$ (D) $(-2, 1/2)$

121. The maximum number of normal that can be drawn from a point to a parabola is
(A) 0 (B) 1
(C) 2 (D) 3

122. The point on the parabola $y^2 = 8x$ at which the normal is inclined at 60° to x-axis has the coordinates
(A) $(6, -4\sqrt{3})$ (B) $(6, 4\sqrt{3})$
(C) $(-6, -4\sqrt{3})$ (D) $(-6, 4\sqrt{3})$

123. The slope of the normal at the point $(at^2, 2at)$ of the parabola $y^2 = 4ax$ is
(A) $1/t$ (B) t
(C) $-t$ (D) $-1/t$

124. The equation of the normal at the point $\left(\frac{a}{4}, a\right)$ to the parabola $y^2 = 4ax$ is

(A) $4x+8y+9a=0$ (B) $4x+8y-9a=0$
(C) $4x+y-a=0$ (D) $4x-y+a=0$

125. The equation of normal to the parabola at the point $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$ is
(A) $y=m^2x-2mx-am^3$
(B) $m^3y=m^2x-2am^2-a$
(C) $m^3y=2am^2-m^2x+a$
(D) None of these

126. If the line $2x+y+k=0$ is normal to the parabola $y^2=-8x$, then the value of k is
(A) –16 (B) –8
(C) –24 (D) 24

127. If a normal drawn to the parabola $y^2=4ax$ at the point $(a, 2a)$ meets parabola again on $(at^2, 2at)$, then the value of t is
(A) 1 (B) 3
(C) –1 (D) –3

128. In the parabola $y^2=6x$, the equation of the chord through the vertex and the negative end of the latus rectum, is
(A) $y=2x$ (B) $y+2x=0$
(C) $x=2y$ (D) $x+2y=0$

129. The length of chord of contact of the tangents drawn from the point (2, 5) to the parabola $y^2=8x$, is
(A) $\frac{1}{2}\sqrt{41}$ (B) $\sqrt{41}$
(C) $\frac{3}{2}\sqrt{41}$ (D) $2\sqrt{41}$

130. If a and c are the segments of a focal chord of a parabola and b the semi-latus rectum, then
(A) a, b, c are in AP. (B) a, b, c are in GP.
(C) a, b, c are in HP. (D) None of these

131. If the segment intercepted by the parabola $y^2=4ax$ with the line $lx+my+n=0$ subtends a right angle at the vertex, then
(A) $4al+n=0$ (B) $4al+4am+n=0$
(C) $4am+n=0$ (D) $al+n=0$

132. A set of the parallel chords of the parabola $y^2=4ax$ have their mid-point on
(A) Any straight line through the vertex
(B) Any straight line through the focus
(C) Any straight line parallel to the axis
(D) Another parabola

133. The equations of the normals at the ends of latus rectum of the parabola $y^2=4ax$ are given by
(A) $x^2-y^2-6ax+9a^2=0$
(B) $x^2-y^2-6ax-6ay+9a^2=0$
(C) $x^2-y^2-6ay+9a^2=0$
(D) None of these

134. If the normals at two points P and Q of a parabola $y^2=4ax$ intersect at a third point R on the curve, then the product of ordinates of points P and Q is
(A) $4a^2$ (B) $2a^2$
(C) $-4a^2$ (D) $8a^2$

135. If $x=my+c$ is a normal to the parabola $x^2=4ay$, then the value of c is
(A) $-2am-am^3$ (B) $2am+am^3$
(C) $-\dfrac{2a}{m}-\dfrac{a}{m^3}$ (D) $\dfrac{2a}{m}+\dfrac{a}{m^3}$

136. If PSQ is the focal chord of the parabola $y^2=8x$ such that $SP=6$, then the length of SQ is
(A) 6 (B) 4
(C) 3 (D) None of these

137. At what point on the parabola $y^2=4x$, the normal makes equal angles with the coordinate axes
(A) (4, 4) (B) (9, 6)
(C) (4, –4) (D) (1, –2)

138. The equation of any normal to the parabola $y^2=4a(x-a)$ is
(A) $y=mx-2am-am^3$
(B) $y=m(x+a)-2am-am^3$
(C) $y=m(x-a)+\dfrac{a}{m}$
(D) $y=m(x-a)-2am-am^3$

139. The tangents drawn at the ends of any focal chord of a parabola $y^2=4ax$ intersect in the line
(A) $y-a=0$ (B) $y+a=0$
(C) $x-a=0$ (D) $x+a=0$

140. The centroid of the triangle formed by joining the feet of the normals drawn from any point to the parabola $y^2=4ax$, lies on
(A) Axis (B) Directrix
(C) Latus rectum (D) Tangent at vertex

141. If the normal to $y^2=12x$ at the point (3, 6) meets the parabola once again at the point (27, –18), and the circle on the normal chord as diameter is
(A) $x^2+y^2+30x+12y-27=0$
(B) $x^2+y^2+30x+12y+27=0$
(C) $x^2+y^2-30x-12y-27=0$
(D) $x^2+y^2-30x+12y-27=0$

142. The length of the normal chord to the parabola $y^2=4x$, which subtends right angle at the vertex is
(A) $6\sqrt{3}$ (B) $3\sqrt{3}$
(C) 2 (D) 1

143. If $x+y=k$ is a normal to the parabola $y^2=12x$, then k is
(A) 3 (B) 9
(C) –9 (D) –3

144. The normal at the point $(bt_1^2, 2bt_1)$ on a parabola meets the parabola again in the point $(bt_2^2, 2bt_2)$, then

(A) $t_2 = -t_1 - \frac{2}{t_1}$ **(B)** $t_2 = -t_1 + \frac{2}{t_1}$

(C) $t_2 = t_1 - \frac{2}{t_1}$ **(D)** $t_2 = t_1 + \frac{2}{t_1}$

145. The focal chord to $y^2 = 16x$ is tangent to $(x-6)^2 + y^2 = 2$, then the possible values of the slope of this chord are

(A) {–1, 1} **(B)** {–2, 2}

(C) {–2, 1/2} **(D)** {2, –1/2}

146. The normal to the parabola $y^2 = 8x$ at the point (2, 4) meets the parabola again at the point

(A) {–18, –12} **(B)** {–18, 12}

(C) {18, 12} **(D)** (18, –12)

147. The polar of the focus of the parabola is

(A) *x*-axis **(B)** *y*-axis

(C) Directrix **(D)** Latus rectum

148. The equation of diameter of parabola $y^2 = x$ corresponding to the chord $x - y + 1 = 0$ is

(A) $2y = 3$ **(B)** $2y = 1$

(C) $2y = 5$ **(D)** $y = 1$

149. The area of the triangle formed by the lines joining the vertex of the parabola $x^2 = 12y$ to the ends of its latus rectum is

(A) 12 sq. units **(B)** 16 sq. units

(C) 18 sq. units **(D)** 24 sq. units

150. The area of triangle formed inside the parabola $y^2 = 4x$ and whose ordinates of vertices are 1, 2 and 4 is

(A) 7/2 **(B)** 5/2

(C) 3/2 **(D)** 3/4

151. An equilateral triangle is inscribed in the parabola $y^2 = 4ax$ whose vertices are at the parabola, then the length of its side is equal to

(A) $8a$ **(B)** $8a\sqrt{3}$

(C) $a\sqrt{2}$ **(D)** None of these

152. The ordinates of the triangle inscribed in parabola $y^2 = 4ax$ are y_1, y_2 and y_3. Then the area of the triangle is

(A) $\frac{1}{8a}(y_1 + y_2)(y_2 + y_3)(y_3 + y_1)$

(B) $\frac{1}{4a}(y_1 + y_2)(y_2 + y_3)(y_3 + y_1)$

(C) $\frac{1}{8a}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)$

(D) $\frac{1}{4a}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)$

153. From the point (–1, 2), the tangent lines are drawn to the parabola $y^2 = 4x$, then the equation of chord of contact is

(A) $y = x + 1$ **(B)** $y = x - 1$

(C) $y + x = 1$ **(D)** None of these

154. In Question 153, the area of triangle formed by chord of contact and the tangents is given by

(A) 8 **(B)** $8\sqrt{3}$

(C) $8\sqrt{2}$ **(D)** None of these

155. The point on parabola $2y = x^2$ which is nearest to the point (0, 3) is

(A) (±4, 8) **(B)** (±1, 1/2)

(C) (±2, 2) **(D)** None of these

156. From the point (–1, –60), two tangents are drawn to the parabola $y^2 = 4x$. Then, the angle between the two tangents is

(A) 30° **(B)** 45°

(C) 60° **(D)** 90°

157. The ends of the latus rectum of the conic $x^2 + 10x - 16y + 25 = 0$ are

(A) (3, –4), (13, 4) **(B)** (–3, –4), (13, –4)

(C) (3, 4), (–13, 4) **(D)** (5, –8), (–5, 8)

158. Tangent to the parabola $y = x^2 + 6$ at (1, 7) touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at the point

(A) (–6, –9) **(B)** (–13, –9)

(C) (–6, –7) **(D)** (13, 7)

159. The angle of intersection between the curves $x^2 = 8y$ and $y^2 = 8x$ at origin is

(A) $\pi/4$ **(B)** $\pi/3$

(C) $\pi/6$ **(D)** $\pi/2$

160. If the line $y = 2x + k$ is a tangent to the curve $x^2 = 4y$, then k is equal to

(A) 4 **(B)** 1/2

(C) –4 **(D)** –1/2

161. The equation to a parabola which passes through the intersection of a straight line $x + y = 0$ and the circle $x^2 + y^2 + 4y = 0$ is

(A) $y^2 = 4x$ **(B)** $y^2 = x$

(C) $y^2 = 2x$ **(D)** None of these

162. Let a circle tangent to the directrix of a parabola $y^2 = 2ax$ has its centre coinciding with the focus of the parabola. Then the point of intersection of the parabola and circle is

(A) $(a, -a)$ **(B)** $(a/2, a/2)$

(C) $(a/2, \pm a)$ **(D)** $(\pm a, a/2)$

163. The length intercepted by the curve $y^2 = 4x$ on the line satisfying $dy/dx = 1$ and passing through point (0, 1) is given by

(A) 1 **(B)** 2

(C) 0 **(D)** None of these

164. The equation of a straight line drawn through the focus of the parabola $y^2 = -4x$ at an angle of 120° to *x*-axis is

(A) $y + \sqrt{3}(x-1) = 0$ **(B)** $y - \sqrt{3}(x-1) = 0$

(C) $y + \sqrt{3}(x+1) = 0$ **(D)** $y - \sqrt{3}(x+1) = 0$

165. The number of parabolas that can be drawn if two ends of the latus rectum are given

(A) 1 **(B)** 2

(C) 4 **(D)** 3

166. The normal meet the parabola $y^2 = 4ax$ at that point where the abscissa of the point is equal to the ordinate of the point is

(A) $(6a, -9a)$ **(B)** $(-9a, 6a)$

(C) $(-6a, 9a)$ **(D)** $(9a, -6a)$

Practice Exercise 2

Single/Multiple Correct Choice Type Questions

1. A circle is drawn having the centre at $C(0, 2)$ and passing through the focus (S) of the parabola $y^2 = 8x$, if the radius (CS) intersects the parabola at point P, then

(A) distance of point P from the directrix is $(8 - 4\sqrt{2})$

(B) distance of point C from point P is $(6\sqrt{2} - 8)$

(C) angle subtended by the intercept made by the circle on the directrix at its centre is '$\pi/2$'

(D) point P is the midpoint of C and S

2. If $f(x+y) = f(x)f(y)$ for all $x, y \in R$, $f(1) = 2$ and $a_r = f(r)$ for $r \in N$, then the coordinates of a point on the parabola $y^2 = 8x$ whose focal distance is 4, may be

(A) (a_1, a_2) **(B)** $(a_1, -a_2)$

(C) $(a_3, -a_3)$ **(D)** (a_3, a_3)

3. Let V be the vertex and L be the latus rectum of the parabola $x^2 = 2y + 4x - 4$. Then the equation of the parabola whose vertex is at V, latus rectum is $L/2$ and the axis is perpendicular to the axis of the given parabola is

(A) $y^2 = x - 2$ **(B)** $y^2 = x - 4$

(C) $y^2 = 2 - x$ **(D)** $y^2 = 4 - x$

4. Let P, Q and R are three co-normal points on the parabola $y^2 = 4ax$. Then the correct statement(s) is/are

(A) algebraic sum of the slopes of the normal at P, Q and R vanishes.

(B) algebraic sum of the ordinates of the points P, Q and R vanishes.

(C) centroid of the triangle PQR lies on the axis of the parabola.

(D) circle circumscribing the triangle PQR passes through the vertex of the parabola.

5. If the equation of the tangent at P, Q and vertex A of a parabola are $3x + 4y - 7 = 0$, $2x + 3y - 10 = 0$ and $x - y = 0$, respectively, then

(A) focus is (4, 5)

(B) length of latus rectum is $2\sqrt{2}$

(C) axis is $x + y - 9 = 0$

(D) vertex is $\left(\frac{9}{2}, \frac{9}{2}\right)$

6. If A and B are the points on the parabola $y^2 = 4ax$ with vertex O such that OA is perpendicular to OB and having lengths r_1 and r_2, respectively, then the value of $\dfrac{r_1^{4/3} r_2^{4/3}}{r_1^{2/3} + r_2^{2/3}}$ is

(A) $16a^2$ **(B)** a^2

(C) $4a$ **(D)** None of these

7. The locus of the midpoint of the focal radii of a variable point moving on the parabola, $y^2 = 4ax$ is a parabola whose

(A) latus rectum is half the latus rectum of the original parabola

(B) vertex is $(a/2, 0)$

(C) directrix is y-axis

(D) focus has the coordinates $(a, 0)$

Comprehension Type Questions

Paragraph for Questions 8–10: $y = f(x)$ is a parabola of the form $y = x^2 + ax + 1$, its tangent at the point of intersection of y-axis and parabola also touches the circle $x^2 + y^2 = r^2$. It is known that no point of the parabola is below x-axis.

8. The radius of the circle, when 'a' attains its maximum value is

(A) $\frac{1}{\sqrt{10}}$ **(B)** $\frac{1}{\sqrt{5}}$

(C) 1 **(D)** $\sqrt{5}$

9. The slope of the tangent, when radius of the circle is maximum, is

(A) 0 **(B)** 1

(C) –1 **(D)** Not defined

10. The minimum area bounded by the tangent and the coordinate axes is

(A) $\frac{1}{4}$ **(B)** $\frac{1}{3}$

(C) $\frac{1}{2}$ **(D)** 1

Paragraph for Questions 11–13: If the locus of the circumcentre of a variable triangle having sides y-axis, $y = 2$ and $\ell x + my = 1$, where (ℓ, m) lies on the parabola $y^2 = 4ax$ is a curve C, then

11. Coordinates of the vertex of this curve C is

(A) $\left(2a, \frac{3}{2}\right)$ **(B)** $\left(-2a, -\frac{3}{2}\right)$

(C) $\left(-2a, \frac{3}{2}\right)$ **(D)** $\left(-2a, -\frac{3}{2}\right)$

12. The length of the smallest focal chord of this curve C is

(A) $\frac{1}{12a}$ **(B)** $\frac{1}{4a}$

(C) $\frac{1}{16a}$ **(D)** $\frac{1}{8a}$

13. The curve C is symmetric about the line

(A) $y = -\frac{3}{2}$ **(B)** $y = \frac{3}{2}$

(C) $x = -\frac{3}{2}$ **(D)** $x = \frac{3}{2}$

Paragraph for Questions 14–16: In general, three normals can be drawn from a point to a parabola and the point where they meet the parabola are called co-normal points.

The equation of any normal to $y^2 = 4ax$ is $y = mx - 2am - am^3$. If it passes through (h, k), then $k = mh - 2am - am^3$ or $am^3 + m(2a - h) + k = 0$

This is cubic in m; and it has three roots m_1, m_2 and m_3

Therefore, $m_1 + m_2 + m_3 = 0$, $m_1m_2m_3 = \frac{-k}{a}$, $m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a-h}{a}$

14. Minimum distance between the curves $y^2 = x - 1$ and $x^2 = y - 1$ is equal to

(A) $\frac{3\sqrt{2}}{4}$ **(B)** $\frac{5\sqrt{2}}{4}$

(C) $\frac{7\sqrt{2}}{4}$ **(D)** $\frac{\sqrt{2}}{4}$

15. If the normals from any point to the parabola $x^2 = 4y$ cuts the line $y = 2$ in points whose abscissas are in AP, then the slopes of the tangents at the three co-normal points are in

(A) AP **(B)** GP

(C) HP **(D)** None of these

16. If the normals at three points. P, Q, R of the parabola $y^2 = 4ax$ meet in a point O and S be its focus, then $|SP|\cdot|SQ|\cdot|SR|$ is equal to

(A) a^2 **(B)** $a(SO)^3$

(C) $a(SO)^2$ **(D)** None of these

Matrix Match Type Questions

17. Match the following:

List I	List II
(A) Area of a triangle formed by the tangents drawn from a point (–2, 2) to the parabola $y^2 = 4(x + y)$ and their corresponding chord of contact is	**(p)** 8
(B) Length of the latus rectum of the conic $25\{(x - 2)^2 + (y - 3)^2\} = (3x + 4y - 6)^2$ is	**(q)** $4\sqrt{3}$
(C) If focal distance of a point on the parabola $y = x^2 - 4$ is 25/4 and points are of the form $(\pm\sqrt{a}, b)$ then value of $a + b$ is	**(r)** 4
(D) Length of side of an equilateral triangle inscribed in a parabola $y^2 - 2x - 2y - 3 = 0$ whose one angular point is vertex of the parabola, is	**(s)** $\frac{12}{5}$
	(t) $\frac{24}{5}$

18. Match the following:

List I	List II
(A) Radius of the largest circle which passes through the focus of the parabola $y^2 = 4x$ and contained in it, is	**(p)** 16
(B) Two perpendicular tangents PA and PB are drawn to the parabola $y^2 = 16x$ then minimum value of AB is	**(q)** 5
(C) The shortest distance between parabolas $y^2= 4x$ and $y^2 = 2x - 6$ is d then $d^2 =$	**(r)** 8
(D) The harmonic mean of the segments of a focal chord of the parabola $y^2 = 8x$	**(s)** 4
	(t) 12

19. Match the following:

List I	List II
(A) Parabola $y^2 = 4x$ and the circle having its centre at (6, 5) intersects at right angle, at the point (a, a) then one value of a is equal to	**(p)** 13
(B) The angle between the tangents drawn to $(y - 2)^2 = 4(x + 3)$ at the points where it is intersected by the line $3x - y + 8 = 0$ is $\frac{4\pi}{p}$, then p has the value equal to	**(q)** 8
(C) If the line $x - 1 = 0$ is the directrix of the parabola $y^2 - kx + 8 = 0$, then one of the value of k is	**(r)** $10\sqrt{5}$
(D) Length of the normal chord of the parabola $y^2 = 8x$ at the point where abscissa and ordinate are equal is	**(s)** 4
	(t) 12

Integer Type Questions

20. A (0, 2), B and C are points on parabola $y^2 = x + 4$ and such that $\angle CBA = \frac{\pi}{2}$, then find the least positive value of ordinate of C.

21. The chord of the parabola $y^2 = 4ax$, whose equation is $y - x\sqrt{2} + 4a\sqrt{2} = 0$, is a normal to the curve, and its length is $\lambda\sqrt{3}a$, then find λ.

22. The two parabolas $y^2 = 4ax$ and $y^2 = 4(a - 1)(x - b)$ cannot have common normal other than axis unless $b > \lambda$, then find λ.

23. From a point A common tangents are drawn to the circle $x^2 + y^2 = a^2/2$ and the parabola $y^2 = 4ax$. Find the area of the quadrilateral formed by the common tangents, the chords of contact of the point A, w.r.t. the circle and the parabola is $\frac{\lambda a^2}{4}$, then find λ.

Answer Key

Practice Exercise 1

1. (C)	**2.** (A)	**3.** (C)	**4.** (B)	**5.** (B), (C)	**6.** (A)
7. (B)	**8.** (B)	**9.** (B)	**10.** (B)	**11.** (D)	**12.** (C)
13. (C)	**14.** (C)	**15.** (D)	**16.** (C)	**17.** (A)	**18.** (C)
19. (D)	**20.** (A)	**21.** (D)	**22.** (D)	**23.** (D)	**24.** (A)
25. (D)	**26.** (C)	**27.** (A)	**28.** (A)	**29.** (A)	**30.** (D)
31. (B)	**32.** (C)	**33.** (A)	**34.** (A)	**35.** (A)	**36.** (D)
37. (C)	**38.** (B)	**39.** (B)	**40.** (C)	**41.** (A)	**42.** (D)

43. (A)	**44.** (B)	**45.** (D)	**46.** (B)	**47.** (C)	**48.** (A)
49. (C)	**50.** (A)	**51.** (A)	**52.** (C)	**53.** (B)	**54.** (B)
55. (B)	**56.** (A)	**57.** (C)	**58.** (C)	**59.** (C)	**60.** (D)
61. (D)	**62.** (C)	**63.** (A)	**64.** (D)	**65.** (B)	**66.** (D)
67. (A)	**68.** (A)	**69.** (A)	**70.** (A)	**71.** (A)	**72.** (D)
73. (A)	**74.** (D)	**75.** (B)	**76.** (C)	**77.** (A)	**78.** (C)
79. (B)	**80.** (A)	**81.** (B)	**82.** (A)	**83.** (D)	**84.** (A)
85. (A)	**86.** (A)	**87.** (C)	**88.** (A)	**89.** (C)	**90.** (D)
91. (C)	**92.** (B)	**93.** (A)	**94.** (A)	**95.** (B)	**96.** (C)
97. (B)	**98.** (B)	**99.** (C)	**100.** (D)	**101.** (A)	**102.** (B)
103. (C)	**104.** (D)	**105.** (B)	**106.** (D)	**107.** (C), (D)	**108.** (B)
109. (C)	**110.** (A)	**111.** (D)	**112.** (A)	**113.** (C)	**114.** (B)
115. (D)	**116.** (A), (B)	**117.** (B)	**118.** (B)	**119.** (B)	**120.** (B)
121. (D)	**122.** (A)	**123.** (C)	**124.** (B)	**125.** (C)	**126.** (D)
127. (D)	**128.** (B)	**129.** (C)	**130.** (C)	**131.** (A)	**132.** (C)
133. (A)	**134.** (D)	**135.** (A)	**136.** (C)	**137.** (D)	**138.** (D)
139. (D)	**140.** (A)	**141.** (D)	**142.** (A)	**143.** (B)	**144.** (A)
145. (A)	**146.** (D)	**147.** (C)	**148.** (B)	**149.** (C)	**150.** (D)
151. (B)	**152.** (C)	**153.** (B)	**154.** (C)	**155.** (C)	**156.** (D)
157. (C)	**158.** (C)	**159.** (D)	**160.** (C)	**161.** (C)	**162.** (C)
163. (C)	**164.** (C)	**165.** (B)	**166.** (D)		

Practice Exercise 2

1. (A), (B), (C) **2.** (A), (B) **3.**(A), (C) **4.** (A), (B), (C), (D) **5.** (A), (B), (C), (D) **6.** (A)
7. (A), (B), (C), (D) **8.** (B) **9.** (A) **10.** (A) **11.** (C) **12.** (D)
13. (B) **14.** (A) **15.** (B) **16.** (C) **17.** (A) $\to$ (r), (B) $\to$ (t), (C) $\to$ (p), (D) $\to$ (q)
18. (A) $\to$ (s), (B) $\to$ (p), (C) $\to$ (q), (D) $\to$ (s) **19.** (A) $\to$ (s), (B) $\to$ (q), (C) $\to$ (s), (D) $\to$ (r) **20.** 4 **21.** 6
22. 2 **23.** $\dfrac{15a^2}{4}$

Solutions

Practice Exercise 1

1.

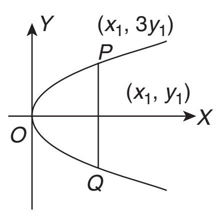


Figure 13.58

Length of double ordinate is $4at = 8a$, $t = 2$.
End point of double ordinates are $(4a, 4a)$ and $(4a, -4a)$.
Product of slope of line joining end points to vertex is equal to -1.
Hence, angle between the lines is right angle.

2. The required locus is $(3y)^2 = 4ax$. That is, $9y^2 = 4ax$.

3. We have $S \equiv (5, 0)$. Therefore, the latus rectum is $4a = 20$.

4. The distance between the focus and the directrix is

$$\left|\frac{3-4-2}{\sqrt{2}}\right| = \frac{\pm 3}{\sqrt{2}}$$

Hence, the latus rectum is $3\sqrt{2}$ (since latus rectum is two times the distance between the focus and the directrix).

5. We have

$$y^2 = 6.24 \Rightarrow y = \pm 12$$

Therefore, the points are $(24,12)$ and $(24,-12)$. Hence, the lines are

$$y = \pm\frac{12}{24}x \Rightarrow 2y = \pm x$$

6. We have $y_1 = 3x_1$. According to the given condition,

$$9x_1^2 = 36x_1$$

$$\Rightarrow x_1 = 4,0 \Rightarrow y_1 = 12,0$$

Hence, the points are (0, 0) and (4,12).

7. We have $a = 3$, which implies that the abscissa is $4 - 3 = 1$ and $y^2 = 12$, $y = \pm 2\sqrt{3}$. Hence, the points are $(1, 2\sqrt{3}), (1, -2\sqrt{3})$.

8. Let the point be (h, k). However, we have $2h = k$. Therefore

$$k^2 = 16h$$

$$\Rightarrow 4h^2 = 16h \Rightarrow h = 0, h = 4 \Rightarrow k = 0, k = 8$$

Therefore, the points are (0, 0), (4, 8). Hence, the focal distances are, respectively,

$$0 + a = 4 \text{ and } 4 + 4 = 8 \qquad (\because a = 4)$$

9. We have

$$y^2 = 4\left(\frac{1}{5}\right)x;\ a = \frac{1}{5}$$

The focus is $\left(\frac{1}{5}, 0\right)$ and the coordinates of the latus rectum are

$$y^2 = \frac{4}{25} \Rightarrow y = \pm\frac{2}{5}$$

or the end points of the latus rectum are

$$\left(\frac{1}{5}, \pm\frac{2}{5}\right)$$

10. Let the equation of parabola be $x^2 = 4ay$. However, we have

$$a = \frac{4}{-2} = -2$$

Then the equation is $x^2 = -8y$ and the latus rectum is $4a = 8$.

11. We have $a = 4$; the vertex is (0, 0); the focus is (0,–4).

12. The vertex is (2, 0). Therefore, the focus is $(2+2, 0) = (4, 0)$.

13. The point (–3,2) satisfies the equation $y^2 = 4ax$. Therefore,

$$4 = -12a$$

$$\Rightarrow 4a = -\frac{4}{3} = \frac{4}{3} \qquad \text{(Considering the positive sign)}$$

14. We have

$$x^2 = -8y \Rightarrow a = -2$$

Thus, the focus is (0,–2). The ends of the latus rectum is (4,–2),(–4,–2).

Trick: Since the ends of latus rectum lie on parabola, so only points (–4,–2)and (4,–2) satisfy the parabola.

15. It is a fundamental concept.

16. Since the axis of parabola is *y*-axis, the equation of parabola is

$$x^2 = 4ay$$

It passes through (6, – 3). Therefore,

$$36 = -12a \Rightarrow a = -3$$

Therefore, the equation of the parabola is

$$x^2 = -12y$$

17. The given equation is

$$x^2 = -8ay$$

Here, $A = 2a$. Thus, the focus of the parabola is $(0, -A)$, that is, $(0, -2a)$. The directrix is $y = A$, that is, $y = 2a$.

18. See Fig. 13.59.

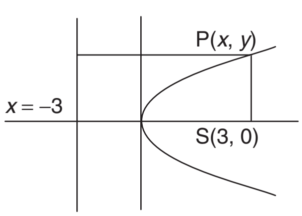


Figure 13.59

Since we have

$$SP^2 = PM^2$$

$$\Rightarrow (x-3)^2 + y^2 = \left|\frac{x+3}{\sqrt{1}}\right|^2$$

$$\Rightarrow x^2 + 9 - 6x + y^2 = x^2 + 9 + 6x$$

$$\Rightarrow y^2 = 12x$$

19. Tangents at the end points of focal chord meets at the directrix so locus of poles of focal chord is directrix.

20. Obviously, the parabola $y^2 = x$ is symmetric about *x*-axis (Fig. 13.60).

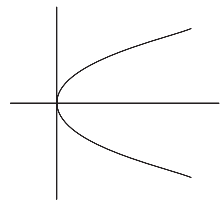


Figure 13.60

21. Let $y = 3x$. Then

$$(3x)^2 = 18x$$

$$\Rightarrow 9x^2 = 18x$$

That is, $x = 2$ and $y = 6$.

22. We have

$$a = \left|\frac{-8}{\sqrt{1+1}}\right| - \left|\frac{-12}{\sqrt{1+1}}\right| = \frac{4}{\sqrt{2}}$$

The length of the latus rectum is 4*a*, that is,

$$4 \times \frac{4}{\sqrt{2}} = 8\sqrt{2}$$

23. The given parabola can be expressed as

$$(y+1)^2 = -(x-1)$$

Hence, the vertex is (1, –1) which lies in IV quadrant.

24. We have

$$\Delta = (1)(1)(2) + 2\left(\frac{3}{2}\right)(0)(-1) - (1)(0)^2 - (1)\left(\frac{3}{2}\right)^2 - 2(-1)^2$$

$$= 2 - \frac{9}{4} - 2 < 0 \text{ and } h^2 - ab = 1 - 1 = 0$$

That is, $h^2 = ab$ is a parabola.

25. We have

$$\frac{y}{2} = t \text{ and } x - 2 = t^2$$

$$\Rightarrow (x-2) = \left(\frac{y}{2}\right)^2$$

$$\Rightarrow y^2 = 4(x-2)$$

26. We have

$$(x+2)^2 = -2(y-2)$$

The equation of the latus rectum is

$$y - 2 = -\frac{1}{2} \Rightarrow y = \frac{3}{2}$$

27. The equation can be written as

$$(3x-1)^2 = -4(9y+2)$$

Hence, the vertex is

$$\left(\frac{1}{3}, -\frac{2}{9}\right)$$

28. Check the equation of parabola for the given points.

29. We have

$$(x+1)^2 = 4a(y+2)$$

Which passes through (3, 6). Therefore,

$$16 = 4a \cdot 8 \Rightarrow a = \frac{1}{2}$$

$$\Rightarrow (x+1)^2 = 2(y+2)$$

$$\Rightarrow x^2 + 2x - 2y - 3 = 0$$

30. The parabola is $(x-2)^2 = (3y-6)$. Hence, the axis is $x-2=0$.

31. Let any point on it be (*x*, *y*). Then from the definition of parabola, we get

$$\frac{\sqrt{(x+8)^2+(y+2)^2}}{\left|(2x-y-9)/\sqrt{5}\right|} = 1$$

On squaring this and simplifying further, we get

$$x^2 + 4y^2 + 4xy + 116x + 2y + 259 = 0$$

32. The directrix is

$$x + 5 = 0$$

The focus is (–3, 0). Therefore,

$$2a = (-5+3) = 2 \Rightarrow a = 1$$

The vertex is

$$\left(\frac{-3+(-5)}{2}, 0\right) = (-4, 0)$$

Therefore, the equation is

$$(y-0)^2 = 4(x+4)$$

33. The equation is of the form $y^2 = 4A(x-a)$, where $A = (a' - a)$ or $y^2 = 4(a'-a)(x-a)$.

34. We have

$$(y-2)^2 = -4x + 4 \Rightarrow (y-2)^2 = -4(x-1)$$

The vertex is (1, 2) and the focus is (0, 2).

35. We have

$$(x+2)^2 = -2y + 7 + 4 \Rightarrow (x+2)^2 = -2\left(y - \frac{11}{2}\right)$$

Hence, the vertex is

$$\left(-2, \frac{11}{2}\right)$$

36. **Trick:** There is no constant term in a curve which passes through (0, 0). Therefore, none is correct.

37. We have

$$(y+B)^2 = -2Ax - C + B^2 = -2A\left(x + \frac{C}{2A} - \frac{B^2}{2A}\right)$$

The equation of latus rectum is

$$x + \lambda = 0$$

The vertex is

$$\left(\frac{-C+B^2}{2A}, B\right)$$

and the focus is

$$\left(\frac{-C+B^2}{2A} - \frac{A}{2}, B\right)$$

The equation of latus rectum is

$$x = \frac{-C+B^2}{2A} - \frac{A}{2} = \frac{B^2 - A^2 - C}{2A}$$

38. The parametric equations of $y^2 = 4ax$ are $x = at^2$ and $y = 2at$. Hence, if the equation is

$$y^2 = 8x$$

Then, the parametric equations are

$$x = 2t^2,\ y = 4t$$

39. On eliminating *t*, we get

$$16x^2 = 4y \Rightarrow x^2 = \frac{1}{4}y$$

which is a parabola.

40. The vertex is (0,4) and the focus is (0, 2). Therefore, $a = 2$.

Hence, parabola is $(x-0)^2 = -4\cdot 2(y-4)$ i.e., $x^2 + 8y = 32$.

41. We know that $\Delta \neq 0$, $h^2 = ab$. Therefore, it is a parabola.

42. We have

$$9x^2 - 6x + 19 = -36y$$

$$\Rightarrow (3x-1)^2 = -36y - 18 = -36\left(y + \frac{1}{2}\right)$$

$$\Rightarrow 9\left(x - \frac{1}{3}\right)^2 = -36\left(y + \frac{1}{2}\right)$$

Hence, length of latus rectum is 4.

43. We know that

$$9y^2 - 16x - 12y - 57 = 0$$

$$\Rightarrow \left(y - \frac{2}{3}\right)^2 = \frac{16}{9}\left(x + \frac{61}{16}\right)$$

Substituting $y - \frac{2}{3} = Y$ and $x + \frac{61}{16} = X$, we get

$$Y^2 = 4\left(\frac{4}{9}\right)X$$

The axis of this parabola is

$$Y = 0 \Rightarrow y - \frac{2}{3} = 0 \Rightarrow 3y = 2$$

44. The equation of the parabola referred to its vertex as the origin is $X^2 = lY$, where $x = X + a$, $y = Y + b$. Therefore, the equation of the parabola referred to the point (*a*, *b*) as the vertex is

$$(x-a)^2 = l(y-b) \text{ or } (x-a)^2 = \frac{l}{2}(2y-2b)$$

45. The given equation can be written as $(x-4)^2 = y - (c-16)$. Therefore, the vertex of the parabola is (4, $c-16$). The point lies on *x*-axis. Therefore,

$$c - 16 = 0 \Rightarrow c = 16$$

46. On eliminating t from $x = t^2 + 1$, $y = 2t$, we obtain

$$y^2 = 4x - 4$$

Similarly, on eliminating s from $x = 2s$, $y = \frac{2}{s}$, we get

$$xy = 4$$

Hence, the point of intersection is (2, 2).

47. We have

$$y^2 - 4y + 4 = 5x + 5 \Rightarrow (y-2)^2 = 5(x+1)$$

Obviously, the latus rectum is 5.

48. As we know,

$$(h-a)^2 + k^2 = h^2 \Rightarrow -2ah + a^2 + k^2 = 0$$

On replacing (h, k) by (x, y), then $y^2 - 2ax + a^2 = 0$ is the required locus.

49. The parabola is

$$x^2 - 2x = 2y$$

or $$x^2 - 2x + 1 = 2y + 1 \Rightarrow (x-1)^2 = 2\left(y + \frac{1}{2}\right)$$

Here,

$$4a = 2 \Rightarrow a = \frac{1}{2}$$

Now, the focus is

$$\left(x - 1 = 0,\ y + \frac{1}{2} = \frac{1}{2}\right)$$

That is, (1,0).

50. The given equation is $y^2 - 4y - 2x - 8 = 0$. Therefore,

$$(y-2)^2 = 2(x+6)$$

Therefore, the length of the latus rectum is 2.

51. We know that

$$(x-a)^2 + (y-b)^2 = \left(\frac{bx + ay - ab}{\sqrt{a^2 + b^2}}\right)^2$$

On solving, we get

$$(ax - by)^2 - 2a^3x - 2b^3y + a^4 + a^2b^2 + b^4 = 0$$

52. We have

$$4y^2 + 2x - 20y + 17 = 0$$

Therefore,

$$2\left(y - \frac{5}{2}\right)^2 = -(x-4) \Rightarrow 4a = \frac{1}{2}$$

53. We know that the eccentricity of parabola is $e = 1$.

54. The given equation can be written as

$$(y+1)^2 = \frac{3}{2}(x+3)$$

Therefore, the vertex is $(-3, -1)$.

55. Given equation of parabola written in standard form, we get

$$4\left(y - \frac{1}{2}\right)^2 = 6(x+1) \Rightarrow \left(y - \frac{1}{2}\right)^2 = \frac{3}{2}(x+1) \Rightarrow Y^2 = \frac{3}{2}X$$

where

$$Y = y - \frac{1}{2},\ X = x + 1$$

Therefore,

$$y = Y + \frac{1}{2},\ x = X - 1 \qquad (1)$$

For the focus, we have $X = a$ and $Y = 0$. Also, since

$$4a = \frac{3}{2}$$

we get

$$a = \frac{3}{8} \Rightarrow x = \frac{3}{8} - 1 = -\frac{5}{8}$$

Now,

$$y = 0 + \frac{1}{2} = \frac{1}{2}$$

Therefore, the focus is

$$\left(-\frac{5}{8}, \frac{1}{2}\right)$$

56. It is given that the parabola is

$$x^2 + 8x + 12y + 4 = 0$$

which can be written as

$$(x+4)^2 = -12y + 12$$

$$\Rightarrow (x+4)^2 = -12(y-1)$$

Therefore, the vertex is $(-4, 1)$.

57. We know that the standard equation of a parabola is $(y-\beta)^2 = 4a(x-\alpha)$. On comparing the given equation with the standard equation, we get the vertex as $(\alpha, \beta) = (-3, 2)$ and $a = 5$. Therefore, the focus of the parabola is $(\alpha + a, \beta) = (2, 2)$.

58. The given equation of parabola is

$$x^2 - 4x - 8y + 12 = 0$$

Therefore,

$$x^2 - 4x = 8y - 12 \Rightarrow (x-2)^2 = 8(y-1)$$

Hence, the length of latus rectum is

$$4a = 8$$

59. The given equation of parabola is

$$y = 2x^2 + x \Rightarrow x^2 + \frac{x}{2} = \frac{y}{2}$$

$$\Rightarrow \left(x + \frac{1}{4}\right)^2 = \frac{y}{2} + \frac{1}{16} \Rightarrow \left(x + \frac{1}{4}\right)^2 = \frac{1}{2}\left(y + \frac{1}{8}\right)$$

which can be written as

$$X^2 = \frac{1}{2}Y \qquad (1)$$

Here, $A = 1/8$. Therefore, the focus of Eq. (1) is

$$\left(0, \frac{1}{8}\right)$$

That is,

$$X = 0,\ Y = \frac{1}{8}$$

$$\Rightarrow x + \frac{1}{4} = 0,\ y + \frac{1}{8} = \frac{1}{8}$$

$$\Rightarrow x = -\frac{1}{4},\ \ y = 0$$

That is, the focus of given parabola is $\left(-\frac{1}{4}, 0\right)$.

60. The equation of the parabola is

$$y^2 - 2y - x + 2 = 0 \Rightarrow (y-1)^2 = (x-1)$$

Let us consider $y - 1 = Y$ and $x - 1 = X$. Therefore, $Y^2 = X$, $a = 1/4$ and the focus is $(1/4, 0)$.

Therefore, the required focus is

$$\left(\frac{1}{4}+1,\ 0+1\right) = (5/4, 1)$$

61. We know that the standard equation of parabola is $(y-k)^2 = 4a(x-h)$. On comparing the given equation with the standard equation, we get $h = 1, k = 2$ and $4a = 16$ or $a = 4$. Therefore, the vertex of the parabola is $(h, k) \equiv (1, 2)$.

62. It is given that the vertex of the parabola $(h, k) \equiv (1, 1)$ and its focus $(a+h, k) \equiv (3, 1)$ or $a + h = 3$ or $a = 2$. We know that as the y-coordinates of the vertex and the focus are the same. Therefore, the axis of the parabola is parallel to x-axis. Thus, the equation of the parabola is

$$(y-k)^2 = 4a(x-h)$$

or $$(y-1)^2 = 4 \times 2(x-1)$$

or $$(y-1)^2 = 8(x-1)$$

63. See Fig. 13.61. We have

$$PM^2 = PS^2$$

$$\Rightarrow (x-5)^2 + (y-3)^2 = \left(\frac{3x-4y+1}{\sqrt{9+16}}\right)^2$$

$$\Rightarrow 25(x^2 + 25 - 10x + y^2 + 9 - 6x)$$

$$= 9x^2 + 16y^2 + 1 - 12xy + 6x - 8y - 12xy$$

$$\Rightarrow 16x^2 + 9y^2 - 256x - 142y + 24xy + 849 = 0$$

$$\Rightarrow (4x+3y)^2 - 256x - 142y + 849 = 0.$$

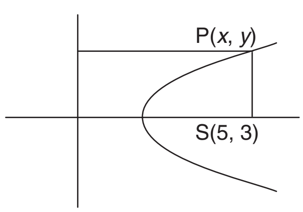


Figure 13.61

64. Point $x = 2at$, $y = at^2$ satisfies the equation of parabola so this point lies on the given parabola or locus of this point is given parabola.

65. See Fig. 13.62. We have

$$VS = \sqrt{(2-2)^2 + (-3+1)^2} = 2$$

From the equation, $(x-h)^2 = -4a(y-k)$, the parabola is

$$(x-2)^2 = -4 \cdot 2(y+1)$$

$$\Rightarrow (x-2)^2 = -8(y+1)$$

$$\Rightarrow x^2 + 4 - 4x = -8y - 8$$

$$\Rightarrow x^2 - 4x + 8y + 12 = 0$$

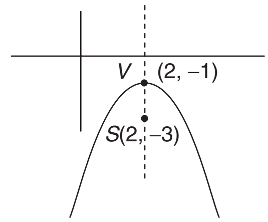


Figure 13.62

66. The equation of the parabola is

$$x^2 - 4x - 8y + 12 = 0$$

$$\Rightarrow x^2 - 4x + 4 = 8y - 8$$

$$\Rightarrow (x-2)^2 = 8(y-1)$$

$$\Rightarrow X^2 = 8Y$$

On comparing with $X^2 = 4aY$, we get $a = 2$. Therefore, the directrix is

$$Y = -a \Rightarrow y - 1 = -2 \Rightarrow y = -1$$

67. We know that

$$SP = PM \Rightarrow SP^2 = PM^2$$

$$\Rightarrow x^2 + y^2 = \left(\frac{x+y-4}{\sqrt{2}}\right)^2$$

$$\Rightarrow x^2 + y^2 - 2xy + 8x + 8y - 16 = 0$$

68. See Fig. 13.63. Here, the is vertex (0, 6) and the focus is (0, 3). Therefore, $Z \equiv (0, 9)$, that is, $y = 9$.

Therefore, the equation of the parabola is

$$SP = PM$$

$$\Rightarrow \sqrt{(x-0)^2 + (y-3)^2} = |y-9|$$

$$\Rightarrow x^2 + y^2 - 6y + 9 = y^2 - 18y + 81$$

or $$x^2 + 12y = 72$$

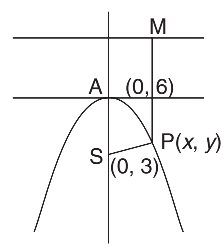


Figure 13.63

69. It is given that equation of parabola is

$$x^2 + 8y - 2x = 7$$

$$\Rightarrow x^2 - 2x + 8y - 7 = 0$$

$$\Rightarrow x^2 - 2x + 1 + 8y - 7 - 1 = 0$$

$$\Rightarrow (x-1)^2 + 8y = 8$$

$$\Rightarrow (x-1)^2 = -8(y-1) \Rightarrow (x-1)^2 = -4(2)(y-1)$$

Here, $a = 2$. Therefore, the equation of the directrix is

$$y - 1 = 2$$

That is, $y = 3$.

70. The given equation of parabola is

$$2x^2+5y-3x+4=0$$

$$\Rightarrow x^2-\frac{3}{2}x=-\frac{5}{2}y-2$$

$$\Rightarrow \left(x-\frac{3}{4}\right)^2=-\frac{5}{2}y-\frac{23}{16}$$

Therefore, the equation of the axis is

$$x-\frac{3}{4}=0 \Rightarrow x=\frac{3}{4}$$

71. The given equation of parabola is

$$x^2+6x+20y-51=0$$

$$\Rightarrow x^2+6x=-20y+51$$

$$\Rightarrow (x+3)^2=-20y+60 \Rightarrow (x+3)^2=-20(y-3)$$

$$\Rightarrow (x+3)^2=-4(5)(y-3)$$

Therefore, the axis of the parabola is $x+3=0$.

72. The point (1, 0) implies that

$$\left(\frac{dy}{dx}\right)_{(1,0)}=2-1=1$$

Hence, the tangent is

$$y-0=x-1$$

73. The required point is simply the focus of the parabola. Therefore,

$$(y+1)^2=-(4x-9)=-4\left(x-\frac{9}{4}\right)$$

Thus,

$$S\equiv\left(-1+\frac{9}{4},-1\right) \text{ or } \left(\frac{5}{4},-1\right)$$

74. Let point of contact be (h, k). Then the tangent at this point is $ky=x+h$.

$$x-ky+h=0\equiv 18x-6y+1=0$$

or

$$\frac{1}{18}=\frac{k}{6}=\frac{h}{1} \text{ or } k=\frac{1}{3}, h=\frac{1}{18}$$

75. We have

$$S_1\equiv x^2-108y=0$$

$$T\equiv xx_1-2a(y+y_1)=0 \Rightarrow xx_1-54\left(y+\frac{x_1^2}{108}\right)=0$$

$$S_2\equiv y^2-32x=0$$

$$T\equiv yy_2-2a(x+x_2)=0 \Rightarrow yy_2-16\left(x+\frac{y_2^2}{32}\right)=0$$

Therefore,

$$\frac{x_1}{16}=\frac{54}{y_2}=\frac{-x_1^2}{y_2^2}=r$$

$$\Rightarrow x_1=16r \text{ and } y_2=\frac{54}{r}$$

Therefore,

$$\frac{-(16r)^2}{(54/r)^2}=r \Rightarrow r=-\frac{9}{4}$$

$$x_1=-36, y_2=-24, y_1=\frac{(36)^2}{108}=12, x_2=18$$

Therefore, the equation of common tangent is

$$(y-12)=\frac{-36}{54}(x+36)\Rightarrow 2x+3y+36=0$$

Alternate solution: On using direct formula of common tangent $yb^{1/3}+xa^{1/3}+(ab)^{2/3}=0$, where $a=8$ and $b=27$. Hence, the required tangent is

$$3y+2x+36=0$$

76. We have

$$y=-\frac{l}{m}x-\frac{n}{m}$$

The condition for the above line to be tangent to $y^2=4ax$ is

$$-\frac{n}{m}=\frac{am}{-l} \text{ or } nl=am^2$$

77. We have

$$x\cos\alpha+y\sin\alpha-p=0 \quad (1)$$

$$2ax-yy_1+2a(x_1+2a)=0 \quad (2)$$

From Eqs. (1) and (2), we get

$$\frac{\cos\alpha}{2a}=\frac{\sin\alpha}{-y}=\frac{-p}{2a(x+2a)}$$

$$\Rightarrow y=-2\tan\alpha \text{ and } x=-p\sec\alpha-2a$$

Therefore,

$$y^2=4a(x+a)$$

$$\Rightarrow 4a^2\tan^2\alpha=-4a(p\sec\alpha+a)$$

$$\Rightarrow p\cos\alpha+a=0$$

78. We have $m=\tan\theta$. The tangent to $y^2=4ax$ is $y=x\tan\theta+c$. Hence,

$$c=\frac{a}{\tan\theta}=a\cot\theta$$

Therefore, the equation of tangent is

$$y=x\tan\theta+a\cot\theta$$

79. The equation of parabola is

$$Y^2=4X$$

where

$$X=x+\frac{5}{4}$$

The tangent parallel to $Y=2X+7$ is

$$Y=2X+\frac{a}{m}$$

$$\Rightarrow y=2\left(x+\frac{5}{4}\right)+\frac{1}{2}\Rightarrow y=2x+3$$

That is,

$$2x-y+3=0$$

80. We have

$$m=\tan\theta=\tan 60^\circ=\sqrt{3}$$

The equation of the tangent at (h, k) to $y^2=4ax$ is $yk=2a(x+h)$. On comparing, we get

$$m=\sqrt{3}=\frac{2a}{k} \text{ or } k=\frac{2a}{\sqrt{3}} \text{ and } h=\frac{a}{3}$$

81. We know that $y = 2x + \lambda$ does not meet if

$$\lambda > \frac{a}{m} = \frac{1}{2.2} = \frac{1}{4} \Rightarrow \lambda > \frac{1}{4}$$

82. Any point on $y^2 = 4ax$ is $(at^2, 2at)$. Therefore, the tangent is

$$2aty = 2a(x + t^2) \Rightarrow yt = x + at^2$$

83. According to the condition, $c = \frac{4}{2} = 2$.

84. From the condition for tangent to a parabola, we get

$$1 = \frac{1}{m} \Rightarrow m = 1$$

85. The tangent at (16, 8) to both are

$$8y = 2(x + 16) \quad (1)$$

and

$$16x = 16(y + 8) \quad (2)$$

Therefore,

$$m_1 = \frac{1}{4}, m_2 = 1$$

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1} = \left(\frac{3}{5}\right) \Rightarrow \theta = \tan^{-1}\left(\frac{3}{5}\right)$$

Alternate solution: Using direct formula, we get

$$\theta = \tan^{-1}\left(\frac{3a^{1/3}b^{1/3}}{2(a^{2/3} + b^{2/3})}\right)$$

where $a = 1$ and $b = 8$. Therefore,

$$\tan^{-1}\left(\frac{6}{2(1+4)}\right) = \tan^{-1}\left(\frac{3}{5}\right)$$

86. The tangent to the parabola is

$$y = mx + \frac{a}{m} \quad (1)$$

A line perpendicular to the tangent and passing from focus $(a, 0)$ is

$$y = -\frac{x}{m} + \frac{a}{m} \quad (2)$$

On solving both lines [Eqs. (1) and (2)], we get

$$x = 0$$

87. Now, m of tangent is $= -1$. Also from the equation of parabola, we get the gradient at (h, k) as the slope of parabola, that is,

$$\frac{dy}{dx} = \frac{-1}{2y - 1} = \frac{-1}{2k - 1}$$

Since the line and the parabola touch at (h, k), we get

$$\frac{-1}{2k-1} = -1 \Rightarrow -2k + 1 = -1 \Rightarrow k = 1$$

Substituting this value in $x + y = 1$, we have $h = 0$. Thus, the point of contact is $(0, 1)$.

88. The tangent at $y^2 = 4ax$ is

$$y = mx + \frac{a}{m}$$

Therefore, the tangent at $y^2 = 4a(x + a)$ is

$$y = m(x + a) + \frac{a}{m}$$

or

$$y = mx + ma + \frac{a}{m} \Rightarrow ma + \frac{a}{m} = c$$

89. We have

$$m_1 = \tan 45° = 1,\ m_2 = 3$$

The slope of the tangent is

$$\frac{3 \pm 1}{1 \mp 3} = -2 \text{ or } \frac{1}{2}$$

The tangent is

$$y = -2x + \frac{2}{-2} \text{ or } 2x + y + 1 = 0$$

90. The end points are $(a, \pm 2a)$. Therefore, the tangents are

$$\pm 2ay = 2a(x + a) \text{ or } m = \pm\frac{2a}{2a} = \pm 1$$

Hence, the angle between them is $\pi/2$.

91. We know that

$$x^2 = 4a(mx + c) \Rightarrow x^2 - 4amx - 4ac = 0$$

It is concluded that it touches. Then

$$B^2 - 4AC = 0$$

$$\Rightarrow 16a^2m^2 + 16ac \Rightarrow ac = -a^2m^2 \Rightarrow c = -am^2$$

92. It is a fundamental property.

93. Any line through the origin is $y = mx$. Since it is a tangent to $y^2 = 4a(x - a)$, it cuts it off in two coincident points. Therefore, the roots of $m^2x^2 - 4ax + 4a^2 = 0$ are equal. Therefore,

$$16a^2 - 16a^2m^2 = 0 \text{ or } m^2 = 1 \text{ or } m = 1, -1$$

The product of slopes is -1. Hence, it is a right-angled triangle.

94. If we replace x by y and y by x, then the line is $y = mx + k$ and the parabola is $y^2 = 4ay$. Hence,

$$k = \frac{a}{m}$$

Alternate solution: If $x = my + k$ touches $x^2 = 4ay$, then the quadratic $(my + k)^2 = 4ay$ has two real and equal roots, that is, $B^2 - 4AC = 0$, which gives

$$k = \frac{a}{m}$$

95. Let the coordinates of P and Q be $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$, respectively. Then $y_1 = 2at_1$ and $y_2 = 2at_2$. The coordinates of the point of intersection of the tangents at Pand Q are $\{at_1t_2, a(t_1 + t_2)\}$. Therefore,

$$y_3 = a(t_1 + t_2)$$

$$\Rightarrow y_3 = \frac{y_1 + y_2}{2}$$

which implies that y_1, y_3 and y_2 are in AP.

96. Solving $x^2 = 4y$ and $y^2 = 4x$, we get $x = 0, y = 0$ and $x = 4, y = 4$. Therefore, the coordinates of P are (4, 4). The equations of the tangents to the two parabolas at (4, 4) are

$$2x - y - 4 = 0 \quad (1)$$

$$x - 2y + 4 = 0 \quad (2)$$

Now, m_1 = Slope of Eq. (1) = 2 and m_2 = Slope of Eq. (2) = $\frac{1}{2}$.

Therefore, $m_1m_2 = 1$, that is,

$$\tan\theta_1 \tan\theta_2 = 1$$

97. Due to the reason that $c = a/m$, we get $c = 1/2$.

98. It is a formula.

99. Since the parabola passes through the point (1, –2), we get

$$4 = 4a \Rightarrow a = 1$$

The formula for tangent is

$$yy_1 = 2a(x + x_1) \Rightarrow -2y = 2(x+1)$$

The required tangent is

$$x + y + 1 = 0$$

100. The line perpendicular to the given line is

$$3y + x = \lambda$$

$$y = \frac{-1}{3}x + \frac{\lambda}{3}$$

Here,

$$m = \frac{-1}{3}, c = \frac{\lambda}{3}$$

If we compare $y^2 = 16x$ with $y^2 = 4ax$, then $a = 4$. The condition for tangency is

$$c = \frac{a}{m} \Rightarrow \frac{\lambda}{3} = \frac{4}{(-1/3)} \Rightarrow \lambda = -36$$

The required equation is

$$x + 3y + 36 = 0$$

101. The equation of the tangent to the parabola $y^2 = 4ax$ is

$$yy_1 = 2a(x + x_1) \Rightarrow y\left(\frac{2a}{t}\right) = 2a\left(x + \frac{a}{t^2}\right)$$

$$\Rightarrow \frac{y}{t} = \left(x + \frac{a}{t^2}\right) \Rightarrow \frac{y}{t} = \frac{t^2x + a}{t^2} \Rightarrow ty = t^2x + a$$

102. We know that $y^2 = 8x$. Therefore,

$$4a = 8 \Rightarrow a = 2$$

Any tangent of parabola is given by

$$y = mx + \frac{a}{m} \text{ or } mx - y + \frac{2}{m} = 0$$

If it is a tangent to the circle $x^2 + y^2 = 2$, then the perpendicular that is drawn from the centre (0,0) is equal to the radius $\sqrt{2}$. Therefore,

$$\frac{2/m}{\sqrt{m^2+1}} = \sqrt{2} \text{ or } \frac{4}{m^2} = 2(m^2+1)$$

$$\Rightarrow m^4 + m^2 - 2 = 0 \Rightarrow (m^2+2)(m^2-1) = 0 \text{ or } m = \pm 1$$

Hence, the common tangents are

$$y = \pm(x+2)$$

Therefore,

$$y = x + 2$$

103. The equation of tangent to parabola

$$ty = x + at^2 \quad (1)$$

Clearly, $lx + my + n = 0$ is also a chord of contact of the tangents. Therefore, $ty = x + at^2$ and $lx + my + n = 0$ represents the same line. Hence,

$$\frac{1}{l} = -\frac{t}{m} = \frac{at^2}{n} \Rightarrow t = \frac{-m}{l}, t^2 = \frac{n}{la}$$

On eliminating t, we get

$$m^2 = \frac{nl}{a}$$

which is an equation of parabola.

104. The intersection of $x - y + 2 = 0$ with $y^2 = 8x$ is given by

$$(x+2)^2 = 8x \Rightarrow x^2 + 4 - 4x = 0 \Rightarrow (x-2)^2 = 0$$

Therefore,

$$x = 2 \text{ and } y = x + 2 = 4$$

105. The equation of the tangent at point (a, 2a) of the parabola $y^2 = 4ax$ is

$$yy_1 = 2a(x + x_1)$$

$$\Rightarrow 2ay = 2a(x+a) \Rightarrow y = x + a$$

This line makes an angle of $\pi/4$ with the x-axis, as

$$m = \tan\theta = 1$$

106. It is given that

$$lx + my + n + 0 \quad (1)$$

$$x^2 = y \quad (2)$$

The point of intersection of the line and the parabola are obtained by solving Eqs. (1) and (2) simultaneously by substituting the values of x from Eq. (1) in Eq. (2), we get

$$\left(\frac{my+n}{l}\right)^2 = y \Rightarrow m^2y^2 + n^2 + 2mny = yl^2$$

$$\Rightarrow m^2y^2 + (2mn - l^2)y + n^2 = 0 \quad (3)$$

If the lines expressed in Eq. (3) touches the parabola expressed in Eq. (2), then the discriminant is 0.

$$(2mn - l^2)^2 = 4m^2n^2$$

$$\Rightarrow 4m^2n^2 + l^4 - 4mnl^2 = 4m^2n^2 \Rightarrow l^2 = 4mn$$

107. It is given that $y^2 = 9x$. Here, $a = 9/4$. Now, the equation of tangent to the parabola $y^2 = 9x$ is

$$y = mx + \frac{9/4}{m}$$

If this tangent goes through the point (4,10), then

$$10 = 4m + \frac{9}{4m} \Rightarrow (4m-9)(4m-1) = 0 \Rightarrow m = \frac{9}{4}, \frac{1}{4}$$

Therefore, the equation of tangents are

$$4y = x + 36$$

and $\quad y = -2x - k$

or $\quad x - 4y + 36 = 0$

and $\quad 9x - 4y + 4 = 0$

108. We know that tangent to the parabola at points t_1 and t_2 are $t_1y = x + at_1^2$ and $t_2y = x + at_2^2$. Since the tangents are perpendicular to the parabola, we get

$$\frac{1}{t_1}\cdot\frac{1}{t_2} = -1 \text{ or } t_1t_2 = -1$$

We also know that their point of intersection is

$$[at_1t_2, a(t_1+t_2)] = [-a, a(t_1+t_2)]$$

Thus, these points lie on the directrix $x = -a$ or $x + a = 0$.

109. See Fig. 13.64. Any tangent to $y^2 = 4x$ is $y = mx + (1/m)$. It touches the circle if

$$3 = \left|\frac{3m + (1/m)}{\sqrt{1+m^2}}\right|$$

or $$9(1+m^2) = \left(3m + \frac{1}{m}\right)^2$$

or $$\frac{1}{m^2} = 3$$

Therefore,

$$m = \pm\frac{1}{\sqrt{3}}$$

For the common tangent to be above the x-axis, we have

$$m = \frac{1}{\sqrt{3}}$$

Therefore, the common tangent is

$$y = \frac{1}{\sqrt{3}}x + \sqrt{3} \Rightarrow \sqrt{3}y = x + 3$$

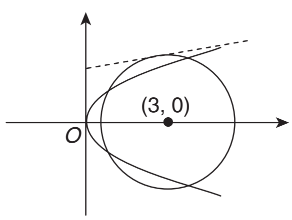


Figure 13.64

110. If $y = mx + c$ is tangent to the given parabola, then $c = a/m$. So, now, equation of tangent will be $y = mx + a/m$.
Let point $p(x_1, y_1)$ lies on the parabola and equation of tangent at p point is $yy_1 = 2a(x + x_1)$.
Lines $y = mx + a/m$ and $yy_1 = 2a(x + x_1)$ are identical line.
Now,

$$yy_1 = 2a(x + x_1)$$

Therefore,

$$\frac{y_1}{1} = \frac{2a}{m} = \frac{2ax_1}{a/m} \Rightarrow (x_1, y_1) = \left(\frac{a}{m^2}, \frac{2a}{m}\right)$$

111. See Fig. 13.65. Here, P$(at^2, 2at)$ and S$(a, 0)$. If the tangent at P, $ty = x + at^2$, meets the directrix $x = -a$ at k, then

$$k = \left(-a, \frac{at^2 - a}{t}\right)$$

Now,

$$m_1 = \text{slope of SP} = \frac{2at}{a(t^2-1)}$$

$$m_2 = \text{slope of SK} = \frac{a(t^2-1)}{-2at}$$

Therefore,

$$m_1m_2 = -1$$

Hence, $\angle PSK = 90°$.

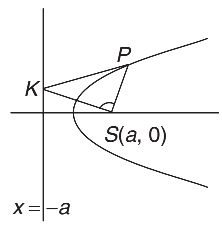


Figure 13.65

112. Tangents at t_1 and t_2 meet at $[at_1t_2, a(t_1+t_2)]$.

113. The point of intersection is $(0, -1)$. Therefore,

$$\frac{dy}{dx} = \frac{2x}{4} \text{ and } \frac{-2x}{4}$$

Therefore,

$$m_1 = 0, m_2 = 0 \Rightarrow \theta = 0°$$

114. The principal axes of the parabolas are x-axis and y-axis. Therefore, the angle between them is 90°.

115. The parabola is $y^2 = ax$. That is,

$$y^2 = 4\left(\frac{a}{4}\right)x \quad (1)$$

Let the point of contact be (x_1, y_1). Therefore, the equation of tangent is

$$y - y_1 = \frac{2(a/4)}{y_1}(x - x_1)$$

$$\Rightarrow y = \frac{a}{2y_1}(x) - \frac{ax_1}{2y_1} + y_1$$

Now,

$$m = \frac{a}{2y_1} = \tan 45° \Rightarrow \frac{a}{2y_1} = 1 \Rightarrow y_1 = \frac{a}{2}$$

From Eq. (1), we get

$$x_1 = \frac{a}{4}$$

Therefore, the point is $\left(\frac{a}{4}, \frac{a}{2}\right)$.

116. It is a fundamental concept.

117. See Fig. 13.66. The equation of the tangent at (x_1, y_1) on the parabola $y^2 = 4ax$ is

$$yy_1 = 2a(x + x_1)$$

Therefore, in this case, $a=1$. The coordinates at the ends of the latus rectum of the parabola $y^2=4x$ are $L(1, 2)$ and $L_1(1,-2)$. The equation of tangent at L and L_1 are $2y=2(x+1)$ and $-2y=2(x+1)$, which gives $x=-1, y=0$. Thus, the required point of intersection is (–1, 0).

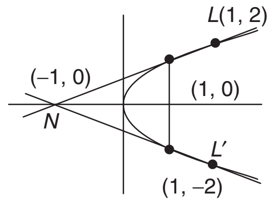


Figure 13.66

118. Any tangent to $y^2=4x$ is

$$y=mx+\frac{1}{m}$$

Since it passes through (1, 4), we have

$$4=m+\frac{1}{m}$$

$$\Rightarrow m^2-4m+1=0$$

$$\Rightarrow m_1+m_2=4,\ m_1m_2=1$$

$$\Rightarrow |m_1-m_2|=2\sqrt{3}$$

If θ is the required angle, then

$$\tan\theta=\frac{2\sqrt{3}}{1+1}=\sqrt{3}$$

$$\Rightarrow \theta=\frac{\pi}{3}$$

119. Any line through origin (0, 0) is $y=mx$. It intersects $y^2=4ax$ at the point $\left(\frac{4a}{m^2},\frac{4a}{m}\right)$. The mid-point of the chord is $\left(\frac{2a}{m^2},\frac{2a}{m}\right)$.

Therefore,

$$x=\frac{2a}{m^2}, y=\frac{2a}{m}\Rightarrow \frac{2a}{x}=\frac{4a^2}{y^2}$$

or $y^2=2ax$ which is a parabola.

120. Let the point be (h, k). The normal is

$$y-k=\frac{-k}{4}(x-h) \text{ or } -kx-4y+kh+4k=0$$

The gradient is

$$-\frac{k}{4}=\frac{1}{2}\Rightarrow k=-2$$

Substituting (h,k) and $k=-2$, we get $h = 1/2$. Hence, the point is $\left(\frac{1}{2},-2\right)$.

Trick: Here, the only point $\left(\frac{1}{2},-2\right)$ satisfies the parabola $y^2=8x$.

121. General equation of normal to the parabola $y^2=4ax$ is given by $y=mx-2am-am^3$.

This normal is passing through any point $p(h, k)$. Then, the cubic equation

$$k=mh-2am-am^3$$

can have a maximum of three real roots. So, from any given point, we can draw a maximum of three normals to any parabola.

122. Normal at (h,k) to the parabola $y^2=8x$ is

$$y-k=-\frac{k}{4}(x-h)$$

The gradient is

$$\tan 60°=\sqrt{3}=-\frac{k}{4}\Rightarrow k=-4\sqrt{3} \text{ and } h=6$$

Hence, the required point is $(6,-4\sqrt{3})$.

123. The normal is

$$y-2at=\frac{-2at}{2a}(x-at^2)$$

Therefore, the slope is $-t$.

124. We have

$$y-a=-\frac{a}{2a}\left(x-\frac{a}{4}\right)$$

$$\Rightarrow 2y+x=2a+\frac{a}{4}=\frac{9a}{4}\Rightarrow 2y+x-\frac{9a}{4}=0$$

That is,

$$4x+8y-9a=0$$

125. We have

$$y-\frac{2a}{m}=-\frac{2a/m}{2a}\left(x-\frac{a}{m^2}\right)$$

$$\Rightarrow y-\frac{2a}{m}=\frac{-1}{m}\left(x-\frac{a}{m^2}\right)$$

$$\Rightarrow m^3y+m^2x-2am^2-a=0$$

126. Here, $y=-2x-k$ is normal to $y^2=-8x$ or

$$-k=-\{-4(-2)-2(-2)^3\}=-(8+16)\Rightarrow k=24$$

127. We know that

$$t_2=-t_1-\frac{2}{t_1}$$

Substituting $t_1=1$ and $t_2=t$. Hence, $t=-3$.

128. The vertex is (0, 0). The end points of latus rectum are $(a, \pm 2a)$. Here $a=6/4$. Therefore, the negative end of the latus rectum is $\left(\frac{3}{2},-3\right)$. The line through the point is

$$y=\frac{-3}{3/2}x \text{ or } y+2x=0$$

129. The equation of chord of contact of the tangent drawn from a point (x_1, y_1) to the parabola $y^2=4ax$ is $yy_1=2a(x+x_1)$ so that

$$5y=2\times 2(x+2)\Rightarrow 5y=4x+8$$

The point of intersection of chord of contact with parabola $y^2 = 8x$ are $\left(\frac{1}{2}, 2\right), (8, 8)$, so that the length is $\frac{3}{2}\sqrt{41}$.

130. The semi-latus rectum is harmonic mean between segments of focal chords of a parabola. Therefore,

$$b = \frac{2ac}{a+c} \Rightarrow a, b \text{ and } c \text{ are in HP.}$$

131. The combined equation of the lines joining the vertex to the points of intersection of the line $lx + my + n = 0$ and the parabola $y^2 = 4ax$ is

$$y^2 = 4ax\left(\frac{lx+my}{-n}\right) \text{ or } 4alx^2 + 4amxy + ny^2 = 0$$

This represents a pair of perpendicular lines if $4al + n = 0$.

132. Let $y = mx + c$ be the chord and c is the variable. Then

$$x = \left(\frac{y-c}{m}\right) \qquad (\text{by } y^2 = 4ax)$$

For getting the points of intersection, we have

$$y^2 = 4a\left(\frac{y-c}{m}\right) \Rightarrow y^2 - \frac{4ay}{m} + \frac{4ac}{m} = 0$$

$$\Rightarrow y_1 + y_2 = \frac{4a}{m} \Rightarrow \frac{y_1 + y_2}{2} = \frac{2a}{m}$$

which is a constant and independent of c.

133. The coordinates of the ends of the latus rectum of the parabola $y^2 = 4ax$ are $(a, 2a)$ and $(a, -2a)$, respectively. The equation of the normal at $(a, 2a)$ to $y^2 = 4ax$ is

$$y - 2a = -\frac{2a}{2a}(x-a)\left[\text{by using } y - y_1 = -\frac{y_1}{2a}(x - x_1)\right]$$

or $$x + y - 3a = 0 \qquad (1)$$

Similarly, the equation of the normal at $(a, -2a)$ is

$$x - y - 3a = 0 \qquad (2)$$

The combined equation of Eqs. (1) and (2) is

$$x^2 - y^2 - 6ax + 9a^2 = 0$$

134. As we know,

$$t_1 \times t_2 = 2 \Rightarrow 2at_1 \times 2at_2 = 8a^2$$

135. The equation of the normal to $x^2 = 4ay$ is of the form $x = my - 2am - am^3$. Therefore,

$$c = -2am - am^3$$

136. Since the semi-latus rectum of a parabola is the harmonic mean between the segments of any focal chord of a parabola. Therefore, SP, 4, SQ are in HP.

$$4 = 2\left[\frac{(\text{SP})(\text{SQ})}{\text{SP}+\text{SQ}}\right] \Rightarrow 4 = \frac{2(6)(\text{SQ})}{6+\text{SQ}} \Rightarrow \text{SQ} = 3$$

137. The equation of a normal to $y^2 = 4x$ at $(m^2, -2m)$ is $y = mx - 2m - m^3$. If the normal makes equal angles with the coordinates axes, then $m = \tan(\pi/4) = 1$. Thus, the required point is $(1, -2)$.

138. Let normal at (h, k) be $y = mx + c$. Then

$$k = mh + c \text{ and } k^2 = 4a(h-a)$$

The slope of tangent at (h, k) is m_1, then on differentiating equation of parabola.

$$2ym_1 = 4a \Rightarrow m_1 = \frac{2a}{k}$$

Also $mm_1 = -1$. Therefore,

$$m = -\frac{k}{2a}$$

Solving and replacing (h, k) by (x, y), we get

$$y = m(x-a) - 2am - am^3$$

139. The tangents (at the end points of focal chord) cut orthogonally at the directrix, that is, $x = -a$ or $x + a = 0$.

140. The normal to parabola is

$$y = mx - 2am - am^3$$

For three values of m, three normals can be drawn on parabola $y^2 = 4ax$. Hence, the three feet of the normals can be obtained. Thus, the centroid of the triangle lies on the axis of parabola.

141. The equation of circle with points (3, 6) and (27, –18) on diameter is

$$(x-3)(x-27) + (y-6)(y+18) = 0$$

$$x^2 + y^2 - 30x + 12y - 27 = 0$$

142. The normal at $P(t_1^2, 2t_1)$ on the parabola

$$y^2 = 4x \qquad (1)$$

which meets it again at the point $Q(t_2^2, 2t_2)$ where

$$t_2 = -t_1 - \frac{2}{t_1} \qquad (2)$$

If PQ subtends a right angle at the vertex (0, 0), then

$$(\text{Slope of } OP) \times (\text{Slope of } OQ) = -1$$

$$\Rightarrow \frac{2t_1}{t_1^2}\left(\frac{2t_2}{t_2^2}\right) = -1 \Rightarrow t_2 = -\frac{4}{t_1} \qquad (3)$$

From Eqs. (2) and (3), we get

$$-t_1 - \frac{2}{t_1} = -\frac{4}{t_1} \Rightarrow -t_1 = -\frac{2}{t_1}$$

$$\Rightarrow t_1^2 = 2 \Rightarrow t_1 = \pm\sqrt{2}$$

Therefore,

$$t_2 = \pm 2\sqrt{2}$$

Therefore, the points P and Q are $(2, \pm 2\sqrt{2})$ and $(8, \pm 4\sqrt{2})$. Thus,

$$PQ = \sqrt{(8-2)^2 + (\pm 4\sqrt{2} \pm 2\sqrt{2})^2} = \sqrt{36+72}$$

$$\Rightarrow \sqrt{108} = 6\sqrt{3}$$

143. Any normal is expressed as $y + tx = 6t + 3t^3$. It is identical with $x + y = k$ if

$$\frac{t}{1} = \frac{1}{1} = \frac{6t + 3t^3}{k}$$

Therefore, $t = 1$ and

$$1 = \frac{6+3}{k} \Rightarrow k = 9$$

144. It is a fundamental theorem.

145. From Fig. 13.67.

$$\theta = 45° \Rightarrow \text{Slope} = \pm 1$$

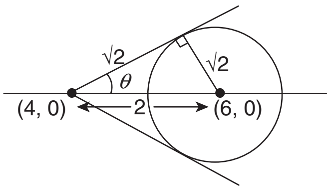


Figure 13.67

146. We have

$$t_2 = -t_1 - \frac{2}{t_1}$$

Since $a = 2$ and $t_1 = 1$, we get $t_2 = -3$. Therefore, the other end is $(at_2^2, 2at_2)$, that is, (18, – 12).

147. Tangents at the end points of focal chord meets at directrix. Hence, polar of focus of parabola is directrix.

148. The equation of the diameter of parabola is

$$y = \frac{2a}{m}$$

Here, $a = 1/4$ and $m = 1$. So,

$$y = \frac{2 \times (1/4)}{1} \Rightarrow 2y = 1$$

149. See Fig. 13.68. We have

$$\Delta = \frac{1}{2}(12 \times 3) = 18 \text{ sq. unit}$$

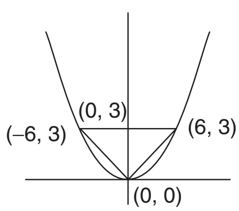


Figure 13.68

150. Let the coordinates of the vertices be $(a, 1)$, $(b, 2)$ and $(c, 4)$.

We have $a = 1/4$, $b = 1$ and $c = 4$. The area of the triangle formed by $\left(\frac{1}{4}, 1\right)$, (1, 2) and (4, 4) is

$$\frac{1}{2}\begin{vmatrix} 1/4 & 1 & 1 \\ 1 & 2 & 1 \\ 4 & 4 & 1 \end{vmatrix} = \frac{3}{4}$$

151. See Fig. 13.69. We have

$$L_1 = \sqrt{3}y - x = 0$$

On solving L_1 and $S_1 \equiv y^2 - 4ax = 0$, we get $y = 4a\sqrt{3}$ and $x = 12a$. Hence,

$$L = \sqrt{144a^2 + 48a^2} = a\sqrt{192} = 8a\sqrt{3}$$

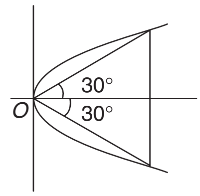


Figure 13.69

152. The points are

$$\left(\frac{y_1^2}{4a}, y_1\right), \left(\frac{y_2^2}{4a}, y_2\right) \text{ and } \left(\frac{y_3^2}{4a}, y_3\right)$$

On using area formula, we get

$$\Delta = \frac{1}{8a}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)$$

153. The chord of contact of (–1, 2) is $yy_1 = 2a(x + x_1)$ or $y = x - 1$.

154. Solving the equation with parabola $y^2 = 4x$, we get the points

$$P(3 + 2\sqrt{2}, 2 + 2\sqrt{2}) \text{ and } Q(3 - 2\sqrt{2}, 2 - 2\sqrt{2})$$

Therefore,

$$PQ^2 = 32 + 32 = 64 \Rightarrow PQ = 8$$

Also, if p is perpendicular from (–1, 2) on PQ, then the area of triangle is

$$\left(\frac{1}{2}PQ\right)p = \frac{1}{2}(8)\left(\frac{4}{\sqrt{2}}\right) = 8\sqrt{2}$$

155. Let the point $\left(x, \frac{x^2}{2}\right)$ be the nearest point on parabola from (0, 3). Therefore, $D = \sqrt{x^2 + \left(\frac{x^2}{2} - 3\right)^2}$ should be min, that is,

$$x^2 + \left(\frac{x^2}{2} - 3\right)^2 = z \text{ should be min.}$$

$$\frac{dz}{dx} = 2x + 2x\left(\frac{x^2}{2} - 3\right) = 0 \Rightarrow x = 0, 2, -2$$

$$\frac{d^2z}{dx^2} = +\text{ve for } x = \pm 2$$

Hence, D is min at $(\pm 2, 2)$.

156. The given point (–1, –60) lies on the directrix $x = -1$ of the parabola $y^2 = 4x$. Thus, the tangents are at right angle.

157. The equation of the conic is $x^2 + 10x - 16y + 25 = 0$. That is, $(x + 5)^2 = 16y$. Therefore, the conic is parabola with the focus (–5, 4) since the focus is the mid-point of the latus rectum. Hence, only the points given in option (C) can be the end points of the latus rectum.

158. The equation of the tangent at (1, 7) to $y = x^2 + 6$ is

$$\frac{1}{2}(y + 7) = x(1) + 6 \Rightarrow y = 2x + 5 \quad (1)$$

This tangent also touches the circle

$$x^2 + y^2 + 16x + 12y + c = 0 \quad (2)$$

On solving Eqs. (1) and (2), we get

$$x^2 + (2x+5)^2 + 16x + 12(2x+5) + c = 0$$

$$\Rightarrow 5x^2 + 60x + 85 + c = 0$$

Since the roots are equal, we get

$$b^2 - 4ac = 0 \Rightarrow (60)^2 - 4 \times S \times (85 + c) = 0$$

$$\Rightarrow 85 + c = 180 \Rightarrow 5x^2 + 60x + 180 = 0$$

$$\Rightarrow x = -\frac{60}{10} = -6 \Rightarrow y = -7$$

Hence, the point of contact is (–6, –7).

159. From Fig. 13.70, it is obvious that the angle between the curve is equal to the angle between x-axis and y-axis, that is $\pi/2$.

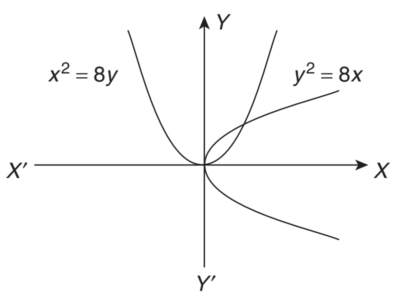


Figure 13.70

160. It is given that $m = 2$, $c = k$ and $a = 1$. Hence, $k = -4$.

161. It is given that

$$x + y = 0 \quad (1)$$

$$x^2 + y^2 + 4y = 0 \quad (2)$$

On solving Eqs. (1) and (2), we get $x = 0$; $y = 0$ and $x = 2$; $y = -2$ which means that the parabola pass through points (0, 0) and (2, –2) and these points satisfy the parabola $y^2 = 2x$.

162. See Fig. 13.71. The given parabola is $y^2 = 2ax$. Therefore, the focus (a/2, 0) and the directrix is given by $x = -a/2$ since the circle touches the directrix.

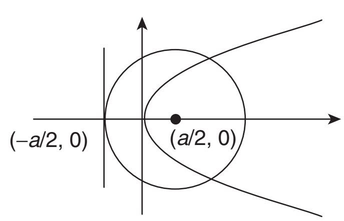


Figure 13.71

Therefore, the radius of circle is equal to the distance from the point (a/2, 0) to the line.

$$x = -\frac{a}{2} = \frac{|(a/2) + (a/2)|}{\sqrt{1}} = a$$

Therefore, the equation of circle is

$$\left(x - \frac{a}{2}\right)^2 + y^2 = a^2 \quad (1)$$

Also

$$y^2 = 2ax \quad (2)$$

On solving Eqs. (1) and (2), we get

$$x = \frac{a}{2}, -\frac{3a}{2}$$

Substituting these values in $y^2 = 2ax$, we get $y = \pm a$ and $x = -3a/2$ gives imaginary values of y. Therefore, the required points are $(a/2, \pm a)$.

163. Let the equation of the line be

$$y = ax + b$$

Therefore,

$$\frac{dy}{dx} = a = 1 \quad \text{(given)}$$

Also, $y = ax + b$ passing through (0, 1). Therefore, $b = 1$. Thus, the required line is $y = x + 1$. Now, the point of the intersection of the line and the parabola gives

$$x^2 + 2x + 1 = 4x \Rightarrow x^2 - 2x + 1 \Rightarrow x = 1$$

Therefore, $y = 2$. Hence, the line touches the parabola. Thus, the intercepted length is equal to 0.

164. See Fig. 13.72. Now, $m = \tan(120°) = -\sqrt{3}$ = Slope of the line which passes through (–1, 0).

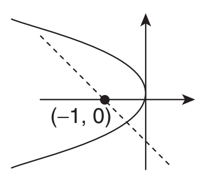


Figure 13.72

The required equation is

$$y - 0 = -\sqrt{3}(x+1)$$

$$\Rightarrow y + \sqrt{3}(x+1) = 0$$

165. We know that only two parabolas can be drawn with a given latus rectum (Fig. 13.73).

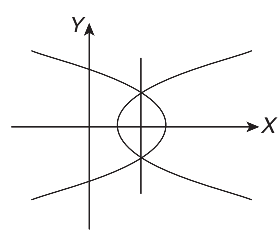


Figure 13.73

166. If the normal drawn to point $(at_1^2, 2at_1)$ of a parabola $y^2 = 4ax$ meets at point $(at_2^2, 2at_2)$ of same parabola (Fig. 13.74), then

$$t_2 = -t_1 - \frac{2}{t_1}$$

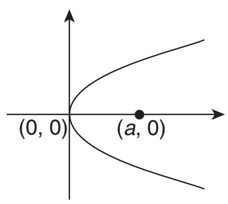


Figure 13.74

It is given that $x = y$ since the abscissa and the ordinate are equal. Therefore,

$$y^2 = 4ax \Rightarrow y^2 = 4ay \qquad \text{(on using } x = y\text{)}$$

$$\Rightarrow y^2 = 4ay = 0 \Rightarrow y(y - 4a) = 0 \Rightarrow y = 0 \text{ or } y = 4a$$

Therefore, the points are $(x = 0, y = 0)$ and $(x = 4a, y = 4a)$.

$$2at = 4a \Rightarrow t_1 = \frac{4a}{2a} = 2;\ t_2 = -2 - \frac{2}{2} = -2 - 1 = -3$$

Therefore,

$$(at_2^2, 2at_2) = [a \times (-3)^2, 2a(-3)) = (9a, -6a)$$

Practice Exercise 2

1. See Fig. 13.75. According to the given coordinates of C and coordinates of focus, we can see if we plot the diameter through C and S the other end (say Q) lies on the directrix of parabola. Hence, the circle must pass through R.

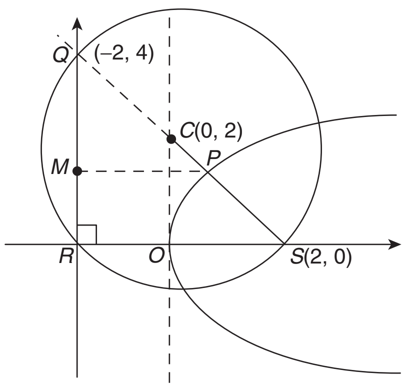


Figure 13.75

As equation of (CS) $x + y = 2$ and parabola is $y^2 = 8x$, so

x-coordinate of 'P' is $(6 - 4\sqrt{2})$

$$\Rightarrow PM = 2 + 6 - 4\sqrt{2} = 8 - 4\sqrt{2}$$

$$CP = (\text{Radius of the circle} - (SP))$$

$$= 2\sqrt{2} - (8 - 4\sqrt{2}) = (6\sqrt{2} - 8)$$

Due to geometry of circle (C) is also correct.

2. Here, $a = 2$, let p$(2t^2, 4t)$ be a point on parabola

$$SP = 4$$

$$\Rightarrow a(1 + t^2) = 4 \Rightarrow t = \pm 1$$

$$P = (2, 4) \text{ or } (2, -4)$$

$$f(x + y) = f(x) f(y);\ f(1) = 2;\ f(2) = 2^2$$

$$f(n) = 2^n;\ a_1 = 2;\ a_2 = 4$$

Therefore, $p \equiv (a_1, a_2)$ or $(a_1, -a_2)$

3. Here, $x^2 - 4x = 2y - 4$

$$x^2 - 4x + 4 = 2y - 4 + 4$$

$$(x - 2)^2 = 2y$$

Vertex of parabola (2, 0) and length of latus rectum = 2
Length of latus rectum of the required parabola = 1
Therefore, equation of the required parabola is

$$(y - 0)^2 = \pm 1(x - 2)$$

$$y^2 = x - 2 \text{ or } y^2 = 2 - x$$

4. Equation of normal to the parabola $y^2 = 4ax$ from co-normal points $p(am_1^2, -2am_1)$, $Q(am_1^2, -2am_2)$ *and* $R(am_1^2, -2am_3)$ meets at the point A (h, k).

$y = mx - 2am - am^3$ passing through point A.

$k = mh - 2am - am^3 \Rightarrow am^3 + (2a - h)m + k = 0$ having roots m_1, m_2, m_3.

Sum of roots one at a time = $(m_1 + m_2 + m_3) = 0$.

It means algebraic sum of the slopes of the normal at P, Q and R vanishes $(m_1 + m_2 + m_3) = 0$.

It means algebraic sum of the ordinates of the points P, Q and R vanishes $\left(-\sum 2am_1 = 0\right)$.

Coordinate of centroid of triangle PQR is

$$\left(\frac{\sum am_1^2}{3}, -\frac{\sum 2am_1}{3}\right) = \left(\frac{\sum am_1^2}{3}, 0\right)$$

Centroid of the triangle PQR lies on the axis of the parabola.

General equation of circle through co-normal points is given by $a^2m^4 + 2a(2a + g)m^2 - 4afm + c = 0$ having roots m_1, m_2, m_3, m_4.

Sum of roots one at a time = $m_1 + m_2 + m_3 + m_4 = 0 \Rightarrow m_4$ $0(m_1 + m_2 + m_3 = 0)$

Circle circumscribing the triangle PQR passes through the vertex of the parabola.

5. See Fig. 13.76.

$$R \equiv (1, 1)$$

$$T \equiv (2, 2)$$

Equation of RS is $4x - 3y - 1 = 0$.
Equation of TS is $3x - 2y - 2 = 0$.
Therefore, focus S $\equiv$ (4, 5)
Length of latus rectum $= 4 \times \frac{1}{\sqrt{2}} = 2\sqrt{2}$

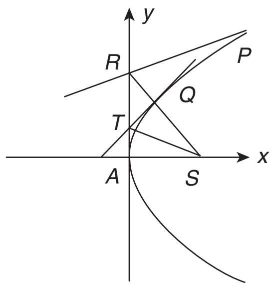


Figure 13.76

Axis is $x + y - 9 = 0$

$$\text{Vertex} \equiv \left(\frac{9}{2}, \frac{9}{2}\right)$$

6. See Fig. 13.77. Let point $A(r_1\cos\theta, r_1\sin\theta)$ lies on the parabola.

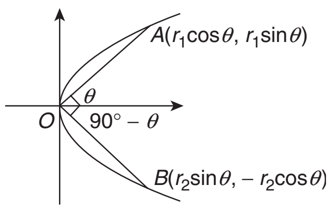


Figure 13.77

Therefore, $r_1^2\sin^2\theta = 4ar_1\cos\theta$ and point B $(r_2\sin\theta, -r_2\cos\theta)$ also lies on the parabola. (1)

Therefore, $r_2^2\cos^2\theta = 4ar_2\sin\theta$ (2)

From Eq. (1), $r_1^2\sin^4\theta = 16a^2\cos^2\theta$ (3)

From Eqs. (3) and (2), we get

$$\sin^3\theta = \frac{64a^3}{r_1^2 r_2} \quad (4)$$

Similarly, $\cos^3\theta = \dfrac{64a^3}{r_1 r_2^2}$ (5)

$$\sin^2\theta + \cos^2\theta = 1 \Rightarrow \frac{r_1^{4/3} r_2^{4/3}}{r_1^{2/3} + r_2^{2/3}} = 16a^2$$

7. Any point on the parabola is $(at^2, 2at)$.

Therefore, midpoint of $(a, 0)$ and $(at^2, 2at)$ is $\left(\dfrac{a+at^2}{2}, at\right)$.

Therefore, locus is given by $x = \dfrac{a+at^2}{2}$, $y = at$.

$$2x = a\left(1+\frac{y^2}{a^2}\right) = a + \frac{y^2}{a}$$

$$\Rightarrow 2ax = a^2 + y^2$$

$$\Rightarrow y^2 = 2a(x - a/2)$$

It is a parabola with the vertex at $(a/2, 0)$, latus rectum $= 2a$

Directrix is $x - a/2 = -a/2$

That is, $x = 0$

Therefore, focus is

$$x - a/2 = a/2$$

$$\Rightarrow x = a$$

That is, $(a, 0)$.

8. Since no point of the parabola is below x-axis.

Therefore, $a^2 - 4 \le 0$

Therefore, maximum value of a is 2.

Equation of the parabola, when $a = 2$ is $y = x^2 + 2x + 1$.

It intersects y-axis at (0, 1).

Equation of the tangent at (0, 1) is $y = 2x + 1$.

Since, $y = 2x + 1$ touches the circle $x^2 + y^2 = r^2$.

Therefore, $r = \dfrac{1}{\sqrt{5}}$.

9. Equation of the tangent at (0, 1) to the parabola $y = x^2 + ax + 1$ is $\dfrac{y+1}{2} = \dfrac{a}{2}(x+0)+1$

That is, $ax - y + 1 = 0$

Therefore,

$$r = \frac{1}{\sqrt{a^2+1}}$$

Radius is maximum when $a = 0$.

Therefore, equation of the tangent is $y = 1$.

Hence, slope of the tangent is 0.

10. Equation of tangent is $y = ax + 1$.

Intercepts are $-\dfrac{1}{a}$ and 1.

Therefore, area of the Δ bounded by tangent and the axes

$$= \frac{1}{2}\left|-\frac{1}{a}\cdot 1\right| = \frac{1}{2|a|}$$

It is minimum when $a = 2$.

Therefore, minimum area $= \dfrac{1}{4}$.

11. See Fig. 13.78.

$$C \equiv \left(0, \frac{1}{m}\right)$$

$$B \equiv \left(\frac{1-2m}{\ell}, 2\right), A(0, 2)$$

Let (h, k) be the circumcentre of ΔABC. Therefore,

$$h = \frac{1-2m}{2\ell};\ k = \frac{1+2m}{2m}$$

or $$2h = \frac{1-2m}{\ell};\ k = 1 + \frac{1}{2m}$$

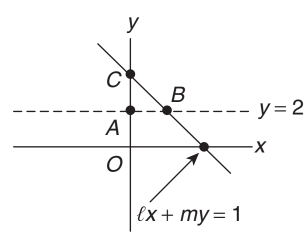


Figure 13.78

$$m=\frac{1}{2k-2};\ \ell=\frac{k-2}{2h(k-1)}$$

As (ℓ, m) lies on $y^2 = 4ax$. Therefore,

$$m^2 = 4al$$

$$\Rightarrow \left(\frac{1}{2k-2}\right)^2 = 4a\left\{\frac{k-2}{2h(k-1)}\right\}$$

$$h = 8a\,(k^2 - 3k + 2)$$

So, locus of (h, k) is

$$x = 8a\,(y^2 - 3y + 2)$$

$$\Rightarrow \left(y-\frac{3}{2}\right)^2 = \frac{1}{8a}(x+2a)$$

Hence, vertex is $\left(-2a, \frac{3}{2}\right)$.

12. Length of smallest focal chord = length of latus rectum $=\frac{1}{8a}$.

13. From the equation of curve C, it is clear that it is symmetric about the line $y=\frac{3}{2}$.

14. Minimum distance is along common normal. So, first find out the common normal. Length of LR,

$$4a = 1$$

$$\Rightarrow a=\frac{1}{4}$$

Equation of normal for $y_2 = x - 1$ at point P

$$y=mx-\frac{3m}{2}-\frac{m^3}{4} \qquad (1)$$

Equation of normal for $x^2 = y - 1$ at point Q

$$y=mx+\frac{3}{2}+\frac{1}{4m^2} \qquad (2)$$

Eqs. (1) and (2) are similar so comparing its coefficient, we have

$$1=\frac{\frac{-3m}{2}-\frac{m^3}{4}}{\frac{3}{2}+\frac{1}{4m^2}} \Rightarrow \frac{3}{2}+\frac{1}{4m^2}=-\frac{3}{2}m-\frac{m^3}{4}$$

$$\Rightarrow m^5 + 6m^3 + 6m^2 + 1 = 0$$

Its one root is $m = -1$ and remaining roots are imaginary.

So, coordinates of $P\equiv\left(1+\frac{1}{4}, 2\times\frac{1}{4}\right)\equiv\left(\frac{5}{4}, \frac{1}{2}\right)$

$$Q\equiv\left(-\frac{2\times\frac{1}{4}}{-1}, 1+\frac{1}{4}\right)=\left(\frac{1}{2}, \frac{5}{4}\right)$$

$$\text{Distance} = \sqrt{\left(\frac{5}{4}-\frac{1}{2}\right)^2+\left(\frac{5}{4}-\frac{1}{2}\right)^2}$$

$$= \left(\frac{5}{4}-\frac{1}{2}\right)\sqrt{2}=\frac{5-2}{4}\sqrt{2}$$

$$= \frac{3\sqrt{2}}{4}$$

15. See Fig. 13.79.

Let any point on parabola $x^2 = 4y$ be $(2t, t^2)$. Then slope of tangent is 't' and slope of normal is $-\frac{1}{t}$.

Equation of normal

$$t^3 - (y - 2)t - x = 0 \to t_1, t_2, t_3 \qquad (1)$$

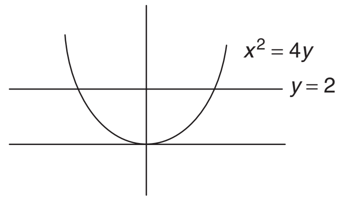


Figure 13.79

$$t_1 + t_2 + t_3 = 0,\ t_1 \cdot t_2 \cdot t_3 = x$$

Solving $y = 2$ and Eq. (1), we get

$$x = t_3$$

t_1^3, t_2^3, t_3^3 are in AP.

$$2t_2^3 = t_1^3 + t_3^3$$

$$= (t_1 + t_3)^3 - 3t_1t_3(t_1 + t_3) = (-t_2)^3 - 3t_1t_3(-t_2)$$

$$\Rightarrow 3t_2^3 = 3t_1t_2t_3$$

$$\Rightarrow t_2^2 = t_1t^3$$

Therefore, t_1, t_2, t_3 are in GP.

16. See Fig. 13.80.

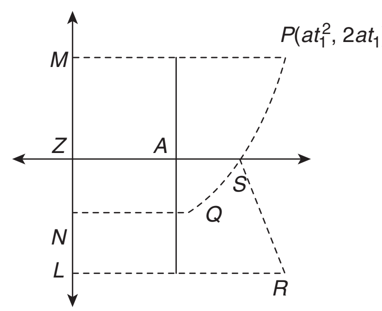


Figure 13.80

$$|SP|\cdot|SQ|\cdot|SR| = |PM|\cdot|QN|\cdot|RL|$$

$$= (a + at_1^2)\,(a + at_2^2)\,(a + at_3^2)$$

$$= a^3[(1 + t_1^2)\,(1 + t_2^2)\,(1 + t_3^2)\,]$$

$$= a^3[(1 + t_1^2 + t_2^2 + t_1^2\,t_2^2)\,(1 + t_3^2)]$$

$$=a^3\left[1+\sum t_1^2+\sum (t_1t_2)^2+t_1^2t_2^2t_3^2\right]$$

$$=a^3\left[1+\left(\sum t_1\right)^2-2\left(\sum t_1t_2\right)+\left(\sum t_1t_2\right)^2-2t_1t_2t_3\left(\sum t_1\right)+(t_1t_2t_3)^2\right]$$

$$=a^3\left[1+0-2\left(\frac{2a-x}{a}\right)+\left(\frac{2a-x}{a}\right)^2+0+\frac{y^2}{a^2}\right]$$

$$=a^3\left[1-\frac{2}{a}(2a-x)+\frac{(2a-x)^2}{a^2}+\frac{y^2}{a^2}\right]$$

$$= a[a^2 - 4a^2 + 2ax + 4a^2 - 4ax + x^2 + y^2]$$

$$= a[(x - a)^2 + y^2] = a(SO)^2$$

17. (A) Point of contact of tangent drawn from (– 2, 2) on $y^2 = 4(x+y)$ are (0, 4) and (0, 0), then area of triangle formed by (– 2, 2) and point of contact of chord (0, 4) and (0, 0) is given by 4.

(B) The conic is a parabola having focus is (2, 3) and the directrix $3x + 4y - 6 = 0$.
Therefore,
latus rectum = 2 (⊥ distance of focus from the directrix)

$$=2\left(\frac{6+12-6}{5}\right)=\frac{24}{5}$$

(C)
$$y + 4 = x^2$$
$$x^2=4\cdot\frac{1}{4}(y+4)$$
$$\text{Focal distance }=\frac{25}{4}$$

Therefore, distance from the directrix $\left(y=\frac{-17}{4}\right)$ is equal to ordinate of points on the parabola whose focal distance is equal to $\frac{25}{4}$.

Thus,

$$\frac{-17}{4}+\frac{25}{4}=2$$
$$\Rightarrow \text{Points are } (\pm\sqrt{6},\ 2)$$
$$\Rightarrow a + b = 8$$

(D) Length of side $=8\sqrt{3}\,a=8\sqrt{3}\,\frac{1}{2}=4\sqrt{3}$

18. (A) Equation of the circle is $(x - r - 1)^2 + y^2 = r^2$
Putting $y^2 = 4x$ and then
$$D=0$$
$$\Rightarrow r=4$$

(B) AB is focal chord.
Therefore, min AB = latus rectum = $4a$ = 16.

(C) Shortest distance along common normal.
Slope of common normal = 0, ±2
So, feet of normals will be (4, 4) and (5, 2).
Or, (4, – 4) and (5, – 2). Therefore,
$$d=\sqrt{1+4}=\sqrt{5}\Rightarrow d^2=5$$

(D) Harmonic mean = $2a$ = 4

19. (A) Point (a, a) lies on $y^2 = 4x$
$$a^2 = 4a$$
That is, a = 0, 4
$$a = 4$$

(B) The line $3x - y + 8 = 0$ passes through the focus (– 2, 2), so the tangents at the end points on the chord is $\frac{\pi}{2}$.
$$p = 8$$

(C) $y^2 = k(x - 8/k)$

Equation of the directrix is
$$x-\frac{8}{k}=-\frac{k}{4}\Rightarrow x=\frac{8}{k}-\frac{k}{4}$$
Compare with
$$x=1\Rightarrow\frac{8}{k}-\frac{k}{4}=1\Rightarrow k=4$$

(D) End points of the normal chord will be (8, 8) and

$$\left(2\left(\frac{-5}{2}\right)^2,\ 2\cdot 2\cdot\left(\frac{-5}{2}\right)\right)$$

Therefore, length of the chord will be $10\sqrt{5}$.

20. See Fig. 13.81. Now, $\left(\frac{t_2-t_1}{t_2^2-t_1^2}\right)\left(\frac{t_1-2}{t_1^2-4}\right)=-1$

Put $t_1 \neq t_2$ and $t_1 \neq 2$, then

$$\frac{1}{(t_1+t_2)}\frac{1}{(t_2+2)}=-1$$
$$\Rightarrow t_1^2+t_1t_2+2t_1+2t_2+1=0$$

Quadratic in t_1 should have real roots. So,

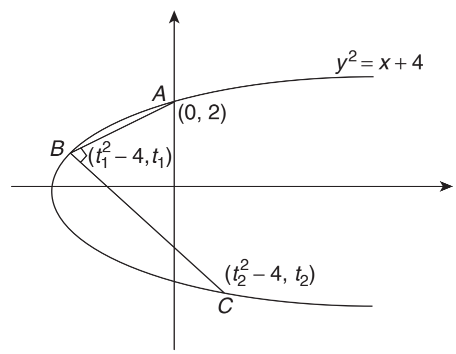


Figure 13.81

$$(t_2+2)^2-4(2t_2+1)\geq 0$$
$$\Rightarrow t_2^2-4t_2\geq 0$$
$$\Rightarrow t_2 \leftarrow (-\infty,\ 0]\cup[4,\ \infty)$$

Therefore, the least positive value of t_2 is 4.

21. Equation of any normal to the parabola $y^2 = 4ax$ is
$$y = mx - 2am - am^3 \qquad (1)$$
If the slope $m=\sqrt{2}$ then the Eq. (1) becomes
$$y=\sqrt{2}\,x-2a\sqrt{2}-a(\sqrt{2})^3$$
$$\Rightarrow\quad y=\sqrt{2}\,x-2a\sqrt{2}-2a\sqrt{2}$$
$$\Rightarrow\quad y-x\sqrt{2}+4a\sqrt{2}=0$$
So, the given line is a normal to the parabola.
$$\text{Length of chord }=\frac{4}{m^2}\sqrt{(1+m^2)(a-mc)}$$
$$=\frac{4}{2}\sqrt{(1+2)(a-8a)} = 6\sqrt{3a}$$

22. For $y^2 = 4ax$, equation of normal is
$$y = mx - 2am - am^3 \qquad (1)$$
For $y^2 = 4(a - 1)(x - b)$, equation of normal is
$$y = m(x - b) - 2(a - 1)m - (a - 1)m^3 \qquad (2)$$
For common normal equation, Eqs. (1) and (2) are same, so compare the coefficients
$$1=\frac{2am+am^3}{mb+2cm+cm^2} \qquad (\text{where } c = (a - 1))$$

$$\Rightarrow 1 = \frac{2a+am^2}{b+2(a-1)+(a-1)m^2}$$

$\Rightarrow b + 2a - 2 + am^2 - m^2 = 2a + am^2$
$\Rightarrow m^2 = b - 2$
$b - 2 > 0 \Rightarrow b > 2$

23. See Fig. 13.82. Let a common tangent through A meet the circle at $B_1\left(\frac{a}{\sqrt{2}}\cos\theta, \frac{a}{\sqrt{2}}\sin\theta\right)$ and the parabola at A_1 $(at^2, 2at)$.

Equation of the tangent to the parabola at A_1 is

$$ty = x + at^2 \qquad (1)$$

Equation of the tangent to the circle at B_1 is

$$x\cos\theta + y\sin\theta = \frac{a}{\sqrt{2}} \qquad (2)$$

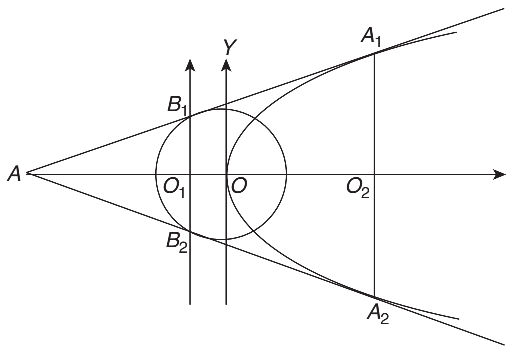


Figure 13.82

Since Eqs. (1) and (2) represent the same line

$$-\frac{1}{\cos\theta} = \frac{1}{\sin\theta} = \sqrt{2}\,t^2 \qquad (3)$$

$$\Rightarrow \frac{1}{2t^4} + \frac{1}{2t^2} = 1$$

$\Rightarrow 1 + t^2 = 2t^4$
$\Rightarrow 2t^4 - t^2 - 1 = 0$
$\Rightarrow (t^2 - 1)(2t^2 + 1) = 0$

which gives two real values of t, equal to ± 1, giving two common tangents through A to the given circle and the parabola. Let the other common tangent meet the circle at B_2 and the parabola at A_2. Then coordinate of A_1 are $(a, 2a)$ and coordinate of A_2 are $(a, -2a)$, so

$$A_1A_2 = 4a$$

From Eq. (3), we get coordinate of B_1 as $\left(-\frac{a}{2}, \frac{a}{2}\right)$ and the coordinate of B_2 as $\left(-\frac{a}{2}, -\frac{a}{2}\right)$, so

$$B_1B_2 = a$$

The quadrilateral $A_1B_1B_2A_2$ formed by the common tangents and the chords of contact B_1B_2 of the circle and A_1A_2 of the parabola is a trapezium whose area is

$$\frac{1}{2}(A_1A_2 + B_1B_2) \times \left(\frac{a}{2} + a\right) = \frac{1}{2} \times 5a \times \frac{3a}{2} = \frac{15a^2}{4}$$

Solved JEE 2017 Questions

JEE Main 2017

1. If the common tangents to the parabola, $x^2 = 4y$ and the circle, $x^2 + y^2 = 4$ intersect at the point P, then the distance of P from the origin, is

(A) $2(\sqrt{2}+1)$ **(B)** $3+2\sqrt{2}$

(C) $2(3+2\sqrt{2})$ **(D)** $\sqrt{2}+1$

(ONLINE)

Solution: Let us consider equation $x^2 + y^2 = 4$. Then the tangent to the equation is given as

$$y = mx \pm 2\sqrt{1+m^2} \quad (1)$$

Also, let us consider,

$$x^2 = 4y \quad (2)$$

Substituting the value of y from Eq. (1) to Eq. (2), we get

$$x^2 = 4(mx \pm 2\sqrt{1+m^2})$$

$$\Rightarrow x^2 = 4mx \pm 8\sqrt{1+m^2}$$

That is, $x^2 = 4mx + 8\sqrt{1+m^2}$ $\Rightarrow x^2 = 4mx - 8\sqrt{1+m^2}$

$\Rightarrow x^2 - 4mx - 8\sqrt{1+m^2} = 0$. Therefore,

$$x = \frac{-(-4m) \pm \sqrt{(-4m)^2 - 4(-8\sqrt{1+m^2})}}{2}$$

Now, the discriminant is $D = 0$. Therefore,

$$(-4m)^2 - 4(-8\sqrt{1+m^2}) = 0$$

$$\Rightarrow 16m^2 + 4.8\sqrt{1+m^2} = 0$$

$$\Rightarrow m^2 + 2\sqrt{1+m^2} = 0$$

$$\Rightarrow m^2 = -2\sqrt{1+m^2}$$

$$\Rightarrow \frac{m^2}{-2} = \sqrt{1+m^2}$$

$$\Rightarrow \frac{m^4}{4} = 1+m^2$$

$$\Rightarrow m^4 = 4 + 4m^2 \Rightarrow m^4 - 4m^2 - 4 = 0$$

$$\Rightarrow m^2 = \frac{4 \pm \sqrt{16 - 4(-4)}}{2}$$

$$\Rightarrow m^2 = \frac{4 \pm \sqrt{32}}{2} = \frac{4 \pm 4\sqrt{2}}{2}$$

$$\Rightarrow m^2 = 2 \pm 2\sqrt{2} = 2(1 \pm \sqrt{2})$$

Hence, the correct answer is option (A).

2. If $y = mx + c$ is the normal at a point on the parabola $y^2 = 8x$ whose focal distance is 8 units then $|c|$ is equal to

(A) $8\sqrt{3}$ **(B)** $2\sqrt{3}$

(C) $16\sqrt{3}$ **(D)** $10\sqrt{3}$

(ONLINE)

Solution: We have

$$y = mx + c$$

and

$$y^2 = 8x$$

$$a = 2$$

$$2yy' = 8$$

$$y' = \frac{4}{y}$$

The focal distance of point $(2t^2, 4t)$ is $2(1 + t^2)$. Therefore,

$$2(1 + 1^2) = 8$$

$$\Rightarrow 1 + t^2 = 4 \Rightarrow t^2 = 3 \Rightarrow t = \pm\sqrt{3}$$

The slope of the tangent at $2(\pm\sqrt{3})^2,\ 4(\pm\sqrt{3})$:

$$(6, \pm 4\sqrt{3})$$

$$\left.\frac{4}{y}\right|_{y=\pm 4\sqrt{3}} = \pm\frac{1}{\sqrt{3}}$$

The slope of the normal is $\pm\sqrt{3}$. That is,

$$m = \pm\sqrt{3}$$

$$y = \pm\sqrt{3}x + c$$

These lines pass through $(6, \pm 4\sqrt{3})$. Therefore,

$$\pm 4\sqrt{3} = \pm 6\sqrt{3} + c$$

$$\Rightarrow c = \pm 10\sqrt{3} \Rightarrow |c| = 10\sqrt{3}$$

Hence, the correct answer is option (D).

JEE Advanced 2017

1. If a chord, which is not a tangent, of the parabola $y^2 = 16x$ has the equation $2x + y = p$, and midpoint (h, k), then which of the following is(are) possible value(s) of p, h and k?

(A) $p = -2, h = 2, k = -4$ **(B)** $p = -1, h = 1, k = -3$

(C) $p = 2, h = 3, k = -4$ **(D)** $p = 5, h = 4, k = -3$

Solution: It is given that the equation of parabola, $y^2 = 16x$ and the equation of the chord is

$$2x + y = p \quad (1)$$

Equation of chord with middle point (h, k) is given as

$$ky - 16\left(\frac{x+h}{2}\right) = k^2 - 16h$$

$$\Rightarrow ky - 8(x + h) = k^2 - 16h$$

$$\Rightarrow ky - 8x - 8h = k^2 - 16h$$

$$\Rightarrow ky - 8x - 8h - k^2 + 16h = 0$$

$$\Rightarrow -8x + ky + 8h - k^2 = 0$$

$$\Rightarrow 8x - ky = 8h - k^2 \quad (2)$$

Dividing Eq. (2) by 4, we get

$$\frac{8x}{4} - \frac{ky}{4} = \frac{8h}{4} - \frac{k^2}{4}$$

$$\Rightarrow 2x - \frac{ky}{4} = 2h - \frac{k^2}{4}$$

On comparing this with Eq. (1), we get

$$\frac{-k}{4} = 1 \text{ and } p = 2h - \frac{k^2}{4}$$

$$\Rightarrow k = -4 \text{ and } p = 2h - \frac{(-4)^2}{4} = 2h - 4$$

$$\Rightarrow p = 2h - 4$$

$$\Rightarrow 2h - p = 4$$

Only the values $p = 2$ and $h = 3$ satisfy this equation. Therefore, $p = 2$, $h = 3$ and $k = -4$

Hence, the correct answer is option (C).

14 Ellipse

14.1 Ellipse – Fundamentals

1. **Definition 1:** An ellipse is the locus of a point, which moves such that the ratio of its distances from a fixed point to a fixed line is constant and the value of the constant is less than unity. Here, the fixed point is called the 'focus', the fixed line is called the 'directrix' and the constant is known as the 'eccentricity'.

   Generally, the equation of an ellipse, whose focus is the point (h, k) and directrix $lx + my + n = 0$ and whose eccentricity is e, is

$$(x-h)^2 + (y-k)^2 = e^2\left[\frac{(lx+my+n)^2}{l^2+m^2}\right] \qquad (e < 1)$$

2. **Definition 2:** An ellipse is the locus of a point, which moves such that the sum of its distances from two fixed points is constant and the value of the constant is more than the distance between these two fixed points. These two fixed points are the two foci of the ellipse and the sum of distances is equal to the length of its major axis.
3. **Standard equation of ellipse:** If we consider the focus as $(\pm ae, 0)$ and the directrix as $x = \pm a/e$ (Fig. 14.1), we get the equation of ellipse as follows:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

   where $b^2 = a^2(1 - e^2)$.

Results:

(i) The equation of the directrix corresponding to focus $(-ae, 0)$ is $x = -a/e$.

(ii) If $a > b$, then $2a$ is the length of major axis and $2b$ is the length of minor axis.

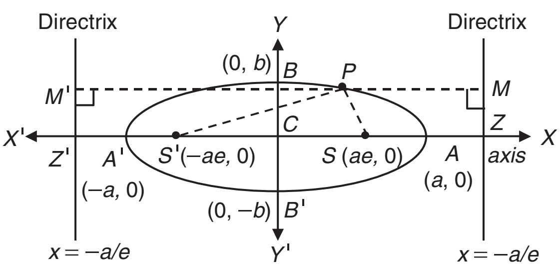


Figure 14.1

4. **General equation of second degree:** The general equation of second degree, that is,

$$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

   represents an ellipse if

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$$

   and $$h^2 < ab$$

   To find the centre of ellipse in general equation of second degree, differentiate the general equation partially with respect to x and with respect to y to get two equations,

$$\frac{\partial f}{\partial x} = 2ax + 2hy + 2g = 0$$

   and

$$\frac{\partial f}{\partial y} = 2hx + 2by + 2f = 0$$

   Solving these two equations, centre of ellipse can be obtained.

5. **Centre of ellipse:** The point of intersection of the major axis and the minor axis is the 'centre of the ellipse'. In the case discussed here the centre is (0, 0).
6. **Focal chord:** Any chord of the ellipse passing through the focus is called 'focal chord'.
7. **Latus rectum:** A focal chord, which is perpendicular to the major axis of the ellipse is known as 'latus rectum'. In the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (a > b)$$

   Note:

   (i) The end points of the latus rectum are $\left(\pm ae, \pm\dfrac{b^2}{a}\right)$.

   (ii) The length of the latus rectum is $\dfrac{2b^2}{a}$.

8. **Focal distance of a point on ellipse:** Let P be a point on the ellipse, where S and S' are the foci, then PS and PS' are known as the focal distance of point P.

$$PS = ePM = e\left(\frac{a}{e} - x\right) = a - ex$$

   and $$PS' = ePM'$$

$$PS' = e\left(\frac{a}{e} + x\right) = a + ex$$

   Remark: In an ellipse, the sum of the focal distances of a point is constant which is equal to the length of the major axis.

9. **Double ordinates:** See Fig. 14.2. If P is any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and PN is perpendicular to major axis, which when produced meets the curve again at P' then PP' is called a double ordinate. If coordinates of P are (h, k), then coordinates of P' will be $(h, -k)$, where $\frac{h^2}{a^2} + \frac{k^2}{y^2} = 1$, that is,

$$k = \pm\frac{b}{a}\sqrt{a^2 - h^2}$$

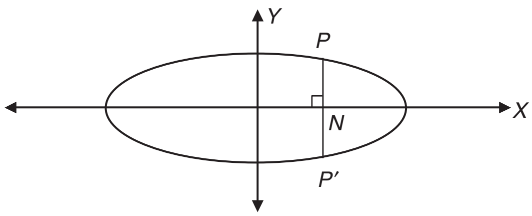


Figure 14.2

Hence, coordinates of P and P' are $\left(h, \frac{b}{a}\sqrt{a^2-h^2}\right)$ and $\left(h, -\frac{b}{a}\sqrt{a^2-h^2}\right)$.

Illustration 14.1 Find the equation of an ellipse whose focus is (−1, 1), eccentricity is 1/2 and the directrix is $x - y + 3 = 0$.

Solution: See Fig. 14.3. Let P(x, y) be any point on the ellipse whose focus is S(−1, 1) and the directrix is $x - y + 3 = 0$.

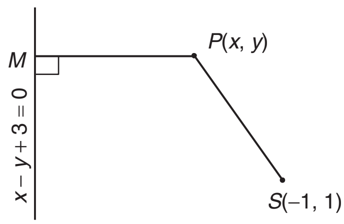


Figure 14.3

Now, let us draw PM which is perpendicular from P(x, y) on the directrix $x - y + 3 = 0$. Then, by definition, we have

$$\text{SP} = e\text{PM}$$
$$\Rightarrow (\text{SP})^2 = e^2(\text{PM})^2$$
$$(x+1)^2 + (y-1)^2 = \frac{1}{4}\left\{\frac{x-y+3}{\sqrt{2}}\right\}^2$$
$$\Rightarrow 8(x^2+y^2+2x-2y+2) = x^2+y^2+9-2xy+6x-6y$$
$$\Rightarrow 7x^2+7y^2+2xy+10x-10y+7=0$$

which is the equation of the given ellipse.

Illustration 14.2 Find the equation of the ellipse referred to its centre as origin and whose minor axis is equal to the distance between the foci and whose latus rectum is 10.

Solution: Let the equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{(say, } a > b\text{)}$$

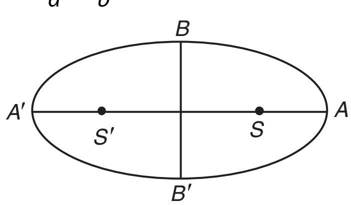


Figure 14.4

Then the foci are $S(ae, 0)$ and $S'(-ae, 0)$ (Fig. 14.4); the length of minor axis is BB′ = $2b$; the length of latus rectum is $2b^2/a$. Now,

$$\text{BB}' = \text{SS}'$$
$$\Rightarrow 2b = 2ae$$
$$\Rightarrow b = ae \quad (1)$$

and

$$\frac{2b^2}{a} = 10$$
$$\Rightarrow b^2 = 5a \quad (2)$$

Also, we have

$$b^2 = a^2(1-e^2) \quad (3)$$

Substituting the value of b [from Eq. (1)] in Eq. (3), we have

$$a^2e^2 = a^2(1-e^2)$$
$$\Rightarrow e^2 = 1 - e^2$$
$$\Rightarrow 2e^2 = 1$$

Therefore,

$$e = \frac{1}{\sqrt{2}}$$

From Eq. (1), we have

$$b = \frac{a}{\sqrt{2}}$$

Therefore,

$$b^2 = \frac{a^2}{2}$$
$$\Rightarrow 5a = \frac{a^2}{2} \quad \text{[from Eq. (2)]}$$
$$\Rightarrow a = 10$$

From Eq. (2),

$$b^2 = 5 \times 10 = 50$$

Substituting the values of a and b in $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the equation of the ellipse is

$$\frac{x^2}{100}+\frac{y^2}{50}=1$$

or

$$x^2 + 2y^2 = 100$$

Illustration 14.3 Find the eccentricity of the ellipse, whose foci are (−3, 4) and (3, −4) and which passes through the point (1, 2).

Solution: The sum of the focal distance of (1, 2) is

$$\sqrt{16+4}+\sqrt{4+36} = 2(\sqrt{5}+\sqrt{10}) = 2a$$

The distance between foci is

$$\sqrt{36+64} = 10 = 2ae$$

where a is the length of semi-major axis and e is the eccentricity of the ellipse.

$$e = \frac{10}{2(\sqrt{5}+\sqrt{10})}$$

Key Points:

1. For an ellipse, we have $\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)^2}{b^2}=1$, $(a > b)$.

2. The eccentricity of an ellipse is $e=\sqrt{1-\frac{b^2}{a^2}}$.
3. The equation of a major axis is $y = b$.
4. The equation of a minor axis is $x = a$.
5. The centre of an ellipse is (a, b).
6. The vertices of an ellipse are $(a \pm \text{a}, b)$.
7. The foci of an ellipse are $(a \pm ae, b)$.
8. The directrices of an ellipse are $x=\alpha\pm\frac{a}{e}$.
9. The ends of a minor axis is $(\alpha, \beta \pm b)$.
10. The length of a latus rectum is $\frac{2b^2}{a}$.

Illustration 14.4 Show that the following equation represents an ellipse: $4(x - 2y + 1)^2 + 9(2x + y + 2)^2 = 25$. Also find its centre and eccentricity.

Solution: On simplification, the equation becomes

$$4(x^2+4y^2+1-4xy+2x-4y)+9(4x^2+y^2+4+4xy+8x+4y)=25$$

or $$40x^2+20xy+25y^2+80x+20y+15=0$$

or $$8x^2+4xy+5y^2+16x+4y+3=0$$

On comparing with the general equation of the second degree, we get

$$h^2-ab=2^2-8.5=4-40<0$$

$$\Delta = 8(5)(3) + (2)(2)(8)(2) - (8)(2)^2 - (5)(8)^2 - (3)(2)^2$$

$$= 120 + 64 - 32 - 320 - 12 \neq 0, h \neq 0$$

Therefore, the curve is an ellipse. For the equation $4(x - 2y + 1)^2 + 9(2x + y + 2)^2 = 25$, we find that $x - 2y + 1 = 0$ and $2x + y + 2 = 0$ are mutually perpendicular lines. Therefore, substituting

$$\frac{x-2y+1}{\sqrt{1^2+(-2)^2}}=X \quad (1)$$

and $$\frac{2x+y+2}{\sqrt{2^2+1^2}}=Y \quad (2)$$

The equation changes to

$$4(\sqrt{5}X)^2+9(\sqrt{5}Y)^2=25$$

or $$4X^2 + 9Y^2 = 5$$

or $$\frac{X^2}{5/4}+\frac{Y^2}{5/9}=1$$

which is in the standard form of equation of an ellipse. Therefore, $a^2=5/4$ and $b^2=5/9$. However,

$$b^2 = a^2(1 - e^2)$$

$$\frac{5}{9}=\frac{5}{4}(1-e^2)$$

or $$\frac{4}{9}=1-e^2$$

That is,

$$e^2=1-\frac{4}{9}=\frac{5}{9} \quad \Rightarrow e=\frac{\sqrt{5}}{3}$$

Now, the centre is $(0, 0)_{X, Y}$. When $X = 0$, $Y = 0$, from Eqs. (1) and (2), we have

$$x - 2y + 1 = 0 \text{ and } 2x + y + 2 = 0$$

On solving these two equations, we get

$$5x + 5 = 0$$

Therefore, $x = -1$ and hence $y = 0$. Therefore, the centre is $(-1, 0)$.

14.2 Position of Point Relative to Ellipse

Let the equation of the ellipse be

$$S\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Let us consider a point $P(h, k)$ lies outside the ellipse (Fig. 14.5) if

$$PM > QM \Rightarrow |k| > |y|$$

$$\Rightarrow k^2 > y^2$$

$$\Rightarrow k^2>b^2\left(1-\frac{h^2}{a^2}\right)$$

$$\Rightarrow \frac{h^2}{a^2}+\frac{k^2}{b^2}-1>0$$

That is, $S_1 > 0$. Similarly, point R lies inside the ellipse if RM < QM.

$$\Rightarrow \frac{h^2}{a^2}+\frac{k^2}{b^2}-1<0$$

That is, $S_1 < 0$.

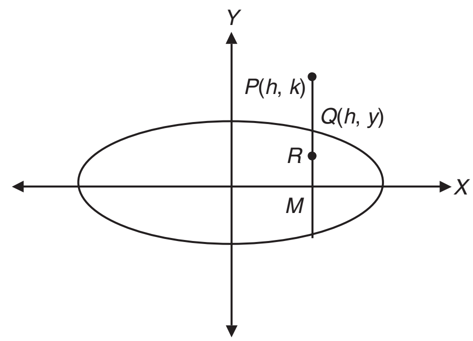


Figure 14.5

Illustration 14.5 Find the position of the point (4, –3) relative to the ellipse $5x^2 + 7y^2 = 140$.

Solution: Since $5(4)^2 + 7(-3)^2 - 140 = 80 + 63 - 140 = 3 > 0$, the point (4, –3) lies outside the ellipse $5x^2 + 7y^2 = 140$.

14.3 Parametric Equation of Ellipse

If we draw a circle with major axis of any ellipse as diameter, then this circle is known as auxiliary circle for that ellipse. Now, let us consider any point P on the ellipse and draw a line through it parallel to the minor axis. The point where this line cuts the auxiliary circle such that P and Q lies on the same side of the major axis, is known as corresponding point. If the line joining Q to the center of the ellipse, makes an angle θ with the major axis, θ is known as the 'eccentric angle' of point P.

If an ellipse be given as $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $(a > b)$, then the auxiliary circle is $x^2 + y^2 = a^2$ and the point Q is $(a\cos\theta, a\sin\theta)$; hence, point P is $(a\cos\theta, b\sin\theta)$ (Fig. 14.6), where $x = a\cos\theta$, $y = b\sin\theta$ is called the parametric equation of the ellipse.

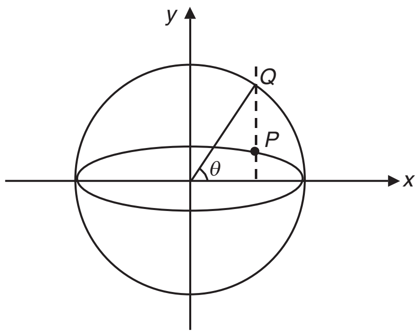


Figure 14.6

The parametric coordinates of ellipse with centre as (α, β), that is, of $\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)^2}{b^2}=1$, $(a > b)$ are

$(a \cos \theta + \alpha, b \sin \theta + \beta)$ where $0 \le \theta < 2\pi$.

Illustration 14.6 Let point P be a variable point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and point Q be its corresponding point on the auxiliary circle. If point R moves on the line segment PQ such that $PR:RQ = r:s$. Find the locus of point R.

Solution: See Fig. 14.7. Let Q be $(a\cos\theta, a\sin\theta)$. Then point P is $(a\cos\theta, b\sin\theta)$.

Let point R be (h, k). Therefore,

$$h = a\cos\theta$$

$$k = \frac{rb\sin\theta + sa\sin\theta}{r+s}$$

Hence,

$$\cos\theta = \frac{h}{a};\ \sin\theta = \frac{k(r+s)}{br+as}$$

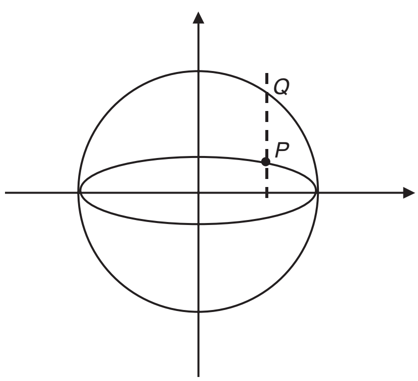


Figure 14.7

Hence,

$$\cos\theta = \frac{h}{a};\ \sin\theta = \frac{k(r+s)}{br+as}$$

On squaring and adding these two equations, we get

$$\frac{h^2}{a^2}+\frac{k^2(r+s)^2}{(br+as)^2}=1$$

Thus, the locus is

$$\frac{x^2}{a^2}+\frac{y^2(r+s)^2}{(br+as)^2}=1$$

14.4 Another Form of Ellipse (When $b > a$)

In this case, the major axis and the minor axis of the ellipse are along y-axis and x-axis (Fig. 14.8), respectively. Then

$AA' = 2b$ (length of major axis)

$BB' = 2a$ (length of minor axis)

The foci S and S' are $(0, be)$ and $(0, -be)$, respectively. The directrices are MZ and $M'Z'$ given by the equation $y = b/e$ and $y = -b/e$, respectively.

$$e^2 = 1-\frac{a^2}{b^2}$$

The length of latus rectum is $2a^2/b$.

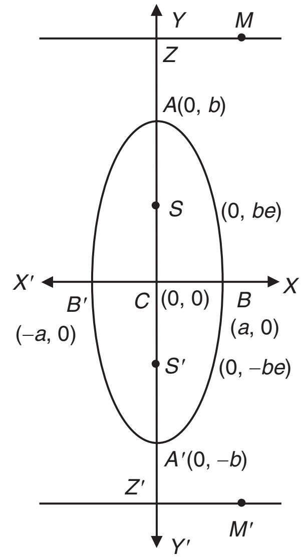


Figure 14.8

Illustration 14.7 Find the lengths and the equations of the focal radii that are drawn from the point $(4\sqrt{3}, 5)$ on the ellipse $25x^2 + 16y^2 = 1600$.

Solution: The equation of the ellipse is

$$25x^2 + 16y^2 = 1600$$

or

$$\frac{x^2}{64}+\frac{y^2}{100}=1$$

Here, $b > a$. Since $a^2 = 64$ and $b^2 = 100$, we get

$$a^2 = b^2(1-e^2) \Rightarrow 64 = 100(1-e^2) \Rightarrow e = \frac{3}{5}$$

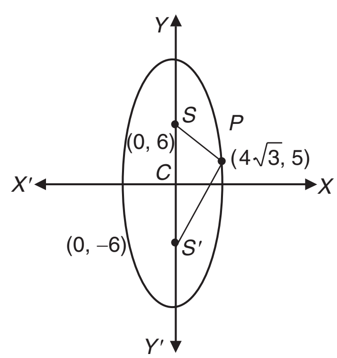


Figure 14.9

See Fig. 14.9. Let $P(x_1, y_1) = (4\sqrt{3}, 5)$ be a point on the ellipse. Then, SP and S′P are the focal radii. Therefore,

$$SP = b - ey_1$$

and $$S'P = b + ey_1$$

That is,

$$SP = 10 - \frac{3}{5} \times 5$$

and $$S'P = 10 + \frac{3}{5} \times 5$$

Therefore,

$$SP = 7 \text{ and } S'P = 13$$

Also, S is $(0, be)$. That is,

$$S \equiv \left(0, 10 \times \frac{3}{5}\right)$$

That is, S is (0, 6) and S' is $(0, -be)$. Therefore,

$$S' \equiv \left(0, -10 \times \frac{3}{5}\right)$$

Therefore, $S' \equiv (0, -6)$ and thus the equation of SP is

$$y - 5 = \frac{6-5}{0-4\sqrt{3}}(x - 4\sqrt{3})$$

or $$-4\sqrt{3}y + 20\sqrt{3} = x - 4\sqrt{3}$$

or $$x + 4\sqrt{3}y - 24\sqrt{3} = 0$$

and the equation of $S'P$ is

$$y - 5 = \frac{-6-5}{0-4\sqrt{3}}(x - 4\sqrt{3})$$

$$\Rightarrow -4\sqrt{3}y + 20\sqrt{3} = -11x + 44\sqrt{3}$$

or $$11x - 4\sqrt{3}y - 24\sqrt{3} = 0$$

Your Turn 1

1. Find the centre, the lengths of the axes and the eccentricity of the ellipse, $2x^2 + 3y^2 - 4x - 12y + 13 = 0$.

   Ans. $(1, 2), \sqrt{2}, \frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

2. Find the equation of the ellipse which has the centre at (1, 2), one focus at (6, 2) and passing through the point (4, 6).

   Ans. $\frac{(x-1)^2}{45} + \frac{(y-2)^2}{20} = 1$

3. Find the centre, the length of axes, the eccentricity and the foci of the ellipse $12x^2 + 4y^2 + 24x - 16y + 25 = 0$.

   Ans. $(-1, 2), \sqrt{3}, 1, \sqrt{\frac{2}{3}}, \left(-1, 2 \pm \frac{1}{\sqrt{2}}\right)$

4. Find the equation to the ellipse whose one vertex is (3, 1), the nearer focus is (1, 1) and the eccentricity is 2/3.

   Ans. $\frac{(x+3)^2}{36} + \frac{(y-1)^2}{20} = 1$

5. A point P moves such that the sum of its distances from two fixed points is a constant. Prove that the locus of the point P is an ellipse.

6. If $3x^2 + 2y^2 - 6xy + 2x + 4y - k = 0$, then find the range of k for which the given equation represents an ellipse.

   Ans. $k \in \phi$

7. Find out the distance between the directrices of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

   Ans. $18/\sqrt{5}$

8. State true or false: The length of the major axis of the ellipse $(5x - 10)^2 + (5y + 15)^2 = \frac{(3x - 4y + 7)^2}{4}$ is 20/3.

   Ans. True

9. State true or false: If P is any point on the ellipse $16x^2 + 25y^2 = 100$ and F_1 and F_2 are the foci of the ellipse, then $PF_1 + PF_2 = 8$.

   Ans. False

14.5 Tangent to Ellipse

1. **Point form:** Let $\frac{y^2}{a^2} + \frac{y^2}{b^2} = 1$ be an ellipse and (x_1, y_1) be a point on it. Then equation of tangent to the ellipse at the point (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$ written as $T = 0$. For the general form of ellipse $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, the tangent is

   $$axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

2. **Parametric form:** Let $(a\cos\theta, b\sin\theta)$ be a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Then the equation of tangent to the ellipse is

   $$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta - 1 = 0$$

3. **Slope form:** The line $y = mx + c$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if and only if $c^2 = a^2 m^2 + b^2$. Hence, the equation of tangent to the ellipse with slope m is $y = mx \pm \sqrt{a^2m^2 + b^2}$. Here, '±' denotes that there can be two tangents to the ellipse with the same slope m.

4. **Director circle:** The locus of the point, where the perpendicular tangents of the ellipse meet, is called the director circle of the ellipse. Let the tangent $y = mx \pm \sqrt{a^2m^2 + b^2}$ passes through the point (h, k). Then

   $$k = mh \pm \sqrt{a^2m^2 + b^2}$$

   $$\Rightarrow (k - mh)^2 = a^2 m^2 + b^2$$

   $$\Rightarrow (h^2 - a^2) m^2 - 2kmh + k^2 - b^2 = 0 \quad (1)$$

   The roots of the equation gives slope of the tangents which are intersecting at (h, k). For the director circle, $m_1m_2 = -1$.

   $$\frac{k^2 - b^2}{h^2 - a^2} = -1 \Rightarrow h^2 + k^2 = a^2 + b^2$$

   Thus, the locus of the point (h, k) is $x^2 + y^2 = a^2 + b^2$, which is the equation of the director circle of the ellipse. Thus, in general, the director circle of any ellipse is a circle with the same centre as the centre of the ellipse and radius as square root of sum of the squares of length of semi-axes.

5. **Equation of tangents from an external point (x_1, y_1):** Let (x_1, y_1) be a point outside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Then the combined equation of the tangents to the ellipse is

$$\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2=\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$$

which is written as $T^2 = SS_1$.

6. **Chord of contact:** Let (x_1, y_1) be a point outside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Then the equation of chord of contact of the point with respect to the ellipse is $\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)=0$, written as $T = 0$.

Illustration 14.8 If $2x + 3y = 6\sqrt{2}$ is a tangent to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$, then find the eccentric angle of the point of tangency.

Solution: The tangent can be written as

$$\frac{x}{3}\left(\frac{1}{\sqrt{2}}\right)+\frac{y}{2}\left(\frac{1}{\sqrt{2}}\right)=1 \quad (1)$$

The equation of tangent at point θ is given as

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta-1=0 \quad (2)$$

From Eqs. (1) and (2), the eccentric angle $\theta = 45°$.

Illustration 14.9 For what value of λ does the line $y = x + \lambda$ touches the ellipse $9x^2 + 16y^2 = 144$.

Solution: The equation of ellipse is given by

$$9x^2 + 16y^2 = 144$$

or

$$\frac{x^2}{16}+\frac{y^2}{9}=1$$

On comparing this with

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

we get $a^2 = 16$ and $b^2 = 9$ and on comparing the line $y = x + \lambda$ with $y = mx + c$, we get $m = 1$ and $c = \lambda$. If the line $y = x + \lambda$ touches the ellipse $9x^2 + 16y^2 = 144$, then

$$c^2 = a^2m^2 + b^2$$

$$\Rightarrow \lambda^2 = 16\times 1^2 + 9\times \lambda^2 = 25$$

Therefore, $\lambda = \pm 5$.

Illustration 14.10 Find the equations of the tangents to the ellipse $3x^2 + 4y^2 = 12$, which are perpendicular to the line $y + 2x = 4$.

Solution: Let m be the slope of the tangent, since the tangent is perpendicular to the line $y + 2x = 4$. Therefore,

$$mx - 2 = -1 \Rightarrow m = \frac{1}{2}$$

Since

$$3x^2 + 4y^2 = 12$$

or

$$\frac{x^2}{4}+\frac{y^2}{3}=1$$

We can compare this with

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Therefore, $a^2 = 4$ and $b^2 = 3$. Thus, the equations of the tangents are

$$y=\left(\frac{1}{2}\right)x\pm\sqrt{4\times\frac{1}{4}+3}$$

$$\Rightarrow y=\left(\frac{1}{2}\right)x\pm 2$$

or

$$x-2y\pm 4=0$$

Illustration 14.11 Find the equation of pair of tangents drawn from the point (1, 2) and (2, 1) to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$.

Solution: Since the point (1, 2) lies outside the ellipse, the equation of the pair of tangents drawn from it is

$$\left(\frac{x}{9}+\frac{2y}{4}-1\right)^2=\left(\frac{x^2}{9}+\frac{y^2}{4}-1\right)\left(\frac{1}{9}+\frac{4}{4}-1\right)$$

$$\Rightarrow \left(\frac{x}{9}+\frac{2y}{4}-1\right)^2=\frac{1}{9}\left(\frac{x^2}{9}+\frac{y^2}{4}-1\right)$$

Since the point (2, 1) lies inside the ellipse, no tangent can be drawn from it.

Illustration 14.12 If tangent at a point P on the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ intersects ellipse $\frac{x^2}{15}+\frac{y^2}{10}=1$ at the point A and B, then prove that the tangents at point A and B intersect at the right angle.

Solution: Let point P be ($3\cos\theta$, $2\sin\theta$) and let the tangent at points A and B to the ellipse $\frac{x^2}{15}+\frac{y^2}{10}=1$ intersect at the point $Q(h, k)$ (Fig. 14.10).

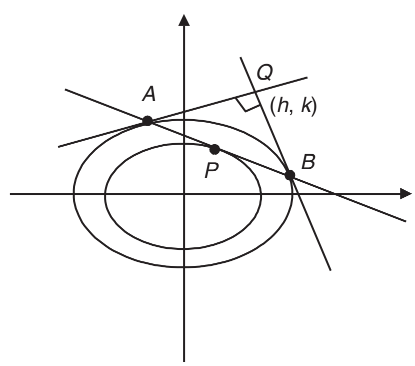


Figure 14.10

Thus, the equation of AB as a tangent to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ is

$$\frac{x\cos\theta}{3}+\frac{y\sin\theta}{2}=1 \quad (1)$$

and the equation of AB as chord of contact of the point Q with respect to the ellipse $\frac{x^2}{15}+\frac{y^2}{10}=1$ is

$$\frac{xh}{15}+\frac{yk}{10}=1 \quad (2)$$

Since Eqs. (1) and (2) are representing same line, we have

$$\frac{\cos\theta/3}{h/15}=\frac{\sin\theta/2}{k/10}=1$$

$$\Rightarrow \cos\theta=\frac{h}{5},\ \sin\theta=\frac{k}{5}$$

$$\Rightarrow \frac{h^2}{25}+\frac{k^2}{25}=1$$

Thus, the locus of (h, k) is $x^2+y^2=25=15+10$, which is the director circle of ellipse $\frac{x^2}{15}+\frac{y^2}{10}=1$. Hence, the tangents at point A and B which intersect at the point Q meets at right angle.

Illustration 14.13 A tangent to the circle $x^2+y^2=5$ at the point (−2, 1) intersect the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ at points A and B. If the tangents to the ellipse at points A and B intersect at point C, find the coordinates of point C.

Solution: The equation of tangent to the circle $x^2+y^2=5$ at point (−2, 1) is

$$-2x+y-5=0$$

$$\Rightarrow 2x-y+5=0 \quad (1)$$

Let us consider that the point C be (h, k). The equation of line AB as a chord of contact from the point C w.r.t. the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ is

$$\frac{xh}{9}+\frac{yk}{4}-1=0 \quad (2)$$

Since Eqs. (1) and (2) represents same line, we get

$$\frac{h/9}{2}=\frac{k/4}{-1}=\frac{-1}{5}$$

Thus,

$$h=\frac{-18}{5} \text{ and } k=\frac{4}{5}$$

Therefore, the coordinates of point C are $\left(\frac{-18}{5},\frac{4}{5}\right)$.

14.6 Normal to Ellipse

1. **Point form:** Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be an ellipse and $P(x_1, y_1)$ be a point on it. Then the equation of normal to the ellipse at point P is

$$\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$$

or

$$\frac{x-x_1}{x_1/a^2}=\frac{y-y_1}{y_1/b^2}$$

2. **Parametric form:** Let $P(a\cos\theta, b\sin\theta)$ be a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Then the equation of normal to the ellipse at the point P is $ax\sec\theta - by\operatorname{cosec}\theta=a^2-b^2$.

3. **Slope form:** The line $y=mx+c$ is a normal to the ellipse if and only if $c=\frac{\pm m(a^2-b^2)}{\sqrt{b^2m^2+a^2}}$. Thus, the equation of normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with slope m is given by

$$y=mx\pm\frac{m(a^2-b^2)}{\sqrt{b^2m^2+a^2}}.$$

4. **Normals from a given point:** The equation of normal at $(a\cos\theta, b\sin\theta)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$ax\sec\theta - by\operatorname{cosec}\theta=a^2-b^2$$

$$\Rightarrow \frac{ax[1+\tan^2(\theta/2)]}{[1-\tan^2(\theta/2)]}-\frac{by[1+\tan^2(\theta/2)]}{2\tan(\theta/2)}=a^2-b^2$$

Let $\tan(\theta/2)=t$; therefore,

$$byt^4+2(ax+a^2-b^2)t^3+2(ax-a^2+b^2)t-by=0$$

If this normal passes through a given point (h, k), then

$$bkt^4+2(ah+a^2-b^2)t^3+2(ah-a^2+b^2)t-bk=0$$

This is a four-degree equation in t which means that maximum four normals can be drawn from a given point to the ellipse. Let t_1, t_2, t_3 and t_4 be the roots of this equation. Therefore,

$$t_1+t_2+t_3+t_4=-2\frac{ah+a^2e^2}{bk} \quad (2)$$

$$\sum t_1t_2=0 \quad (3)$$

$$\sum t_1t_2t_3=-2\frac{ah-a^2e^2}{bk} \quad (4)$$

and

$$t_1t_2t_3t_4=-1 \quad (5)$$

We know that

$$\tan\left(\frac{\theta_1}{2}+\frac{\theta_2}{2}+\frac{\theta_3}{2}+\frac{\theta_4}{2}\right)=\frac{s_1-s_3}{1-s_2+s_4}=\frac{s_1-s_3}{0}=\infty$$

Therefore,

$$\frac{\theta_1+\theta_2+\theta_3+\theta_4}{2}=n\pi+\frac{\pi}{2}$$

and hence $\theta_1+\theta_2+\theta_3+\theta_4=(2n+1)\pi=$ an odd multiple of two right angles.

5. **Conormal points lie on a fixed curve:** Let $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$ be the conormal points. Let the normals at points P, Q, R and S meet at $T(h, k)$. Then, the equation of normal at point $P(x_1, y_1)$ is

$$\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$$

The point $T(h, k)$ lies on it

$$(a^2-b^2)x_1y_1+b^2kx_1-a^2hy_1=0 \quad (6)$$

Similarly, for the points Q, R and S, x_1 and y_1 are replaced by (x_2, y_2), (x_3, y_3) and (x_4, y_4), respectively, in Eq. (6). Therefore, the points P, Q, R and S lie on the curve

$$(a^2-b^2)xy+b^2kx-a^2hy=0$$

which is called 'Apollonius' – a rectangular hyperbola.

Illustration 14.14 A normal inclined at an angle 45° to x-axis of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is drawn. The normal meets the major and the minor axes at points P and Q, respectively. If C is the centre of the ellipse, prove that the area of ΔCPQ is $\frac{(a^2-b^2)^2}{2(a^2+b^2)}$ sq. unit.

Solution: Let $R(a\cos\phi, b\sin\phi)$ be any point on the ellipse. Then the equation of normal at point R is

$$ax\sec\phi - by\,\text{cosec}\,\phi = a^2-b^2$$

or

$$\frac{x}{\cos\phi(a^2-b^2)/a}+\frac{y}{-\sin\phi(a^2-b^2)/b}=1$$

which meets the major and the minor axes, respectively, at points

$$P\left(\frac{a^2-b^2}{a}\cos\phi, 0\right) \text{ and } Q\left(0, -\frac{a^2-b^2}{b}\sin\phi\right)$$

Therefore,

$$CP=\left(\frac{a^2-b^2}{a}\right)|\cos\phi|$$

and

$$CQ=\left(\frac{a^2-b^2}{b}\right)|\sin\phi|$$

Therefore, the area of ΔCPQ is

$$\frac{1}{2}\times CP\times CQ = \frac{(a^2-b^2)^2|\sin\phi\cos\phi|}{2ab} \quad (1)$$

However, the slope of the normal is given as

$$\frac{a}{b}\tan\phi=\tan 45°$$

Therefore,

$$\frac{a}{b}\tan\phi=1$$

Now,

$$\tan\phi=\frac{b}{a}$$

Since,

$$\sin 2\phi=\frac{2\tan\phi}{1+\tan^2\phi}=\frac{2ab}{a^2+b^2}$$

Therefore, from Eq. (1), we get the area of ΔCPQ as

$$\frac{(a^2-b^2)^2|\sin 2\phi/2|}{2ab}=\frac{(a^2-b^2)^2(ab/a^2+b^2)}{2ab}$$

$$=\frac{(a^2-b^2)^2}{2(a^2+b^2)} \text{ sq. unit}$$

Illustration 14.15 If the line $3y = 3x + 1$ is a normal to the ellipse $\frac{x^2}{5}+\frac{y^2}{b^2}=1$, then find the length of the minor axis of the ellipse.

Solution: The equation of the normal with slope m to the given ellipse is

$$y=mx\pm\frac{m(5-b^2)}{\sqrt{b^2m^2+5}}$$

Since $m = 1$, we get

$$y=x\pm\frac{(5-b^2)}{\sqrt{b^2+5}} \quad (1)$$

$$y=x+\frac{1}{3} \quad (2)$$

Both Eqs. (1) and (2) represent the same line. Therefore,

$$\pm\frac{(5-b^2)}{\sqrt{b^2+5}}=\frac{1}{3}$$

$$\Rightarrow (5-b^2)^2=\frac{1}{9}(5+b^2)$$

$$\Rightarrow 9b^4-91b^2+220=0$$

$$\Rightarrow b^2=4$$

or $$b^2=55/9$$

Hence the length of the minor axis is either 4 or $\frac{2\sqrt{55}}{3}$.

14.7 Chords of Ellipse

Consider a line $y = mx + c$ and the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. On eliminating y from both of these equations, we get

$$(a^2m^2+b^2)x^2+2a^2mcx+a^2c^2-a^2b^2=0 \quad (7)$$

The roots of Eq. (1) gives abscissa of the points where the straight line cuts the ellipse. Now, the discriminant of Eq. (7) is

$$D=4a^4m^2c^2-4a^2(c^2-b^2)(a^2m^2+b^2)$$

$$=4a^2\left\{a^2m^2c^2-a^2m^2c^2+b^2a^2m^2+b^4-b^2c^2\right\}$$

$$=4a^2b^2(a^2m^2+b^2-c^2)$$

Let us consider the following three cases:

Case 1: If $c^2 < a^2m^2 + b^2$, then the straight line $y = mx + c$ is a real chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Case 2: If $c^2=a^2m^2+b^2$, then the straight line is a tangent to the ellipse.

Case 3: If $c^2>a^2m^2+b^2$, then the straight line is an imaginary chord of the ellipse.

14.7.1 Chord with Mid-Point

Let (x_1, y_1) be the mid-point of a chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then its equation is given by

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$$

or $$T=S_1$$

Illustration 14.16 Show that the locus of the middle points of chords of an ellipse, which pass through a fixed point, is another ellipse.

Solution: Let $P(x_1, y_1)$ be the middle point of any chord AB of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Then the equation of chord AB is

$$T = S_1$$

$$\Rightarrow \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$$

$$\Rightarrow \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2} \quad (1)$$

However, it passes through a fixed point $Q(h, k)$; its coordinates must satisfy Eq. (1). Therefore,

$$\frac{hx_1}{a^2}+\frac{ky_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$$

which can be rewritten as

$$\frac{[x_1-(h/2)]^2}{a^2}+\frac{[y_1-(k/2)]^2}{b^2}=\frac{1}{4}\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)$$

Hence, the locus of point $P(x_1, y_1)$ is

$$\frac{[x-(h/2)]^2}{a^2}+\frac{[y-(k/2)]^2}{b^2}=\frac{1}{4}\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)$$

which, obviously, is an ellipse with its centre at $\left(\frac{h}{2},\frac{k}{2}\right)$ and the axes parallel to coordinate axes (Fig. 14.11).

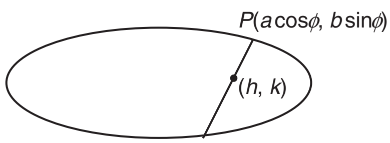


Figure 14.11

14.7.2 Parametric Form of Chord

Let $A(a\cos\theta_1, b\sin\theta_1)$ and $B(a\cos\theta_2, b\sin\theta_2)$ be two points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Then the equation of chord AB is given by

$$\frac{x}{a}\cos\left(\frac{\theta_1+\theta_2}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right)=\cos\left(\frac{\theta_1-\theta_2}{2}\right)$$

If $\theta_1 = \theta_2 = \theta$, then this chord becomes a tangent at point θ, whose equation is given by

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$$

Illustration 14.17 If the three of the sides of a quadrilateral inscribed in an ellipse, respectively, are parallel to three given straight lines, show that the fourth side also is parallel to a fixed straight line.

Solution: Let $PQRS$ be a quadrilateral inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Let PQ, QR and RS be the three sides parallel to the given lines (Fig. 14.12). The equation of PQ is

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$$

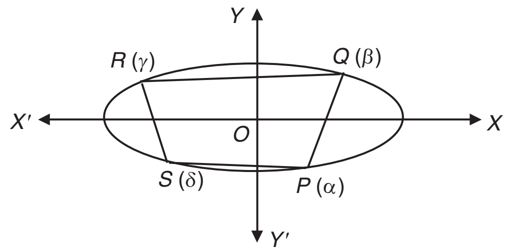


Figure 14.12

Its slope is

$$-\frac{b}{a}\cot\left(\frac{\alpha+\beta}{2}\right)$$

which is constant by hypothesis. Therefore,

$$\alpha+\beta=2\lambda_1 \quad \text{(a constant, say)}$$

Similarly,

$$\beta+\gamma=2\lambda_2 \quad \text{(a constant, say)}$$

and

$$\gamma+\delta=2\lambda_3 \quad \text{(a constant, say)}$$

Now, the equation of SP is

$$\frac{x}{a}\cos\left(\frac{\alpha+\delta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\delta}{2}\right)=\cos\left(\frac{\alpha-\delta}{2}\right)$$

Its slope is

$$m=-\frac{b}{a}\cot\left(\frac{\alpha+\delta}{2}\right)$$

However,

$$\alpha+\delta=(\alpha+\beta)-(\beta+\gamma)+(\gamma+\delta)=2\lambda_1-2\lambda_2+2\lambda_3=\text{constant}$$

Hence, the slope of the fourth side of the quadrilateral, PS, is a constant. Hence, the fourth side of the quadrilateral is also parallel to a fixed straight line.

14.8 Diameter of Ellipse

The locus of the midpoints of parallel chords of an ellipse is called a diameter of the ellipse. It always passes through the centre of the ellipse.

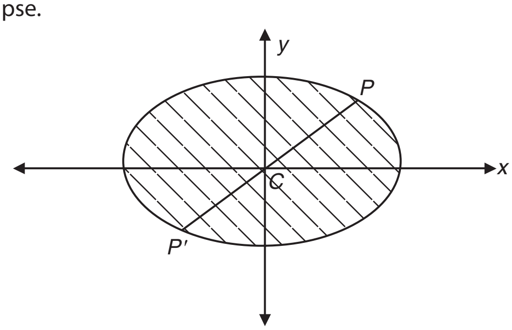


Figure 14.13

In Fig. 14.13, PCP' is a diameter of the ellipse.

If $y = mx + c$ be the system of parallel chords to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where c is different for different chords, then diameter of ellipse corresponding to given system of parallel chords is given by

$$y=\frac{-b^2}{a^2m}x$$

14.8.1 Conjugate Diameters

Two diameters are said to be conjugate when each bisects all chords parallel to the other. If $y = m_1x$ and $y = m_2x$ are two conjugate diameters, then $m_1m_2 = -\frac{b^2}{a^2}$.

14.8.2 Property of Conjugate Diameters

The eccentric angles of the ends of a pair of conjugate diameters differ by a right angle.

See Fig. 14.14. Let P be the point $(a\cos\phi, b\sin\phi)$ and Q be the point $(a\cos\phi', b\sin\phi')$. Therefore, equation of CP is

$$y = x\frac{b}{a}\tan\phi$$

and equation of CQ is

$$y = x\frac{b}{a}\tan\phi'$$

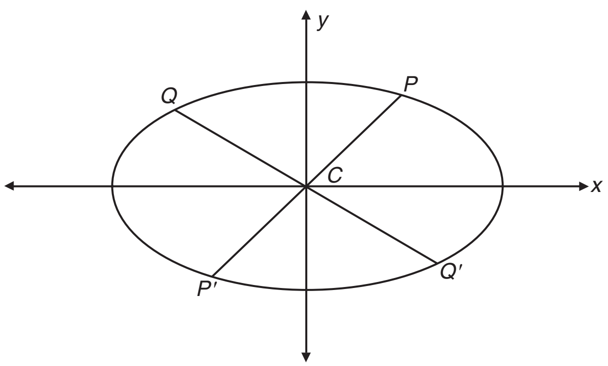


Figure 14.14

These diameters are conjugate, therefore

$$\left(\frac{b}{a}\tan\phi\right)\left(\frac{b}{a}\tan\phi'\right) = -\frac{b^2}{a^2}$$

$$\Rightarrow \tan\phi = -\cot\phi' = \tan(\phi' \pm 90°)$$

$$\Rightarrow \phi - \phi' = \pm 90°$$

Thus, coordinates of Q are $(a\cos(\phi + 90°), b\sin(\phi + 90°))$, that is, $(-a\sin\phi, b\cos\phi)$ and that of Q' are $(a\cos(\phi - 90°), b\sin(\phi - 90°))$, that is, $(a\sin\phi, -b\cos\phi)$.

Illustration 14.18 Show that the tangents at the ends of conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ intersect on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$.

Solution: Let PP' and QQ' be two conjugate diameters of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Then coordinates of P are $(a\cos\phi, b\sin\phi)$ and that of Q are $(a\cos(90 + \phi), b\sin(90 + \phi))$, that is, $(-a\sin\phi, b\cos\phi)$.

Equations of tangents at P and Q are

$$\frac{x\cos\phi}{a} + \frac{y\sin\phi}{b} = 1 \quad (1)$$

and

$$-\frac{x\sin\phi}{a} + \frac{y\cos\phi}{b} = 1 \quad (2)$$

To get the locus of intersection point of these two tangents, we eliminate ϕ by squaring and adding the Eqs. (1) and (2). Therefore,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$$

which is the required locus.

14.9 Geometric Properties of Ellipse

1. **Property 1:** Property 1 is given as Illustration 14.19.
2. **Property 2:** The tangent and normal at any point P of an ellipse are the angle bisectors of the lines joining the point P to the foci.
3. **Property 3:** The foot of perpendiculars from foci to any tangent of any ellipse lie on the auxiliary circle of the ellipse; the product of length of perpendiculars is constant and also equal to the square of length of semi minor axis. Also, the line joining the point of tangency to any focus is parallel to the line joining the centre of the ellipse to the foot of perpendicular from other focus.

Remark: Students themselves can prove Properties 2 and 3.

Illustration 14.19 Prove that the distance of any focus from the point where the normal at point P of the ellipse cuts the major axis is equal to e times the distance of the same focus from the point P, where e is the eccentricity.

Solution: See Fig. 14.15.

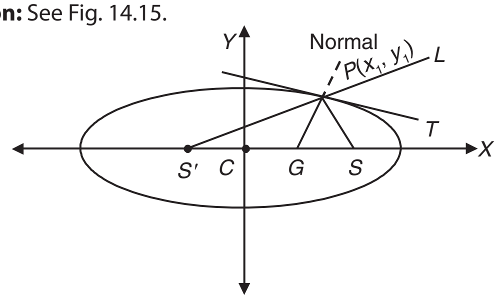


Figure 14.15

Let P be any point on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Therefore, the equation of the normal PG is

$$(x - x_1)\frac{a^2}{x_1} = (y - y_1)\frac{b^2}{y_1}$$

Substituting $y = 0$, for the point G, we have

$$(x - x_1)\frac{a^2}{x_1} = -b^2$$

Hence,

$$x = CG = \left(\frac{a^2 - b^2}{a^2}\right)x_1 = \frac{a^2e^2}{a^2}x_1 = e^2x_1$$

Therefore,

$$SG = CS - CG = ae - e^2x_1 = e(a - ex_1) = eSP$$

Similarly,

$$S'G = eS'P$$

$$\Rightarrow \frac{SG}{S'G} = \frac{eSP}{eS'P} = \frac{SP}{S'P}$$

Therefore, the normal *PG* bisects the internal $\angle SPS'$ between the focal distances, but the tangent and the normal are at right angles; hence, the tangent *PT* bisects the external angle *SPL* between them.

Your Turn 2

1. Find the points of contact of the line $x + 2y - 4 = 0$ with $3x^2 + 4y^2 = 12$. **Ans.** (1, 3/2)
2. Find the angle between the pair of tangents drawn to the ellipse $3x^2 + 2y^2 = 5$ from the point (1, 2). **Ans.** $\tan^{-1}(12/\sqrt{5})$
3. For the ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$, find whether the point (1, 1) lies inside or outside the ellipse. **Ans.** Inside the ellipse
4. Any tangent to an ellipse is cut by the tangents at the ends of major axis in the point *T* and *T′*. Prove that the circle whose diameter is *TT′* which passes through the foci of the ellipse.
5. An ellipse passes through the point (4, –1) and touches the line $x + 4y - 10 = 0$. Find its equation if its axes coincide with coordinate axes.

   Ans. $\frac{x^2}{80}+\frac{4y^2}{5}=1$ and $\frac{x^2}{20}+\frac{y^2}{5}=1$
6. Find the locus of the point of intersection of the tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. It is given that slope of one tangent is three times the others slope.

   Ans. $3x^2y^2 = 4(x^2 - a^2)(y^2 - b^2)$
7. The distance of a point on the ellipse $\frac{x^2}{6}+\frac{y^2}{2}=1$ from the centre is 2 unit. Find the eccentric angle of the point.

   Ans. $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ or $\frac{7\pi}{4}$
8. State true or false: If a tangent of slope *m* at a point of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through (2*a*, 0) and if *e* denotes eccentricity of ellipse, then $3m^2 + e^2 = 1$.

   Ans. True
9. State true or false: The maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ is $3\sqrt{3}(ab/4)$.

   Ans. True
10. State true or false: If the line $3x + 5y = k$ touches the ellipse $16x^2 + 25y^2 = 400$, then the only possible value of *k* is 25.

    Ans. False
11. State true or false. Only two tangents can be drawn from a point *P*(–2, 1) to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$.

    Ans. False

Additional Solved Examples

1. The equations of tangents to the ellipse $x^2 + 4y^2 = 4$, which are inclined at 60° to *x*-axis, are

   (A) $y = \sqrt{3}\,x \pm \sqrt{13}$ **(B)** $y = \sqrt{3}\,x \pm 2/\sqrt{3}$
   (C) $y = \sqrt{3}\,x \pm 2\sqrt{3}$ **(D)** None of these

Solution: The equation of tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ in slope form is

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

Here, $m = \sqrt{3}$ and $a^2m^2 + b^2 = 13$.

Hence, the correct answer is option (A).

2. If the normal at the point $P(\theta)$ to the ellipse $\frac{x^2}{14}+\frac{y^2}{5} = 1$ intersects it once again at the point $Q(2\theta)$, then $\cos\theta$ is equal to

   (A) 2/3 **(B)** –2/3 **(C)** 3/2 **(D)** –3/2

Solution: The equation of normal at $(a\cos\theta, b\sin\theta)$ is

$$ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2$$

Now, this normal passes through $(a\cos2\theta, b\sin2\theta)$. Therefore,

$$a\left(a\frac{\cos2\theta}{\cos\theta}\right) - b\left(b\frac{\sin2\theta}{\sin\theta}\right) = a^2 - b^2$$

$$\Rightarrow 18\cos^2\theta - 9\cos\theta - 14 = 0$$

$$\Rightarrow \cos\theta = -\frac{2}{3} \text{ or } \cos\theta = \frac{7}{6}$$

which is not possible.

Hence, the correct answer is option (B).

3. If F_1 and F_2 be two foci of an ellipse and *PT* and *PN* be the tangent and normal, respectively, to the ellipse at point *P*, then

   (A) *PN* bisects $\angle F_1PF_2$
   (B) *PT* bisects $\angle F_1PF_2$
   (C) *PT* bisects angle $180° - \angle F_1PF_2$
   (D) None of these

Solution: See Fig. 14.16. If *PN* bisects $\angle F_1PF_2$, then in ΔPF_1F_2, we have

$$\frac{PF_1}{PF_2} = \frac{F_1N}{F_2N}$$

which can be proved. Now, we can say that *PN* is angle bisector of lines PF_1 and PF_2 and *PT* is perpendicular to *P* and hence it is another bisector of PF_1 and PF_2.

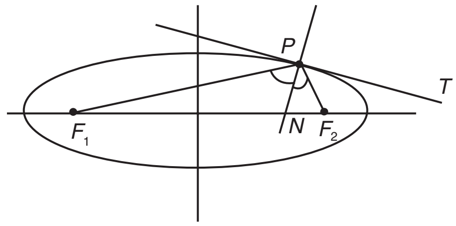


Figure 14.16

Hence, the correct answers are options (A) and (C).

4. If $(13x-1)^2+(13y-2)^2=c(5x+12y-1)^2$ represents an ellipse, then the set of values of c is

(A) $(1,\infty)$ **(B)** $(0, 1)$

(C) 1 **(D)** None of these

Solution: We have

$$\left(x-\frac{1}{13}\right)^2+\left(y-\frac{2}{13}\right)^2=c\left(\frac{5x+12y-1}{13}\right)^2$$

From the definition, we get the focus as $\left(\frac{1}{13},\frac{2}{13}\right)$ and directrix as $5x+12y-1=0$. Here, c is same as the square of eccentricity which belongs to (0, 1).

Hence, the correct answer is option (B).

5. The centre of the ellipse $14x^2-4xy+11y^2-44x-58y+71=0$ is

(A) $(-2, 3)$ **(B)** $(2, -3)$

(C) $(2, 3)$ **(D)** $(-2, -3)$

Solution: Let us consider

$$S\equiv 14x^2-4xy+11y^2-44x-58y+71=0$$

For finding the center of any conic section, first, we differentiate w.r.t. x, then we differentiate w.r.t. y. We get two linear equations in x and y. On solving them, we get the center of the conic.

$$\frac{\partial S}{\partial x}=28x-4y-44=0 \quad (1)$$

$$\frac{\partial S}{\partial y}=-4x+22y-58=0 \quad (2)$$

On solving, we get $x=2$ and $y=3$.

Hence, the correct answer is option (C).

6. The minimum area of the triangle formed by any tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with the coordinate axis is

(A) $\frac{a^2+b^2}{2}$ **(B)** $\frac{(a+b)^2}{2}$

(C) ab **(D)** $\frac{(a-b)^2}{2}$

Solution: See Fig. 14.17. The equation of tangent at $(a\cos\theta, b\sin\theta)$ is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$$

where

$$P\equiv\left(\frac{a}{\cos\theta},0\right);\ Q\equiv\left(0,\frac{b}{\sin\theta}\right)$$

The area of ΔOPQ is

$$\frac{1}{2}\left|\left(\frac{a}{\cos\theta}\right)\left(\frac{b}{\sin\theta}\right)\right|=\frac{ab}{|\sin 2\theta|}$$

That is,

$$(\text{Area})_{\min}=ab$$

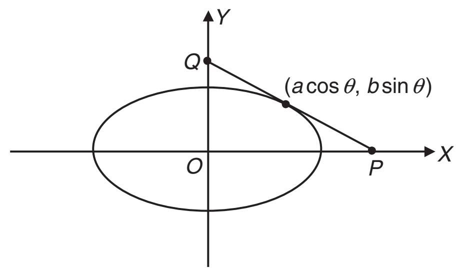


Figure 14.17

Hence, the correct answer is option (C).

7. Tangents are drawn from the points on the line $x-y-5=0$ to $x^2+4y^2=4$. Then, all the chords of contact pass through a fixed point whose coordinates are

(A) $\left(\frac{4}{5},-\frac{1}{5}\right)$ **(B)** $\left(\frac{1}{5},-\frac{4}{5}\right)$

(C) $\left(\frac{4}{5},\frac{1}{5}\right)$ **(D)** $\left(-\frac{4}{5},\frac{1}{5}\right)$

Solution: Any point on the line $x-y-5=0$ is of the form $(t, t-5)$. The chord of contact of this point w.r.t. the curve $x^2+4y^2=4$ is

$$tx+4(t-5)y-4=0$$

or

$$(-20y-4)+t(x+4y)=0$$

which is a family of straight lines and each member of this family passes through the point of intersection of straight lines $-20y-4=0$ and $x+4y=0$.

Hence, the correct answer is option (A).

8. The distance of the centre of the ellipse $x^2+2y^2-2=0$, from those tangents of the ellipse which are equally inclined to both the axes, is

(A) $\frac{2}{\sqrt{2}}$ **(B)** $\frac{\sqrt{3}}{2}$ **(C)** $\frac{\sqrt{2}}{3}$ **(D)** $\sqrt{\frac{3}{2}}$

Solution: The equation of the ellipse is $\frac{x^2}{2}+\frac{y^2}{1}=1$. The general tangent to the ellipse of slope m is

$$ym=mx\pm\sqrt{2m^2+1}$$

Since this is equally inclined to the axes, $m = \pm 1$. Thus, the tangents are

$$y = \pm x \pm \sqrt{2+1} = \pm x \pm \sqrt{3}$$

Then, the distance of any tangent from origin is equal to $\sqrt{3/2}$.

Hence, the correct answer is option (D).

9. If $\sqrt{3}\,bx + ay = 2ab$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then the eccentric angle θ of the point of contact is

(A) $\frac{\pi}{6}$ **(B)** $\frac{\pi}{4}$
(C) $\frac{\pi}{3}$ **(D)** $\frac{\pi}{2}$

Solution: The equation of tangent is

$$\frac{x}{a}\frac{\sqrt{3}}{2}+\frac{y}{b}\frac{1}{2}=1$$

and the equation of tangent at the point ($a\cos\phi$, $b\sin\phi$) is

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1$$

Both are same, thus,

$$\cos\phi=\frac{\sqrt{3}}{2};\quad \sin\phi=\frac{1}{2}$$

$$\Rightarrow \phi=\frac{\pi}{6}$$

Hence, the correct answer is option (A).

10. If PQ be a chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, which subtends right angle at the centre, then find its distance from the centre.

(A) $\frac{ab}{\sqrt{a^2+b^2}}$

(B) $\sqrt{a^2+b^2}$

(C) $\sqrt{ab}$

(D) It depends on the slope of the chord

Solution: Let $x\cos\alpha + y\sin\alpha = p$ be the chord PQ, then p is the desired distance. Homogenizing the equation of the ellipse with the help of this equation, we get the combined equation of OP and OQ

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{x\cos\alpha+y\sin\alpha}{p}\right)^2$$

$$\Rightarrow\left(\frac{1}{a^2}-\frac{\cos^2\alpha}{p^2}\right)x^2+\left(\frac{1}{b^2}-\frac{\sin^2\alpha}{p^2}\right)y^2-\frac{2xy\sin\alpha\cos\alpha}{p^2}=0$$

Since $OP\perp OQ$, we get

$$\frac{1}{a^2}-\frac{\cos^2\alpha}{p^2}+\frac{1}{b^2}-\frac{\sin^2\alpha}{p^2}=0$$

$$\Rightarrow p=\frac{ab}{\sqrt{a^2+b^2}}$$

Hence, the correct answer is option (A).

11. The point of contact of the line $3x + 2y = 8$ with $3x^2 + 4y^2 = 16$ is

(A) (–3, 4) **(B)** (2, 1) **(C)** (5, 4) **(D)** (2, 0)

Solution: Substituting $3x + 2y = 8$ in the equation $3x^2 + 4y^2 = 16$, we get

$$(x-2)^2=0$$

$$\Rightarrow x=2;\ y=1$$

Hence, the correct answer is option (B).

12. If the tangent at the point $\left(4\cos\phi, \frac{16}{\sqrt{11}}\sin\phi\right)$ to the ellipse $16x^2 + 11y^2 = 256$ is also the tangent to the circle $x^2 + y^2 - 2x = 15$, then the values of ϕ are

(A) $\pm\frac{\pi}{2}$ **(B)** $\pm\frac{\pi}{4}$ **(C)** $\pm\frac{\pi}{3}$ **(D)** $\pm\frac{\pi}{6}$

Solution: The equation of the tangent at $\left(4\cos\phi, \frac{16}{\sqrt{11}}\sin\phi\right)$ to the given ellipse is

$$64\cos\phi\, x + 16\sqrt{11}\sin\phi\, y = 256$$

or

$$4\cos\phi\, x + \sqrt{11}\sin\phi\, y = 16$$

Since it touches the given circle $(x-1)^2 + y^2 = 16$, we get

$$\left|\frac{4\cos\phi-16}{\sqrt{16\cos^2\phi+11\sin^2\phi}}\right|=4$$

That is,

$$\cos^2\phi - 8\cos\phi + 16 = 16\cos^2\phi + 11\sin^2\phi$$

$$4\cos^2\phi + 8\cos\phi - 5 = 0$$

$$\Rightarrow (2\cos\phi - 1)(2\cos\phi + 5) = 0$$

$$\Rightarrow \cos\phi = \frac{1}{2}$$

$$\Rightarrow \phi = \pm\frac{\pi}{3}$$

Hence, the correct answer is option (C).

13. A tangent to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ is met by the tangents at the end of the major axis at the points P and Q. If the circle on PQ as diameter passes through R, then R may be

(A) $(0,\sqrt{5})$ **(B)** $(\sqrt{5},0)$
(C) (3, 2) **(D)** (0, 0)

Solution: A tangent to ellipse is

$$\frac{x}{3}\cos\phi+\frac{y}{2}\sin\phi=1$$

It intersects $x = 3$ and $x = -3$ at points

$$P\left(3,\frac{2(1-\cos\phi)}{\sin\phi}\right)\text{and } Q\left(-3,\frac{2(1+\cos\phi)}{\sin\phi}\right)$$

The circle with PQ as diameter is

$$(x+3)(x-3)+\left(y-\frac{2(1-\cos\phi)}{\sin\phi}\right)\left(y-\frac{2(1+\cos\phi)}{\sin\phi}\right)=0$$

$$\Rightarrow x^2+y^2-5-\left(\frac{4}{\sin\phi}\right)y=0$$

It is a family of circles passing through the intersection of the circle $x^2 + y^2 - 5 = 0$ and the line $y = 0$, which is $(\pm\sqrt{5}, 0)$.

Hence, the correct answer is option (B).

14. The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the straight line $\frac{x}{7}+\frac{y}{2}=1$ on the x-axis and the straight line $\frac{x}{3}-\frac{y}{5}=1$ on y-axis. Its eccentricity is

(A) $\frac{3\sqrt{2}}{7}$ **(B)** $\frac{2\sqrt{3}}{7}$

(C) $\frac{\sqrt{3}}{7}$ **(D)** None of these

Solution: The line $\frac{x}{7}+\frac{y}{7}=1$ cuts x-axis at (7, 0) and the line $\frac{x}{3}-\frac{y}{5}=1$ cuts y-axis at (0, –5). Since the ellipse passes through these points, $a^2 = 49$ and $b^2 = 25$, we get

$$25 = 49(1 - e^2)$$

$$\Rightarrow e^2 = \frac{24}{49} \text{ or } e = \frac{2\sqrt{6}}{7}$$

Hence, the correct answer is option (D).

15. The tangent at a point P on an ellipse intersects the major axis at point T, and N is the foot of the perpendicular from P to the same axis. Show that the circle drawn on NT as diameter intersects the auxiliary circle orthogonally.

Solution: Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Let P $(a\cos\theta, b\sin\theta)$ be a point on the ellipse. The equation of the tangent at P is

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$

where it meets the major axis, $y = 0$. Therefore, T is $(a\sec\theta, 0)$. The coordinates of N are $(a\cos\theta, 0)$. The equation of the circle with NT as its diameter is

$$(x - a\sec\theta)(x - a\cos\theta) + y^2 = 0$$

$$\Rightarrow x^2 + y^2 - ax(\sec\theta + \cos\theta) + a^2 = 0$$

It cuts the auxiliary circle $x^2 + y^2 - a^2 = 0$ orthogonally if

$$2g(0) + 2f(0) = a^2 - a^2 = 0$$

which is true.

16. The tangents at any point $P(a\cos\theta, b\sin\theta)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the auxiliary circle at two points which subtends a right angle at the centre. Prove that the eccentricity of the ellipse is $1/\sqrt{1+\sin^2\theta}$.

Solution: The tangent to the ellipse is

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1 \quad (1)$$

Homogenizing the equation of auxiliary circle $x^2 + y^2 = a^2$ with the help of Eq. (1), we get

$$x^2+y^2=a^2\left(\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta\right)^2$$

$$\Rightarrow (1-\cos^2\theta)x^2-\frac{2a}{b}\sin\theta\cos\theta xy+\left(1+\frac{a^2}{b^2}\sin^2\theta\right)y^2=0$$

It represents a pair of perpendicular lines if

$$(1-\cos^2\theta)+\left(1+\frac{a^2}{b^2}\sin^2\theta\right)=0$$

$$\Rightarrow \frac{a^2}{b^2}=\frac{\sin^2\theta}{1+\sin^2\theta} \Rightarrow 1-e^2=\frac{\sin^2\theta}{1+\sin^2\theta}$$

Therefore,

$$e^2=\frac{1}{1+\sin^2\theta} \text{ or } e=\frac{1}{\sqrt{1+\sin^2\theta}}$$

17. If the tangent drawn at a point $(t^2, 2t)$ on the parabola $y^2 = 4x$ is the same as the normal drawn at a point $(\sqrt{5}\cos\phi, 2\sin\phi)$ on the ellipse $4x^2 + 5y^2 = 20$, find the value of t and ϕ.

Solution: The equation of the tangent at $(t^2, 2t)$ to the parabola $y^2 = 4x$ is

$$x - ty + t^2 = 0 \quad (1)$$

The equation of the normal at $(\sqrt{5}\cos\phi, 2\sin\phi)$ to the ellipse $\frac{x^2}{5}+\frac{y^2}{4}=1$ is

$$\sqrt{5}\sec\phi\, x - 2\operatorname{cosec}\phi\, x = 1$$

or

$$x-\frac{2}{\sqrt{5}}\cot\phi\, y-\frac{1}{\sqrt{5}}\cos\phi=0 \quad (2)$$

Since Eqs. (1) and (2) represent the same line, we get

$$1=\frac{2\cot\phi}{\sqrt{5}t}=\frac{-\cos\phi}{\sqrt{5}t^2}$$

On eliminating t from these equations, we get

$$4\cot^2\phi=-\sqrt{5}\cos\phi \Rightarrow \cos\phi=0$$

or

$$\sqrt{5}\cos^2\phi-4\cos\phi-\sqrt{5}=0$$

$$\Rightarrow \phi=\frac{\pi}{2},\frac{3\pi}{2}$$

or

$$\cos\phi=-\frac{1}{\sqrt{5}}$$

That is,

$$\phi=\pi\pm\cos^{-1}\frac{1}{\sqrt{5}}$$

so that

$$t=0,0,\frac{1}{\sqrt{5}},-\frac{1}{\sqrt{5}}$$

18. Prove that the tangent and the normal at any point of an ellipse bisect the angles between the focal radii of that point.

Solution: Let PT and PN be the tangent and the normal at $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The equation of the tangent is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$$

Therefore, the slope of the tangent is

$$m_1 = -\frac{b^2 x_1}{a^2 y_1}$$

and the slope of the normal is

$$m_2 = \frac{a^2 y_1}{b^2 x_1}$$

If $S(ae, 0)$ and $S'(-ae, 0)$ are the foci, then the slope of SP is

$$m_3 = \frac{y_1}{x_1 - ae}$$

and the slope of $S'P$ is

$$m_4 = \frac{y_1}{x_1 + ae}$$

$$\Rightarrow \tan\angle SPN) = \frac{m_3 - m_2}{1 + m_2 m_3} = \frac{aey_1}{b^2} \qquad (\text{using } a^2 - b^2 = a^2e^2)$$

Similarly,

$$\tan(\angle S'PN) = \frac{aey_1}{b^2} \Rightarrow \angle SPN = \angle S'PN$$

which implies that PN bisects the angle between the focal radii SP and $S'P$. The tangent PT, being perpendicular to PN, is the other bisector.

19. A man who runs around a racecourse notes that the sum of the distance of two flag-posts from him is 10 m and the distance between the flag-posts is 8 m. Prove that the area of the path encloses in sq. m. is 15π.

Solution: Let $P(x, y)$ be the position of the man at any time. Let $S(4, 0)$ and $S'(-4, 0)$ be the fixed flag-posts, with C as the origin. Since $SP + S'P = 10$ m, which is a constant, the locus of P is an ellipse with S and S' as foci and the length of the major axis as 10 m. Therefore, $ae = 4$ and $2a = 10$. So,

$$e = \frac{4}{5}$$

$$\Rightarrow b^2 = a^2(1 - e^2)$$

$$\Rightarrow b^2 = 25\left(1 - \frac{16}{52}\right) = 9 \Rightarrow b = 3$$

Hence, the area of the ellipse is $\pi ab = \pi(5)(3) = 15\pi$.

20. A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at points P and Q. Prove that the tangents at P and Q to the ellipse $x^2 + 2y^2 = 6$ are at right angles.

Solution: Let (h, k) be the point of intersection of the tangents to $x^2 + 2y^2 = 6$ at points P and Q. Hence, the equation of the line PQ is

$$hx + 2yk = 6 \qquad (1)$$

The tangent at any point $(2\cos\theta, \sin\theta)$ to the ellipse $x^2 + 4y^2 = 4$ is

$$2\cos\theta x + 4\sin\theta y = 4 \qquad (2)$$

$$\Rightarrow \cos\theta x + 2\sin\theta y = 2$$

Since Eqs. (1) and (2) represent the same line, we have

$$\frac{\cos\theta}{h} = \frac{2\sin\theta}{2k} = \frac{2}{6}$$

$$\Rightarrow \cos\theta = \frac{h}{3};\ \sin\theta = \frac{k}{3}$$

$$\Rightarrow \frac{h^2}{9} + \frac{k^2}{9} = 1$$

$$\Rightarrow h^2 + k^2(9) = 6 + 3 = a^2 + b^2$$

where the semi-axes of the ellipse $x^2 + 2y^2 = 6$ are

$$a = \sqrt{6} \text{ and } b = \sqrt{3}$$

Hence, the locus of (h, k) is $x^2 + y^2 = a^2 + b^2$ which is the director circle of the ellipse $x^2 + 2y^2 = 6$. Hence, the tangents at points P and Q are at right angles.

21. A variable point P on an ellipse of eccentricity e is joined to the foci S and S'. Prove that the locus of the in-centre of the triangle PSS' is an ellipse whose eccentricity is $\sqrt{2e/(1+e)}$.

Solution: Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose foci are $S(ae, 0)$ and $S'(-ae, 0)$. If $P(a\cos\theta, b\sin\theta)$ is any point on the ellipse, then $SP = a(1 - e\cos\theta)$, $S'P = a(1 + e\cos\theta)$ and $SS' = 2ae$. Let (h, k) be the in-centre of the triangle PSS'. Then

$$h = \frac{-ae[a(1 - e\cos\theta)] + ae[e(1 + e\cos\theta)] + a\cos\theta(2ae)}{a(1 - e\cos\theta) + a(1 + e\cos\theta) + 2ae} = ae\cos\theta$$

$$k = \frac{b\sin\theta(2ae)}{a(1 - e\cos\theta) + a(1 + e\cos\theta) + 2ae} = \frac{b\sin\theta e}{1 + e}$$

$$\Rightarrow \cos\theta = \frac{h}{ae};\ \sin\theta = eb \Rightarrow \frac{h^2}{a^2e^2} + \frac{(1+e)^2k^2}{a^2b^2} = 1$$

Hence, the locus of (h, k) is

$$\frac{x^2}{a^2e^2} + \frac{y^2}{e^2b^2/(1+e)^2} = 1$$

which is an ellipse, whose eccentricity is given by

$$e_1 = \sqrt{1 - \frac{e^2b^2}{(1+e)^2a^2e^2}}$$

$$= \sqrt{1 - \frac{(1-e^2)}{(1+e)^2}} = \sqrt{\frac{(1+e)^2 - (1-e^2)}{(1+e)^2}}$$

$$= \sqrt{\frac{2e + 2e^2}{(1+e)^2}} = \sqrt{\frac{2e(1+e)}{(1+e)^2}} = \sqrt{\frac{2e}{(1+e)}}$$

22. Find a point on the curve $x^2 + 2y^2 = 6$ whose distance from the line $x + y = 7$ is minimum.

Solution: The given equation can be written as $\frac{x^2}{6} + \frac{y^2}{3} = 1$, which represents an ellipse. Any point on this ellipse is $P(\sqrt{6}\cos\theta, \sqrt{3}\sin\theta)$. The shortest distance between the ellipse and the given line is along the common normal to both. The slope of the normal at point P is

$$\frac{\sqrt{6}\sec\theta}{\sqrt{3}\,\text{cosec}\,\theta} = \sqrt{2}\tan\theta$$

which is the slope of the normal to the line $x + y = 7$. Hence,

$$\sqrt{2}\tan\theta = 1$$

$$\Rightarrow \cos\theta = \sqrt{\frac{2}{3}},\ \sin\theta = \frac{1}{\sqrt{3}}$$

so that point P is (2, 1).

23. Find the coordinates of all the points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ for which the area of ΔPON is maximum where O is the origin and N is the foot of the perpendicular from O to the tangent at P.

Solution: See Fig. 14.18. Let P be the point $(a\cos\theta, b\sin\theta)$.

Tangent at P is $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$.

Therefore,

$$ON=\frac{ab}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}$$

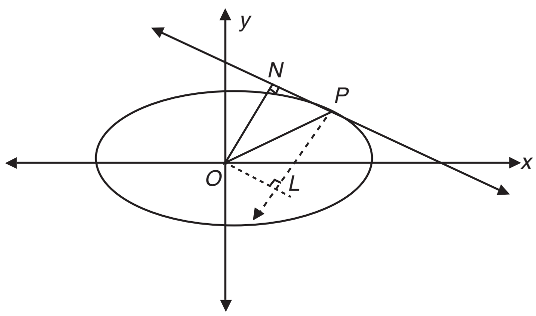


Figure 14.18

Also equation of normal at P is

$$ax\sec\theta - by\operatorname{cosec}\theta = a^2-b^2$$

Therefore,

$$NP=OL=\frac{a^2-b^2}{\sqrt{a^2\sec^2\theta+b^2\operatorname{cosec}^2\theta}}=\frac{(a^2-b^2)\sin\theta\cos\theta}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}$$

and

$$Z=\text{Area of }\Delta OPN=\frac{1}{2}ON\times NP$$

$$=\frac{1}{2}ab(a^2-b^2)\frac{\sin\theta\cos\theta}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}$$

Let $U=\frac{a^2\sin^2\theta+b^2\cos^2\theta}{\sin\theta\cos\theta}=a^2\tan\theta+b^2\cot\theta$

Hence, Z will be maximum when U is minimum.

Therefore,

$$\frac{dU}{d\theta}=a^2\sec^2\theta-b^2\operatorname{cosec}^2\theta=0 \Rightarrow \tan\theta=\frac{b}{a}$$

$$\left(\frac{d^2U}{d\theta^2}\right)_{\theta=\tan^{-1}\frac{b}{a}}>0 \Rightarrow U\text{ is min at }\theta=\tan^{-1}\frac{b}{a}$$

Therefore, Z is maximum when $\tan\theta=\frac{b}{a}$ and P is

$$\left(\frac{a^2}{\sqrt{a^2+b^2}},\frac{b^2}{\sqrt{a^2+b^2}}\right)$$

By symmetry, there can be four such points one in each quadrant. Therefore, points are

$$\left(\pm\frac{a^2}{\sqrt{a^2+b^2}},\pm\frac{b^2}{\sqrt{a^2+b^2}}\right)$$

Previous Years' Solved JEE Main/AIEEE Questions

1. The ellipse $x^2+4y^2=4$ is inscribed in a rectangle aligned with the coordinate axes, which in turn is inscribed in another ellipse that passes through the point (4, 0). Then the equation of the ellipse is

(A) $x^2+16y^2=16$ **(B)** $x^2+12y^2=16$

(C) $4x^2+48y^2=48$ **(D)** $4x^2+64y^2=48$

[AIEEE 2009]

Solution: We have

$$x^2+4y^2=4\Rightarrow\frac{x^2}{4}+\frac{y^2}{1}=1\Rightarrow a=2,\ b=1\Rightarrow P=(2,1)$$

The ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\Rightarrow\frac{x^2}{4^2}+\frac{y^2}{b^2}=1$$

Point (2, 1) lies on the ellipse as shown in Fig. 14.19. Therefore

$$\frac{4}{16}+\frac{1}{b^2}=1\Rightarrow\frac{1}{b^2}=1-\frac{1}{4}=\frac{3}{4}\Rightarrow b^2=\frac{4}{3}$$

Therefore,

$$\frac{x^2}{16}+\frac{y^2}{4/3}=1\Rightarrow\frac{x^2}{16}+\frac{3y^2}{4}=1\Rightarrow x^2+12y^2=16$$

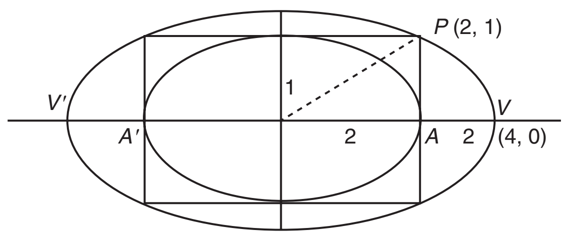


Figure 14.19

Hence, the correct answer is option (B).

2. Equation of the ellipse whose axes are the axes of coordinates and which passes through the point (–3, 1) and has eccentricity $\sqrt{\frac{2}{5}}$ is

(A) $5x^2+3y^2-48=0$ **(B)** $3x^2+5y^2-15=0$

(C) $5x^2+3y^2-32=0$ **(D)** $3x^2+5y^2-32=0$

[AIEEE 2011]

Solution: We have

$$b^2=a^2(1-e^2)=a^2\left(1-\frac{2}{5}\right)=a^2\frac{3}{5}=\frac{3a^2}{5}$$

$$\Rightarrow\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\Rightarrow\frac{9}{a^2}+\frac{5}{3a^2}=1\Rightarrow a^2=\frac{32}{3}\Rightarrow b^2=\frac{32}{5}$$

Therefore, the equation of ellipse is $3x^2+5y^2-32=0$.

Hence, the correct answer is option (D).

3. The equation of the circle passing through the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$, and having centre at (0, 3) is

(A) $x^2+y^2-6y+7=0$ **(B)** $x^2+y^2-6y-5=0$
(C) $x^2+y^2-6y+5=0$ **(D)** $x^2+y^2-6y-7=0$

[JEE MAIN 2013]

Solution: Foci of the ellipse is given by $(\pm ae, 0)$. We have radius of the circle as

$$r=\sqrt{(ae)^2+b^2} \quad (1)$$

where $a=4$; $b=3$;

$$e=\sqrt{1-(9/16)}=\sqrt{7}/4 \Rightarrow ae=\sqrt{7} \Rightarrow \text{foci are } (\pm\sqrt{7}, 0)$$

Therefore, from Eq. (1), we get,

$$r=\sqrt{(ae)^2+b^2}=\sqrt{7+9}=4$$

Therefore, equation of circle with centre $(0, 3)$ and radius 4 is

$$(x-0)^2+(y-3)^2=(4)^2$$

That is,

$$x^2+y^2-6y-7=0$$

Hence, the correct answer is option (D).

4. The locus of the foot of perpendicular drawn from the centre of the ellipse $x^2+3y^2=6$ on any tangent to it is

(A) $(x^2+y^2)^2=6x^2+2y^2$ **(B)** $(x^2+y^2)^2=6x^2-2y^2$
(C) $(x^2-y^2)^2=6x^2+2y^2$ **(D)** $(x^2-y^2)^2=6x^2-2y^2$

[JEE MAIN 2014 (OFFLINE)]

Solution: See Fig. 14.20.

Ellipse is $$\frac{x^2}{(\sqrt{6})^2}+\frac{y^2}{(\sqrt{2})^2}=1$$

Equation of tangent at P is $$\frac{x\cos\theta}{\sqrt{6}}+\frac{y\sin\theta}{\sqrt{2}}=1 \quad (1)$$

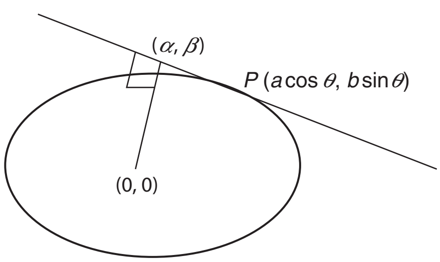


Figure 14.20

Equation of tangent at (α, β) is

$$y-\beta=-\frac{\alpha}{\beta}(x-\alpha) \text{ or } \beta y-\beta^2=-\alpha x+\alpha^2$$

or $$\alpha x+\beta y=\alpha^2+\beta^2 \quad (2)$$

Comparing Eq. (1) and Eq. (2), we get

$$\frac{\cos\theta}{\sqrt{6}\alpha}=\frac{\sin\theta}{\sqrt{2}\beta}=\frac{1}{\alpha^2+\beta^2}$$

or $$\sqrt{\frac{\cos^2\theta+\sin^2\theta}{6\alpha^2+2\beta^2}} \Rightarrow \frac{1}{\sqrt{6\alpha^2+2\beta^2}}=\frac{1}{\alpha^2+\beta^2}$$

$$\Rightarrow 6\alpha^2+2\beta^2=(\alpha^2+\beta^2)^2$$

Therefore, locus is $(x^2+y^2)^2=6x^2+2y^2$.

Hence, the correct answer is option (A).

5. If OB is the semi-minor axis of an ellipse, F_1 and F_2 are its foci and the angle between F_1B and F_2B is a right angle, then the square of the eccentricity of the ellipse is

(A) $\frac{1}{2}$ **(B)** $\frac{1}{\sqrt{2}}$ **(C)** $\frac{1}{2\sqrt{2}}$ **(D)** $\frac{1}{4}$

[JEE MAIN 2014 (ONLINE SET-1)]

Solution: See Fig. 14.21.

$$b^2=a^2(1-e^2) \quad (1)$$

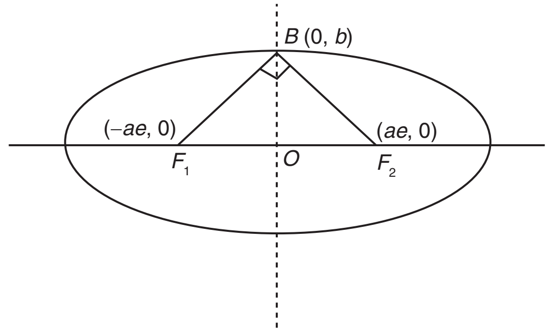


Figure 14.21

Since, $F_1BF_2=90^\circ$

Therefore,

$$(F_1F_2)^2=(F_1B)^2+(F_2B)^2$$

$$(2ae)^2=(ae)^2+b^2+(ae)^2+b^2 \Rightarrow 4a^2e^2-2a^2e^2=2b^2$$

or $$2a^2e^2=2b^2 \Rightarrow b^2=a^2e^2 \quad (2)$$

From Eqs. (1) and (2), we get

$$a^2e^2=a^2(1-e^2) \Rightarrow e^2+e^2=1$$

$$e^2=\frac{1}{2}$$

Hence, the correct answer is option (A).

6. The minimum area of a triangle formed by any tangent to the ellipse $\frac{x^2}{16}+\frac{y^2}{81}=1$ and the coordinate axes is

(A) 12 **(B)** 18 **(C)** 26 **(D)** 36

[JEE MAIN 2014 (ONLINE SET-3)]

Solution: See Fig. 14. 22.

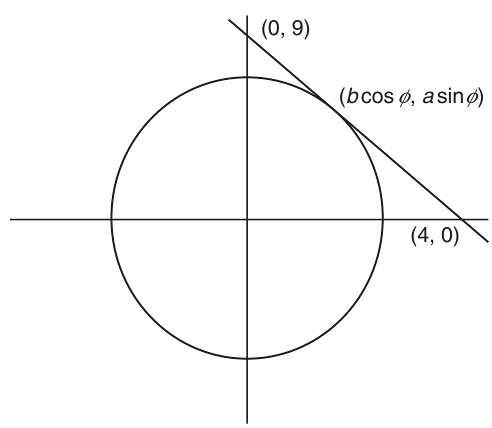


Figure 14.22

Equation of tangent is $\frac{x\cos\phi}{b}+\frac{y\sin\phi}{a}=1$

Here, $a=9$, $b=4$

Therefore, x-intercept $=\frac{4}{\cos\phi}$ and y-intercept $=\frac{9}{\sin\phi}$

Thus,

Area of required triangle $=\frac{1}{2}\times\frac{4}{\cos\phi}\times\frac{9}{\sin\phi}\Rightarrow\frac{36}{\sin 2\phi}$

Therefore,

Minimum area $=\frac{36}{1}=36.$

Hence, the correct answer is option (D).

7. The area (in sq. units) of the quadrilateral formed by the tangents at the end points of the latus rectum to the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ is

(A) 18 **(B)** $\frac{27}{2}$

(C) 27 **(D)** $\frac{27}{4}$

[JEE MAIN 2015 (OFFLINE)]

Solution: Equation of given ellipse is

$$\frac{x^2}{9}+\frac{y^2}{5}=1$$

$$\Rightarrow a^2=9, b^2=5$$

Now,

$$b^2=a^2(1-e^2)\Rightarrow 5=9(1-e^2)\Rightarrow e=2/3$$

One of the end points of the latus rectum is $P(ae, b^2/a)\equiv(2, 5/3)$.

Equation of tangent to the ellipse at P is

$$\frac{2x}{9}+\frac{5y}{15}=1 \text{ or } 2x+3y=9 \text{ or } \frac{x}{9/2}+\frac{y}{3}=1$$

Therefore,

$$\text{area of quadrilateral } =4\left(\frac{1}{2}\right)\left(\frac{9}{2}\right)(3)=27 \text{ sq. units}$$

Hence, the correct answer is option (C).

8. If the distance between the foci of an ellipse is half the length of its latus rectum, then the eccentricity of the ellipse is

(A) $\frac{1}{2}$ **(B)** $\frac{2\sqrt{2}-1}{2}$

(C) $\sqrt{2}-1$ **(D)** $\frac{\sqrt{2}-1}{2}$

[JEE MAIN 2015 (ONLINE SET-2)]

Solution:

$$S_1S_2=\text{Distance between foci}=2ae$$

$$=\frac{1}{2}\left(\text{length of latus rectum}=\frac{2b^2}{a}\right)$$

Therefore,

$$2ae=\frac{1}{2}\left(\frac{2b^2}{a}\right)\Rightarrow 2a^2e=b^2$$

$$\Rightarrow e=\frac{b^2}{2a^2}=\frac{a^2(1-e^2)}{2a^2}=\frac{1-e^2}{2}\Rightarrow e^2+2e-1=0$$

$$\Rightarrow e=\frac{-2+\sqrt{4+4}}{2}=\frac{\sqrt{8}-2}{2}=\sqrt{2}-1$$

Hence, the correct answer is option (C).

9. If the tangent at a point on the ellipse $\frac{x^2}{27}+\frac{y^2}{3}=1$ meets the coordinates axes at A and B, and O is the origin, then the minimum area (in sq. units) of the triangle OAB is

(A) $3\sqrt{3}$ **(B)** $\frac{9}{2}$

(C) 9 **(D)** $9\sqrt{3}$

[JEE MAIN 2016 (ONLINE SET-1)]

Solution: See Fig. 14.23. The point on the ellipse $\frac{x^2}{27}+\frac{y^2}{3}=1$ is $P(3\sqrt{3}\cos\theta, \sqrt{3}\sin\theta)$. The equation of the tangent is

$$\frac{3\sqrt{3}\cos\theta x}{27}+\frac{\sqrt{3}y\sin\theta}{3}=1$$

$$\frac{x\cos\theta}{3\sqrt{3}}+\frac{y\sin\theta}{\sqrt{3}}=1$$

Therefore, the points A and B are given by

$$A\left(\frac{3\sqrt{3}}{\cos\theta},0\right) \text{ and } B\left(0,\frac{\sqrt{3}}{\sin\theta}\right)$$

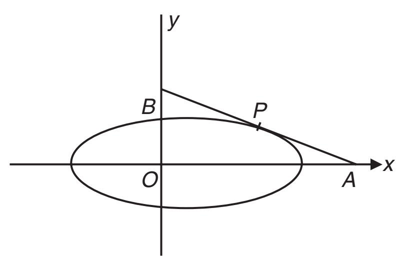


Figure 14.23

The area of ΔOAB is

$$\Delta OAB=\frac{1}{2}\times OA\times OB$$

$$=\frac{1}{2}\times\frac{9}{\sin\theta\cos\theta}=\frac{9}{\sin 2\theta}$$

Therefore, the minimum area of the triangle is 9 sq. units.

Hence, the correct answer is option (C).

Previous Years' Solved JEE Advanced/ IIT-JEE Questions

1. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$, $y_1<0$, $y_2<0$, be the end points of the latus rectum of the ellipse $x2+4y^2=4$. The equations of parabolas with latus rectum PQ are

(A) $x^2+2\sqrt{3}y=3+\sqrt{3}$ **(B)** $x^2-2\sqrt{3}y=3+\sqrt{3}$

(C) $x^2+2\sqrt{3}y=3-\sqrt{3}$ **(D)** $x^2-2\sqrt{3}y=3-\sqrt{3}$

[IIT-JEE 2008]

Solution: We have

$$x^2 + 4y^2 = 4$$

$$\frac{x^2}{4} + \frac{y^2}{1} = 1$$

and

$$b^2 = a^2(1-e^2)$$

$$1 = 4(1-e^2)$$

$$\Rightarrow e = \frac{\sqrt{3}}{2}$$

$$\Rightarrow P\left(\sqrt{3}, \frac{-1}{2}\right) \text{ and } Q\left(-\sqrt{3}, \frac{-1}{2}\right)$$

Because P and Q are the end points of latus rectum of parabola, mid-point of PQ is the focus of parabola. So,

$$\text{focus} = \left(0, \frac{-1}{2}\right)$$

Because length of latus rectum $= 4a$, we have

$$4a = 2\sqrt{3} \Rightarrow a = \frac{\sqrt{3}}{2}$$

Therefore, two parabola are possible whose vertices are

$$\left(0, \frac{-\sqrt{3}}{2} - \frac{1}{2}\right) \text{ and } \left(0, \frac{\sqrt{3}}{2} - \frac{1}{2}\right)$$

Hence, the equations of parabola are

$$x^2 = 2\sqrt{3}\left[y - \left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)\right]$$

and

$$x^2 = 2\sqrt{3}\left[y - \left(\frac{-\sqrt{3}}{2} - \frac{1}{2}\right)\right]$$

Hence, the correct answers are options (B) and (C).

2. The line passing through the extremity A of the major axis and extremity B of the minor axis of the ellipse $x^2 + 9y^2 = 9$ meets its auxiliary circle at the point M. Then the area of the triangle with vertices at A, M and the origin O is

(A) $\frac{31}{10}$ **(B)** $\frac{29}{10}$ **(C)** $\frac{21}{10}$ **(D)** $\frac{27}{10}$

[IIT-JEE 2009]

Solution: See Fig. 14.24. Equation of line AM is $x + 3y - 3 = 0$.

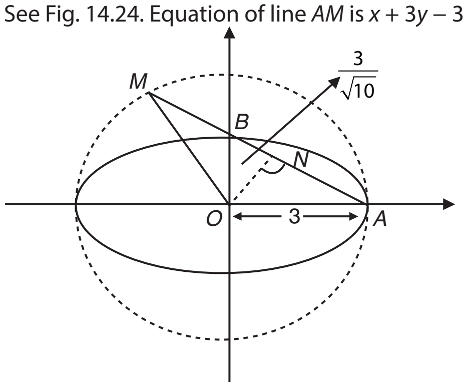


Figure 14.24

Now,

$$\text{Perpendicular distance of line from origin} = \frac{3}{\sqrt{10}}$$

$$\text{Length of } AM = 2\sqrt{9 - \frac{9}{10}} = 2 \times \frac{9}{\sqrt{10}}$$

Therefore,

$$\text{Area} = \frac{1}{2} \times 2 \times \frac{9}{\sqrt{10}} \times \frac{3}{\sqrt{10}} = \frac{27}{10} \text{sq. units}$$

Hence, the correct answer is option (D).

3. The normal at a point P on the ellipse $x^2 + 4y^2 = 16$ meets the x-axis at Q. If M is the mid point of the line segment PQ, then the locus of M intersects the latus rectums of the given ellipse at the points

(A) $\left(\pm\frac{3\sqrt{5}}{2}, \pm\frac{2}{7}\right)$ **(B)** $\left(\pm\frac{3\sqrt{5}}{2}, \pm\frac{\sqrt{19}}{4}\right)$

(C) $\left(\pm 2\sqrt{3}, \pm\frac{1}{7}\right)$ **(D)** $\left(\pm 2\sqrt{3}, \pm\frac{4\sqrt{3}}{7}\right)$

[IIT-JEE 2009]

Solution: See Fig. 14.25.

Normal is $4x\sec\phi - 2y\operatorname{cosec}\phi = 12$

$$Q \equiv (3\cos\phi, 0)$$

$$M \equiv (\alpha, \beta)$$

$$\alpha = \frac{3\cos\phi + 4\cos\phi}{2} = \frac{7}{2}\cos\phi$$

$$\Rightarrow \cos\phi = \frac{2}{7}\alpha$$

$$\beta = \sin\phi$$

$$\cos^2\phi + \sin^2\phi = 1$$

$$\Rightarrow \frac{4}{49}\alpha^2 + \beta^2 = 1 \Rightarrow \frac{4}{49}x^2 + y^2 = 1$$

$$\Rightarrow \text{latus rectum } x = \pm 2\sqrt{3}$$

$$\frac{48}{49} + y^2 = 1 \Rightarrow y = \pm\frac{1}{7}$$

$$(\pm 2\sqrt{3}, \pm 1/7).$$

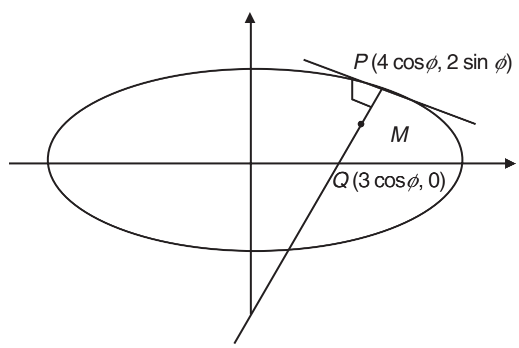


Figure 14.25

Hence, the correct answer is option (C).

Paragraph for Questions 4 to 6: Tangents are drawn from the point $P(3, 4)$ to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ touching the ellipse at points A and B.

[IIT-JEE 2010]

4. The coordinates of A and B are

(A) (3, 0) and (0, 2)

(B) $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and $\left(-\frac{9}{5}, \frac{8}{5}\right)$

(C) $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and (0, 2)

(D) (3, 0) and $\left(-\frac{9}{5}, \frac{8}{5}\right)$

Solution: See Fig. 14.26.

Tangent to $\frac{x^2}{3^2}+\frac{y^2}{2^2}=1$ at the point (3 cosθ, 2 sinθ) is

$$\frac{x\cos\theta}{3}+\frac{y\sin\theta}{2}=1$$

As it passes through (3, 4), we get

$$\cos\theta + 2\sin\theta = 1$$
$$\Rightarrow 4\sin^2\theta = 1+\cos^2\theta - 2\cos\theta$$
$$\Rightarrow 5\cos^2\theta - 2\cos\theta - 3 = 0$$
$$\Rightarrow \cos\theta = 1, -\frac{3}{5}$$
$$\Rightarrow \sin\theta = 0, \frac{4}{5}$$

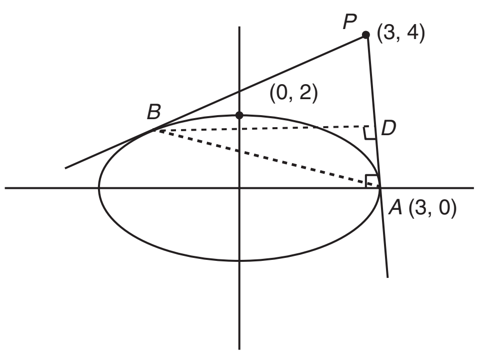


Figure 14.26

Therefore, required points are $A(3, 0)$ and $B\left(\frac{-9}{5}, \frac{8}{5}\right)$.

Hence, the correct answer is option (D).

5. The orthocentre of the triangle PAB is

(A) $\left(5, \frac{8}{7}\right)$ **(B)** $\left(\frac{7}{5}, \frac{25}{8}\right)$

(C) $\left(\frac{11}{5}, \frac{8}{5}\right)$ **(D)** $\left(\frac{8}{25}, \frac{7}{5}\right)$

Solution: Slope of BD must be 0. So,

$$y-\frac{8}{5}=0\left(x+\frac{9}{5}\right) \Rightarrow y=\frac{8}{5}$$

Hence, y coordinate of D and the orthocentre is 8/5.

Hence, the correct answer is option (C).

6. The equation of the locus of the point whose distances from the point P and the line AB are equal is

(A) $9x^2+y^2-6xy-54x-62y+241=0$

(B) $x^2+9y^2+6xy-54x+62y-241=0$

(C) $9x^2+9y^2-6xy-54x-62y-241=0$

(D) $x^2+y^2-2xy+27x+31y-120=0$

Solution: Equation of AB is

$$\frac{3x}{9}+\frac{4y}{4}=1 \Rightarrow \frac{x}{3}+y=1 \Rightarrow x+3y-3=0$$

Therefore, the required locus is

$$(x-3)^2+(y-4)^2=\frac{(x+3y-3)^2}{10}$$
$$\Rightarrow 10x^2+90-60x+10y^2+160-80y$$
$$=x^2+9y^2+9+6xy-6x-18y$$
$$\Rightarrow 9x^2+y^2-6xy-54x-62y+241=0$$

Hence, the correct answer is option (A).

7. Match the statements in Column I with those in Column II. [**Note:** Here z takes the values in the complex plane and Im z and Re z denote, respectively, the imaginary part and the real part of z]

Column I	Column II
(A) The set of points z satisfying $\lvert z-i\lvert z\rvert\rvert=\lvert z+i\lvert z\rvert\rvert$ is contained in or equal to	**(p)** An ellipse with eccentricity $\frac{4}{5}$.
(B) The set of points z satisfying $\lvert z+4\rvert+\lvert z-4\rvert=10$ is contained in or equal to	**(q)** The set of points z satisfying Im $z=0$.
(C) If $\lvert\omega\rvert=2$, then the set of points $z=\omega-1/\omega$ is contained in or equal to	**(r)** The set of points z satisfying $\lvert\text{Im } z\rvert\le 1$.
(D) If $\lvert\omega\rvert=1$, then the set of points $z=\omega+1/\omega$ is contained in or equal to	**(s)** The set of points z satisfying $\lvert\text{Re } z\rvert\le 1$.
	(t) The set of points z satisfying $\lvert z\rvert\le 3$.

[IIT-JEE 2010]

Solution:

$$\left|\frac{z}{|z|}-i\right|=\left|\frac{z}{|z|}+i\right|, z\ne 0$$

$\frac{z}{|z|}$ is unimodular complex number and lies on perpendicular bisector of i and $-i$. So,

$$\frac{z}{|z|}=\pm 1 \Rightarrow z=\pm 1|z| \Rightarrow a \text{ is real number} \Rightarrow \text{Im}(z)=0$$

(A) → (q)

$$|z+4|+|z-4|=10$$

z lies on an ellipse whose focus are (4, 0) and (–4, 0) and length of major axis is 10. So,

$$2ae=8 \text{ and } 2a=10 \Rightarrow e=4/5$$

Therefore,

$$|\text{Re}(z)|=5$$

$$(\text{B}) \rightarrow (\text{p})$$

$$|w|=2 \Rightarrow w=2(\cos\theta+i\sin\theta)$$

Therefore,

$$Z=x+iy=2(\cos\theta+i\sin\theta)-\frac{1}{2}(\cos\theta-i\sin\theta)$$

$$=\frac{3}{2}\cos\theta+i\frac{5}{2}\sin\theta \Rightarrow \frac{x^2}{(3/2)^2}+\frac{y^2}{(5/2)^2}=1$$

Now,

$$e^2=1-\frac{9/4}{25/4}=1-\frac{9}{25}=\frac{16}{25} \Rightarrow e=\frac{4}{5}$$

$$(\text{C}) \rightarrow (\text{p}), (\text{t})$$

$$|w|=1 \Rightarrow z=x+iy=\cos+i\sin\theta+\cos\theta-i\sin\theta$$

$$\Rightarrow x+iy=2\cos\theta$$

Therefore,

$$|\text{Re}(z)| \le 2, |\text{Im}(z)|=0$$

$$(\text{D}) \rightarrow (\text{q}), (\text{t})$$

Hence, the correct matches are (A) → (q), (B) → (p), (C) → (p, t), (D) → (q, t).

8. The ellipse $E_1: \frac{x^2}{9}+\frac{y^2}{4}=1$ is inscribed in a rectangle R whose sides are parallel to the coordinate axes. Another ellipse E_2 passing through the point (0, 4) circumscribes the rectangle R. The eccentricity of the ellipse E_2 is

(A) $\frac{\sqrt{2}}{2}$ **(B)** $\frac{\sqrt{3}}{2}$

(C) $\frac{1}{2}$ **(D)** $\frac{3}{4}$

[IIT-JEE 2012]

Solution: The given situation is shown in Fig. 14.27.

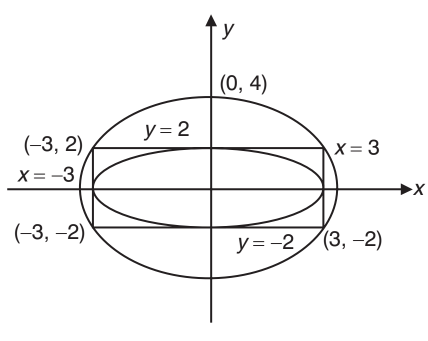


Figure 14.27

Equation of ellipse is $(y+2)(y-2)+\lambda(x+3)(x-3)=0$

It passes through

$$(0,4) \Rightarrow \lambda=\frac{4}{3}$$

Equation of ellipse is $\frac{x^2}{12}+\frac{y^2}{16}=1$

$$e=\frac{1}{2}$$

Alternate solution: Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ as it is passing through (0, 4) and (3, 2). So,

$$b^2=16 \text{ and } \frac{9}{a^2}+\frac{4}{16}=1$$

$$\Rightarrow a^2=12$$

So,

$$12=16(1-e^2)$$

$$\Rightarrow e=1/2$$

Hence, the correct answer is option (C).

9. A vertical line passing through the point (h, 0) intersects the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$ at the points P and Q. Let the tangents to the ellipse at P and Q meet at the point R. If $\Delta(h)$ = area of the triangle PQR, $\Delta_1=\max\limits_{1/2\le h\le 1}\Delta(h)$ and $\Delta_2=\min\limits_{1/2\le h\le 1}\Delta(h)$, then $\frac{8}{\sqrt{5}}\Delta_1-8\Delta_2=$ ______.

[JEE ADVANCED 2013]

Solution:

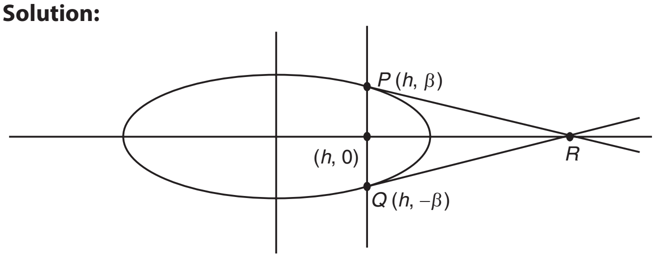


Figure 14.28

We have

$$S \equiv \frac{x^2}{4}+\frac{y^2}{3}=1$$

Let P and Q be (h, β) and (h, $-\beta$), respectively (see Fig. 14.28). Therefore, R is

$$\left(\frac{4}{h},0\right)$$

Therefore,

$$\Delta=\frac{1}{2}\times 2\beta\times\left(\frac{4}{h}-h\right)$$

$$=\sqrt{3}\sqrt{1-\frac{h^2}{4}}\times\left(\frac{4}{h}-h\right)$$

$$=\frac{\sqrt{3}}{2}\frac{(4-h^2)^{3/2}}{h}$$

That is, $\frac{d\Delta}{dh} < 0$, from which it is clear that Δ is decreasing. That is,

$$\Delta_1 = \Delta\left(\frac{1}{2}\right) = \frac{15\sqrt{45}}{8}$$

$$\Delta_2 = \Delta(1) = \frac{9}{2}$$

Therefore,

$$\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 = 9$$

Hence, the correct answer is (9).

10. Suppose that the foci of the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$ are $(f_1, 0)$ and $(f_2, 0)$ where $f_1 > 0$ and $f_2 < 0$. Let P_1 and P_2 be two parabolas with a common vertex at (0, 0) and with foci at $(f_1, 0)$ and $(2f_2, 0)$, respectively. Let T_1 be a tangent to P_1 which passes through $(2f_2, 0)$ and T_2 be a tangent to P_2 which passes through $(f_1, 0)$. If m_1 is the slope of T_1 and m_2 is the slope of T_2, then the value of $\left(\frac{1}{m_1^2} + m_2^2\right)$ is ______.

[JEE ADVANCED 2015]

Solution: See Fig. 14.29. Given ellipse is

$$\frac{x^2}{9} + \frac{y^2}{5} = 1$$

Foci are $(f_1, 0), (f_2, 0), f_1 > 0, f_2 < 0$

Foci of parabola, $P_1 \equiv (f_1, 0)$,

Focus of parabola, $P_2 \equiv (2f_2, 0)$

Tangent to P_1 passes through $(2f_2, 0)$ and tangent to P_2 through $(f_1, 0)$.

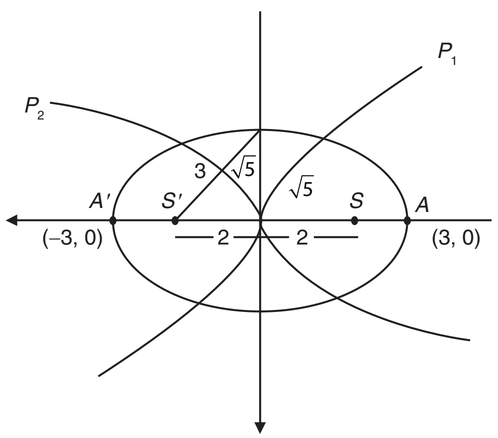


Figure 14.29

Clearly $f_1 = 2, f_2 = 2$.

Focus of $P_1 \equiv (2, 0)$, so equation of P_1 is

$$y^2 = 8x$$

Focus of $P_2 \equiv (-4, 0)$, so equation of P_2 is

$$y^2 = -16x$$

Equation of $T_1: y = m_1 x + \frac{2}{m_1}$,

It passes through (–4, 0), so

$$-4m_1 + \frac{2}{m_1} = 0 \quad (1)$$

$$\Rightarrow m_1^2 = \frac{1}{2} \quad (2)$$

Equation of T_2 is $y = m_2 x + \frac{(-4)}{m_2}$, which passes through (2, 0). So,

$$2m_2 - \frac{4}{m_2} = 0 \quad (3)$$

$$\Rightarrow m_2^2 = 2 \quad (4)$$

$$\Rightarrow \frac{1}{m_1^2} + m_2^2 = 2 + 2 = 4 \quad \text{[Using Eqs. (2) and (4)]}$$

Hence, the correct answer is (4).

11. Let E_1 and E_2 be two ellipses whose centres are at the origin. The major axes of E_1 and E_2 lie along the x-axis and the y-axis, respectively. Let S be the circle $x^2 + (y - 1)^2 = 2$. The straight line $x + y = 3$ touches the curves S, E_1 and E_2 at P, Q and R, respectively. Suppose that $PQ = PR = \frac{2\sqrt{2}}{3}$. If e_1 and e_2 are the eccentricities of E_1 and E_2, respectively, then the correct expression(s) is (are)

(A) $e_1^2 + e_2^2 = \frac{43}{40}$ **(B)** $e_1 e_2 = \frac{\sqrt{7}}{2\sqrt{10}}$

(C) $\left|e_1^2 - e_2^2\right| = \frac{5}{8}$ **(D)** $e_1 e_2 = \frac{\sqrt{3}}{4}$

[JEE ADVANCED 2015]

Solution: See Figs. 14.30 and 14.31.

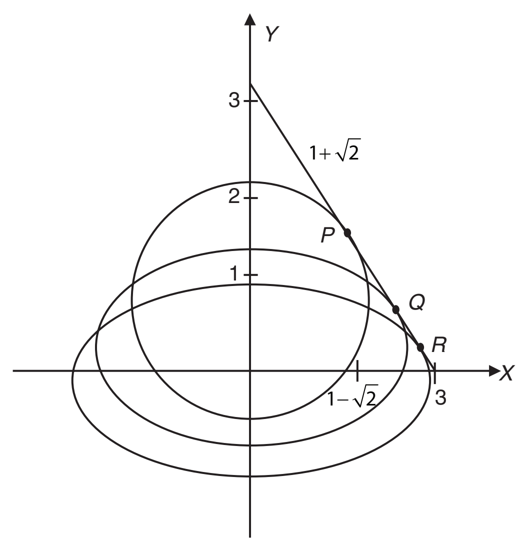


Figure 14.30

$$PQ = PR = \frac{2\sqrt{2}}{3}$$

Let

$$E_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (1)$$

and

$$E_2 = \frac{x^2}{c^2} + \frac{y^2}{d^2} = 1 \quad (2)$$

$x + y = 3$ is a common tangent to $S = 0$, $E_1 = 0$ and $E_2 = 0$. So,

$$P \equiv (1, 2)$$

Also equation of tangent to E_1 is

$$y = -x + \sqrt{a^2 + b^2} \Rightarrow \sqrt{a^2 + b^2} = 3$$

Similarly

$$\sqrt{c^2 + d^2} = 3$$

$$\Rightarrow a^2 + b^2 = c^2 + d^2 = 9 \quad (3)$$

Equation of $x + y = 3$ is given by

$$\frac{x-1}{-1/\sqrt{2}} = \frac{y-2}{+1/\sqrt{2}} = r = \pm\frac{2\sqrt{2}}{3}$$

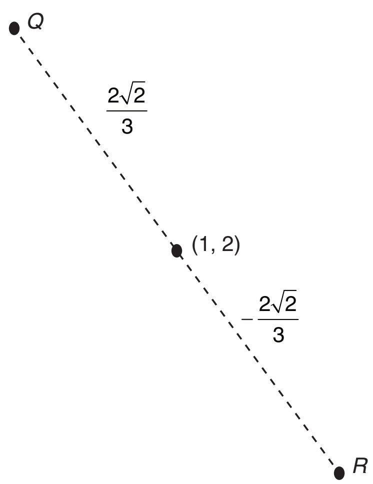


Figure 14.31

Coordinates of Q and R can be obtained by using

$$r = \frac{2\sqrt{2}}{3} \text{ and } r = -\frac{2\sqrt{2}}{3}$$

Therefore,

For Q, $x = 1 - \frac{2}{3}$, $y = 2 + \frac{2}{3}$, i.e. $\left(\frac{1}{3}, \frac{8}{3}\right)$

and for R, $x = 1 + \frac{2}{3}$, $y = 2 - \frac{2}{3}$, i.e. $\left(\frac{5}{3}, \frac{4}{3}\right)$

Point $Q\left(\frac{1}{3}, \frac{8}{3}\right)$ lies on E_1, so

$$b^2 + 64a^2 = 9a^2b^2$$

$\Rightarrow (9 - a^2) + 64a^2 = 9a^2(9 - a^2)$ [using Eq. (3)]

$\Rightarrow (a^2 - 1)^2 = 0$

$\Rightarrow a^2 = 1$

$\Rightarrow b^2 = 8$

Also point $R\left(\frac{5}{3}, \frac{4}{3}\right)$ lies on E_2, so

$$25d^2 + 16c^2 = 9c^2d^2$$

$\Rightarrow (d^2 - 4)^2 = 0$

$\Rightarrow d^2 = 4, c^2 = 5$

$\Rightarrow e_1^2 = 1 - \frac{1}{8} = \frac{7}{8}$, $e_2^2 = 1 - \frac{4}{5} = \frac{1}{5}$

$\Rightarrow e_1^2 + e_2^2 = \frac{7}{8} + \frac{1}{5} = \frac{43}{40}$, $e_1^2 \cdot e_2^2 = \frac{7}{40}$

$\Rightarrow e_1 e_2 = \frac{\sqrt{7}}{2\sqrt{10}}$

Hence, the correct answers are options (A) and (B).

Paragraph for Questions 12 and 13: Let $F_1(x_1, 0)$ and $F_2(x_2, 0)$ for $x_1 < 0$ and $x_2 > 0$, be the foci of the ellipse $\frac{x^2}{9} + \frac{y^2}{8} = 1$. Suppose a parabola having vertex at the origin and focus at F_2 intersects the ellipse at point M in the first quadrant and at point N in the fourth quadrant.

[JEE ADVANCED 2016]

12. The orthocentre of the triangle F_1MN is

(A) $\left(-\frac{9}{10}, 0\right)$ **(B)** $\left(\frac{2}{3}, 0\right)$

(C) $\left(\frac{9}{10}, 0\right)$ **(D)** $\left(\frac{2}{3}, \sqrt{6}\right)$

Solution: See Fig. 14. 32.

It is given that $F_1(x_1, 0)$ and $F_2(x_2, 0)$ are the foci of the ellipse

$$\frac{x^2}{9} + \frac{y^2}{8} = 1$$

Therefore, $a^2 = 9$ and $b^2 = 8$.

$$b^2 = a^2(1 - e^2)$$

$$\frac{8}{9} = 1 - e^2 \Rightarrow e^2 = 1 - \frac{8}{9} = \frac{1}{9} \Rightarrow e = \frac{1}{3}$$

The focus is

$$f_1\left(-3 \cdot \frac{1}{3}, 0\right) \text{ and } f_2\left(3 \cdot \frac{1}{3}, 0\right)$$

That is,

$$f_1(-1, 0) \text{ and } f_2(1, 0)$$

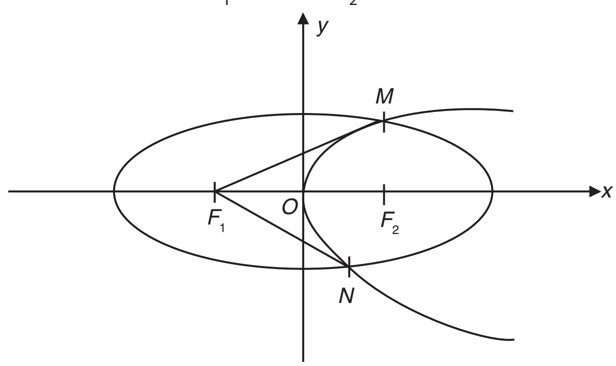


Figure 14.32

The equation of parabola is

$$y^2 = 4(OF_2)x$$

$$y^2 = 4x \quad (\text{Since } OF_2 = 1)$$

The point of intersection of ellipse and parabola is

$$\frac{x^2}{9}+\frac{4x}{8}=1 \Rightarrow \frac{x^2}{9}+\frac{x}{2}=1$$

$$\Rightarrow 2x^2+9x-18=0$$

$$\Rightarrow 2x^2+12x-3x-18=0$$

$$\Rightarrow 2x(x+6)-3(x+6)=0$$

$$\Rightarrow x=\frac{3}{2} \qquad \text{(Since } x=-6 \text{ is rejected)}$$

Now,

$$y^2(4)\frac{3}{2}=6$$

$$y=\pm\sqrt{6}$$

That is, the points M and N are, respectively, $M\left(\frac{3}{2},\sqrt{6}\right)$ and $N\left(\frac{3}{2},-\sqrt{6}\right)$.

See Fig. 14.33. Let the orthocentre be (h, k).

The slope of $OM = \dfrac{k-\sqrt{6}}{h-(3/2)}$

The slope of $ON = \dfrac{\sqrt{6}}{-1-(3/2)} = \dfrac{-2\sqrt{6}}{5}$

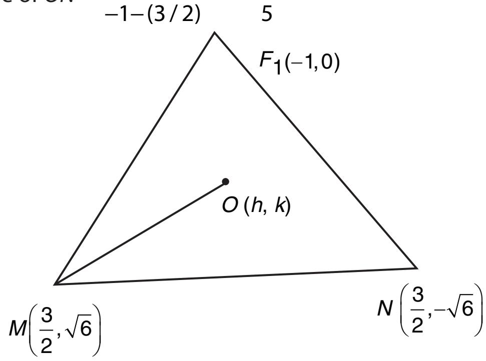


Figure 14.33

Now,

$$\left(\frac{k-\sqrt{6}}{h-(3/2)}\right)\left(\frac{-2\sqrt{6}}{5}\right)=-1$$

$$2\sqrt{6}k-12=5h-\frac{15}{2}$$

$$5h-2\sqrt{6}k=\frac{15}{2}-12=\frac{-9}{2}$$

The slope of $ON = \dfrac{k+\sqrt{6}}{h-(3/2)}$

The slope of $F_1M = \dfrac{\sqrt{6}}{(3/2)+1} = \dfrac{2\sqrt{6}}{5}$

$$\frac{k+\sqrt{6}}{h-(3/2)}\times\frac{2\sqrt{6}}{5}=-1$$

$$2\sqrt{6}k+12=-5h+\frac{15}{2}$$

$$5h+2\sqrt{6}k=\frac{15}{2}-12=\frac{-9}{2}$$

$$5h+2\sqrt{6}k=\frac{-9}{2} \qquad (1)$$

$$5h-2\sqrt{6}k=\frac{-9}{2} \qquad (2)$$

Solving Eqs. (1) and (2), we get

$$10h=-9 \Rightarrow h=\frac{-9}{10} \text{ and } k=0$$

Hence, the orthocentre of the triangle F_1MN is $\left(-\frac{9}{10},0\right)$.

Hence, the correct answer is option (A).

13. If the tangents to the ellipse at M and N meet at R and the normal to the parabola at M meets the x-axis at Q, then the ratio of area of the triangle MQR to area of the quadrilateral MF_1NF_2 is

(A) 3:4 **(B)** 4:5 **(C)** 5:8 **(D)** 2:3

Solution: From the common data given in the Solution of Question 12, the equation of tangent to ellipse at $M\left(\frac{3}{2},\sqrt{6}\right)$ is

$$\frac{3x}{2(9)}+\frac{y\sqrt{6}}{8}=1$$

Now, $\dfrac{x}{6}+\dfrac{y\sqrt{6}}{8}=1$ meets the x-axis, $y=0$

That is,

$$\frac{x}{6}=1 \Rightarrow x=6$$

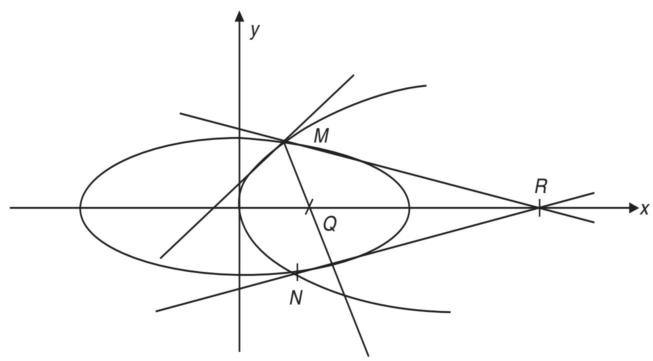


Figure 14.34

See Fig. 14.34. The point R is (6, 0).

The normal to the parabola at m is

$$y+tx=2at+at^3$$

where $a=1$, $t^2=\dfrac{3}{2}$ and $2t=\sqrt{6}$. Therefore, $t=\sqrt{\dfrac{3}{2}}$.

Now,

$$y+\sqrt{\frac{3}{2}}x=2\sqrt{\frac{3}{2}}+\frac{3}{2}\sqrt{\frac{3}{2}}$$

which cuts the x-axis at $y=0$.

$$\sqrt{\frac{3}{2}}x = 2\sqrt{\frac{3}{2}} + \frac{3}{2}\sqrt{\frac{3}{2}}$$

$$x = 2 + \frac{3}{2} = \frac{7}{2}$$

That is, we have $Q\left(\frac{7}{2}, 0\right)$.

The area of MQR is

$$\left|\frac{1}{2}\begin{vmatrix} 3/2 & \sqrt{6} & 1 \\ 6 & 0 & 1 \\ 7/2 & 0 & 1 \end{vmatrix}\right| = \left|\frac{\sqrt{6}}{2}\left(6 - \frac{7}{2}\right)\right| = \frac{5\sqrt{6}}{4}$$

The area of the quadrilateral MF_1NF_2 is

$$2(\Delta MF_1F_2) = 2\sqrt{6}$$

and the required ratio is

$$\frac{5\sqrt{6}}{4.2\sqrt{6}} = \frac{5}{8}$$

Hence, the correct answer is option (C).

Practice Exercise 1

1. If the latus rectum of an ellipse be equal to half of its minor axis, then its eccentricity is

   (A) 3/2 **(B)** $\sqrt{3}/2$ **(C)** 2/3 **(D)** $\sqrt{2}/3$

2. If the distance between the directrices be thrice the distance between the foci, then the eccentricity of ellipse is

   (A) 1/2 **(B)** 2/3 **(C)** $1/\sqrt{3}$ **(D)** 4/5

3. The equation of the ellipse whose centre is at origin and which passes through the points (–3, 1) and (2, –2) is

   (A) $5x^2+3y^2=32$ **(B)** $3x^2+5y^2=32$
   (C) $5x^2-3y^2=32$ **(D)** $3x^2+5y^2+32=0$

4. If the eccentricity of an ellipse be 5/8 and the distance between its foci be 10, then its latus rectum is

   (A) 39/4 **(B)** 12 **(C)** 15 **(D)** 37/2

5. If the foci and vertices of an ellipse be $(\pm 1, 0)$ and $(\pm 2, 0)$, then the minor axis of the ellipse is

   (A) $2\sqrt{5}$ **(B)** 2 **(C)** 4 **(D)** $2\sqrt{3}$

6. The equations of the directrices of the ellipse $16x^2+25y^2=400$ are

   (A) $2x=\pm 25$ **(B)** $5x=\pm 9$
   (C) $3x=\pm 10$ **(D)** None of these

7. The eccentricity of an ellipse is 2/3, its latus rectum is 5 and its centre is (0, 0). The equation of the ellipse is

   (A) $\frac{x^2}{81}+\frac{y^2}{45}=1$ **(B)** $\frac{4x^2}{81}+\frac{4y^2}{45}=1$
   (C) $\frac{x^2}{9}+\frac{y^2}{5}=1$ **(D)** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

8. The latus rectum of an ellipse is 10 and the minor axis is equal to the distance between the foci. The equation of the ellipse is

   (A) $x^2+2y^2=100$ **(B)** $x^2+\sqrt{2}y^2=10$
   (C) $x^2-2y^2=100$ **(D)** None of these

9. The distance between the directrices of the ellipse $\frac{x^2}{36}+\frac{y^2}{20}=1$ is

   (A) 8 **(B)** 12 **(C)** 18 **(D)** 24

10. The distance between the foci of the ellipse $3x^2+4y^2=48$ is

    (A) 2 **(B)** 4 **(C)** 6 **(D)** 8

11. The equation of the ellipse, whose vertices are $(\pm 5, 0)$ and foci are $(\pm 4, 0)$, is

    (A) $9x^2+25y^2=225$ **(B)** $25x^2+9y^2=225$
    (C) $3x^2+4y^2=192$ **(D)** None of these

12. The equation of the ellipse, whose foci are $(\pm 5, 0)$ and one of its directrix is $5x=36$, is

    (A) $\frac{x^2}{36}+\frac{y^2}{11}=1$ **(B)** $\frac{x^2}{6}+\frac{y^2}{\sqrt{11}}=1$
    (C) $\frac{x^2}{6}+\frac{y^2}{11}=1$ **(D)** None of these

13. If the eccentricity of an ellipse be $1/\sqrt{2}$, then its latus rectum is equal to its

    (A) Minor axis **(B)** Semi-minor axis
    (C) Major axis **(D)** Semi-major axis

14. The length of the latus rectum of the ellipse $5x^2+9y^2=45$ is

    (A) $\sqrt{5}/4$ **(B)** $\sqrt{5}/2$ **(C)** 5/3 **(D)** 10/3

15. If the distance between a focus and corresponding directrix of an ellipse be 8 and the eccentricity be 1/2, then length of the minor axis is

    (A) 3 **(B)** $4\sqrt{2}$ **(C)** 6 **(D)** None of these

16. The eccentricity of the conic $16x^2+7y^2=112$ is

    (A) $3/\sqrt{7}$ **(B)** 7/16 **(C)** 3/4 **(D)** 4/3

17. If the distance between the foci of an ellipse is equal to its minor axis, then its eccentricity is

    (A) 1/2 **(B)** $1/\sqrt{2}$ **(C)** 1/3 **(D)** $1/\sqrt{3}$

18. An ellipse passes through the point (–3, 1) and its eccentricity is $\sqrt{2/5}$. The equation of the ellipse is

    (A) $3x^2+5y^2=32$ **(B)** $3x^2+5y^2=25$
    (C) $3x^2+y^2=4$ **(D)** $3x^2+y^2=9$

19. The length of major and minor axis of an ellipse are 10 and 8, respectively, and its major axis is along *y*-axis. The equation of the ellipse referred to its centre as origin is

(A) $\frac{x^2}{25}+\frac{y^2}{16}=1$ **(B)** $\frac{x^2}{16}+\frac{y^2}{25}=1$

(C) $\frac{x^2}{100}+\frac{y^2}{64}=1$ **(D)** $\frac{x^2}{64}+\frac{y^2}{100}=1$

20. If the centre, one of the foci and semi-major axis of an ellipse be (0, 0), (0, 3) and 5, then its equation is

(A) $\frac{x^2}{16}+\frac{y^2}{25}=1$ **(B)** $\frac{x^2}{25}+\frac{y^2}{16}=1$

(C) $\frac{x^2}{9}+\frac{y^2}{25}=1$ **(D)** None of these

21. The equation of the ellipse, whose one of the vertices is (0,7) and the corresponding directrix is $y=12$, is

(A) $95x^2+144y^2=4655$ **(B)** $144x^2+95y^2=4655$

(C) $95x^2+144y^2=13,680$ **(D)** None of these

22. The equation $2x^2+3y^2=30$ represents

(A) A circle **(B)** An ellipse

(C) A hyperbola **(D)** A parabola

23. The equation of the ellipse, whose latus rectum is 8 and whose eccentricity is $1/\sqrt{2}$ referred to the principal axes of coordinates, is

(A) $\frac{x^2}{18}+\frac{y^2}{32}=1$ **(B)** $\frac{x^2}{8}+\frac{y^2}{9}=1$

(C) $\frac{x^2}{64}+\frac{y^2}{32}=1$ **(D)** $\frac{x^2}{16}+\frac{y^2}{24}=1$

24. The eccentricity of the ellipse, whose latus rectum is equal to the distance between two focus points, is

(A) $\frac{\sqrt{5}+1}{2}$ **(B)** $\frac{\sqrt{5}-1}{2}$ **(C)** $\frac{\sqrt{5}}{2}$ **(D)** $\frac{\sqrt{3}}{2}$

25. For the ellipse $3x^2+4y^2=12$, the length of latus rectum is

(A) $\frac{3}{2}$ **(B)** 3 **(C)** $\frac{8}{3}$ **(D)** $\sqrt{\frac{3}{2}}$

26. For the ellipse $\frac{x^2}{64}+\frac{y^2}{28}=1$, the eccentricity is

(A) $\frac{3}{4}$ **(B)** $\frac{4}{3}$ **(C)** $\frac{1}{\sqrt{7}}$ **(D)** $\frac{1}{3}$

27. If the length of the major axis of an ellipse is three times the length of its minor axis, then its eccentricity is

(A) $\frac{1}{3}$ **(B)** $\frac{1}{\sqrt{3}}$ **(C)** $\frac{1}{\sqrt{2}}$ **(D)** $\frac{2\sqrt{2}}{3}$

28. The length of the latus rectum of an ellipse is $1/3$ of the major axis. Its eccentricity is

(A) $\frac{2}{3}$ **(B)** $\sqrt{\frac{2}{3}}$ **(C)** $\frac{5\times4\times3}{7^3}$ **(D)** $\left(\frac{3}{4}\right)^4$

29. An ellipse is described by using an endless string which is passed over two pins. If the axes are 6 cm and 4 cm, then the necessary length of the string and the distance between the pins, respectively, in cm, are

(A) $6, 2\sqrt{5}$ **(B)** $6, \sqrt{5}$

(C) $4, 2\sqrt{5}$ **(D)** None of these

30. The equation $\frac{x^2}{2-r}+\frac{y^2}{r-5}+1=0$ represents an ellipse if

(A) $r>2$ **(B)** $2<r<5$

(C) $r>5$ **(D)** None of these

31. The locus of the point of intersection of perpendicular tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

(A) $x^2+y^2=a^2-b^2$ **(B)** $x^2-y^2=a^2-b^2$

(C) $x^2+y^2=a^2+b^2$ **(D)** $x^2-y^2=a^2+b^2$

32. The length of the latus rectum of the ellipse $\frac{x^2}{36}+\frac{y^2}{49}=1$ is

(A) 98/6 **(B)** 72/7

(C) 72/14 **(D)** 98/12

33. The distance of the point θ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ from a focus is

(A) $a(e+\cos\theta)$ **(B)** $a(e-\cos\theta)$

(C) $a(1+e\cos\theta)$ **(D)** $a(1+2e\cos\theta)$

34. The equation of the ellipse, whose one focus is at (4, 0) and whose eccentricity is 4/5, is

(A) $\frac{x^2}{3^2}+\frac{y^2}{5^2}=1$ **(B)** $\frac{x^2}{5^2}+\frac{y^2}{3^2}=1$

(C) $\frac{x^2}{5^2}+\frac{y^2}{4^2}=1$ **(D)** $\frac{x^2}{4^2}+\frac{y^2}{5^2}=1$

35. The foci of $16x^2+25y^2=400$ are

(A) $(\pm3, 0)$ **(B)** $(0, \pm3)$

(C) $(3, -3)$ **(D)** $(-3, 3)$

36. The eccentricity of the ellipse $9x^2+25y^2=225$ is

(A) $\frac{3}{5}$ **(B)** $\frac{4}{5}$ **(C)** $\frac{9}{25}$ **(D)** $\frac{\sqrt{34}}{5}$

37. The eccentricity of the ellipse $25x^2+16y^2=100$ is

(A) $\frac{5}{14}$ **(B)** $\frac{4}{5}$ **(C)** $\frac{3}{5}$ **(D)** $\frac{2}{5}$

38. The length of the latus rectum of the ellipse $9x^2+4y^2=1$ is

(A) $\frac{3}{2}$ **(B)** $\frac{8}{3}$ **(C)** $\frac{4}{9}$ **(D)** $\frac{8}{9}$

39. The locus of a variable point, whose distance from (–2, 0) is $2/3$ times its distance from the line $x=-9/2$, is

(A) Ellipse **(B)** Parabola

(C) Hyperbola **(D)** None of these

40. If $P(x, y)$, $F_1(3, 0)$, $F_2(-3, 0)$ and $16x^2+25y^2=400$, then PF_1+PF_2 equals
(A) 8 **(B)** 6 **(C)** 10 **(D)** 12

41. If P is any point on the ellipse $9x^2+36y^2=324$, whose foci are S and S', then $SP + S'P$ equals
(A) 3 **(B)** 12 **(C)** 36 **(D)** 324

42. What is the equation of the ellipse with foci $(\pm 2, 0)$ and eccentricity $1/2$?
(A) $3x^2+4y^2=48$ **(B)** $4x^2+3y^2=48$
(C) $3x^2+4y^2=0$ **(D)** $4x^2+3y^2=0$

43. The eccentricity of the ellipse $4x^2+9y^2=36$, is
(A) $\frac{1}{2\sqrt{3}}$ **(B)** $\frac{1}{\sqrt{3}}$ **(C)** $\frac{\sqrt{5}}{3}$ **(D)** $\frac{\sqrt{5}}{6}$

44. The eccentricity of the ellipse $25x^2+16y^2=400$ is
(A) 3/5 **(B)** 1/3 **(C)** 2/5 **(D)** 1/5

45. The distance between the foci of an ellipse is 16 and eccentricity is $1/2$. The length of the major axis of the ellipse is
(A) 8 **(B)** 64 **(C)** 16 **(D)** 32

46. If the eccentricity of the two ellipses $\frac{x^2}{169}+\frac{y^2}{25}=1$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are equal, then the value of a/b is
(A) 5/13 **(B)** 6/13 **(C)** 13/5 **(D)** 13/6

47. In the ellipse, the minor axis is 8 and eccentricity is $\sqrt{5}/3$. Then major axis is
(A) 6 **(B)** 12 **(C)** 10 **(D)** 16

48. In an ellipse $9x^2+5y^2=45$, the distance between the foci is
(A) $4\sqrt{5}$ **(B)** $3\sqrt{5}$ **(C)** 3 **(D)** 4

49. The equation of the ellipse with eccentricity $1/2$ and foci at $(\pm 1, 0)$ is
(A) $\frac{x^2}{3}+\frac{y^2}{4}=1$ **(B)** $\frac{x^2}{4}+\frac{y^2}{3}=1$
(C) $\frac{x^2}{3}+\frac{y^2}{4}=\frac{4}{3}$ **(D)** None of these

50. The sum of focal distances of any point on the ellipse with major and minor axes, $2a$ and $2b$, respectively, is equal to
(A) $2a$ **(B)** $\frac{2a}{b}$ **(C)** $\frac{2b}{a}$ **(D)** $\frac{b^2}{a}$

51. The equation of ellipse, whose distance between the foci is equal to 8 and the distance between the directrices is 18, is
(A) $5x^2-9y^2=180$ **(B)** $9x^2+5y^2=180$
(C) $x^2+9y^2=180$ **(D)** $5x^2+9y^2=180$

52. In an ellipse, the distance between its foci is 6 and its minor axis is 8. Then its eccentricity is
(A) $\frac{4}{5}$ **(B)** $\frac{1}{\sqrt{52}}$ **(C)** $\frac{3}{5}$ **(D)** $\frac{1}{2}$

53. If a bar of given length moves with its extremities on two fixed straight lines at right angles, then the locus of any point on bar marked on the bar describes a/an
(A) Circle **(B)** Parabola
(C) Ellipse **(D)** Hyperbola

54. The centre of the ellipse $4x^2+9y^2-16x-54y+61=0$ is
(A) (1, 3) **(B)** (2, 3) **(C)** (3, 2) **(D)** (3, 1)

55. The latus rectum of the ellipse $4x^2+9y^2-8x-36y+4=0$ is
(A) 8/3 **(B)** 4/3 **(C)** $\sqrt{5}/3$ **(D)** 16/3

56. The eccentricity of the ellipse $4x^2+y^2-8x+2y+1=0$ is
(A) $1/\sqrt{3}$ **(B)** $\sqrt{3}/2$
(C) $1/2$ **(D)** None of these

57. The equation of an ellipse, whose eccentricity is 1/2 and the vertices are (4, 0) and (10, 0) is
(A) $3x^2+4y^2-42x+120=0$
(B) $3x^2+4y^2+42x+120=0$
(C) $3x^2+4y^2+42x-120=0$
(D) $3x^2+4y^2-42x-120=0$

58. The equation of the ellipse, whose centre is (2, –3), one of the foci is (3, –3) and the corresponding vertex is (4, –3) is
(A) $\frac{(x-2)^2}{3}+\frac{(y+3)^2}{4}=1$ **(B)** $\frac{(x-2)^2}{4}+\frac{(y+3)^2}{3}=1$
(C) $\frac{x^2}{3}+\frac{y^2}{4}=1$ **(D)** None of these

59. The equation $14x^2-4xy+11y^2-44x-58y+71=0$ represents
(A) A circle
(B) An ellipse
(C) A hyperbola
(D) A rectangular hyperbola

60. The centre of the ellipse $\frac{(x+y-2)^2}{9}+\frac{(x-y)^2}{16}=1$ is
(A) (0, 0) **(B)** (1, 1) **(C)** (1, 0) **(D)** (0, 1)

61. The equation of an ellipse, whose focus is (–1, 1), directrix is $x-y+3=0$ and eccentricity is $1/2$, is given by
(A) $7x^2+2xy+7y^2+10x-10y+7=0$
(B) $7x^2-2xy+7y^2-10x+10y+7=0$
(C) $7x^2-2xy+7y^2-10x-10y-7=0$
(D) $7x^2-2xy+7y^2+10x+10y-7=0$

62. The foci of the ellipse $25(x+1)^2+9(y+2)^2=225$ are at
(A) (–1, 2) and (–1, –6)
(B) (–1, 2) and (6, 1)
(C) (1, –2) and (1, –6)
(D) (–1, –2) and (1, 6)

63. The eccentricity of the ellipse $9x^2+5y^2-30y=0$ is
(A) 1/3 **(B)** 2/3 **(C)** 3/4 **(D)** None of these

64. The curve represented by $x=3(\cos t+\sin t)$, $y=4(\cos t-\sin t)$ is an/a

(A) Ellipse **(B)** Parabola
(C) Hyperbola **(D)** Circle

65. The equation $x=a\cos\theta, y=b\sin\theta (a>b)$ represents a conic section whose eccentricity e is given by

(A) $e^2=\dfrac{a^2+b^2}{a^2}$ **(B)** $e^2=\dfrac{a^2+b^2}{b^2}$
(C) $e^2=\dfrac{a^2-b^2}{a^2}$ **(D)** $e^2=\dfrac{a^2-b^2}{b^2}$

66. The eccentricity of the ellipse $4x^2+9y^2+8x+36y+4=0$ is

(A) 5/6 **(B)** 3/5 **(C)** $\sqrt{2}/3$ **(D)** $\sqrt{5}/3$

67. The coordinates of the foci of the ellipse $3x^2+4y^2-12x-8y+4=0$ are

(A) (1, 2); (3, 4) **(B)** (1, 4); (3, 1)
(C) (1, 1); (3, 1) **(D)** (2, 3); (5, 4)

68. The eccentricity of the curve represented by the equation $x^2+2y^2-2x+3y+2=0$ is

(A) 0 **(B)** 1/2 **(C)** $1/\sqrt{2}$ **(D)** $\sqrt{2}$

69. For the ellipse $25x^2+9y^2-150x-90y+225=0$, the eccentricity is

(A) 2/5 **(B)** 3/5 **(C)** 4/5 **(D)** 1/5

70. The eccentricity of the ellipse $\dfrac{(x-1)^2}{9}+\dfrac{(y+1)^2}{25}=1$ is

(A) 4/5 **(B)** 3/5 **(C)** 5/4 **(D)** Imaginary

71. The length of the axes of the conic $9x^2+4y^2-6x+4y+1=0$ are

(A) $\dfrac{1}{2}, 9$ **(B)** $3, \dfrac{2}{5}$ **(C)** $1, \dfrac{2}{3}$ **(D)** 3, 2

72. The eccentricity of the ellipse $9x^2+5y^2-18x-2y-16=0$ is

(A) 1/2 **(B)** 2/3 **(C)** 1/3 **(D)** 3/4

73. The eccentricity of the conic $4x^2+16y^2-24x-32y=1$ is

(A) $\dfrac{\sqrt{3}}{2}$ **(B)** $\dfrac{1}{2}$ **(C)** $\dfrac{\sqrt{3}}{4}$ **(D)** $\sqrt{3}$

74. If the line $y=2x+c$ be a tangent to the ellipse $\dfrac{x^2}{8}+\dfrac{y^2}{4}=1$, then what is the value of c?

(A) ± 4 **(B)** ± 6 **(C)** ± 1 **(D)** ± 8

75. The position of the point (4, –3) with respect to the ellipse $2x^2+5y^2=20$ is

(A) Outside the ellipse **(B)** On the ellipse
(C) On the major axis **(D)** None of these

76. The equation of the tangent to the ellipse $x^2+16y^2=16$ which is making an angle of 60° with x-axis is

(A) $\sqrt{3}x-y+7=0$ **(B)** $\sqrt{3}x-y-7=0$
(C) $\sqrt{3}x-y\pm 7=0$ **(D)** None of these

77. The position of the point (1, 3) w.r.t. the ellipse $4x^2+9y^2-16x-54y+61=0$ is

(A) Outside the ellipse **(B)** On the ellipse
(C) On the major axis **(D)** On the minor axis

78. The line $lx+my-n=0$ is tangent to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ if

(A) $a^2l^2+b^2m^2=n^2$ **(B)** $al^2+bm^2=n^2$
(C) $a^2l+b^2m=n$ **(D)** None of these

79. The locus of the point of intersection of mutually perpendicular tangent to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is

(A) A straight line
(B) A parabola
(C) A circle
(D) None of these

80. The equation of the tangent at the point $\left(\dfrac{1}{4},\dfrac{1}{4}\right)$ of the ellipse $\dfrac{x^2}{4}+\dfrac{y^2}{12}=1$ is

(A) $3x+y=48$ **(B)** $3x+y=3$
(C) $3x+y=16$ **(D)** None of these

81. The angle between the pair of tangents that are drawn to the ellipse $3x^2+2y^2=5$ from the point (1, 2) is

(A) $\tan^{-1}\left(\dfrac{12}{5}\right)$ **(B)** $\tan^{-1}(6\sqrt{5})$
(C) $\tan^{-1}\left(\dfrac{12}{\sqrt{5}}\right)$ **(D)** $\tan^{-1}(12\sqrt{5})$

82. The equations of the tangents of the ellipse $9x^2+16y^2=144$ which passes through the point (2, 3) is

(A) $y=3, x+y=5$ **(B)** $y=-3, x-y=5$
(C) $y=4, x+y=3$ **(D)** $y=-4, x-y=3$

83. If any tangent to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ cuts off intercepts of length h and k on the axes, then find $\dfrac{a^2}{h^2}+\dfrac{b^2}{k^2}$.

(A) 0 **(B)** 1 **(C)** –1 **(D)** None of these

84. If the line $y=mx+c$ touches the ellipse $\dfrac{x^2}{b^2}+\dfrac{y^2}{a^2}=1$, then find the value of c.

(A) $\pm\sqrt{b^2m^2+a^2}$ **(B)** $\pm\sqrt{a^2m^2+b^2}$
(C) $\pm\sqrt{b^2m^2-a^2}$ **(D)** $\pm\sqrt{a^2m^2-b^2}$

85. The ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ and the straight line $y=mx+c$ intersect in real points only if

(A) $a^2m^2<c^2-b^2$ **(B)** $a^2m^2>c^2-b^2$
(C) $a^2m^2\geq c^2-b^2$ **(D)** $c\geq b$

86. If $y=mx+c$ is the tangent on the ellipse $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$, then the value of c is

(A) 0 **(B)** $3/m$

(C) $\pm\sqrt{9m^2+4}$ (D) $\pm 3\sqrt{1+m^2}$

87. The locus of the point of intersection of the perpendicular tangents to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ is

(A) $x^2+y^2=9$ (B) $x^2+y^2=4$
(C) $x^2+y^2=13$ (D) $x^2+y^2=5$

88. The eccentric angles of the extremities of latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are given by

(A) $\tan^{-1}\left(\pm\frac{ae}{b}\right)$ (B) $\tan^{-1}\left(\pm\frac{be}{a}\right)$
(C) $\tan^{-1}\left(\pm\frac{b}{ae}\right)$ (D) $\tan^{-1}\left(\pm\frac{a}{be}\right)$

89. The eccentric angle of a point on the ellipse $x^2+3y^2=6$ at a distance 2 unit from the centre of the ellipse is

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{3}$ (C) $\frac{3\pi}{4}$ (D) $\frac{2\pi}{3}$

90. The equation of the tangents which are drawn at the ends of the major axis of the ellipse $9x^2+5y^2-30y=0$ are

(A) $y=\pm 3$ (B) $x=\pm\sqrt{5}$
(C) $y=0, y=6$ (D) None of these

91. The equation of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $(a\cos\theta, b\sin\theta)$ is

(A) $\frac{ax}{\sin\theta}-\frac{by}{\cos\theta}=a^2-b^2$ (B) $\frac{ax}{\sin\theta}-\frac{by}{\cos\theta}=a^2+b^2$
(C) $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$ (D) $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2+b^2$

92. If the normal at the point $P(\theta)$ to the ellipse $\frac{x^2}{14}+\frac{y^2}{5}=1$ intersects it again at the point $Q(2\theta)$, then $\cos\theta$ is equal to

(A) $\frac{2}{3}$ (B) $-\frac{2}{3}$ (C) $\frac{3}{2}$ (D) $-\frac{3}{2}$

93. The line $y=mx+c$ is a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ for which value of c?

(A) $-(2am+bm^2)$ (B) $\frac{(a^2+b^2)m}{\sqrt{a^2+b^2m^2}}$
(C) $-\frac{(a^2-b^2)m}{\sqrt{a^2+b^2m^2}}$ (D) $\frac{(a^2-b^2)m}{\sqrt{a^2+b^2}}$

94. The equation of normal at the point (0, 3) of the ellipse $9x^2+5y^2=45$ is

(A) $y-3=0$ (B) $y+3=0$
(C) x-axis (D) y-axis

95. The equation of the normal at the point (2, 3) on the ellipse $9x^2+16y^2=180$ is

(A) $3y=8x-10$ (B) $3y-8x+7=0$
(C) $8y+3x+7=0$ (D) $3x+2y+7=0$

96. If the line $x\cos\alpha+y\sin\alpha=p$ be normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then

(A) $p^2(a^2\cos^2\alpha+b^2\sin^2\alpha)=a^2-b^2$
(B) $p^2(a^2\cos^2\alpha+b^2\sin^2\alpha)=(a^2-b^2)^2$
(C) $p^2(a^2\sec^2\alpha+b^2\text{cosec}^2\alpha)=a^2-b^2$
(D) $p^2(a^2\sec^2\alpha+b^2\text{cosec}^2\alpha)=(a^2-b^2)^2$

97. The line $lx+my+n=0$ is a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if

(A) $\frac{a^2}{m^2}+\frac{b^2}{l^2}=\frac{(a^2-b^2)}{n^2}$ (B) $\frac{a^2}{l^2}+\frac{b^2}{m^2}=\frac{(a^2-b^2)^2}{n^2}$
(C) $\frac{a^2}{l^2}-\frac{b^2}{m^2}=\frac{(a^2-b^2)^2}{n^2}$ (D) None of these

98. The equation of tangent and normal at point (3, –2) of ellipse $4x^2+9y^2=36$ are

(A) $\frac{x}{3}-\frac{y}{2}=1, \frac{x}{2}+\frac{y}{3}=\frac{5}{6}$ (B) $\frac{x}{3}+\frac{y}{2}=1, \frac{x}{2}-\frac{y}{3}=\frac{5}{6}$
(C) $\frac{x}{2}+\frac{y}{3}=1, \frac{x}{3}-\frac{y}{2}=\frac{5}{6}$ (D) None of these

99. The value of λ, for which the line $2x-\frac{8}{3}\lambda y=-3$ is a normal to the conic $x^2+\frac{y^2}{4}=1$ is

(A) $\frac{\sqrt{3}}{2}$ (B) $\frac{1}{2}$ (C) $-\frac{\sqrt{3}}{2}$ (D) $\frac{3}{8}$

100. The pole of the straight line $x+4y=4$ with respect to ellipse $x^2+4y^2=4$ is

(A) (1, 4) (B) (1, 1) (C) (4, 1) (D) (4, 4)

101. In the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the equation of diameter conjugate to the diameter $y=\frac{b}{a}x$ is

(A) $y=-\frac{b}{a}x$ (B) $y=-\frac{a}{b}x$
(C) $x=-\frac{b}{a}y$ (D) None of these

102. An ellipse has *OB* as semi-minor axis, *F* and *F′* its foci and ∠*FBF′* is a right angle. Then the eccentricity of the ellipse is

(A) $\frac{1}{4}$ (B) $\frac{1}{\sqrt{3}}$ (C) $\frac{1}{\sqrt{2}}$ (D) $\frac{1}{2}$

103. If the foci of an ellipse are $(\pm\sqrt{5},0)$ and its eccentricity is $\frac{\sqrt{5}}{3}$, then the equation of the ellipse is

(A) $9x^2+4y^2=36$ (B) $4x^2+9y^2=36$
(C) $36x^2+9y^2=4$ (D) $9x^2+36y^2=4$

104. The sum of the focal distances of any point on the conic $\frac{x^2}{25}+\frac{y^2}{16}=1$ is

(A) 10 (B) 9 (C) 41 (D) 18

105. The eccentricity of the ellipse $25x^2+16y^2-150x-175=0$ is

(A) 2/5 **(B)** 2/3 **(C)** 4/5 **(D)** 3/5

106. The point (4, –3) w.r.t. the ellipse $4x^2+5y^2=1$

(A) lies on the curve **(B)** is inside the curve
(C) is outside the curve **(D)** is focus of the curve

107. A point ratio of whose distances from a fixed point (–2, 0) and line $x=9/2$ is always 2:3. Then locus of the point is

(A) Hyperbola **(B)** Ellipse
(C) Parabola **(D)** Circle

Practice Exercise 2

Single/Multiple Correct Choice Type Questions

1. The curve $y=(|x|-1)\text{sgn}(x-1)$ divides $\frac{9x^2}{64}+\frac{4}{25}y^2=\frac{1}{\pi}$ in two parts having areas A_1 and A_2 (where $A_1<A_2$), then

(A) $\frac{A_1}{A_2}=\frac{7}{13}$ **(B)** $\frac{A_1}{A_2}=\frac{3}{7}$
(C) $A_1=\frac{7}{3}$ **(D)** $A_2=\frac{13}{7}$

2. Let $S_1\equiv 3x^2+4y^2-1=0$; $S_2\equiv|x+y|-1=0$ and $S_3\equiv|y-x|-1=0$ are the given curves. Then which of the following is/are not true?

(A) Area enclosed by the exterior to the auxiliary circle of $S_1=0$, $S_2\le 0$ and $S_3\le 0$ is $2-\frac{\pi}{3}$ sq. units.

(B) Area enclosed by the exterior to the director circle of $S_1=0$, $S_2\le 0$ and $S_3\le 0$ is $2-\frac{\pi}{4}$ sq. units.

(C) Area enclosed by $S_2=0$ and $S_3=0$ is 1 sq. unit.

(D) Area enclosed by $S_1\ge 0$, $S_2\le 0$ and $S_3\le 0$ is $2-\frac{\pi}{3\sqrt{2}}$ sq. units.

3. If P is a point of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose foci are S and S'. Let $\angle PSS'=\alpha$ and $\angle PS'S=\beta$, then

(A) $PS+PS'=2a$, if $a>b$
(B) $PS+PS'=2b$, if $a<b$
(C) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{1-e}{1+e}$
(D) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{\sqrt{a^2-b^2}}{b^2}[a-\sqrt{a^2-b^2}]$ when $a>b$

4. Let F_1, F_2 be two foci of the ellipse and PT and PN be the tangent and the normal respectively to the ellipse at point P, then

(A) PN bisects $\angle F_1PF_2$
(B) PT bisects $\angle F_1PF_2$
(C) PT bisects angle $(180° - \angle F_1PF_2)$
(D) None of these

5. Let $A(\alpha)$ and $B(\beta)$ be the extremities of a chord of an ellipse. If the slope of AB is equal to the slope of the tangent at a point $C(\theta)$ on the ellipse, then the value of θ is

(A) $\frac{\alpha+\beta}{2}$ **(B)** $\frac{\alpha-\beta}{2}$
(C) $\frac{\alpha+\beta}{2}+\pi$ **(D)** $\frac{\alpha-\beta}{2}-\pi$

6. The parametric angle θ, where $-\pi - \theta\le\pi$, of the point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at which the tangent drawn cuts the intercept of minimum length on the coordinate axes, is/are

(A) $\tan{-1}\sqrt{\frac{b}{a}}$ **(B)** $-\tan{-1}\sqrt{\frac{b}{a}}$
(C) $\pi-\tan{-1}\sqrt{\frac{b}{a}}$ **(D)** $\pi+\tan{-1}\sqrt{\frac{b}{a}}$

7. The equation $3x^2+4y^2-18x+16y+43=c$

(A) cannot represent a real pair of straight lines for any value of c
(B) represents an ellipse, if $c>0$
(C) represent empty set, if $c<0$
(D) a point, if $c=0$.

Comprehension Type Questions

Paragraph for Questions 8–10: An ellipse E has its centre C (1, 3) focus at S (6, 3) and passing through the point P (4, 7), then

8. The product of the lengths of the perpendicular segments from the foci on tangent at point P is

(A) 20 **(B)** 45 **(C)** 40 **(D)** cannot be determined

9. The point of intersection of the lines joining each focus to the foot of the perpendicular from the other focus upon the tangent at point P, is

(A) $\left(\frac{5}{3},5\right)$ **(B)** $\left(\frac{4}{3},3\right)$ **(C)** $\left(\frac{8}{3},3\right)$ **(D)** $\left(\frac{10}{3},5\right)$

10. If the normal at a variable point on the ellipse (E) meets its axes in Q and R, then the locus of the midpoint of QR is a conic with an eccentricity (e'), then

(A) $e'=\frac{3}{\sqrt{10}}$ **(B)** $e'=\frac{\sqrt{5}}{3}$ **(C)** $e'=\frac{3}{\sqrt{5}}$ **(D)** $e'=\frac{\sqrt{10}}{3}$

Paragraph for Questions 11–13: Consider an ellipse (E) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, centered at point '$O$' and having AB and CD as its major and minor axes, respectively, if S_1 be one of the foci of the ellipse, radius of in-circle of triangle OCS_1, be 1 unit and $OS_1=6$ units, then

11. The area of ellipse (E) is

(A) $\frac{65\pi}{4}$ **(B)** $\frac{64\pi}{5}$ **(C)** 64π **(D)** 65π

12. Perimeter of ΔOCS_1, is

(A) 20 units **(B)** 10 units
(C) 15 units **(D)** 25 units

13. If S be the director circle of ellipse (E), then the equation of director circle of S is

(A) $x^2 + y^2 = (48.5)$ **(B)** $x^2 + y^2 = \sqrt{97}$
(C) $x^2 + y^2 = 97$ **(D)** $x^2 + y^2 = \sqrt{48.5}$

Paragraph for Questions 14–16: Second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents an ellipse if $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0$ and $h^2 < ab$. Intersection of major axis and minor axis gives centre of ellipse

14. There are exactly 'n' integral values of λ for which equation $x^2 + \lambda xy + y^2 = 1$ represents an ellipse then 'n' must be
(A) 0 **(B)** 1 **(C)** 2 **(D)** 3

15. Length of the longest chord of the ellipse $x^2 + y^2 + xy = 1$ is
(A) $\sqrt{2}$ **(B)** $\frac{1}{\sqrt{2}}$ **(C)** $2\sqrt{2}$ **(D)** 1

16. Length of the chord perpendicular to the longest chord as in the above question and passing through the centre of the ellipse is
(A) $\frac{1}{\sqrt{2}}$ **(B)** $\frac{\sqrt{3}}{2}$ **(C)** $\sqrt{\frac{2}{3}}$ **(D)** $\frac{1}{\sqrt{3}}$

Paragraph for Questions 17–19: A bird flies on ellipse $ax^2 + by^2 = 1$ and $z = 5\sqrt{3}$ $(b > a > 0)$ whose eccentricity is $\frac{1}{\sqrt{2}}$. An observer stands at a point $P(\alpha, \beta, 0)$ where the maximum and minimum angle of elevation of the bird are 60° and 30° when the bird is at Q and R, respectively, on its path and Q' and R' are projection of Q and R on xy plane, P, Q', R' are collinear and the distance between Q' and R' is maximum. Let θ be the angle of elevation of the bird when it is at a point on the arc of the ellipse exactly midway between Q and R. It is given that $a\alpha^2 + b\beta^2 - 1 > 0$

17. If $\alpha > 0$, then the equation of the line along which the minimum angle of elevation is observed, is
(A) $\frac{x - \sqrt{3}}{\sqrt{3}} = \frac{z + \sqrt{3}}{-1}$
(B) $\frac{x - 13}{\sqrt{3}} = \frac{z + \sqrt{3}}{-1}, y = 0$
(C) $\frac{x - 10}{\sqrt{3}} = \frac{z + \sqrt{3}}{-1}, y = 0$
(D) $\frac{x - 10}{\sqrt{3}} = \frac{y}{0} = \frac{z + \sqrt{3}}{-1}$

18. Equation of the plane which touches the ellipse at Q and passes through P $(\alpha > 0)$ is
(A) $-\sqrt{3}\,x + y + z - 10\sqrt{3} = 0$
(B) $\sqrt{3}\,x + y + z - 10\sqrt{3} = 0$
(C) $\sqrt{3}\,x + z - 10\sqrt{3} = 0$
(D) $\sqrt{3}\,x + y - 10\sqrt{3} = 0$

19. Value of $\tan\theta$, is
(A) $\frac{2}{3}$ **(B)** $\frac{3}{2}$ **(C)** $\sqrt{\frac{2}{3}}$ **(D)** $\sqrt{\frac{6}{5}}$

Matrix Match Type Questions

20. Match the following:

List I	List II
(A) Length of common tangent to the hyperbola $x^2 - 9y^2 = 9$ and $y^2 - 9x^2 = 9$ is	**(p)** 9/8
(B) A line drawn through the focus of hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ and parallel to one of its asymptotes meets the curve at P, then SP is equal to	**(q)** $\sqrt{3}$
(C) Tangent to the ellipse $\frac{x^2}{12} + \frac{y^2}{324} = 1$ at $(2\sqrt{3}\cos\theta, 18\sin\theta)$, $(\theta \in (0, \pi))$ is drawn. The value of $\tan\theta$ such that the sum of length of the intercepts on the axes made by this tangent is minimum, is	**(r)** 5
(D) If the line $2x + \sqrt{6}y = 2$ touches the hyperbola $x^2 - 2y^2 = 4$, and the slope of the line joining the origin and point of contact is m, then $\sqrt{8}m$ is	**(s)** $-\sqrt{3}$
	(t) 1

21. Match the following:

List I	List II
(A) A tangent to the ellipse $\frac{x^2}{27} + \frac{y^2}{48} = 1$ having slope $-\frac{4}{3}$ cuts the x- and y-axis at the points A and B, respectively. If O is the origin then the area of triangle OAB is equal to	**(p)** 36
(B) Product of the perpendiculars drawn from the points (±3, 0) to the line $y = mx - \sqrt{25m^2 + 16}$ is	**(q)** 72
(C) An ellipse passing through the origin has its foci (3, 4) and (6, 8), then the length of its minor axis is	**(r)** 10
(D) If PQ is focal chord of ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ which passes through $S \equiv (3, 0)$ and $PS = 2$, then the length of the chord PQ is	**(s)** 16
	(t) $10\sqrt{2}$

22. Match the following:

List I	List II
(A) A stick of length 10 m slides on coordinate axes, then locus of a point dividing this stick reckoning from x-axis in the ratio 6:4 is a curve whose eccentricity is e, then $9e$ is equal to	**(p)** $\sqrt{6}$
(B) AA' is major axis of an ellipse $3x^2 + 2y^2 + 6x - 4y - 1 = 0$ and P is a variable point on it then the greatest area of triangle APA' is	**(q)** $2\sqrt{7}$
(C) Distance between the foci of the curve represented by the equation $x = 1 + 4\cos\theta, y = 2 + 3\sin\theta$ is	**(r)** $\frac{128}{3}$
(D) Tangents are drawn to the ellipse $\frac{x^2}{16} + \frac{y^2}{7} = 1$ at end points of the latus rectum. The area of the quadrilateral so formed is	**(s)** $3\sqrt{5}$
	(t) $\frac{\sqrt{5}}{3}$

23. Match the following:

List I	List II
(A) If the angle between the straight lines joining the foci and one of the ends of the minor axis of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ is 90°. Find its eccentricity.	**(p)** **4**
(B) For an ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ with vertices A and A', tangent drawn at the point P in the first quadrant meets the y-axis in Q and the chord $A'P$ meets the y-axis in M. If 'O' is the origin, then $OQ_2 - MQ_2$ equals to	**(q)** 2
(C) The x-coordinate of points on the axis of the parabola $4y^2 - 32x + 4y + 65 = 0$ from which all the three normals to the parabola are real is	**(r)** $\frac{1}{\sqrt{2}}$
(D) The area of the parallelogram inscribed in the ellipse $\frac{x^2}{2^2}+\frac{y^2}{(1/2)^2} = 1$ whose diagonals are the conjugate diameters of the ellipse are given by	**(s)** 7
	(t) 8

Integer Type Questions

24. Let P, Q be two points on the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ whose eccentric angles differ by a right angle. Tangents are drawn at P and Q to meet at R. If the chord PQ divides the join of C and R in the ratio $m{:}n$ (C being the centre of the ellipse), then find $m + n$ ($m{:}n$ is in simplified form).

25. Find the number of tangent(s) which can be drawn to the ellipse $16x^2 + 25y^2 = 400$, such that the sum of perpendicular distances from the foci to the tangent is 8.

26. The tangent at the point having an eccentric angle of $\frac{7\pi}{6}$ on the ellipse $b^2x^2 + a^2y^2 = a^2b^2, a > b$ with eccentricity e meets the auxiliary circle in two points which subtend a right angle at the centre. Let $f(x, y)=0$ represents family of hyperbolas having transverse axes along y-axis, center at origin and eccentricity $e+\sqrt{5}$. Then the integral part of reciprocal of absolute value of the slope of asymptotes of family $f(x, y)=0$ is equal to ______.

27. Number of distinct normal lines that can be drawn to ellipse $\frac{x^2}{169}+\frac{y^2}{25} = 1$ from the point P (0, 6) is ______.

28. Origin O is the centre of two concentric circles whose radii are a and b, respectively, $a < b$. A line OPQ is drawn to cut the inner circle in P and the outer circle in Q. PR is drawn parallel to the y–axis and QR is drawn parallel to the x–axis. The locus of R is an ellipse touching the two circles. If the foci of this ellipse lie on the inner circle, if eccentricity is $\sqrt{2}\,\lambda$, then find λ.

29. Number of points on the ellipse $\frac{x^2}{50} + \frac{y^2}{20} = 1$ from which a pair of perpendicular tangents may be drawn to the ellipse $\frac{x^2}{16}+\frac{y^2}{9} = 1$ is _____.

Answer Key

Practice Exercise 1

1. (B)	**2.** (C)	**3.** (B)	**4.** (A)	**5.** (D)	**6.** (D)
7. (B)	**8.** (A)	**9.** (C)	**10.** (B)	**11.** (A)	**12.** (A)
13. (D)	**14.** (D)	**15.** (D)	**16.** (C)	**17.** (B)	**18.** (A)
19. (B)	**20.** (A)	**21.** (B)	**22.** (B)	**23.** (C)	**24.** (B)

25. (B)	**26.** (A)	**27.** (D)	**28.** (B)	**29.** (D)	**30.** (B)
31. (C)	**32.** (B)	**33.** (C)	**34.** (B)	**35.** (A)	**36.** (B)
37. (C)	**38.** (C)	**39.** (A)	**40.** (C)	**41.** (B)	**42.** (A)
43. (C)	**44.** (A)	**45.** (D)	**46.** (C)	**47.** (B)	**48.** (D)
49. (B)	**50.** (A)	**51.** (D)	**52.** (C)	**53.** (C)	**54.** (B)
55. (A)	**56.** (B)	**57.** (A)	**58.** (B)	**59.** (B)	**60.** (B)
61. (A)	**62.** (A)	**63.** (B)	**64.** (A)	**65.** (C)	**66.** (D)
67. (C)	**68.** (C)	**69.** (C)	**70.** (A)	**71.** (C)	**72.** (B)
73. (A)	**74.** (B)	**75.** (A)	**76.** (C)	**77.** (C)	**78.** (A)
79. (C)	**80.** (D)	**81.** (C)	**82.** (A)	**83.** (B)	**84.** (A)
85. (C)	**86.** (C)	**87.** (C)	**88.** (C)	**89.** (A), (C)	**90.** (C)
91. (C)	**92.** (B)	**93.** (C)	**94.** (D)	**95.** (B)	**96.** (D)
97. (B)	**98.** (A)	**99.** (A), (C)	**100.** (B)	**101.** (A)	**102.** (C)
103. (B)	**104.** (A)	**105.** (D)	**106.** (C)	**107.** (B)	

Practice Exercise 2

1. (A), (C)	**2.** (B), (C), (D)	**3.** (A), (B), (C)	**4.** (A), (C)	**5.** (A), (C)	**6.** (A), (B), (C)
7. (A), (B), (C), (D)	**8.** (A)	**9.** (D)	**10.** (B)	**11.** (A)	**12.** (C)
13. (C)	**14.** (D)	**15.** (C)	**16.** (C)	**17.** (B)	**18.** (C)
19. (C)	**20.** (A) → (r); (B) → (p); (C) → (q), (s); (D) → (s)	**21.** (A) → (p); (B) → (s); (C) → (t); (D) → (r)	**22.** (A) → (s); (B) → (p); (C) → (q); (D) → (r)	**23.** (A) → (r); (B) → (p); (C) → (s), (t); (D) → (q)	**24.** 2
25. 2	**26.** 2	**27.** 3	**28.** 1/2	**29.** 4	

Solutions

Practice Exercise 1

1. We have

$$\frac{2b^2}{a}=b$$

$$\Rightarrow \frac{b}{a}=\frac{1}{2} \Rightarrow \frac{b^2}{a^2}=\frac{1}{4}$$

Hence,

$$e=\sqrt{1-\frac{b^2}{a^2}}=\frac{\sqrt{3}}{2}$$

2. According to the condition,

$$\frac{2a}{e}=6ae \Rightarrow e=\frac{1}{\sqrt{3}}$$

3. We have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Since this passes through (–3, 1) and (2, –2), we get

$$\frac{9}{a^2}+\frac{1}{b^2}=1 \text{ and } \frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{4}$$

$$\Rightarrow a^2=\frac{32}{3};\ b^2=\frac{32}{5}$$

Hence, the required equation of ellipse is $3x^2+5y^2=32$.

Trick: Since only the equation $3x^2+5y^2=32$ passes through (–3, 1) and (2, –2). Hence, the result.

4. We have

$$a=\frac{10}{2(5)/8}=8;\ b=8\sqrt{1-\frac{25}{64}}=8\left(\frac{\sqrt{39}}{8}\right)=\sqrt{39}$$

The latus rectum is

$$\frac{2b^2}{a}=\frac{2\times 39}{8}=\frac{39}{4}$$

5. We have $ae=1$, $a=2$, and $e=1/2$. This implies that

$$b=\sqrt{4\left(1-\frac{1}{4}\right)}=\sqrt{3}$$

Hence, the minor axis is $2\sqrt{3}$.

6. We have

$$\frac{x^2}{25}+\frac{y^2}{16}=1 \Rightarrow e=\sqrt{1-\frac{16}{25}}=\frac{3}{5}$$

Therefore, the directrices are

$$x\pm\frac{5}{3/5}=0$$

or $$3x\pm 25=0$$

7. We have

$$\left(\frac{2}{3}\right)^2 = 1 - \frac{b^2}{a^2}$$

and $$\frac{2b^2}{a} = 5 \Rightarrow a = \frac{81}{4},\ b = \frac{45}{4}$$

Hence, the equation is

$$\frac{4x^2}{81} + \frac{4y^2}{45} = 1$$

8. It is given that

$$\frac{2b^2}{a} = 10 \text{ and } 2b = 2ae$$

Also, we have

$$b^2 = a^2(1-e^2)$$

$$\Rightarrow e^2 = (1-e^2) \Rightarrow e = \frac{1}{\sqrt{2}}$$

$$\Rightarrow b = \frac{a}{\sqrt{2}} \text{ or } b = 5\sqrt{2},\ a = 10$$

Hence, the equation of ellipse is

$$\frac{x^2}{(10)^2} + \frac{y^2}{(5\sqrt{2})^2} = 1$$

That is,

$$x^2 + 2y^2 = 100$$

9. We have $a = 6$ and $b = 2\sqrt{5}$.

$$b^2 = a^2(1-e^2)$$

$$\Rightarrow \frac{20}{36} = (1-e^2)$$

$$\Rightarrow e = \sqrt{\frac{16}{36}} = \frac{2}{3}$$

However, the directrices are $x = \pm(a/e)$. Hence, the distance between the directrices is

$$2\left(\frac{6}{2/3}\right) = 18$$

10. We have

$$\frac{x^2}{(48/3)} + \frac{y^2}{(48/4)} = 1$$

Now,

$$a^2 = 16, b^2 = 12 \Rightarrow e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{1}{2}$$

The distance is

$$2ae = 2(4)\left(\frac{1}{2}\right) = 4$$

11. We have the vertices

$$(\pm 5, 0) \equiv (\pm a, 0) \Rightarrow a = 5$$

The foci are

$$(\pm 4, 0) \equiv (\pm ae, 0) \Rightarrow e = \frac{4}{5}$$

Therefore,

$$b = (5)\left(\frac{3}{5}\right) = 3$$

Hence, the equation is

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

$$\Rightarrow 9x^2 + 25y^2 = 225$$

12. The foci are

$$(\pm 5, 0) \equiv (\pm ae, 0)$$

The directrix is

$$\left(x = \frac{36}{5}\right) \equiv x = \frac{a}{e}$$

Hence,

$$\frac{a}{e} = \frac{36}{5},\ ae = 5$$

$$\Rightarrow a = 6 \text{ and } e = \frac{5}{6}$$

Therefore,

$$b = 6\sqrt{1 - \frac{25}{36}} = 6\frac{\sqrt{11}}{6} = \sqrt{11}$$

Hence, the equation is

$$\frac{x^2}{36} + \frac{y^2}{11} = 1$$

13. We have $e = 1/\sqrt{2}$. The latus rectum is

$$\frac{2b^2}{a} = \frac{2a^2}{a}\left(1 - \frac{1}{2}\right) = a$$

which is semi-major axis.

14. The ellipse is

$$\frac{x^2}{9} + \frac{y^2}{5} = 1$$

The latus rectum is

$$\frac{2b^2}{a} = \frac{2(5)}{3} = \frac{10}{3}$$

15. We have

$$\frac{a}{e} - ae = 8$$

Also,

$$e = \frac{1}{2} \Rightarrow a = \frac{8e}{1-e^2} = \frac{8(4)}{2(3)} = \frac{16}{3}$$

Therefore,

$$b = \frac{16}{3}\sqrt{\left(1 - \frac{1}{4}\right)} = \frac{16}{3}\frac{\sqrt{3}}{2} = \frac{8\sqrt{3}}{3}$$

Hence, the length of minor axis is $16\sqrt{3}/3$.

16. We have

$$\frac{x^2}{112/16} + \frac{y^2}{112/7} = 1$$

Therefore,

$$e=\sqrt{1-\frac{112}{16}\left(\frac{7}{112}\right)}=\frac{3}{4}$$

17. The foci are $(\pm ae, 0)$. Therefore, according to the condition, we have $2ae = 2b$ or $ae = b$. Also,

$$b^2 = a^2(1-e^2) \Rightarrow e^2 = (1-e^2) \Rightarrow e = \frac{1}{\sqrt{2}}$$

18. Let the equation of ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Since this passes through (– 3, 1), we get

$$\frac{9}{a^2}+\frac{1}{b^2}=1$$

$$\Rightarrow 9+\frac{a^2}{b^2}=a^2 \quad (1)$$

It is given that the eccentricity is $\sqrt{2/5}$. Therefore,

$$\frac{2}{5}=1-\frac{b^2}{a^2}\Rightarrow\frac{b^2}{a^2}=\frac{3}{5} \quad (2)$$

From Eqs. (1) and (2), we get

$$a^2=\frac{32}{3} \text{ and } b^2=\frac{32}{5}$$

Hence, the required equation of ellipse is

$$3x^2+5y^2=32$$

19. It is given that $2b = 10$ and $2a = 8$. Therefore, $b = 5$ and $a = 4$. Hence, the required equation is

$$\frac{x^2}{16}+\frac{y^2}{25}=1$$

20. The centre is (0, 0), the focus is (0, 3) and $b = 5$. The focus (0, 3) implies that

$$be=3 \Rightarrow e=\frac{3}{5} \Rightarrow a=b\sqrt{1-e^2}=4$$

Hence, the required equation is

$$\frac{x^2}{16}+\frac{y^2}{25}=1$$

21. The vertex is (0,7) and the directrix is $y = 12$. Therefore, $b = 7$. Also, $b/e = 12$. So,

$$e=\frac{7}{12} \text{ and } a=7\sqrt{\frac{95}{144}}$$

Hence, the equation of ellipse is

$$144x^2+95y^2=4655$$

22. We have

$$\frac{x^2}{(30/2)}+\frac{y^2}{(30/3)}=1$$

$$\Rightarrow\frac{x^2}{15}+\frac{y^2}{10}=1$$

Hence, the given equation represents an ellipse.

23. We have

$$\frac{2b^2}{a}=8,\ e=\frac{1}{\sqrt{2}} \Rightarrow a^2=64,\ b^2=32$$

Hence, the equation of ellipse is

$$\frac{x^2}{64}+\frac{y^2}{32}=1$$

24. We have

$$\frac{2b^2}{a}=2ae$$

$$\Rightarrow b^2=a^2e \text{ or } e=\frac{b^2}{a^2}$$

Also,

$$e=\sqrt{1-\frac{b^2}{a^2}} \text{ or } e^2=1-e \text{ or } e^2+e-1=0$$

Therefore,

$$e=\frac{-1\pm\sqrt{5}}{2}$$

Since $e < 1$, we get

$$e=\frac{\sqrt{5}-1}{2}$$

25. We have

$$\frac{x^2}{4}+\frac{y^2}{3}=1$$

The latus rectum is

$$\frac{2b^2}{a}=3$$

26. We have

$$e^2=1-\frac{b^2}{a^2}$$

$$\Rightarrow e^2=\frac{36}{64} \Rightarrow e=\frac{3}{4}$$

27. The major axis is given by

3 × (Minor axis)

Therefore,

$$2a=3(2b)$$

$$\Rightarrow a^2=9b^2=9a^2(1-e^2)\Rightarrow e=\frac{2\sqrt{2}}{3}$$

28. The latus rectum is $\frac{1}{3}$ of major axis, so

$$\frac{2b^2}{a}=\frac{2a}{3}$$

$$\Rightarrow a^2=3b^2=3a^2(1-e^2)$$

$$\Rightarrow e=\sqrt{\frac{2}{3}}$$

29. It is given that $2a = 6$ and $2b = 4$. that is, $a = 3$ and $b = 2$.

$$e^2=1-\frac{b^2}{a^2}=\frac{5}{9}\Rightarrow e=\frac{\sqrt{5}}{3}$$

The distance between the pins is

$$2ae = (2\sqrt{5})\text{ cm}$$

The length of the string is

$$2a + 2ae = (6 + 2\sqrt{5})\text{ cm}$$

30. We have

$$\frac{x^2}{2-r} + \frac{y^2}{r-5} + 1 = 0 \Rightarrow \frac{x^2}{r-2} + \frac{y^2}{5-r} = 1$$

Hence, $r > 2$ and $r < 5$, which implies that $2 < r < 5$.

31. Let point be (h, k); thus, the pair of tangent is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1\right) = \left(\frac{hx}{a^2} + \frac{yk}{b^2} - 1\right)^2$$

Now, the pair of tangents is perpendicular if we have

$$\text{Coefficient of } x^2 + \text{Coefficient of } y^2 = 0$$

$$\Rightarrow \frac{k^2}{a^2b^2} + \frac{h^2}{a^2b^2} = \frac{1}{a^2} + \frac{1}{b^2} \Rightarrow h^2 + k^2 = a^2 + b^2$$

On replacing (h, k) by (x, y), we get

$$x^2 + y^2 = a^2 + b^2$$

32. We have $a^2 = 36$ and $b^2 = 49$. Since $b > a$, the length of the latus rectum is

$$\frac{2a^2}{b} = 2 \times \frac{36}{7} = \frac{72}{7}$$

33. The focal distance of any point $P(x, y)$ on the ellipse is equal to

$$SP = a + ex$$

Now,

$$x = a\cos\theta$$

Here,

$$SP = a + ae\cos\theta = a(1 + e\cos\theta)$$

34. We have, $ae = 4$ and

$$e = \frac{4}{5} \Rightarrow a = 5$$

Now,

$$b^2 = a^2(1-e^2) \Rightarrow b^2 = 25\left(1 - \frac{16}{25}\right) = 9$$

Hence, the equation of ellipse is

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

35. The equation of the ellipse is

$$16x^2 + 25y^2 = 400$$

or

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

Here, we have

$$a^2 = 25 \text{ and } b^2 = 16 \Rightarrow e = \frac{3}{5}$$

Hence, the foci are $(\pm 3, 0)$.

36. The ellipse is

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

Therefore,

$$e = \sqrt{1 - \frac{b^2}{a^2}}$$

That is,

$$\sqrt{1 - \frac{9}{25}} = \frac{4}{5}$$

37. We have

$$\frac{x^2}{4} + \frac{y^2}{(25/4)} = 1$$

Here, $a = 2$ and $b = 5/2$. Therefore, $b > a$; hence,

$$a^2 = b^2(1-e^2)$$

$$\Rightarrow 4 = \frac{25}{4}(1-e^2)$$

$$\Rightarrow \frac{16}{25} = 1 - e^2$$

$$\Rightarrow e^2 = 1 - \frac{16}{25} = \frac{9}{25}$$

Therefore,

$$e = \frac{3}{5}$$

38. The given ellipse is

$$\frac{x^2}{(1/3)^2} + \frac{y^2}{(1/2)^2} = 1$$

Here, $b > a$. Therefore, the latus rectum is

$$\frac{2a^2}{b} = \frac{2 \times (1/9)}{1/2} = \frac{4}{9}$$

39. Let us consider point $P(x_1, y_1)$. Thus,

$$\sqrt{(x_1+2)^2 + y_1^2} = \frac{2}{3}\left(x_1 + \frac{9}{2}\right)$$

$$\Rightarrow (x_1+2)^2 + y_1^2 = \frac{4}{9}\left(x_1 + \frac{9}{2}\right)^2$$

$$\Rightarrow 9[x_1^2 + y_1^2 + 4x_1 + 4] = 4\left(x_1^2 + \frac{81}{4} + 9x_1\right)$$

$$\Rightarrow 5x_1^2 + 9y_1^2 = 45 \Rightarrow \frac{x_1^2}{9} + \frac{y_1^2}{5} = 1$$

The locus of (x_1, y_1) is $\frac{x^2}{9} + \frac{y^2}{5} = 1$, which is the equation of an ellipse.

40. The equation of the curve is

$$\frac{x^2}{5^2}+\frac{y^2}{4^2}=1$$

$$\Rightarrow -5\le x\le 5,\ -4\le y\le 4$$

Now,

$$PF_1+PF_2=\sqrt{[(x-3)^2+y^2]}+\sqrt{[(x+3)^2+y^2]}$$

$$=\sqrt{(x-3)^2+\frac{400-16x^2}{25}}+\sqrt{(x+3)^2+\frac{400-16x^2}{25}}$$

$$=\frac{1}{5}\left\{\sqrt{(9x^2+625-150x)}+\sqrt{(9x^2+625+150x)}\right\}$$

$$=\frac{1}{5}\left\{\sqrt{(3x-25)^2}+\sqrt{(3x+25)^2}\right\}=\frac{1}{5}\{25-3x+3x+25\}$$

$$=10 \qquad (\because 25-3x>0,\ 25+3x>0)$$

41. We have

$$SP+S'P=2a=2\times 6=12$$

42. Since $ae=\pm 2$, we get $a=\pm 4$ because $e=1/2$. Now, we have

$$b^2=a^2(1-e^2)$$

$$\Rightarrow b^2=16(1-1/4)$$

$$\Rightarrow b^2=12$$

Hence, the ellipse is

$$\frac{x^2}{16}+\frac{y^2}{12}=1$$

$$\Rightarrow 3x^2+4y^2=48$$

43. We have

$$4x^2+9y^2=36$$

$$\Rightarrow \frac{x^2}{9}+\frac{y^2}{4}=1$$

Here, $a^2=9$ and $b^2=4$. Now,

$$b^2=a^2(1-e^2)$$

Therefore,

$$e=\frac{\sqrt{5}}{3}$$

44. We have

$$\frac{x^2}{16}+\frac{y^2}{25}=1$$

$$\Rightarrow a^2=b^2\,(1-e^2)$$

$$\Rightarrow e^2=1-\frac{16}{25}$$

$$\Rightarrow e=\frac{3}{5}$$

45. It is given that the distance between the foci is $2ae=16$ and the eccentricity of ellipse is $e=1/2$. We know that the length of the major axis of the ellipse is

$$2a=\frac{2ae}{e}=\frac{16}{1/2}=32$$

46. In the first case, the eccentricity is

$$e=\sqrt{1-\frac{25}{169}}$$

In the second case,

$$e'=\sqrt{1-\frac{b^2}{a^2}}$$

According to the given condition, we get

$$\sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\frac{25}{169}}$$

$$\Rightarrow \frac{b}{a}=\frac{5}{13} \qquad (\because a>0, b>0)$$

$$\Rightarrow \frac{a}{b}=\frac{13}{5}$$

47. The given minor axis of ellipse is $2b=8$ or $b=4$ and the eccentricity is $e=\sqrt{5}/3$. We know that in an ellipse,

$$e^2=1-\frac{b^2}{a^2}\ \Rightarrow\ \frac{5}{9}=1-\frac{16}{a^2}$$

$$\frac{16}{a^2}=1-\frac{5}{9}\ \Rightarrow\ a^2=\frac{16\times 9}{4}=36$$

$$\Rightarrow a=6$$

We also know that the major axis of the ellipse is $2a=2\times 6=12$.

48. The given equation can be written as $\frac{x^2}{5}+\frac{y^2}{9}=1$. On comparing the given equation with the standard equation, we get $a^2=5$ and $b^2=9$. We also know that in an ellipse (where $b^2>a^2$)

$$a^2=b^2(1-e^2)\lambda \text{ or } e^2=\frac{b^2-a^2}{b^2}=\frac{9-5}{9}=\frac{4}{9} \text{ or } e=\frac{2}{3}$$

Therefore, the distance between foci are

$$2be=2\times 3\times\frac{2}{3}=4$$

49. It is given that, $e=1/2$ and $(\pm ae, 0)=(\pm 1, 0)$. So,

$$ae=1\ \Rightarrow a=2$$

Now,

$$b^2=a^2(1-e^2)$$

$$\Rightarrow b^2=4\left(1-\frac{1}{4}\right)\Rightarrow b^2=3$$

Hence, the equation of the ellipse is

$$\frac{x^2}{4}+\frac{y^2}{3}=1$$

50. The sum of the focal distances of a point in an ellipse is equal to the length of the major axis of that ellipse. It is one of the properties of an ellipse.

51. We have

$$2ae=8, \frac{2a}{e}=18 \Rightarrow a=\sqrt{4\times 9}=6$$

That is,

$$e=\frac{2}{3} \text{ and } b=6\sqrt{1-\frac{4}{9}}=\frac{6}{3}\sqrt{5}=2\sqrt{5}$$

Hence, the equation is obtained as

$$\frac{x^2}{36}+\frac{y^2}{20}=1$$

That is,

$$5x^2+9y^2=180$$

52. The distance between the foci is 6. Therefore, $ae=3$. The minor axis is 8. Therefore,

$$2b=8 \Rightarrow b=4 \Rightarrow b^2=16$$
$$\Rightarrow a^2(1-e^2)=16$$
$$\Rightarrow a^2-a^2e^2=16$$
$$\Rightarrow a^2-9=16 \Rightarrow a=5$$

Hence,

$$ae=3 \Rightarrow e=\frac{3}{5}$$

53. See Fig. 14.35. Let AB be a bar of length l and it moves such that A and B always lie on x-axis and y-axis, respectively.

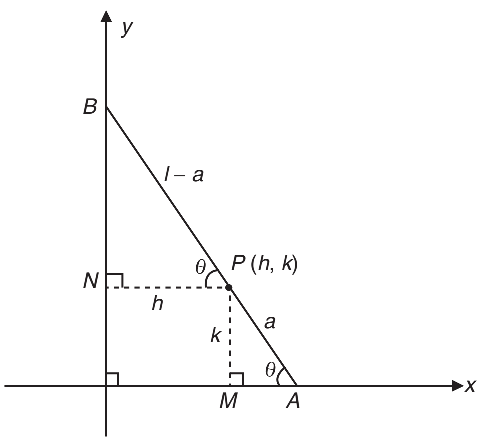


Figure 14.35

Let $P(h, k)$ be any point marked on AB such that $AP=a$. $PM \perp$ x-axis and $PN \perp$ y-axis.

Let $\angle A=\theta$. Then $\angle BPN=\theta$.

Therefore,

$$\sin\theta=\frac{k}{AP} \text{ and } \cos\theta=\frac{h}{BP}$$

$$\Rightarrow \sin\theta=\frac{k}{a} \text{ and } \cos\theta=\frac{h}{l-a}$$

Now,

$$\cos^2\theta+\sin^2\theta=1$$

$$\Rightarrow \frac{h^2}{(l-a)^2}+\frac{k^2}{a^2}=1$$

Therefore, locus of (h, k) is

$$\frac{x^2}{(l-a)^2}+\frac{y^2}{a^2}=1$$

which is an ellipse.

54. We have

$$4(x-2)^2+9(y-3)^2=36$$

Hence, the centre is (2, 3).

55. The ellipse is

$$4(x-1)^2+9(y-2)^2=36$$

or

$$\frac{(x-1)^2}{9}+\frac{(y-2)^2}{4}=1$$

Therefore, the latus rectum is

$$=\frac{2b^2}{a}=\frac{2\times 4}{3}=\frac{8}{3}$$

56. We have

$$4x^2-8x+y^2+2y+1=0$$
$$\Rightarrow (2x-2)^2+(y+1)^2=-1+4+1$$
$$\Rightarrow \frac{(x-1)^2}{1}+\frac{(y+1)^2}{4}=1$$
$$\Rightarrow e=\sqrt{1-\frac{1}{4}}$$
$$\Rightarrow e=\frac{\sqrt{3}}{2}$$

57. The major axis is

$$6=2a \Rightarrow a=3$$

Therefore,

$$e=\frac{1}{2} \Rightarrow b=3\sqrt{1-\frac{1}{4}}=\frac{3\sqrt{3}}{2}$$

Also, the centre is (7, 0). The equation is

$$\frac{(x-7)^2}{9}+\frac{y^2}{(27/4)}=1$$
$$\Rightarrow 3x^2+4y^2-42x+120=0$$

58. The foci are $(3,-3)$. Then

$$ae=3-2=1$$

The vertex is $(4,-3)$. So,

$$a=4-2=2 \Rightarrow e=\frac{1}{2}$$
$$\Rightarrow b=a\sqrt{\left(1-\frac{1}{4}\right)}=\frac{2}{2}\sqrt{3}=\sqrt{3}$$

Therefore, the equation of ellipse with centre $(2,-3)$ is

$$\frac{(x-2)^2}{4}+\frac{(y+3)^2}{3}=1$$

59. Hint: Check $\Delta \neq 0$ and $h^2 < ab$.

60. The centre of the given ellipse is the point of intersection of the lines $x+y-2=0$ and $x-y=0$. That is, (1, 1).

61. Let any point on it be (*x*, *y*). Then

$$\frac{\sqrt{(x+1)^2+(y-1)^2}}{\left|(x-y+3)/\sqrt{2}\right|}=\frac{1}{2}$$

On squaring this and simplifying further, we get

$$7x^2+2xy+7y^2+10x-10y+7=0$$

62. We have

$$\frac{(x+1)^2}{225/25}+\frac{(y+2)^2}{225/9}=1$$

Now,

$$a=\sqrt{\frac{225}{25}}=\frac{15}{5},\ b=\sqrt{\frac{225}{9}}=\frac{15}{3}\ \Rightarrow\ e=\sqrt{1-\frac{9}{25}}=\frac{4}{5}$$

The focus is

$$\left[-1,\ -2\pm\frac{15}{3}\left(\frac{4}{5}\right)\right]=(-1,-2\pm 4)=(-1,2);(-1,-6)$$

63. We have

$$9x^2+(\sqrt{5}y-3\sqrt{5})^2=45$$

or

$$\frac{x^2}{5}+\frac{(y-3)^2}{9}=0$$

$$\Rightarrow a^2=5 \text{ and } b^2=9$$

Therefore,

$$e=\frac{2}{3}$$

64. We have

$$\frac{x^2}{9}+\frac{y^2}{16}=(1+\sin 2t)+(1-\sin 2t)=2$$

Obviously, it is an ellipse.

65. $x=a\cos\theta,\ y=b\sin\theta$ represents the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

for which

$$e=\sqrt{1-\frac{b^2}{a^2}} \text{ or } e^2=\frac{a^2-b^2}{a^2}$$

66. We have

$$4x^2+8x+4+9y^2+36y+36=36$$

$$\Rightarrow \frac{(x+1)^2}{9}+\frac{(y+2)^2}{4}=1$$

Therefore,

$$e=\sqrt{1-\frac{4}{9}}=\frac{\sqrt{5}}{3}$$

67. We have

$$3x^2-12x+4y^2-8y=-4$$

$$\Rightarrow 3(x-2)^2+4(y-1)^2=12$$

$$\Rightarrow \frac{(x-2)^2}{4}+\frac{(y-1)^2}{3}=1$$

$$\Rightarrow \frac{X^2}{4}+\frac{Y^2}{3}=1$$

Therefore,

$$e=\sqrt{1-\frac{3}{4}}=\frac{1}{2}$$

Hence, the foci are

$$\left(X=\pm 2\times\frac{1}{2},\ Y=0\right)$$

That is,

$$(x-2=\pm 1,\ y-1=0)=(3,1) \text{ and } (1,1)$$

68. The equation $x^2+2y^2-2x+3y+2=0$ can be written as

$$\frac{(x-1)^2}{2}+\left(y+\frac{3}{4}\right)^2=\frac{1}{16}$$

$$\Rightarrow \frac{(x-1)^2}{(1/8)}+\frac{[y+(3/4)]^2}{(1/16)}=1$$

which is an ellipse with

$$a^2=\frac{1}{8} \text{ and } b^2=\frac{1}{16}$$

Therefore,

$$\frac{1}{16}=\frac{1}{8}(1-e^2)$$

$$\Rightarrow e^2=1-\frac{1}{2}\Rightarrow e=\frac{1}{\sqrt{2}}$$

69. The given equation of ellipse is

$$25x^2+9y^2-150x-90y+225=0$$

$$\Rightarrow 25(x-3)^2+9(y-5)^2=225$$

$$\Rightarrow \frac{(x-3)^2}{9}+\frac{(y-5)^2}{25}=1$$

Here, $b>a$. Therefore, the eccentricity is

$$e=\sqrt{1-\frac{a^2}{b^2}}=\sqrt{1-\frac{9}{25}}=\sqrt{\frac{16}{25}}=\frac{4}{5}$$

70. We have

$$a^2=b^2(1-e^2) \qquad (\because a<b)$$

That is,

$$9=25(1-e^2)$$

$$\Rightarrow \frac{9}{25}=1-e^2$$

$$\Rightarrow e^2=\frac{16}{25}\Rightarrow e=\frac{4}{5}$$

71. It is given that the equation of conic is

$$9x^2+4y^2-6x+4y+1=0$$

$$\Rightarrow (3x-1)^2+(2y+1)^2=1$$

$$\Rightarrow \frac{[x-(1/3)]^2}{1/9}+\frac{(y+1)^2}{1/2}=1$$

Here, $a=1/3$ and $b=1/2$; therefore,

$$2a=\frac{2}{3} \text{ and } 2b=1$$

The length of the axes are $\left(1,\frac{2}{3}\right)$.

72. The given equation can be written as

$$\frac{(x-1)^2}{5}+\frac{(y-2)^2}{9}=1$$

That is,

$$e=\sqrt{\frac{b^2-a^2}{b^2}}=\sqrt{\frac{9-5}{9}}=\frac{2}{3}$$

73. The given equation of conic is

$$4x^2+16y^2-24x-32y=1$$

$$\Rightarrow (2x-6)^2+(4y-4)^2=53$$

$$\Rightarrow 4(x-3)^2+16(y-1)^2=53$$

$$\Rightarrow \frac{(x-3)^2}{53/4}+\frac{(y-1)^2}{53/16}=1$$

Therefore,

$$e=\sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\frac{53/16}{53/4}}=\sqrt{1-\frac{1}{4}}=\frac{\sqrt{3}}{2}$$

74. We have

$$c=\pm\sqrt{b^2+a^2m^2}=\pm\sqrt{4+8(4)}=\pm 6$$

75. Since $S_1>0$, the point is outside the ellipse.

76. We have

$$m=\tan 60°=\sqrt{3}$$

Therefore, the equation of tangent is

$$y=\sqrt{3}x\pm\sqrt{1+3(16)} \Rightarrow y=\sqrt{3}x\pm 7$$

77. We have

$$E\equiv 4(1)^2+9(3)^2-16(1)-54(3)+61<0$$

Therefore, the point is inside the ellipse on the major axis.

78. We have $y=\frac{-l}{m}x+\frac{n}{m}$ which is tangent to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if

$$\frac{n}{m}=\pm\sqrt{b^2+a^2\left(\frac{l}{m}\right)^2}$$

or

$$n^2=m^2b^2+l^2a^2$$

79. It is a fundamental concept that locus of intersection point of perpendicular tangents to an ellipse is a circle called director circle.

80. The point does not lie on ellipse. Therefore, no tangent can be drawn at the given point.

81. We have equation of pair of tangents

$$SS_1=T^2$$

That is,

$$(3x^2+2y^2-5)(3+8-5)=(3x+4y-5)^2$$

or

$$9x^2-24xy-4y^2+30x+40y-55=0$$

Therefore,

$$\tan\theta=2\frac{\sqrt{h^2-ab}}{a+b},\ a=9,\ b=-4 \text{ and } h=-12$$

$$\Rightarrow \tan\theta=\frac{12}{\sqrt{5}} \text{ or } \theta=\tan^{-1}\frac{12}{\sqrt{5}}$$

82. The tangent is

$$y-3=m(x-2) \Rightarrow y-mx=3-2m$$

However, it is the tangent to the given ellipse. Therefore,

$$3-2m=\pm\sqrt{16m^2+9}$$

$$\Rightarrow m=0,-1$$

Hence, the tangents are $y=3$ and $x+y=5$.

83. The tangent at $(a\cos\theta, b\sin\theta)$ to the ellipse is

$$\frac{(a\cos\theta)x}{a^2}+\frac{(b\sin\theta)y}{b^2}=1$$

or

$$\frac{x}{(a/\cos\theta)}+\frac{y}{(b/\sin\theta)}=1$$

Therefore, the intercepts are

$$h=\frac{a}{\cos\theta},\ k=\frac{b}{\sin\theta} \Rightarrow \frac{a^2}{h^2}+\frac{b^2}{k^2}=1$$

84. $c=\pm\sqrt{b^2m^2+a^2}$ (Due to the reason that a and b are interchanged.)

85. To cut at real points, we have

$$c^2\le a^2m^2+b^2 \Rightarrow a^2m^2\ge c^2-b^2$$

86. Here, $a=3$ and $b=2$. Therefore, by formula, we get

$$c^2=b^2+a^2m^2$$

Therefore,

$$c^2=4+9m^2$$

Thus,

$$c=\pm\sqrt{9m^2+4}$$

87. The locus of the point of intersection of the two perpendicular tangents that are drawn on the ellipse is

$$x^2+y^2=a^2+b^2$$

which is called a 'director circle'. The given ellipse is

$$\frac{x^2}{9}+\frac{y^2}{4}=1$$

Therefore, the locus is

$$x^2+y^2=13$$

88. The coordinates of any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose eccentric angle is θ are

$$(a\cos\theta,\, b\sin\theta)$$

The coordinates of the end points of the latus rectum are $\left(ae, \pm\frac{b^2}{a}\right)$. Therefore,

$$a\cos\theta=ae$$

and

$$b\sin\theta=\pm\frac{b^2}{a}$$

$$\Rightarrow \tan\theta=\pm\frac{b}{ae}\Rightarrow \theta=\tan^{-1}\left(\pm\frac{b}{ae}\right)$$

89. Let the eccentric angle of the point be θ, then its coordinates are $(\sqrt{6}\cos\theta,\sqrt{2}\sin\theta)$. So,

$$6\cos^2\theta+2\sin^2\theta=4$$

or

$$\cos^2\theta=\frac{1}{2}$$

Thus,

$$\cos\theta=\pm\frac{1}{\sqrt{2}}$$

Therefore,

$$\theta=\frac{\pi}{4} \text{ or } \frac{3\pi}{4}$$

90. On changing the equation $9x^2+5y^2-30y=0$ in standard form, we get

$$9x^2+5(y^2-6y)=0$$

$$\Rightarrow 9x^2+5(y^2-6y+9)=45$$

$$\Rightarrow \frac{x^2}{5}+\frac{(y-3)^2}{9}=1$$

Since $a^2 < b^2$, the axis of ellipse is on y-axis. Since this is at y-axis, by substituting $x=0$, we can obtain the vertex. Then,

$$0+5y^2-30y=0\Rightarrow y=0,\ y=6$$

Therefore, the tangents of vertex are $y=0,\ y=6$.

91. We have equation of normal at $(a\cos\theta, b\sin\theta)$ as

$$ax\sec\theta-by\ \text{cosec}\,\theta=a^2-b^2$$

or

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$$

92. The normal at $P(a\cos\theta, b\sin\theta)$ is $ax\sec\theta-by\ \text{cosec}\,\theta=a^2-b^2$, where $a^2=14,\ b^2=5$.

It meets the curve once again at $Q(2\theta)$. That is,

$$(a\cos 2\theta,\ b\sin 2\theta)$$

Therefore,

$$\frac{a}{\cos\theta}a\cos 2\theta-\frac{b}{\sin\theta}(b\sin 2\theta)=a^2-b^2$$

$$\Rightarrow \frac{14}{\cos\theta}\cos 2\theta-\frac{5}{\sin\theta}(\sin 2\theta)=14-5$$

$$\Rightarrow 18\cos^2\theta-9\cos\theta-14=0$$

$$\Rightarrow (6\cos\theta-7)(3\cos\theta+2)=0\Rightarrow \cos\theta=-\frac{2}{3}$$

93. Parametric normal to the ellipse is

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2 \quad (1)$$

Given normal is

$$mx-y=-c \quad (2)$$

Because Eqs. (1) and (2) represent the same line,

$$\frac{m}{a/\cos\theta}=\frac{-1}{-b/\sin\theta}=\frac{-c}{a^2-b^2}$$

$$\Rightarrow \cos\theta=\frac{-ac}{m(a^2-b^2)},\quad \sin\theta=\frac{-bc}{a^2-b^2}$$

Now,

$$\cos^2\theta+\sin^2\theta=1$$

$$\Rightarrow \frac{a^2c^2}{m^2(a^2-b^2)^2}+\frac{b^2c^2}{(a^2-b^2)^2}=1$$

$$\Rightarrow \frac{(a^2+b^2m^2)c^2}{m^2(a^2-b^2)^2}=1$$

$$\Rightarrow c=\pm\frac{m(a^2-b^2)}{\sqrt{a^2+b^2m^2}}$$

94. For $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the equation of normal at point (x_1,y_1) is

$$\frac{(x-x_1)a^2}{x_1}=\frac{(y-y_1)b^2}{y_1}$$

Therefore,

$$(x_1,y_1)\equiv(0,3),\ a^2=5,\ b^2=9$$

$$\Rightarrow \frac{(x-0)}{0}\ 5=\frac{(y-3)(9)}{3}$$

or $x=0$. That is, y-axis.

95. We have

$$\frac{x-x_1}{x_1/a^2}=\frac{y-y_1}{y_1/b^2}$$

which is the standard equation of normal at point (x_1,y_1). In the given ellipse, we have $a^2 = 20$ and $b^2 = 180/16$. Hence, the equation of normal at the point $(2, 3)$ is

$$\frac{x-2}{2/20}=\frac{y-3}{48/180}$$

$$\Rightarrow 40(x-2)=15(y-3)$$

$$\Rightarrow 8x-3y=7$$

$$\Rightarrow 3y-8x+7=0$$

96. The equation of any normal to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$ax\sec\phi - by\operatorname{cosec}\phi = a^2-b^2 \quad (1)$$

The straight line $x\cos\alpha+y\sin\alpha=p$ is a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if Eq. (1) and $x\cos\alpha+y\sin\alpha=p$ represent the same line.

$$\frac{a\sec\phi}{\cos\alpha}=\frac{-b\operatorname{cosec}\phi}{\sin\alpha}=\frac{a^2-b^2}{p}$$

$$\Rightarrow \cos\phi=\frac{ap}{(a^2-b^2)\cos\alpha},\ \sin\phi=\frac{-bp}{(a^2-b^2)\sin\alpha}$$

Since $\sin^2\phi+\cos^2\phi=1$, we have

$$\frac{b^2p^2}{(a^2-b^2)^2\sin^2\alpha}+\frac{a^2p^2}{(a^2-b^2)^2\cos^2\alpha}=1$$

$$\Rightarrow p^2(b^2\operatorname{cosec}^2\alpha+a^2\sec^2\alpha)=(a^2-b^2)^2$$

97. The equation of any normal to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$ax\sec\theta-by\operatorname{cosec}\theta=a^2-b^2 \quad (1)$$

The straight line

$$lx+my+n=0 \quad (2)$$

is a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if Eqs. (1) and (2) represent the same line. Then

$$\frac{a\sec\theta}{l}=\frac{b\operatorname{cosec}\theta}{-m}=\frac{a^2-b^2}{-n}$$

$$\Rightarrow \cos\theta=\frac{-an}{l(a^2-b^2)}$$

and $$\sin\theta=\frac{bn}{m(a^2-b^2)}$$

Since $\cos^2\theta+\sin^2\theta=1$, we get

$$\frac{a^2n^2}{l^2(a^2-b^2)^2}+\frac{b^2n^2}{m(a^2-b^2)^2}=1$$

$$\Rightarrow \frac{a^2}{l^2}+\frac{b^2}{m^2}=\frac{(a^2-b^2)^2}{n^2}$$

98. It is given that the equation of ellipse is $4x^2+9y^2=36$. The tangent at point (3, –2) is

$$\frac{(3)x}{9}+\frac{(-2)y}{4}=1$$

or $$\frac{x}{3}-\frac{y}{2}=1$$

Therefore, the normal is $\frac{x}{2}+\frac{y}{3}=k$ and it passes through point (3, –2). Hence,

$$\frac{3}{2}-\frac{2}{3}=k\Rightarrow k=\frac{5}{6}$$

Therefore, the normal is

$$\frac{x}{2}+\frac{y}{3}=\frac{5}{6}$$

99. We know that the equation of normal at the point ($b\cos\theta$, $a\sin\theta$) of the ellipse $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ is given by

$$bx\sec\theta-ay\operatorname{cosec}\theta=b^2-a^2$$

Thus, let the equation of normal to the ellipse $x^2+\frac{y^2}{4}=1$ be

$$x\sec\theta-2y\operatorname{cosec}\theta=-3 \quad (1)$$

Also given that the equation of normal is

$$2x-\frac{8\lambda}{3}y=-3 \quad (2)$$

As Eqs. (1) and (2) represent the same line, we get

$$\frac{2}{\sec\theta}=\frac{-\frac{8\lambda}{3}}{-2\operatorname{cosec}\theta}=\frac{-3}{-3}$$

$$\Rightarrow \cos\theta=\frac{1}{2} \text{ and } \sin\theta=\frac{3}{4\lambda}$$

But $\cos^2\theta+\sin^2\theta=1$

$$\Rightarrow \frac{1}{4}+\frac{9}{16\lambda^2}=1\Rightarrow \frac{9}{16\lambda^2}=\frac{3}{4}\Rightarrow \lambda^2=\frac{3}{4}\Rightarrow \lambda=\pm\frac{\sqrt{3}}{2}$$

100. We know that equation of polar at point (h, k) is

$$\frac{hx}{a^2}+\frac{ky}{b^2}=1\Rightarrow \frac{hx}{4}+\frac{ky}{1}=1$$

$$\Rightarrow hx+4ky=4 \quad (1)$$

which is similar to the given straight line

$$x+4y=4 \quad (2)$$

On comparing Eqs. (1) and (2), we get $h=1$ and $k=1$. Hence, the point is (1, 1).

101. We have

$$y=\frac{-b}{a}x$$

(Two diameters $y=m_1x$ and $y=m_2x$ are conjugate diameters if $m_1m_2=-\frac{b^2}{a^2}$).

102. See Fig. 14.36. We have

$$\angle F'BF = 90^\circ, F'B \perp FB$$

That is,

$$\text{Slope of } (F'B) \times \text{Slope of } (F'B) = -1$$

$$\Rightarrow \frac{b}{ae} \times \frac{b}{-ae} = -1,\ b^2 = a^2e^2 \quad (1)$$

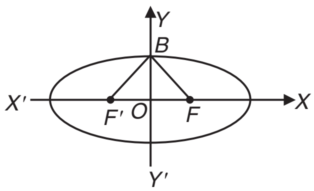


Figure 14.36

We know that

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{a^2e^2}{a^2}} = \sqrt{1 - e^2}$$

$$\Rightarrow e^2 = 1 - e^2$$

$$\Rightarrow 2e^2 = 1$$

$$\Rightarrow e^2 = \frac{1}{2}$$

$$\Rightarrow e = \frac{1}{\sqrt{2}}$$

103. Since $ae = \pm\sqrt{5}$, we have

$$\Rightarrow a = \pm\sqrt{5}\left(\frac{3}{\sqrt{5}}\right) = \pm 3 \Rightarrow a^2 = 9$$

Therefore,

$$b^2 = a^2(1 - e^2) = 9\left(1 - \frac{5}{9}\right) = 4$$

Hence, the equation of ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ is $4x^2 + 9y^2 = 36$.

104. For a point P on the ellipse, there needs to be focus S and S'. Therefore,

$$SP + S'P = 2a$$

Therefore, for $\frac{x^2}{25} + \frac{y^2}{16} = 1$, the sum of the focal distances is $2 \times 5 = 10$.

105. We have

$$25(x-3)^2 + 16y^2 = 400$$

$$\frac{(x-3)^2}{16} + \frac{y^2}{25} = 1$$

Therefore,

$$e = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$$

106. On using the condition, the point (x_1, y_1) lies on or besides the ellipse in three possible cases:

Case 1: On the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ if $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 = 0$.

Case 2: Outside the ellipse if $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 > 0$.

Case 3: Inside the ellipse if $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 < 0$.

The given ellipse is

$$\frac{x^2}{1/4} + \frac{y^2}{1/5} = 1$$

Therefore,

$$\frac{16}{1/4} + \frac{9}{1/5} - 1 = 64 + 45 - 1 > 0$$

The point (4, –3) lies outside the ellipse.

107. See Fig. 14.37. It is given that

$$PS = \frac{2}{3}PM$$

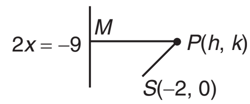


Figure 14.37

The focus is $S(-2, 0)$. The equation of the directrix is

$$2x - 9 = 0$$

Now,

$$(PS)^2 = \frac{4}{9}(PM)^2$$

$$\Rightarrow (h+2)^2 + (k)^2 = \frac{4}{9}\left(\frac{2h-9}{2}\right)^2$$

$$\Rightarrow 9[(h+2)^2 + (k)^2] = \frac{4(2h-9)^2}{4}$$

$$\Rightarrow 9h^2 + 9k^2 + 36h + 36 = 4h^2 + 81 + 36h$$

$$\Rightarrow \frac{5h^2}{45} + \frac{9k^2}{45} = 1 \Rightarrow \frac{h^2}{9} + \frac{k^2}{5} = 1 \Rightarrow 1$$

The locus of point $P(h, k)$ is $\frac{x^2}{9} + \frac{y^2}{5} = 1$, which is an ellipse.

Alternate solution: By definition, when a point moves in such a way that the ratio of its distances from a fixed point and from a fixed line is constant e where $e < 1$, then point traces an ellipse.

Practice Exercise 2

1. See Fig. 14.38.

$$\text{Area of ellipse } = \pi \times \frac{8}{3\sqrt{\pi}} \times \frac{5}{2\sqrt{\pi}} = \frac{20}{3} = A_e$$

Both the shaded areas are equal, then

$$A_1 = \frac{1}{2}A_e - \Delta PQR = \frac{10}{3} - 1 = \frac{7}{3}$$

$$A_2 = \frac{1}{2}A_e + \Delta PQR = \frac{10}{3} + 1 = \frac{13}{3}$$

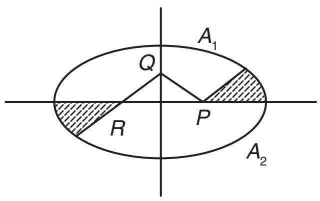


Figure 14.38

Thus, $\dfrac{A_1}{A_2} = \dfrac{7}{13}$

2. See Figs. 14.39 and 14.40.

$$3x^2 + 3y^2 \geq 1$$
$$|y + x| \leq 1$$
$$|y - x| \leq 1$$

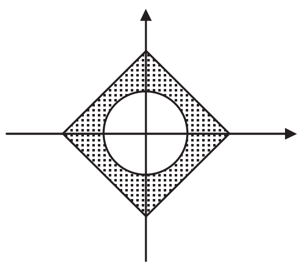


Figure 14.39

$$\text{Required area } = 4\left[\frac{1}{2} - \frac{1}{4}\pi\left(\frac{1}{\sqrt{3}}\right)^2\right] = 2 - \left[\frac{\pi}{3}\right] \text{ sq. units}$$

Now, director circle of $S_1 = 0$ is $12x^2 + 12y^2 - 7 = 0$

Conditions given in (B) are contradictory.

Hence, common region does not exist.

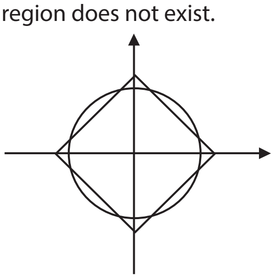


Figure 14.40

3. See Fig. 14.41. Focal property of ellipse

$$PS + PS' = 2a; \text{if } a > b$$
$$PS + PS' = 2b \text{ ; if } a < b$$
$$PS \cos\alpha + PS' \cos\beta = 2ae \quad (1)$$
$$PS \sin\alpha - PS' \sin\beta = 0 \quad (2)$$
$$PS + PS' = 2a \quad (3)$$

From Eqs. (1) and (2), we get

$$PS = \frac{2ae\sin\beta}{\sin(\alpha+\beta)},\ PS' = \frac{2ae\sin\alpha}{\sin(\alpha+\beta)} \quad (4)$$

From Eqs. (3) and (4), we get

$$e(\sin\alpha + \sin\beta) = \sin(\alpha+\beta)$$

Therefore,

$$e \times 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha+\beta}{2}$$

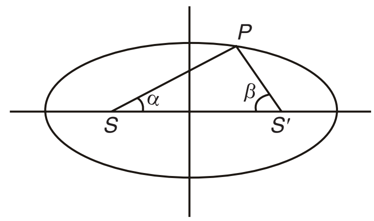


Figure 14.41

Therefore,

$$e\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right) = \cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}$$

Therefore,

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{1-e}{1+e} = \frac{2a\left(a - \sqrt{a^2-b^2}\right) - b^2}{b^2}$$

Therefore, (C) is correct option and (D) is incorrect.

4. The tangent and normal at a point P on the ellipse bisect the external and internal angles between the focal distances of P.

5. We have

Slope of AB = Slope of tangent at C

$$\Rightarrow \frac{b(\sin\beta - \sin\alpha)}{a(\cos\beta - \cos\alpha)} = \frac{-b\cos\theta}{a\sin\theta}\quad\left[\text{for } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \frac{dy}{dx} = \frac{-b^2x}{a^2y}\right]$$

$$\Rightarrow \frac{-\cos\left(\frac{\alpha+\beta}{2}\right)}{\sin\left(\frac{\alpha+\beta}{2}\right)} = \frac{-\cos\theta}{\sin\theta}$$

$$\Rightarrow \tan\left(\frac{\alpha+\beta}{2}\right) = \tan\theta \Rightarrow \theta = \frac{\alpha+\beta}{2} + n\pi\ (n \in \mathrm{I})$$

6. Equation of tangent at θ is

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1 \Rightarrow \frac{x}{a\sec\theta} + \frac{y}{b\,\mathrm{cosec}\,\theta} = 1$$

Therefore, length of the intercept

$$d = \sqrt{a^2\sec^2\theta + b^2\,\mathrm{cosec}^2\theta}$$

Therefore,

$$d^2 = a^2\sec^2\theta + b^2\,\mathrm{cosec}^2\theta$$

$$\Rightarrow \frac{d(d^2)}{d\theta} = 2a^2\frac{\sin\theta}{\cos^3\theta} - 2b^2\frac{\cos\theta}{\sin^3\theta} = 0$$

That is,

$$\frac{a^2 \sin\theta}{\cos^3\theta} = \frac{b^2 \cos\theta}{\sin^3\theta}$$

Therefore,

$$\tan^4\theta = \frac{b^2}{a^2}$$

Therefore, $\tan\theta = \pm\sqrt{\frac{b}{a}}$

$$\Rightarrow \theta = \tan^{-1}\sqrt{\frac{b}{a}}, -\tan^{-1}\sqrt{\frac{b}{a}}$$

$$\Rightarrow \theta = \pi - \tan^{-1}\sqrt{\frac{b}{a}}, -\pi + \tan^{-1}\sqrt{\frac{b}{a}}$$

7. Clearly if $c = 0$, it represents point (3, – 2).
If $c > 0$, then it represents the ellipse and if $c < 0$, no locus of the sum because the sum of two perfect squares cannot be negative.

Hence, for no value of c it represents a straight line.

8. Product of the length of the perpendicular segments from the foci on tangent at $P(4, 7)$ is $b^2 = 20$.

9. A line joining each focus to the foot of the perpendicular from the other focus upon the tangent at P (4, 7) meet the normal PG and bisects it.
Therefore, required point is midpoint of PG.

Therefore, equation of normal at P (4, 7) is given by

$$3x - y - 5 = 0$$

Therefore, $G\left(\frac{8}{3}, 3\right)$

Therefore, required point is midpoint of PG, that is

$$\left(\frac{10}{3}, 5\right)$$

10. Locus of midpoint of QR is another ellipse having the same eccentricity as that of ellipse (e). So,

$$e' = e = \frac{\sqrt{5}}{3}$$

11. $OS_1 = ae = 6$, $OC = b$ (consider)

Also, $CS_1 = a$

Thus,

$$\text{area of } \Delta OCS_1 = \frac{1}{2}(OS_1) \times (OC) = 3b$$

Now, semiperimeter of ΔOCS_1 is given as

$$\Delta OCS_1 = 1/2\,(OS_1 + OC + CS_1) = 1/2\,(6 + a + b) \qquad (1)$$

Therefore,

$$\text{in-radius of } \Delta OCS_1 = 1$$

$$\Rightarrow \frac{3b}{\frac{1}{2}(6 + a + b)} = 1 \Rightarrow 5b = 6 + a \qquad (2)$$

Also,

$$b^2 = a^2 - a^2e^2 = a^2 - 36 \qquad (3)$$

From Eq. (2), we have

$$25\,b^2 = 36 + 12a + a^2$$

Therefore,

$$25\,(a^2 - 36) = 36 + a^2 + 12\,a$$

$$2a^2 - a - 78 = 0$$

$$\Rightarrow a = \frac{13}{2}, -6$$

$$a = \frac{13}{2}$$

Therefore,

$$b = \frac{5}{2}$$

Hence, area of ellipse $= \pi ab = \frac{65\pi}{4}$ sq. units

12. Perimeter of $\Delta OCS_1 = 6 + a + b = 6 + \frac{13}{2} + \frac{5}{2} = 15$ units

13. S: $x^2 + y^2 = a^2 + b^2$
Therefore,

$$x^2 + y^2 = 97$$

14. Using conditions for ellipse

$$-2 < \lambda < 2 \Rightarrow \lambda = -1, 0, 1$$

15. Consider $f = x^2 + xy + y^2 - 1 = 0$
Let $P(r\cos\alpha, r\sin\alpha)$. Now,

$$2r^2 + r^2\sin 2\alpha = 2 \Rightarrow r^2 = \frac{2}{2 + \sin 2\alpha}$$

$$\text{Length of any chord} = 2r = 2\sqrt{\frac{2}{2 + \sin 2\alpha}}$$

$$\text{Maximum length of chord} = 2\sqrt{2}$$

Hence, (C) is the correct answer.

16. Solving $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$ centre of the ellipse comes out to be (0, 0).
The longest chord of ellipse is major axis. While the chord perpendicular to it and through the centre of the ellipse is minor axis, so is the smallest chord. Let P be a point on ellipse, such that $OP = r$ and $\angle POX = \theta$, then P ($r\cos\theta$, $r\sin\theta$) as P lies on ellipse

$$r^2\cos^2\theta + r^2\sin\theta\cos\theta + r^2\sin^2\theta - 1 = 0$$

$$\Rightarrow r^2 = \frac{1}{1 + \sin\theta\cos\theta} = \frac{1}{1 + \frac{1}{2}\sin 2\theta}$$

Therefore, r is smallest if $\sin 2\theta$ is greatest.
That is, $\sin 2\theta = 1$.
Now,

$$r_{\max}^2 = OA^2 = 2 \Rightarrow OA = \sqrt{2}$$

$$r_{\min}^2 = OB^2 = \frac{2}{3} \Rightarrow OB = \sqrt{\frac{2}{3}}$$

17. See Fig. 14.42. Distance between Q' and R' is maximum, so QR is a major axis and the height of the bird's path above the ground level is $5\sqrt{3}$. $PR' = 15$, $PQ' = 5$, $RQ = R'Q' = 10$
Therefore, coordinates of $Q \equiv (5, 0, 5\sqrt{3})$

$$OP = OR' + R'P = 5 + 5 = 10$$

Therefore, coordinates of P are (10, 0, 0).

Equation of the line joining P (10, 0, 0) and R (– 5, 0 , 5 $\sqrt{3}$) is

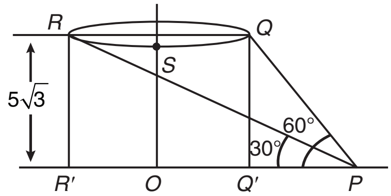


Figure 14.42

$$\frac{x-10}{15} = \frac{y-0}{0} = \frac{z-0}{-5\sqrt{3}}$$

That is,

$$\frac{x-10}{\sqrt{3}} = \frac{z}{-1}, y = 0$$

Since the line passes through the point (13, 0, $-\sqrt{3}$).

Therefore, the equation of the line can be written as

$$\frac{x-13}{\sqrt{3}} = \frac{z+\sqrt{3}}{-1}, y = 0$$

18. Equation of a plane in which the ellipse lies is

$$z = 5\sqrt{3} \quad (1)$$

Equation of a vertical plane tangent at Q is

$$x = 5 \quad (2)$$

Therefore, the required plane is

$$((z - 5\sqrt{3}) + \lambda(x-5)) = 0$$

As this plane passes through P (10, 0, 0).

Therefore, equation of this plane is $\sqrt{3}\,x + z - 10\sqrt{3} = 0$.

19. Let the bird is at S and projection of S on the ground is S'.

So,

$$OS' = \text{semi-minor axis} = \frac{1}{\sqrt{b}}$$

Thus,

$$\left(\frac{1}{\sqrt{2}}\right)^2 = 1 - \frac{\frac{1}{b}}{25} \Rightarrow \frac{1}{\sqrt{b}} = \frac{5}{\sqrt{2}}$$

Therefore,

$$PS' = \sqrt{\left(\frac{5}{\sqrt{2}}\right)^2 + (10)^2} = \sqrt{\frac{25+200}{2}} = \sqrt{\frac{225}{2}}$$

$$= \frac{15}{\sqrt{2}} \text{ and } SS' = 5\sqrt{3}$$

Hence,

$$\tan\theta = \frac{SS'}{PS'} = \sqrt{\frac{2}{3}}$$

20. (A) Common tangent, $y = x + 2\sqrt{2}$.

Points of contact with two hyperbola are $\left(\frac{-9}{2\sqrt{2}}, \frac{-1}{2\sqrt{2}}\right)$ and $\left(\frac{1}{2\sqrt{2}}, \frac{9}{2\sqrt{2}}\right)$.

Hence, length = 5.

(B) For hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, s is $(ae, 0)$ equation of line

$$y = \frac{b}{a}(x - ae)$$

and point $P\left(\frac{a^2(1+e^2)}{2ae}, \frac{-ab(e^2-1)}{2ae}\right)$

Hence,

$$SP = \frac{b^2}{2a} = \frac{9}{8}$$

(C) Equation of tangent is $\frac{x\cos\theta}{2\sqrt{3}} + \frac{y\sin\theta}{18} = 1$

Sum of intercept is $2\sqrt{3}\sec\theta + 6\operatorname{cosec}\theta = f(\theta)$ (say)

$$f'(\theta) = \frac{2\sqrt{3}\sin^3\theta - 18\cos^3\theta}{\sin^2\theta\cos^2\theta} = 0 \Rightarrow \tan^3\theta = 3\sqrt{3}$$

$$\Rightarrow \tan\theta = \sqrt{3}$$

So, by symmetry in second quadrant, $\tan\theta = -\sqrt{3}$.

(D) Equation of tangent at (x_1, y_1) is $xx_1 - 2yy_1 = 4$ which is same as $2x + \sqrt{6}y = 2$

Hence,

$$\frac{x_1}{2} = \frac{-2y_1}{\sqrt{6}} = \frac{4}{2} \Rightarrow x_1 = 4, y_1 = -\sqrt{6}$$

$$\text{Slope, } m = -\frac{\sqrt{6}}{4} \Rightarrow \sqrt{8}m = -\sqrt{3}$$

21. (A) Let $(\sqrt{27}\cos\theta, \sqrt{48}\sin\theta)$ be a point on the ellipse.

Thus, equation of tangent or their points is

$$\frac{\sqrt{27}\cos\theta\, x}{27} + \frac{\sqrt{48}\sin\theta\, y}{48} = 1$$

Now,

$$\text{slope} - \frac{\cos\theta}{\sqrt{27}} \times \frac{\sqrt{48}}{\sin\theta} = -\frac{4}{3}\cot\theta = -\frac{4}{3}$$

That is,

$$\cot\theta = 1$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

Therefore, equation of the tangent is $\frac{x}{\sqrt{54}} + \frac{y}{\sqrt{96}} = 1$

Hence,

$$\text{area of triangle} = \frac{1}{2} \cdot 3\sqrt{6} \times 4\sqrt{6} = 36$$

(B) Product of perpendicular is

$$\left|\frac{3m - \sqrt{25m^2 + 16}}{\sqrt{1+m^2}}\right| \left|\frac{-3m - \sqrt{25m^2+16}}{\sqrt{1+m^2}}\right|$$

$$= \left|\frac{25m^2 + 16 - 9m^2}{1 + m^2}\right| = 16$$

(C) $2a = 5 + 10 = 15$

Therefore, the sum of focal distance of any point on the ellipse is equal to the length of the major axis

$$a = \frac{15}{2}$$

$$2ae = \sqrt{(6-3)^2 + (8-4)^2} = 5$$

$$e = \frac{1}{3}$$

Thus,

$$b^2 = \frac{225}{4}\left(1 - \frac{1}{9}\right) = 50$$

$$\Rightarrow b = 5\sqrt{2}$$

Therefore, $2b = 10\sqrt{2}$.

(D) $a = 5, b = 4, e = \frac{3}{5}$

So, $ae = 3$.

And, $SA = 2$ also $SP = 2$

Since, P coincides with A and Q coincides with A'.

Therefore, $PQ = 2a = 10$.

22. See Fig. 14.43.

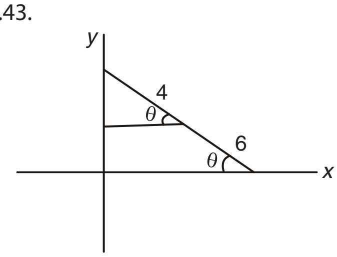


Figure 14.43

(A) $x = 10\cos\theta - 6\cos\theta = 4\cos\theta$

$y = 6\sin\theta$

Thus, locus is

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{6}\right)^2 = 1$$

$$\frac{x^2}{16} + \frac{y^2}{36} = 1$$

And

$$e = \sqrt{\frac{36-16}{36}} = \sqrt{\frac{20}{36}} = \frac{\sqrt{5}}{3}$$

Therefore, $9e = 3\sqrt{5}$.

(B) $3(x^2 + 2x + 1) + 2(y^2 - 2y + 1) = 3 + 2 + 1$

$$\frac{(x+1)^2}{2} + \frac{(y-1)^2}{3} = 1$$

$$e = \sqrt{\frac{3-2}{3}} = \frac{1}{\sqrt{3}}$$

$$a = \sqrt{3}, b = \sqrt{2}$$

Therefore,

$$\text{area} = \frac{1}{2} \times 2\sqrt{2} \times \sqrt{2} = \sqrt{6}$$

(C)

$$\frac{(x-1)^2}{16} + \frac{(y-2)^2}{9} = 1$$

$$a = 4e = \sqrt{\frac{7}{16}} = \frac{\sqrt{7}}{4}$$

So, $ae = \sqrt{7}$.

Therefore, distance between the foci $= 2\sqrt{7}$.

(D) See Fig. 14.44.

$$\frac{x^2}{16} + \frac{y^2}{7} = 1$$

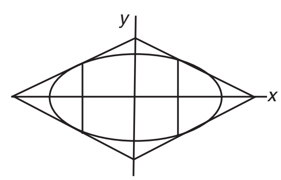


Figure 14.44

$$e = \sqrt{\frac{9}{16}} = \frac{3}{4}$$

One end of the latus rectum is $\left(ae, \frac{b^2}{a}\right) = \left(3, \frac{7}{4}\right)$

Therefore, equation of the tangent is

$$\frac{3x}{16} + \frac{7}{4} \cdot \frac{y}{7} = 1$$

$$\frac{3x}{16} + \frac{y}{4} = 1$$

It meets x-axis at $\left(\frac{16}{3}, 0\right)$ and y-axis at (0, 4).

Hence,

$$\text{area of rhombus} = 2 \times \frac{16}{3} \times 4 = \frac{128}{3}$$

23. See Fig. 14.45.

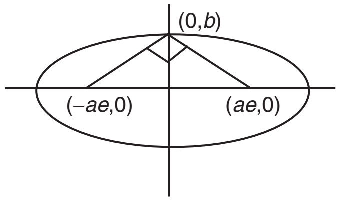


Figure 14.45

(A) $$\frac{b-0}{0-ae} \times \frac{b-0}{0+ae} = -1$$

$$\Rightarrow -\frac{b^2}{a^2e^2} = -1$$

$$\Rightarrow \frac{a^2(1-e^2)}{a^2e^2} = 1 \Rightarrow 1 - e^2 = e^2 \Rightarrow 2e^2 = 1$$

$$\Rightarrow e = \frac{1}{\sqrt{2}}$$

(B) See Fig. 14.46.

Here, $a = 3; b = 2$

and $T: \frac{x\cos\theta}{3} + \frac{y\sin\theta}{2} = 1$

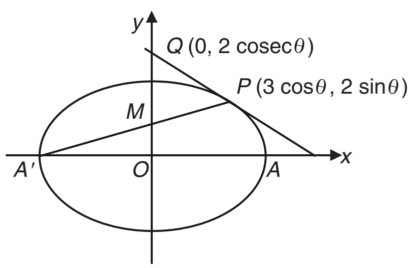


Figure 14.46

$$x = 0; y = 2\operatorname{cosec}\theta$$

Equation of chord $A'P$, $y = \frac{2\sin\theta}{3(\cos\theta+1)}(x+3)$

Put $x = 0$, $y = \frac{2\sin\theta}{1+\cos\theta} = OM$

Now,

$$OQ^2 - MQ^2 = OQ^2 - [OQ - OM]^2$$
$$= 2(OQ)(OM) - OM^2 = OM\{2(OQ) - (OM)\}$$
$$= \frac{2\sin\theta}{1+\cos\theta}\left[\frac{4}{\sin\theta} - \frac{2\sin\theta}{1+\cos\theta}\right] = 4$$

(C) $$\left(y+\frac{1}{2}\right)^2 = 8(x-2) \Rightarrow Y^2 = 8X$$

For three normal

$$X > 2a \Rightarrow x - 2 > 4$$

Therefore, $x > 6$.

(D) Area of a parallelogram $= 2ab$

$$= 2 \times (2) \times \left(\frac{1}{2}\right) = 2$$

24. Let P be $(5\cos\theta, 4\sin\theta)$ and Q be $(-5\sin\theta, 4\cos\theta)$.

Equation of tangent at P,

$$\frac{x}{5}\cos\theta + \frac{y}{4}\sin\theta = 1 \quad (1)$$

Equation of tangent at Q,

$$-\frac{x}{5}\sin\theta + \frac{y}{4}\cos\theta = 1 \quad (2)$$

Solving Eqs. (1) and (2), we get

$$R \equiv \big(5(\cos\theta - \sin\theta),\, 4(\sin\theta + \cos\theta)\big)$$

Therefore, $m:n$ is 1:1. So,

$$m + n = 2$$

Alternate solution: Let

$$P(5, 0); Q(0, 4)$$
$$\Rightarrow R(5, 4)$$

Intersection of CR and PQ is $\left(\frac{5}{2}, 2\right)$, which is the midpoint of CR.

$$m:n = 1:1 \Rightarrow m + n = 2$$

25. If p_1 and p_2 are the length of perpendiculars from foci to any tangent then

$$p_1p_2 = b^2 = 16 \text{ and } p_1 + p_2 = 8 \Rightarrow p_1 = p_2 = 4$$

Hence, only two tangents are possible.

26. See Fig. 14.47. Joint equation of OA and OB is

$$x^2 + y^2 - a^2\left(\frac{x\sqrt{3}}{2a} + \frac{y}{2b}\right)^2 = 0$$

Since,

$$\angle AOB = 90°$$
$$\Rightarrow \text{coefficient of } x^2 + \text{coefficient of } y^2 = 0$$
$$\Rightarrow a^2 = 5b^2$$
$$\Rightarrow e^2 = \frac{4}{5}$$

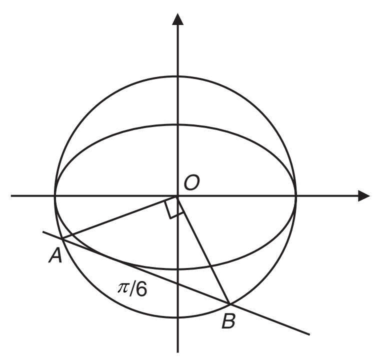


Figure 14.47

Now equation of hyperbola is $\frac{x^2}{a_1^2}-\frac{y^2}{b_1^2}=-1$, with $e_1^2=1+\frac{a_1^2}{b_1^2}$

$$\frac{x^2}{a_1^2}-\frac{y^2}{b_1^2}=-1 \text{ with } a_1^2=\frac{44b_1^2}{5}$$

$$\Rightarrow f(x, y)\equiv 44y^2-5x^2-44b_1^2=0$$

Now,

$$|m_{\text{asymptote}}|=\sqrt{\frac{5}{44}}$$

$$\Rightarrow \left|\frac{1}{m_{\text{asymptote}}}\right|=\sqrt{\frac{44}{5}} \Rightarrow \left[\frac{1}{m_{\text{asymptote}}}\right]=2$$

27. $$\frac{x^2}{169}+\frac{y^2}{25}=1$$

Equation of normal at the point ($13\cos\theta$, $5\sin\theta$)

$(y-5\sin\theta)=\frac{13\sin\theta}{5\cos\theta}(x-13\cos\theta)$ it passes through (0, 6)

Then,

$$\cos\theta(15+72\sin\theta)=0$$

Or, $$\cos\theta=0,\ \sin\theta=-\frac{5}{24}$$

$$\theta=\frac{\pi}{2},\ 2\pi-\sin^{-1}\frac{5}{24},\ \pi+\sin^{-1}\frac{5}{24}$$

Equation has three roots and hence, three normal can be drawn.

28. See Fig. 14.48.

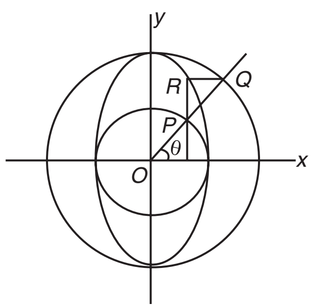


Figure 14.48

Let line *OPQ* makes angle θ with *x*-axis so

$$P\equiv(a\cos\theta, a\sin\theta),\ Q(b\cos\theta, b\sin\theta)$$

And let $R(x, y)$

So,

$$X=a\cos\theta \text{ and } Y=b\sin\theta$$

Eliminating θ, we get

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \text{ locus of } R \text{ is an ellipse.}$$

Also, $a<b$ so vertices are $(0, b)$ and $(0, -b)$ and extremities of minor axis are $(\pm a, 0)$.

So, ellipse touches both inner circle and outer circle (Fig. 14.48).

If foci are $(0, \pm a)$, so $a=be$.

That is, $e=a/b$

Also,

$$e=\sqrt{1-e^2} \Rightarrow e^2=1-e^2 \Rightarrow e=1/\sqrt{2}$$

and ratio of radii is

$$\frac{a}{b}=e=\frac{1}{\sqrt{2}} \quad (1)$$

Given that

$$e=\sqrt{2}\,\lambda \quad (2)$$

Therefore, from Eqs. (1) and (2), we have

$$\lambda=1/2$$

29. For ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ equation of director circle is $x^2+y^2=25$. This director circle will cut the ellipse.

$$\frac{x^2}{50}+\frac{y^2}{20}=1 \text{ at 4 points}$$

Hence, number of points is 4.

Solved JEE 2017 Questions

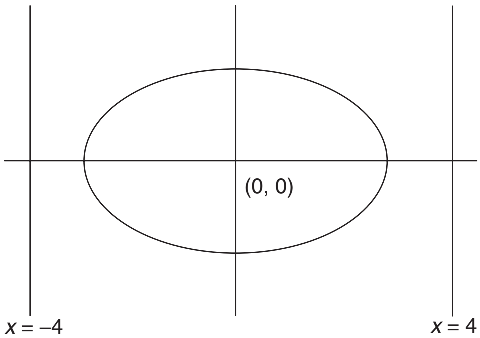
JEE Main 2017

1. The eccentricity of an ellipse whose centre is at the origin is $\frac{1}{2}$. If one of its directrices is $x = -4$, then the equation of the normal to it at $\left(1, \frac{3}{2}\right)$ is

   (A) $4x - 2y = 1$ **(B)** $4x + 2y = 7$
   (C) $x + 2y = 4$ **(D)** $2y - x = 2$

(OFFLINE)

Solution: The given ellipse is depicted in the following figure:



Eccentricity of ellipse is 1/2. Now,

$$\frac{-a}{e} = -4$$

That is,

$$a = 4 \times \frac{1}{2} = 2$$

Therefore,

$$b^2 = a^2(1 - e^2) = a^2\left(1 - \frac{1}{4}\right) = 3$$

Hence, the equation of the given ellipse is written as

$$\frac{x^2}{4} + \frac{y^2}{3} = 1$$

$$\frac{x}{2} + \frac{2y}{3} \times y' = 0$$

$$y' = \frac{-3x}{4y}$$

$$y'|_{(1,3/2)} = \frac{-3}{4} \times \frac{2}{3} = \frac{-1}{2}$$

Therefore, the equation of normal at $\left(1, \frac{3}{2}\right)$ is

$$y - \frac{3}{2} = 2(x - 1)$$

$$2y - 3 = 4x - 4$$

$$\Rightarrow 4x - 2y = 1$$

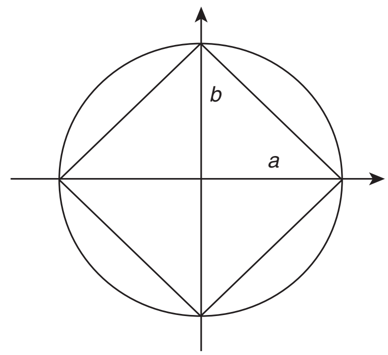
Hence, the correct answer is option (A).

2. Consider an ellipse, whose centre is at the origin and its major axis is along the x-axis. If its eccentricity is 3/5 and the distance between its foci is 6, then the area (in sq. units) of the quadrilateral inscribed in the ellipse, with the vertices as the vertices of the ellipse, is

   (A) 8 **(B)** 32
   (C) 80 **(D)** 40

(ONLINE)

Solution: It is given that that eccentricity is $e = \frac{3}{5}$ and the distance between the foci is $2ae = 6$.



From the equation $2ae = 6$, we get

$$ae = 3$$

or

$$a = \frac{3}{e} = 3 \times \frac{5}{3} = 5$$

Also, we have

$$c = ae = 3$$

Therefore, from the equation of ellipse, we get

$$b^2 = a^2 - c^2$$
$$\Rightarrow b^2 = 25 - 9$$
$$\Rightarrow b^2 = 16$$
$$\Rightarrow b = 4$$

Therefore, the area of quadrilateral is

$$A = 4 \times \frac{1}{2} ab$$

$$= 4 \times \frac{1}{2} \times 5 \times 4 = 40 \text{ sq. units}$$

Hence, the correct answer is option (D).

3. The eccentricity of an ellipse having centre at the origin, axes along the coordinate axes and passing through the points (4, –1) and (–2, 2) is

   (A) $\frac{\sqrt{3}}{2}$ **(B)** $\frac{\sqrt{3}}{4}$
   (C) $\frac{2}{\sqrt{5}}$ **(D)** $\frac{1}{2}$

(ONLINE)

Solution: We have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Let $\frac{1}{a^2}=A$ and $\frac{1}{b^2}=B$. Therefore,

$$Ax + By = 1$$
$$16A + B = 1(4, -1) \quad (1)$$
$$4A + 4B = 1(-2, 2)$$
$$A+B=\frac{1}{4} \quad (2)$$

Subtracting Eq. (2) from Eq. (1), we get

$$A=\frac{1}{20}; \quad a^2 = 20$$

$$B=1-\frac{16}{20}=\frac{4}{20}=\frac{1}{5} \quad b^2 = 5;$$

Therefore, the required eccentricity of the ellipse is obtained as follows:

$$b^2 = a^2(1 - e^2)$$
$$\Rightarrow 5 = 20(1 - e^2)$$
$$\Rightarrow \frac{1}{4}=1-e^2 \Rightarrow e^2=\frac{2}{4} \Rightarrow e=\frac{\sqrt{3}}{2}$$

Hence, the correct answer is option (A).

15 Hyperbola

15.1 Hyperbola – Fundamentals

1. **Definition 1:** A hyperbola is the locus of a point, which moves such that the ratio of its distance from a fixed point to a fixed line is constant and the value of the constant is greater than unity. Here, the fixed point is known as the 'focus', fixed line is called a 'directrix' and the constant is called an 'eccentricity'. Generally, the equation of the hyperbola, whose focus is the point (h, k), its directrix is $lx + my + n = 0$ and its eccentricity is e, is expressed as

$$(x-h)^2 + (y-k)^2 = e^2\left[\frac{(lx+my+n)^2}{l^2+m^2}\right] \qquad (e > 1)$$

2. **Definition 2:** A hyperbola is the locus of a point which moves such that the difference in its distances from two fixed points is constant and the value of the constant is less than the distance between these two fixed points.

3. **Standard equation of hyperbola:** If we consider the focus as $(\pm ae, 0)$ and the directrix as $x = \pm(a/e)$, we get the equation of hyperbola as follows:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $b^2 = a^2(e^2 - 1)$.

Results:

(i) The equation of the directrix corresponding to the focus $(-ae, 0)$ is $x = -a/e$.

(ii) $2a$ is the length of transverse axis and $2b$ is the length of conjugate axis.

4. **Focal chord:** Any chord of the hyperbola which is passing through the focus is called 'focal chord'.

5. **Latus rectum:** A focal chord, which is perpendicular to the transverse axis of the hyperbola, is known as 'latus rectum'. In a hyperbola, we have

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad (a > b)$$

(i) The end points of the latus rectum are

$$\left(\pm ae, \pm\frac{b^2}{a}\right)$$

(ii) The length of the latus rectum is $2b^2/a$.

6. **Focal distance of a point:** Let P be a point on the hyperbola, where S and S′ are the foci, then PS and PS′ are known as the focal distance of the point P (Fig. 15.1).

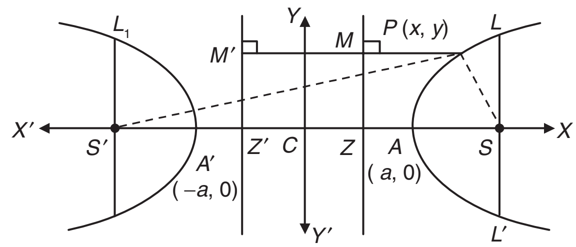


Figure 15.1

Remark: In a hyperbola, the difference in the focal distances of any point is constant which is equal to the length of transverse axis.

Illustration 15.1 Find the equation of the hyperbola whose focus is (2, 2), eccentricity 2 and the equation of directrix $x + y = 9$.

Solution: Let P(x, y) be any point on the hyperbola. Let PM be perpendicular which is drawn from point P on the directrix. Then

$$\frac{SP}{PM} = e$$

$$\Rightarrow (x-2)^2 + (y-2)^2 = 2^2\left[\frac{x+y-9}{\sqrt{2}}\right]^2$$

$$\Rightarrow x^2 + y^2 - 4x - 4y + 8 = 2(x+y-9)^2$$

$$\Rightarrow x^2 + y^2 + 4xy - 32x - 32y + 154 = 0$$

Illustration 15.2 Obtain the equation of a hyperbola with coordinate axes as principal axes where it is given that the distances of one of its vertices from the foci are 9 and 1 unit.

Solution: Let the equation of hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad (1)$$

If the vertices are A$(a, 0)$ and A′$(-a, 0)$ and the foci are S$(ae, 0)$ and S′$(-ae, 0)$, we get

$$l(S'A) = 9 \text{ and } l(SA) = 1$$

$$\Rightarrow a + ae = 9 \text{ and } ae - a = 1$$

or

$$a(1+e) = 9 \text{ and } a(e-1) = 1$$

Therefore,

$$\frac{a(1+e)}{a(e-1)} = \frac{9}{1}$$

$$\Rightarrow 1 + e = 9e - 9$$

$$\Rightarrow e = \frac{5}{4}$$

Since $a(1+e)=9$, we get

$$a\left(1+\frac{5}{4}\right)=9$$

$$\Rightarrow a=4$$

Now, $$b^2=a^2(e^2-1)=16\left(\frac{25}{16}-1\right)$$

Therefore, $b^2=9$. From Eq. (1), the equation of hyperbola is given by

$$\frac{x^2}{16}-\frac{y^2}{9}=1$$

Illustration 15.3 Prove that the equation $\sqrt{(x+4)^2+(y+2)^2}-\sqrt{(x-4)^2+(y-2)^2}=8$ represents a hyperbola.

Solution: Let $a=\sqrt{(x+4)^2+(y+2)^2}$ and $b=\sqrt{(x-4)^2+(y-2)^2}$

We have

$$a^2-b^2=[(x+4)^2+(y+2)^2]-[(x-4)^2+(y-2)^2]=16x+8y \quad (1)$$

$$\sqrt{(x+4)^2+(y+2)^2}-\sqrt{(x-4)^2+(y-2)^2}=8 \quad (2)$$

Dividing Eq. (1) by Eq. (2), we get

$$\sqrt{(x+4)^2+(y+2)^2}+\sqrt{(x-4)^2+(y-2)^2}=\frac{16x+8y}{8} \quad (3)$$

From Eqs. (2) and (3), we get

$$2\sqrt{(x+4)^2+(y+2)^2}=\frac{16x+8y+64}{8}$$

$$\sqrt{(x+4)^2+(y+2)^2}=\frac{8x+4y+32}{8}=\frac{\sqrt{5}}{2}\left(\frac{2x+y+8}{\sqrt{5}}\right)$$

Hence, the given equation is a hyperbola with eccentricity $\sqrt{5}/2$.

Illustration 15.4 Find the equation of the hyperbola of the given transverse axes whose vertex bisects the distance between the centre and the focus.

Solution: Let the vertex be $(a, 0)$ and focus $(ae, 0)$. Then

$$\frac{ae+0}{2}=a \Rightarrow e=2$$

We know that $b^2 = a^2(e^2-1) = 3a^2$. Thus, the equation of the hyperbola is

$$\frac{x^2}{a^2}-\frac{y^2}{3a^2}=1$$

Illustration 15.5 Find the equation of the hyperbola, the distance between whose foci is 16, the eccentricity is $\sqrt{2}$ and the axis is along the x-axis with the origin as its centre.

Solution: We have

$$b^2=a^2(e^2-1)=a^2 \Rightarrow b=a$$

Also,

$$2ae=16 \Rightarrow ae=8 \Rightarrow a=4\sqrt{2}$$

Hence, the equation of the hyperbola is

$$\frac{x^2}{32}-\frac{y^2}{32}=1 \Rightarrow x^2-y^2=32$$

Remarks: For a hyperbola,

$$\frac{(x-\alpha)^2}{a^2}-\frac{(y-\beta)^2}{b^2}=1$$

(1) the eccentricity is $e=\sqrt{1+(b^2/a^2)}$; (2) the equation of the transverse axis is $y=\beta$; (3) the equation of the conjugate axis is $x=\alpha$; (4) the centre is (α, β); (5) the vertices are $(\alpha \pm a, \beta)$; (6) the foci are $(\alpha \pm ae, \beta)$; (7) the directrices are $x=\alpha \pm (a/e)$; (8) the ends of the conjugate axis are $(\alpha, \beta \pm b)$; (9) the length of the latus rectum is $2b^2/a$ and (10) the parametric coordinates are $(a\sec\theta+\alpha, b\tan\theta+\beta)$ where $0 \le \theta < 2\pi$.

Illustration 15.6 Find the centre, eccentricity, foci and directrices of the hyperbola, $16x^2-9y^2+32x+36y-164=0$.

Solution: Here, we have

$$16x^2+32x+16-(9y^2-36y+36)-144=0$$

or $$16(x+1)^2-9(y-2)^2=144$$

Therefore,

$$\frac{(x+1)^2}{9}-\frac{(y-2)^2}{16}=1$$

Substituting $x+1=X$ and $y-2=Y$ in this equation, we get

$$\frac{X^2}{9}-\frac{Y^2}{16}=1$$

which is the standard form. Here, $a^2=9$ and $b^2=16$. Since $b^2=a^2(e^2-1)$, we get $16=9(e^2-1)$. Therefore,

$$e^2-1=\frac{16}{9}$$

That is,

$$e^2=\frac{25}{9} \Rightarrow e=\frac{5}{3}$$

Now, the centre is $(0, 0)_{X,Y}=(-1, 2)$ (this is due to the reason that when $X=0$, $x+1=X$ gives $x=-1$ and when $Y=0$, $y-2=Y$ gives $y=2$). Now, the foci are

$$(\pm ae, 0)_{X,Y}=\left[\pm 3\left(\frac{5}{3}\right), 0\right]_{X,Y}=(\pm 5, 0)_{X,Y}$$

$$=(-1\pm 5, 2)=(4, 2), (-6, 2)$$

The directrices in X, Y coordinates have the equations

$$X \pm \frac{a}{e}=0$$

or $$x+1\pm\frac{3}{5/3}=0$$

That is,

$$x+1\pm\frac{9}{5}=0$$

Therefore,

$$x=-\frac{14}{5} \text{ and } x=\frac{4}{5}$$

15.2 Position of Point Relative to Hyperbola

Let us consider a hyperbola,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad (15.1)$$

Let P ≡ (*h*, *k*) (Fig. 15.2). Now, P lies outside, on or inside the hyperbola [Eq. (15.1)], we can write as

$$\frac{h^2}{a^2}-\frac{k^2}{b^2}-1<,=,>0$$

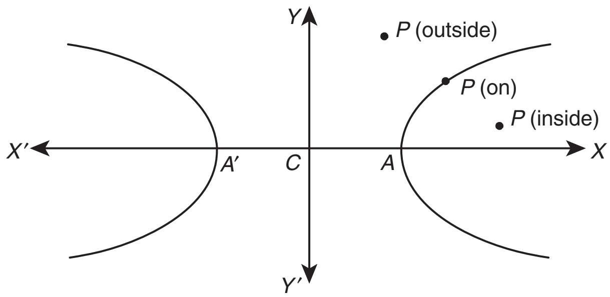


Figure 15.2

Illustration 15.7 Find the position of the point (5, –4) relative to the hyperbola $9x^2 - y^2 = 1$.

Solution: Since $9(5)^2 - (-4)^2 - 1 = 225 - 16 - 1 = 208 > 0$, the point (5, –4) lies inside the hyperbola, $9x^2 - y^2 = 1$ (Fig. 15.2).

15.3 Parametric Equation of Hyperbola

Let us consider a point P on a hyperbola and draw a line through it which is parallel to the conjugate axis (Fig. 15.3). From point P, where it cuts the transverse axis, let us draw a tangent to the auxiliary circle. Now, the angle made by the line joining the point of tangency to the centre of the circle from the positive *x*-axis is known as the 'eccentric angle' of point P. Let the hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad (15.2)$$

Then, the parametric coordinates of point P is ($a\sec\theta$, $b\tan\theta$), where θ is the eccentric angle of point P.

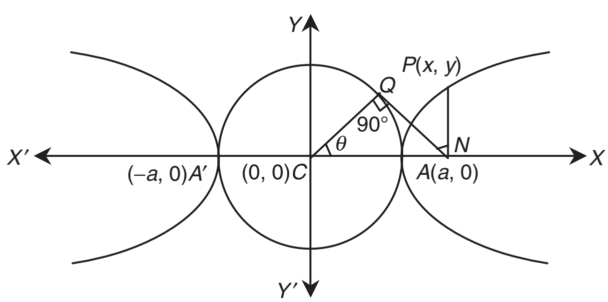


Figure 15.3

Here, $x = a\sec\theta$; $y = b\tan\theta$ is called the parametric equation of the hyperbola (Eq. 15.3).

Illustration 15.8 Let two points P and Q be on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ whose centre C be such that CP is perpendicular to CQ, $a < b$. Prove that $\frac{1}{CP^2}+\frac{1}{CQ^2}=\frac{1}{a^2}-\frac{1}{b^2}$.

Solution: Let $CP = r_1$ be inclined to the transverse axis at an angle θ so that P is ($r_1\cos\theta, r_1\sin\theta$) and P lies on the hyperbola. So,

$$r_1^2\left(\frac{\cos^2\theta}{a^2}-\frac{\sin^2\theta}{b^2}\right)=1$$

On replacing θ by 90° + θ, we get

$$r_2^2\left(\frac{\sin^2\theta}{a^2}-\frac{\cos^2\theta}{b^2}\right)=1$$

$$\Rightarrow \frac{1}{r_1^2}+\frac{1}{r_2^2}=\frac{\cos^2\theta}{a^2}-\frac{\sin^2\theta}{b^2}+\frac{\sin^2\theta}{a^2}-\frac{\cos^2\theta}{b^2}$$

$$\Rightarrow \frac{1}{r_1^2}+\frac{1}{r_2^2}=\frac{1}{a^2}-\frac{1}{b^2}$$

$$\Rightarrow \frac{1}{CP^2}+\frac{1}{CQ^2}=\frac{1}{a^2}-\frac{1}{b^2}$$

15.4 Conjugation of Hyperbola

A hyperbola is called 'conjugate hyperbola' of a given hyperbola when it is obtained by interchanging the transverse and the conjugate axes of the given hyperbola. If $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ be a given hyperbola, then the equation of its conjugate hyperbola is expressed as

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$$

Illustration 15.9 If *e* and *e*′ be the eccentricities of a hyperbola and its conjugate, prove that $\frac{1}{e^2}+\frac{1}{(e')^2}=1$.

Solution: Let the equation of the hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad (1)$$

Therefore, the eccentricity is

$$e=\sqrt{1+\left(\frac{b^2}{a^2}\right)}=\sqrt{\frac{a^2+b^2}{a^2}}$$

or
$$\frac{1}{e^2}=\frac{a^2}{a^2+b^2} \qquad (2)$$

Therefore, the equation of the conjugate hyperbola to that expressed in Eq. (1) is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$$

Thus, the eccentricity is

$$e'=\sqrt{1+\left(\frac{a^2}{b^2}\right)}=\sqrt{\frac{a^2+b^2}{b^2}}$$

or
$$\frac{1}{(e')^2}=\frac{b^2}{a^2+b^2} \qquad (3)$$

Now, adding Eqs. (2) and (3), we get

$$\frac{1}{e^2}+\frac{1}{(e')^2}=1$$

Your Turn 1

1. In the hyperbola, $4x^2 - 9y^2 = 36$, find (a) the length of axes, (b) the coordinates of the foci, (c) eccentricity and (d) the latus rectum.

   Ans. (a) Length of transverse axis is 6, length of conjugate axis is 4

   (b) $(\pm\sqrt{13}, 0)$

   (c) $\frac{\sqrt{13}}{3}$

   (d) 8/3

2. Find the equation to the hyperbola whose eccentricity is 5/4, where its focus is $(a, 0)$ and whose directrix is $4x - 3y = a$.

   Ans. $7y^2 + 24xy - 24ax - 6ay + 15a^2 = 0$

3. Find the length of (a) the transverse and (b) the conjugate axis for the hyperbola $\frac{(x-4)^2}{4} - \frac{(y-2)^2}{9} + 1 = 0$.

   Ans. (a) 6; (b) 4

4. How many tangents can be drawn from the point (1, 2) to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$?

   Ans. Two

5. Find the parametric coordinates of a point with eccentric angle θ on the hyperbola $16x^2 - 9y^2 + 32x + 36y - 164 = 0$.

   Ans. $(3\sec\theta - 1, 4\tan\theta + 2)$

6. If F and F′ are the foci of the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ and P be a point on it. Find |PF – PF′|.

   Ans. 10

7. State true or false: The eccentricity of the hyperbola $\frac{x^2}{5} - \frac{y^2}{3} = 1$ is $\pm\sqrt{8/5}$.

   Ans. False

8. State true or false: The length of the transverse axis exceeds the length of conjugate axis for the hyperbola $4x^2 - 9y^2 = 36$ by 5.

   Ans. False

9. State true or false: Any point on the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ can be written as $(5\operatorname{cosec}\theta, 4\cot\theta)$.

   Ans. True

10. State true or false: The equation $(x-\alpha)^2 + (y-\beta)^2 = k(lx + my + n)^2$ represents a hyperbola for $k > (l^2 + m^2)^{-1}$.

    Ans. True

15.5 Tangent of Hyperbola

1. **Point form:** Let $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be a hyperbola and (x_1, y_1) be a point on it. Then the equation of tangent to the hyperbola at the point (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = 0$ written as $T = 0$.

2. **Parametric form:** Let $(a\sec\theta, b\tan\theta)$ be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Then the equation of tangent to the hyperbola is $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta - 1 = 0$.

3. **Slope form:** The line $y = mx + c$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if and only if $c^2 = a^2m^2 - b^2$. Thus, the equation of tangent to the hyperbola with slope m is $y = mx \pm \sqrt{a^2m^2 - b^2}$. Here, '±' denotes that we can get two tangents to the hyperbola with the same slope m.

4. **Director circle:** The locus of the point, where the perpendicular tangents of the hyperbola meet, is called director circle of the hyperbola. Let the tangent $y = mx \pm \sqrt{a^2m^2 - b^2}$ passes through the point (h, k). Then

$$k = mh \pm \sqrt{a^2m^2 - b^2}$$
$$\Rightarrow (k - mh)^2 = a^2m^2 - b^2$$
$$\Rightarrow (h^2 - a^2)m^2 - 2kmh + k^2 + b^2 = 0$$

The roots of the equation give the slope of the tangents which are intersecting at (h, k). For the director circle,

$$m_1m_2 = -1$$
$$\Rightarrow \frac{k^2 + b^2}{h^2 - a^2} = -1 \Rightarrow h^2 + k^2 = a^2 - b^2$$

Thus, the locus of the point (h, k) is $x^2 + y^2 = a^2 - b^2$, which is the equation of the director circle of the hyperbola. Hence, in general, the director circle of a hyperbola is a circle with the centre same as the centre of the hyperbola and radius as square root of difference in the squares of length of semi-transverse axis and semi-conjugate axis.

5. **Equation of tangents from an external point (x_1, y_1):** Let (x_1, y_1) be a point outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Then the combined equation of tangents to the hyperbola is

$$\left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1\right)^2 = \left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right)\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right)$$

which is written as $T^2 = SS_1$,

where $S = \left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right)$ and $S_1 = \left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right)$

6. **Chord of contact:** Let (x_1, y_1) be a point outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then the equation of chord of contact of the point with respect to the hyperbola is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = 0$, which is written as $T = 0$.

Illustration 15.10 If the line $y = mx + \sqrt{(a^2m^2 - b^2)}$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(a\sec\theta, b\tan\theta)$, show that $\theta = \sin^{-1}\left(\frac{b}{am}\right)$.

Solution: Since $(a\sec\theta, b\tan\theta)$ lies on

$$y = mx + \sqrt{(a^2m^2 - b^2)}$$

we get

$$b\tan\theta = am\sec\theta + \sqrt{(a^2m^2 - b^2)}$$

$$\Rightarrow (b\tan\theta - am\sec\theta)^2 = a^2m^2 - b^2$$

$$\Rightarrow b^2\tan^2\theta + a^2m^2\sec^2\theta - 2abm\tan\theta\sec\theta = a^2m^2 - b^2$$

$$\Rightarrow a^2m^2\tan^2\theta - 2abm\tan\theta\sec\theta + b^2\sec^2\theta = 0$$

or $\quad a^2m^2\sin^2\theta - 2abm\sin\theta + b^2 = 0 \qquad (\because \cos\theta \neq \theta)$

Therefore,

$$\sin\theta = \frac{2abm \pm \sqrt{4a^2b^2m^2 - 4a^2b^2m^2}}{2a^2m^2} = \left(\frac{b}{am}\right)$$

Hence,

$$\theta = \sin^{-1}\left(\frac{b}{am}\right)$$

Illustration 15.11 Let m_1 and m_2 be slopes of tangents from a point (1, 4) on the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$. Then find the point from which the tangents that are drawn on the hyperbola have the slopes $|m_1|$ and $|m_2|$ and positive intercepts on y-axis.

Solution: A tangent to the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ of slope m is

$$y = mx \pm \sqrt{25m^2 - 16}$$

$$\Rightarrow (4 - m)^2 = 25m^2 - 16$$

$$\Rightarrow 3m^2 + m - 4 = 0$$

$$\Rightarrow m_1 = 1, m_2 = -\frac{4}{3}$$

$$\Rightarrow |m_1| = 1, |m_2| = \frac{4}{3}$$

Therefore, the tangents are

$$y = x + 3 \text{ and } y = \frac{4}{3}x + \frac{16}{3}$$

On solving these equations, we get the point (–7, –4).

Illustration 15.12 Find the range of a for which two perpendicular tangents can be drawn to the hyperbola from any point outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{9} = 1$.

Solution: We know that the locus of the point, where the perpendicular tangents of the hyperbola meet, is called the director circle of the hyperbola whose equation is given by

$$x^2 + y^2 = a^2 - b^2 \qquad (1)$$

A point satisfies Eq. (1) only when

$$a^2 - b^2 \geq 0$$

$$\Rightarrow a^2 - 9 \geq 0$$

$$\Rightarrow a \in (-\infty, -3] \cup [3, \infty)$$

15.6 Normal to Hyperbola

1. **Point form:** Let $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be a hyperbola and $P(x_1, y_1)$ be a point on it. Then the equation of the normal to the hyperbola at point P is $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$.
2. **Parametric form:** Let $P(a\sec\theta, b\tan\theta)$ be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Then the equation of normal to the hyperbola at point P is $ax\cos\theta + by\cot\theta = a^2 + b^2$.
3. **Slope form:** The line $y = mx + c$ is a normal to the hyperbola if and only if $c = \frac{\pm m(a^2 + b^2)}{\sqrt{a^2 - b^2m^2}}$. Thus, the equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with slope m is given by

$$y = mx \pm \frac{m(a^2 + b^2)}{\sqrt{a^2 - b^2m^2}}$$

Illustration 15.13 If the normal at ϕ on a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets transverse axis at G, prove that $AG \times A'G = a^2(e^4\sec^2\phi - 1)$, where A and A′ are the vertices of the hyperbola.

Solution: The equation of normal at $(a\sec\phi, b\tan\phi)$ to the given hyperbola is

$$ax\cos\phi + by\cot\phi = (a^2 + b^2)$$

which meets the transverse axis, that is, x-axis at G. So the coordinates of G are $\left(\left(\frac{a^2 + b^2}{a}\right)\sec\phi, 0\right)$ and the coordinates of the vertices A and A′ are A(a, 0) and A′(–a, 0), respectively. Therefore,

$$AG \times A'G = \left(-a + \left(\frac{a^2 + b^2}{a}\right)\sec\phi\right)\left(a + \left(\frac{a^2 + b^2}{a}\right)\sec\phi\right)$$

$$= \left(\frac{a^2 + b^2}{a}\right)^2\sec^2\phi - a^2$$

$$= (ae^2)^2\sec^2\phi - a^2$$

$$= a^2(e^4\sec^2\phi - 1)$$

Illustration 15.14 A straight line is drawn parallel to the conjugate axis of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ to meet it and the conjugate hyperbola at the points P and Q, respectively. Show that the normals at P and Q to the curves meet on x-axis.

Solution: Let $P(a\sec\theta, b\tan\theta)$ be a point on the hyperbola, and $Q(a\tan\phi, b\sec\phi)$ be a point on the conjugate hyperbola. Then

$$a\sec\theta = a\tan\phi \Rightarrow \sec\theta = \tan\phi$$

The equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at point P is

$$y-b\tan\theta=-\frac{a\tan\theta}{b\sec\theta}(x-a\sec\theta)$$

The equation of the normal to the conjugate hyperbola at point Q is

$$y-b\sec\phi=-\frac{a\sec\phi}{b\tan\phi}(x-a\tan\phi)$$

On eliminating x and using $\sec\theta=\tan\phi$, we get

$$y(\sec\phi-\tan\theta)=0$$

$$\Rightarrow y=0$$

Hence, the normals meet on x-axis.

Illustration 15.15 Prove that the line $lx+my-n=0$ is a normal to the hyperbola if $\frac{a^2}{l^2}-\frac{b^2}{m^2}=\frac{(a^2+b^2)^2}{n^2}$.

Solution: The equation of a normal to a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$ax\cos\phi+by\cot\phi=a^2+b^2$$

or $$ax\cos\phi+by\cot\phi-(a^2+b^2)=0 \quad (1)$$

If the straight line $lx+my-n=0$ is a normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then Eq. (1) and $lx+my-n=0$ represent the same line.

$$\frac{a\cos\phi}{l}=\frac{b\cot\phi}{m}=\frac{(a^2+b^2)}{n}$$

or $$\sec\phi=\frac{na}{l(a^2+b^2)} \text{ and } \tan\phi=\frac{nb}{m(a^2+b^2)}$$

Therefore, $$\sec^2\phi-\tan^2\phi=1$$

or $$\frac{n^2a^2}{l^2(a^2+b^2)^2}-\frac{n^2b^2}{m^2(a^2+b^2)^2}=1$$

$$\Rightarrow \frac{a^2}{l^2}-\frac{b^2}{m^2}=\frac{(a^2+b^2)^2}{n^2}$$

Illustration 15.16 Prove that the part of the tangent at any point of a hyperbola intercepted between the point of contact and the transverse axis is a harmonic mean between the lengths of the perpendicular which is drawn from the foci on the normal at the same point.

Solution: The equation of tangent is

$$\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$$

Let this meet the transverse axis at point A. Therefore, we have A($a\cos\theta$, 0) and the point of contact is P($a\sec\theta$, $b\tan\theta$). Thus,

$$AP^2=a^2(\cos\theta-\sec\theta)^2+b^2\tan^2\theta=(a^2\sin^2\theta+b^2)\tan^2\theta \quad (1)$$

The equation of normal is

$$ax\cos\theta+by\cot\theta=a^2+b^2$$

Therefore, the length of perpendicular from foci to the normal is

$$ST=\left|\frac{a^2e\cos\theta-(a^2+b^2)}{\sqrt{a^2\cos^2\theta+b^2\cot^2\theta}}\right|$$

That is,

$$S'T'=\left|\frac{-a^2e\cos\theta-(a^2+b^2)}{\sqrt{a^2\cos^2\theta+b^2\cot^2\theta}}\right|$$

On further calculations, we get

$$AP=2\left(\frac{ST\times S'T'}{ST+S'T'}\right)$$

15.7 Chords of Hyperbola

Consider a line $y=mx+c$ and the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. On eliminating y from both equations we get

$$(a^2m^2-b^2)x^2+2a^2mcx+a^2c^2+a^2b^2=0 \quad (15.3)$$

The roots of Eq. (15.3) give abscissae of the point where the straight line cuts the hyperbola. The discriminant of Eq. (15.3) is

$$D=4a^4m^2c^2-4a^2(c^2+b^2)(a^2m^2-b^2)$$

$$=4a^2\left\{a^2m^2c^2-a^2m^2c^2-b^2a^2m^2+b^4+b^2c^2\right\}$$

$$=4a^2b^2\left\{c^2+b^2-a^2m^2\right\}$$

The following three cases arise:

Case 1: If $c^2>(a^2m^2-b^2)$, then the straight line $y=mx+c$ is a real chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Case 2: If $c^2=(a^2m^2-b^2)$, then the straight line is a tangent to the hyperbola.

Case 3: If $c^2<(a^2m^2-b^2)$, then straight line will be an imaginary chord of the hyperbola.

15.7.1 Chord with Mid-Point

Let (x_1, y_1) be the mid-point of a chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Then its equation is given by $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}$, which is written as $T=S_1$.

15.7.2 Parametric Form of Chord

Let A($a\sec\theta_1$, $b\tan\theta_1$) and B($a\sec\theta_2$, $b\tan\theta_2$) be two points on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Then the equation of chord AB is given by

$$\frac{x}{a}\cos\left(\frac{\theta_1-\theta_2}{2}\right)-\frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right)=\cos\left(\frac{\theta_1+\theta_2}{2}\right)$$

If $\theta_1=\theta_2=\theta$, then this chord become a tangent at point θ and the equation of which is given by

$$\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$$

Illustration 15.17 Find the locus of the mid-points of the chords of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ which subtend a right angle at the origin.

Solution: Let (h, k) be the mid-point of the chord of the hyperbola. Then its equation is

$$\frac{hx}{a^2}-\frac{ky}{b^2}-1=\frac{h^2}{a^2}-\frac{k^2}{b^2}-1$$

or
$$\frac{hx}{a^2}-\frac{ky}{b^2}=\frac{h^2}{a^2}-\frac{k^2}{b^2} \quad (1)$$

The equation of the lines joining the origin to the points of intersection of the hyperbola and the chord [Eq. (1)] is obtained by making homogeneous hyperbola with the help of Eq. (1)

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{[(hx/a^2)-(ky/b^2)]^2}{[(h^2/a^2)-(k^2/b^2)]^2}$$

$$\Rightarrow \frac{1}{a^2}\left(\frac{h^2}{a^2}-\frac{k^2}{b^2}\right)^2 x^2-\frac{1}{b^2}\left(\frac{h^2}{a^2}-\frac{k^2}{b^2}\right)y^2=\frac{h^2}{a^4}x^2+\frac{k^2}{b^4}y^2-\frac{2hk}{a^2b^2}xy \quad (2)$$

The lines represented by Eq. (2) is at right angle if

$$\text{Coefficient of } x^2+\text{Coefficient of } y^2=0$$

$$\Rightarrow \frac{1}{a^2}\left(\frac{h^2}{a^2}-\frac{k^2}{b^2}\right)^2-\frac{h^2}{a^4}-\frac{1}{b^2}\left(\frac{h^2}{a^2}-\frac{k^2}{b^2}\right)-\frac{k^2}{b^4}=0$$

$$\Rightarrow \left(\frac{h^2}{a^2}-\frac{k^2}{b^2}\right)^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right)=\frac{h^2}{a^4}+\frac{k^2}{b^4}$$

Hence, the locus of (h, k) is

$$\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)\left(\frac{1}{a^2}-\frac{1}{b^2}\right)=\frac{x^2}{a^4}+\frac{y^2}{b^4}$$

Illustration 15.18 Find the value of c for which the line $y = 2x + c$ is a real chord of the hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1$.

Solution: Here $a = 3$ and $b = 4$ for the hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1$ and the slope of the line $y = 2x + c$ is $m = 2$. For the line to be a real chord, we have

$$c^2 > a^2m^2 - b^2$$
$$\Rightarrow c^2 > 9\times 4 - 16$$
$$\Rightarrow c^2 > 20$$
$$\Rightarrow c \in (-\infty, -\sqrt{20})\cup(\sqrt{20}, \infty)$$

Your Turn 2

1. Find the equation of tangent of the hyperbola $\frac{x^2}{4}-\frac{y^2}{9}=1$ at point θ.

   Ans. $\frac{x}{2}\sec\theta-\frac{y}{3}\tan\theta-1=0$

2. If the tangent provided in Question 1 is parallel to the line $3x - y + 4 = 0$, then find out value of θ.

   Ans. 30°, 210°

3. Find the locus of the middle points of the chords of hyperbola $3x^2 - 2y^2 + 4x - 6y = 0$ parallel to $y = 2x$.

   Ans. $3x - 4y = 4$

4. Find the equation of the chord of the hyperbola $25x^2 - 16y^2 = 400$ which is bisected at the point (6, 2).

   Ans. $75x - 16y = 418$

5. The tangents are drawn from the point (3, 2) to the hyperbola $\frac{x^2}{25}-\frac{y^2}{16}=1$. Find the sum of the squares of the slopes of the tangents.

   Ans. 89/16

15.8 Asymptotes

1. A straight line is said to be an asymptote of a hyperbola if it touches the hyperbola at two points at infinity; however, the line does not lie completely on it, that is, the distance of the line from the origin is finite. The equation of a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at point (x_1, y_1) is expressed as

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1 \quad (15.4)$$

and since the point lies on the hyperbola, we can write

$$\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1 \quad (15.5)$$

Using Eq. (15.5), Eq. (15.4) can be written as

$$\frac{x}{a}\pm\frac{y}{b}\sqrt{1-\frac{a^2}{x_1}}=\frac{a}{x_1} \quad (15.6)$$

Equation (15.6) represents the equation of asymptotes if the point of tangency is at infinity. Thus, since $x_1 \to \infty$, Eq. (15.6) becomes equation of asymptotes, which is given by

$$\frac{x}{a}\pm\frac{y}{b}=0$$

$$\Rightarrow \frac{x}{a}+\frac{y}{b}=0,\ \frac{x}{a}-\frac{y}{b}=0$$

which are the two asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

The combined equation of the asymptotes as a pair of straight lines can be written as

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0 \quad (15.7)$$

From Eq. (15.7) and from the equation of hyperbola, we can say that the combined equation of asymptotes of a hyperbola differs by a constant with the equation of hyperbola. If $S = 0$ represents equation of a hyperbola, then the combined equation of asymptotes of hyperbola can be written as $S + k = 0$ (where k is a constant) and vice versa. The value of k can be found by substituting $\Delta = 0$ in the combined equation of asymptotes since it represents two straight lines. Also, the equation of conjugate hyperbola can be written as $S + 2k = 0$.

2. **Angle between asymptotes:** Equation of asymptotes for the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is given as $\frac{x}{a}-\frac{y}{b}=0$ and $\frac{x}{a}+\frac{y}{b}=0$.

If θ is the angle included between these two asymptotes, then

$$\theta = 2\tan^{-1}\left(\frac{b}{a}\right)$$

Illustration 15.19 Find the hyperbola whose asymptotes are $2x - y = 3$ and $3x + y - 7 = 0$ and which passes through the point (1, 1).

Solution: The equation of the hyperbola differs from the equation of the asymptotes by a constant. This implies that the equation of the hyperbola with asymptotes $3x + y - 7 = 0$ and $2x - y = 3$ is

$$(3x + y - 7)(2x - y - 3) + k = 0$$

which passes through (1, 1), that is, $k = -6$. Hence, the equation of the hyperbola is

$$(2x - y - 3)(3x + y - 7) = 6$$

15.9 Rectangular Hyperbola

A hyperbola is called rectangular hyperbola if its asymptotes are perpendicular to each other. A hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is a rectangular hyperbola if the angle between its asymptotes is 90°. That is,

$$\theta = 2\tan^{-1}\left(\frac{b}{a}\right) = \frac{\pi}{2}$$

$$\Rightarrow b = a$$

Therefore, the equation of rectangular hyperbola is given by

$$x^2 - y^2 = a^2$$

Results:

(i) A point on a rectangular hyperbola is of the form $x = a\sec\theta$; $y = a\tan\theta$.

(ii) The general tangent of a rectangular hyperbola is of the form $y = mx \pm a\sqrt{m^2 - 1}$.

(iii) A rectangular hyperbola's director circle is a point circle, that is, the only pair of tangent which are at right angle are the asymptotes.

(iv) A rectangular hyperbola's normal is of the form, $x\cos\theta + y\cot\theta = 2a$.

(v) A rectangular hyperbola's asymptotes are $y = x$ and $y = -x$.

(vi) Eccentricity of any rectangular hyperbola is a constant which is equal to $\sqrt{2}$.

Illustration 15.20 Show that the locus of the middle points of the normal chords of the rectangular hyperbola $x^2 - y^2 = a^2$ is $(y^2 - x^2)^3 = 4a^2x^2y^2$.

Solution: Let (h, k) be the mid-point of the chord of the hyperbola $x^2 - y^2 = a^2$. Then its equation (from $T = S_1$) is

$$hx - ky = h^2 - k^2 \quad (1)$$

However, since Eq. (1) is normal to the hyperbola, its equation is

$$x\cos\theta + y\cot\theta = 2a \quad (2)$$

On comparing Eq. (1) and (2), we get

$$\frac{h}{\cos\theta} = \frac{-k}{\cot\theta} = \frac{h^2 - k^2}{2a}$$

This implies that

$$\sec\theta = \frac{h^2 - k^2}{2ah} \quad (3)$$

and

$$\tan\theta = \frac{h^2 - k^2}{-2ak} \quad (4)$$

From Eqs. (3) and (4), we get

$$\sec^2\theta - \tan^2\theta = \frac{(h^2 - k^2)^2}{4a^2}\left(\frac{1}{h^2} - \frac{1}{k^2}\right)$$

$$\Rightarrow 1 = \frac{(h^2 - k^2)^2}{4a^2}\frac{(k^2 - h^2)}{h^2k^2}$$

That is,

$$(h^2 - k^2)^2(k^2 - h^2) = 4a^2h^2k^2$$

Hence, the locus of the mid-point (h, k) is

$$(x^2 - y^2)^2(y^2 - x^2) = 4a^2x^2y^2$$

or

$$(y^2 - x^2)^3 = 4a^2x^2y^2$$

15.9.1 Another Form of Rectangular Hyperbola

In a rectangular hyperbola, $x^2 - y^2 = a^2$, we have two pairs of mutually perpendicular straight lines:

1. Pair of axes of the hyperbola, that is, transverse axis $y = 0$ and conjugate axis $x = 0$.
2. Pair of asymptotes, that is, $y = x$ and $y = -x$.
   If we interchange the role of these two pairs of straight lines, that is, $x = 0$ and $y = 0$ become the pair of asymptotes and $y = x$ becomes the transverse axes; $y = -x$ becomes the conjugate axis. Then, the equation of rectangular hyperbola becomes $xy = c^2$ (Fig. 15.4) where $c^2 = a^2/2$.

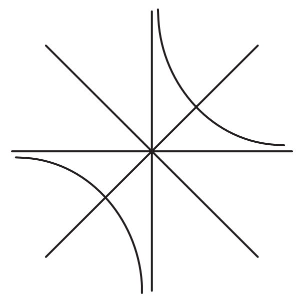


Figure 15.4

Remark: If we make $y = -x$ as transverse axis and $y = x$ as conjugate axis, then the equation of hyperbola becomes $xy = -c^2$ (Fig. 15.5).

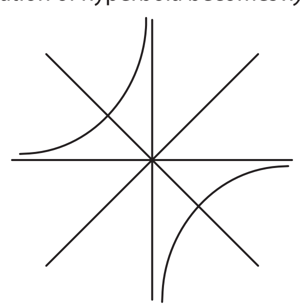


Figure 15.5

Results:

(i) A point on a rectangular hyperbola $xy = c^2$ is of the form of $x = ct$, $y = c/t$, which is also the parametric equation of the rectangular hyperbola, $xy = c^2$.

(ii) The equation of tangent is of the form

$$y = -\frac{1}{t^2}x + \frac{2c}{t}$$

Remarks: Here, the slope of the tangent $m = -1/t^2$, which is negative for any real t. That is, the tangent to the rectangular hyperbola of the form $xy = c^2$ makes obtuse angle with the positive x-axis. On substituting $m = -1/t^2$, we get the equation of tangent in the slope form as follows: $y = mx \pm 2c\sqrt{-m}$.

(iii) The equation of normal will be of the form

$$ty - t^3x + ct^4 - c = 0 \quad (15.8)$$

If this normal passes through the point (h, k), then

$$ct^4 - ht^3 + kt - c = 0 \quad (15.9)$$

Thus, in general, four normals can be drawn from a point (h, k), where the parameters t_1, t_2, t_3 and t_4 of the feet of normals are the roots of Eq. (15.9).

$$t_1 + t_2 + t_3 + t_4 = h/c$$

$$\Sigma t_1t_2 = 0$$

$$\Sigma t_1t_2t_3 = -k/c$$

$$t_1t_2t_3t_4 = -1$$

Illustration 15.21 A triangle is inscribed in a rectangular hyperbola such that the tangent at one of the vertices is perpendicular to the opposite side. Prove that the triangle is a right-angled triangle.

Solution: Let $A\left(ct_1, \frac{c}{t_1}\right), B\left(ct_2, \frac{c}{t_2}\right)$ and $C\left(ct_3, \frac{c}{t_3}\right)$ be the vertices of the triangle inscribed in the rectangular hyperbola $xy = c^2$. Then the slope of the tangent at the vertex A is $-1/t_1^2$. Also, the slope of the side BC is $-1/t_2t_3$. These lines are perpendicular to each other, so

$$t_1^2t_2t_3 = -1$$

$$\Rightarrow \left(-\frac{1}{t_1t_2}\right)\left(-\frac{1}{t_1t_2}\right) = -1$$

Since the slope of AC is $-1/t_1t_3$ and the slope of AB is $-1/t_1t_2$, the triangle is right angled at A.

Illustration 15.22 Find the distance from point A(4, 2) to the points in which the line $4x - y + 4 = 0$ meets the hyperbola, $xy = 24$.

Solution: On solving $4x - y + 4 = 0$, we get

$$xy = 24$$

$$\Rightarrow x(4x + 4) = 24$$

$$\Rightarrow x(x + 1) = 6$$

$$\Rightarrow x = -3, 2$$

$$\Rightarrow y = -8, 12$$

Therefore, the distance is obtained as

$$\sqrt{(2-4)^2 + (12-2)^2} = \sqrt{104} \text{ and } \sqrt{(-3-4)^2 + (-8-2)^2} = \sqrt{149}$$

Illustration 15.23 If a triangle is inscribed in a rectangular hyperbola, prove that its orthocentre also lies on the curve.

Solution: Let the hyperbola be $xy = c^2$. Let the vertices of ΔABC be A$(ct_1, c/t_1)$, B$(ct_2, c/t_2)$ and C$(ct_3, c/t_3)$. Now, the equation of altitude through point A is

$$y + ct_1t_2t_3 = t_2t_3\left(x + \frac{c}{t_1t_2t_3}\right) \quad (1)$$

Similarly, the equation of altitude through point B is

$$y + ct_1t_2t_3 = t_1t_3\left(x + \frac{c}{t_1t_2t_3}\right) \quad (2)$$

Solving Eqs. (1) and (2), we get the orthocentre as follows:

$$\left(-\frac{c}{t_1t_2t_3}, -ct_1t_2t_3\right)$$

which lies on $xy = c^2$.

Illustration 15.24 The chords of the hyperbola $x^2 - y^2 = a^2$ touch the parabola $y^2 = 4ax$. Prove that the locus of their middle point is the curve $y^2(x - a) = x^3$.

Solution: If (h, k) be the mid-point of the chord, then its equation (from $T = S_1$) is

$$hx - ky = h^2 - k^2$$

or
$$y = \frac{h}{k}x + \frac{k^2 - h^2}{k}$$

If it touches the parabola $y^2 = 4ax$, then from $c = a/m$, we get

$$\frac{k^2 - h^2}{k} = \frac{ak}{h}$$

or
$$ak^2 = hk^2 - h^3$$

or
$$ay^2 = xy^2 - x^3$$

Hence, the locus obtained is

$$y^2(x - a) = x^3$$

15.9.2 Intersection of a Circle and a Rectangular Hyperbola

Consider a circle $x^2 + y^2 + 2gx + 2fy + k = 0$ and rectangular hyperbola $xy = c^2$. On eliminating y from both these equations, we get

$$x^4 + 2gx^3 + kx^2 + 2fc^2x + c^4 = 0 \quad (15.10)$$

The roots of Eq. (15.10) give abscissae of the point where the circle cuts it. From Eq. (1), we get

$$x_1 + x_2 + x_3 + x_4 = -2g$$

$$\Sigma x_1 x_2 = k$$

$$\Sigma x_1 x_2 x_3 = -2fc^2$$

$$x_1 x_2 x_3 x_4 = c^4$$

Similarly, if we eliminate x, we get

$$y_1 + y_2 + y_3 + y_4 = -2f$$

$$\Sigma y_1 y_2 = k$$

$$\Sigma y_1 y_2 y_3 = -2gc^2$$

$$y_1 y_2 y_3 y_4 = c^4$$

Here, we observe the following:

(i) The mean point of points of intersection of a rectangular hyperbola and a circle is the mid-point of a line segment joining the centres of both the curves.

(ii) The product of abscissae of points of intersection is always equal to product of ordinates of points of intersection and is constant.

Illustration 15.25 If a variable circle with fixed radius intersects the rectangular hyperbola $xy = c^2$ at the variable points P, Q, R and S. Prove that $OP^2 + OQ^2 + OR^2 + OS^2$ = constant (where O is the origin).

Solution: Let the variable circle be

$$x^2 + y^2 + 2gx + 2fy + k = 0$$

If this circle cuts $xy = c^2$ at the point (x_i, y_i), where $i = 1, 2, 3$ and 4, then

$$OP^2 + OQ^2 + OR^2 + OS^2 = (x_1^2 + x_2^2 + x_3^2 + x_4^2) + (y_1^2 + y_2^2 + y_3^2 + y_4^2)$$

$$= (x_1 + x_2 + x_3 + x_4)^2 - 2\Sigma x_1x_2 + (y_1 + y_2 + y_3 + y_4)^2 - 2\Sigma y_1y_2$$

$$= 4g^2 - 2k + 4f^2 - 2k = 4(g^2 + f^2 - k)$$

$$= 4r^2$$

where r is the radius of variable circle.

Your Turn 3

1. Find the equation of axes of the rectangular hyperbola $(x + 2)(y - 3) = 25$.

   Ans. Transverse axis is $x - y + 5 = 0$; Conjugate axis is $x + y - 1 = 0$

2. Find the equation of asymptotes for the hyperbola in the above question.

   Ans. $x + 2 = 0$; $y - 3 = 0$

3. Find the value of k for which the line $3x - 4y + k = 0$ is a tangent to the hyperbola $xy = 16$.

   Ans. No value for k

4. How many real tangent(s) can be drawn to the hyperbola $x^2 - 2y^2 - 4 = 0$ from the point $(2, \sqrt{2})$?

   Ans. One

5. State true or false: If the line $ax + by + c = 0$ is a normal to the hyperbola $xy = 1$, then $a > 0$ and $b > 0$.

   Ans. False

Additional Solved Examples

1. If $(a\sec\theta, b\tan\theta)$ and $(a\sec\phi, b\tan\phi)$ are the ends of a focal chord of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then

$$\tan\left(\frac{\theta}{2}\right)\tan\left(\frac{\phi}{2}\right) \text{ equals to}$$

(A) $\frac{e-1}{e+1}$ **(B)** $\frac{1-e}{1+e}$ **(C)** $\frac{1+e}{1-e}$ **(D)** $\frac{e-1}{e+1}$

Solution: Equation of chord connecting the points $(a\sec\theta, b\tan\theta)$ and $(a\sec\phi, b\tan\phi)$ is

$$\frac{x}{a}\cos\left(\frac{\theta-\phi}{2}\right) - \frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right) = \cos\left(\frac{\theta+\phi}{2}\right)$$

If it passes through $(ae, 0)$, then

$$e\cos\left(\frac{\theta-\phi}{2}\right) = \cos\left(\frac{\theta+\phi}{2}\right)$$

$$\Rightarrow e = \frac{\cos[(\theta+\phi)/2]}{\cos[(\theta-\phi)/2]} = \frac{[1-\tan(\theta/2)]\times\tan(\phi/2)}{[1+\tan(\theta/2)]\times\tan(\phi/2)}$$

$$\Rightarrow \tan\left(\frac{\theta}{2}\right)\times\tan\left(\frac{\phi}{2}\right) = \frac{1-e}{1+e}$$

Hence, the correct answer is option (B).

2. The equation of a line passing through the centre of a rectangular hyperbola is $x - y - 1 = 0$. If one of its asymptotes is $3x - 4y - 6 = 0$, the equation of the other asymptote is

(A) $4x - 3y + 17 = 0$ **(B)** $-4x - 3y + 17 = 0$
(C) $-4x + 3y + 1 = 0$ **(D)** $4x + 3y + 17 = 0$

Solution: We know that the asymptotes of a rectangular hyperbola are mutually perpendicular; thus, the other asymptote should be $4x + 3y + \lambda = 0$. The intersection point of asymptotes is also the centre of the hyperbola. Hence, the intersection point of $4x + 3y + \lambda = 0$ and $3x - 4y - 6 = 0$ should lie on the line $x - y - 1 = 0$ and using this, λ can be obtained.

Hence, the correct answer is option (D).

3. The point of intersection of the curves, whose parametric equations are $x = t^2 + 1$; $y = 2t$ and $x = 2s$; $y = 2/s$, is given by

(A) (1, –3) **(B)** (2, 2)
(C) (–2, 4) **(D)** (1, 2)

Solution: We have

$$x = t^2 + 1 \text{ and } y = 2t.$$

$$\Rightarrow x - 1 = \frac{y^2}{4}$$

Also, $x = 2s$ and $y = 2/s$

$$\Rightarrow xy = 4$$

For the point of intersection,

$$\frac{4}{y} - 1 = \frac{y^2}{4}$$

$$\Rightarrow y^3 + 4y - 16 = 0$$

$$\Rightarrow y = 2 \text{ and } x = 2$$

Hence, the correct answer is option (B).

4. The area of triangle formed by the lines $x - y = 0$, $x + y = 0$ and any tangent to the hyperbola $x^2 - y^2 = a^2$ is

(A) $4a^2$ **(B)** $5a^2$ **(C)** $3a^2$ **(D)** a^2

Solution: The equation of tangent is $x\sec\theta - y\tan\theta = a$. The lines are $x - y = 0$ and $x + y = 0$. Therefore, the vertices of the triangle are as follows:

$$(0, 0), \left(\frac{a}{\sec\theta - \tan\theta}, \frac{a}{\sec\theta - \tan\theta}\right) \text{ and}$$

$$\left(\frac{a}{\sec\theta + \tan\theta}, \frac{-a}{\sec\theta + \tan\theta}\right)$$

On using

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

we get the area of the triangle as a^2.

Hence, the correct answer is option (D).

5. A circle cuts the rectangular hyperbola $xy = 1$ at points (x_r, y_r), $r = 1, 2, 3$ and 4. Then, the values of $x_1 x_2 x_3 x_4$ and $y_1 y_2 y_3 y_4$, respectively, are

(A) –1, –1 **(B)** –1, 1 **(C)** 1, –1 **(D)** 1, 1

Solution: Let us consider

$$(x_r, y_r) = \left(t_r, \frac{1}{t_r}\right)$$

The circle is $x^2 + y^2 + 2gx + 2fy + k = 0$. On solving this, we get

$$t^4 + 2gt^3 + kt^2 + 2ft + 1 = 0 \Rightarrow \prod_{r=1}^{4} t_r = 1$$

Therefore,

$$\prod_{r=1}^{4} x_r = \prod_{r=1}^{4} y_r = 1$$

Hence, the correct answer is option (D).

6. Tangents are drawn from a point on the curve $x^2 - 4y^2 = 4$ to the curve $x^2 + 4y^2 = 4$ touching it at points Q and R. Prove that the mid-point of QR lies on $4(x^2 - 4y^2) = (x^2 + 4y^2)^2$.

Solution: The given equations are

$$\frac{x^2}{4} - \frac{y^2}{1} = 1 \quad (1)$$

and

$$\frac{x^2}{4} + \frac{y^2}{1} = 1 \quad (2)$$

Let P(h, k) be the mid-point of QR. The equation of chord whose mid-point is (h, k) w.r.t. Eq. (2) is

$$\frac{hx}{4} + \frac{ky}{1} = \frac{h^2}{4} + \frac{k^2}{1} \quad (3)$$

Also, the equation of chord of contact from any point ($2\sec\phi$, $\tan\phi$) lying on Eq. (1) w.r.t. Eq. (2) is

$$\frac{2\sec\phi(x)}{4} + \frac{\tan\phi(y)}{1} = 1 \quad (4)$$

Since Eqs. (3) and (4) represent the same line, we have

$$\frac{h}{2\sec\phi} = \frac{k}{\tan\phi} = \frac{h^2}{4} + \frac{k^2}{1}$$

$$\Rightarrow \sec\phi = \frac{2h}{(h^2 + 4k^2)} \text{ and } \tan\phi = \frac{4k}{(h^2 + 4k^2)}$$

$$\Rightarrow \frac{4h^2}{(h^2 + 4k^2)^2} - \frac{16k^2}{(h^2 + 4k^2)^2} = 1$$

$$\Rightarrow 4(h^2 - 4k^2) = (h^2 + 4k^2)^2$$

Therefore, the locus is given by

$$4(x^2 - 4y^2) = (x^2 + 4y^2)^2$$

7. If the tangent at the point (h, k) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ cuts the auxiliary circle $x^2 + y^2 = a^2$ at points whose ordinates are y_1 and y_2, then show that $\frac{1}{y_1} + \frac{1}{y_2} = \frac{2}{k}$, that is, y_1, k and y_2 are in HP.

Solution: Given that the equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (1)$$

and equation of the auxiliary circle is

$$x^2 + y^2 = a^2 \quad (2)$$

The equation of the tangent at point (h, k) with respect to Eq. (1) is

$$\frac{hx}{a^2} - \frac{ky}{b^2} = 1$$

$$\Rightarrow x = \frac{a^2}{h}\left(1 + \frac{ky}{b^2}\right) \quad (3)$$

From Eqs. (2) and (3), we have

$$y^2 + \frac{a^4}{h^2}\left(1 + \frac{2ky}{b^2}\right)^2 = a^2$$

That is,

$$y^2 + \frac{a^4}{h^2}\left(1 + \frac{k^2}{b^4}y^2 + \frac{2k}{b^2}y\right) = a^2$$

$$\Rightarrow \left(1 + \frac{a^4k^2}{b^4h^2}\right)y^2 + \frac{2a^4k}{b^2h^2}y + \frac{a^4}{h^2} - a^2 = 0$$

If this equation which is quadratic in y has roots y_1 and y_2, then

$$y_1 + y_2 = \frac{-2a^4k}{b^2h^2}\left(\frac{h^2b^4}{b^4h^2 + a^4k^2}\right)$$

and

$$y_1y_2 = \frac{(a^4 - a^2h^2)}{h^2}\left(\frac{b^4h^2}{b^4h^2 + a^4k^2}\right)$$

Now,

$$\frac{1}{y_1} + \frac{1}{y_2} = \frac{y_1 + y_2}{y_1y_2} = -\frac{2a^2k}{b^2}\frac{1}{(a^2 - h^2)} \quad (4)$$

Also, (h, k) lies on the hyperbola. Therefore,

$$\frac{h^2}{a^2} - \frac{k^2}{b^2} = 1$$

$$\Rightarrow \frac{h^2 - a^2}{a^2} = \frac{k^2}{b^2}$$

$$\Rightarrow (a^2 - h^2) = -\frac{a^2k^2}{b^2} \quad (5)$$

From Eqs. (4) and (5), we get

$$\frac{1}{y_1} + \frac{1}{y_2} = \frac{2a^2k}{b^2}\frac{b^2}{a^2k^2} = \frac{2}{k} \Rightarrow \frac{1}{y_1} + \frac{1}{y_2} = \frac{2}{k}$$

8. Five points are given on a circle. A rectangular hyperbola is made to pass through any of the four points. Show that the centres of all such hyperbolas lie on a circle.

Solution: Let the five points on the given circle be (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) and (x_5, y_5). Let the centre of circle be (g, f) and centre of the hyperbola passing through (x_i, y_i), i = 1, 2, 3, 4 be (h, k). Now, it is given that

$$(x_5 - g)^2 + (y_5 - f)^2 = a^2$$

where a being the radius of the circle. Now, we know that

$$\frac{x_1 + x_2 + x_3 + x_4}{4} = \frac{h + g}{2}$$

$$\Rightarrow \frac{x_1 + x_2 + x_3 + x_4 + x_5}{2} = h + g + \frac{x_5}{2}$$

$$\Rightarrow \frac{\sum x_i}{2} - g - h = \frac{x_5}{2}$$

$$\Rightarrow \left(\frac{\sum x_1}{2} - \frac{3}{2}g\right) - h = \frac{x_5 - g}{2} \quad (1)$$

Similarly,

$$\left(\frac{\sum y_1}{2} - \frac{3}{2}f\right) - k = \frac{y_5 - f}{2} \quad (2)$$

On squaring Eqs. (1) and (2) and then adding them, we get

$$\left[\left(\frac{\sum x_i}{2} - \frac{3}{2}g\right) - h\right]^2 + \left[\left(\frac{\sum y_i}{2} - \frac{3}{2}f\right) - k\right]^2 = \left(\frac{x_5 - g}{2}\right)^2 + \left(\frac{y_5 - f}{2}\right)^2$$

Hence, the locus is

$$\left[x-\left(\frac{\sum x_i}{2}-\frac{3}{2}g\right)\right]^2+\left[y-\left(\frac{\sum y_i}{2}-\frac{3}{2}f\right)\right]^2=\frac{a^2}{4}$$

which is a circle.

9. Prove that all conics which are passing through the intersection of two rectangular hyperbolas are themselves rectangular hyperbolas.

Solution: We know that in the equation of rectangular hyperbola, the sum of the coefficients of x^2 and y^2 should be zero. Let the two rectangular hyperbolas be

$$S_1 \equiv ax^2+2hxy-ay^2+2gx+2fy+c=0$$
$$S_2 \equiv a'x^2+2h'xy-a'y^2+2g'x+2f'y+c'=0$$

Therefore, the equation of any curve passing through intersection of S_1 and S_2 is $S_1+\lambda S_2=0$, where λ is a real parameter. So,

$$(a+\lambda a')x \ +2(h+\lambda h')xy-(a+\lambda a')y^2$$
$$+2x(g+\lambda g')+2y(f+\lambda f')+(c+\lambda c')=0$$

Now,

$$\text{Coefficient of } x^2+\text{Coefficient of } y^2=0$$

Thus, for all values of λ, it represents a rectangular hyperbola.

10. A variable straight line of slope 4 intersects the hyperbola $xy=1$ at two points. Find the locus of the point which divides the line segment between these points in the ratio 1:2.

Solution: Let the line be $y=4x+c$ which meets the curve $xy=1$ at $x(4x+c)=1$. Then

$$4x^2+cx-1=0$$
$$\Rightarrow x_1+x_2=-\frac{c}{4}$$

Also,

$$y(y-c)=4\Rightarrow y^2-cy-4=0\Rightarrow y_1+y_2=c$$

Let the point which divides the line segment in the ratio 1:2 be (h, k). Then

$$\frac{x_1+2x_2}{3}=h \Rightarrow x_2=3h+\frac{c}{4} \Rightarrow x_1=-\frac{c}{2}-3h$$

Also,

$$\frac{y_1+2y_2}{3}=k \Rightarrow y_2=3k-c\Rightarrow y_1=-3k+2c$$

Now, (h, k) lies on the line

$$y=4x+c\Rightarrow k=4h+c\Rightarrow c=k-4h$$

$$\Rightarrow x_1=-\frac{k}{2}+2h-3h=-h-\frac{k}{2} \text{ and } y_1=-3k+2k-8h=-k-8h$$

$$\Rightarrow\left(h+\frac{k}{2}\right)(k+8h)=1 \Rightarrow hk+8h^2+\frac{k^2}{2}+4hk=1$$

$$\Rightarrow 16h^2+k^2+10hk=2$$

Hence, the locus of (h, k) is $16x^2+y^2+10xy=2$.

Previous Years' Solved JEE Main/AIEEE Questions

1. The normal to a curve at $P(x, y)$ meets the x-axis at G. If the distance of G from the origin is twice the abscissa of P, then the curve is a

(A) ellipse **(B)** parabola
(C) circle **(D)** hyperbola

[AIEEE 2007]

Solution: Equation of normal is

$$Y-y=-\frac{dx}{dy}(X-x) \quad (1)$$

It cuts x-axis where $Y=0$.

Putting $Y=0$ in Eq. (1), we have

$$-y=-\frac{dx}{dy}(X-x)\Rightarrow y\frac{dy}{dx}+x=X$$

Therefore, coordinates of G are

$$G=\left(x+y\frac{dy}{dx},0\right)$$

According to question,

$$\left|x+y\frac{dy}{dx}\right|=|2x|\Rightarrow y\frac{dy}{dx}=x \text{ or } y\frac{dy}{dx}=-3x$$
$$\Rightarrow ydy=xdx \text{ or } ydy=-3xdx$$
$$\frac{y^2}{2}=\frac{x^2}{2}+c \text{ or } \frac{y^2}{2}=-\frac{3x^2}{2}+c$$
$$\Rightarrow x^2-y^2=-2c \text{ or } 3x^2+y^2=2c$$

Therefore, either hyperbola or ellipse.

Hence, the correct answer is option (A) or (D).

2. For the hyperbola $\frac{x^2}{\cos^2\alpha}-\frac{y^2}{\sin^2\alpha}=1$, which of the following remains constant when α varies?

(A) eccentricity **(B)** directrix
(C) abscissae of vertices **(D)** abscissae of foci

[AIEEE 2007]

Solution: Given

$$\frac{x^2}{\cos^2\alpha}-\frac{y^2}{\sin^2\alpha}=1$$

Now, $\sin^2\alpha=\cos^2\alpha(e^2-1)$

Therefore,

$$e^2-1=\tan^2\alpha\Rightarrow e=\sec\alpha\Rightarrow e=\text{varies}$$

$$\text{Foci } (\pm ae, 0), \text{ i.e. } (\pm e\cos\alpha, 0)$$

Therefore, abscissae of foci remain 0.

Hence, the correct answer is option (D).

3. Let $P(3\sec\theta, 2\tan\theta)$ and $Q(3\sec\phi, 2\tan\phi)$ where $\theta+\phi=\frac{\pi}{2}$, be two distinct points on the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$. Then the ordinate of the point of intersection of the normals at P and Q is

(A) $\frac{11}{3}$ **(B)** $-\frac{11}{3}$ **(C)** $\frac{13}{2}$ **(D)** $-\frac{13}{2}$

[JEE MAIN 2014 (ONLINE SET-2)]

Solution: See Fig. 15.6.

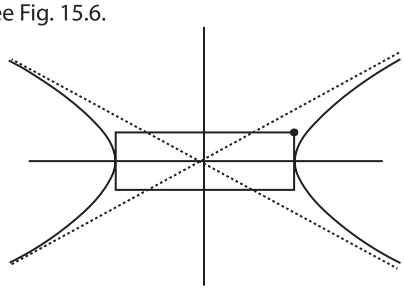


Figure 15.6

Equation of normal at P, is

$$\frac{3x}{\sec\theta}+\frac{2y}{\tan\theta}-13=0 \quad (1)$$

And equation of normal at θ is

$$\frac{3x}{\sec\phi}+\frac{2y}{\tan\phi}-13=0 \quad (2)$$

Eq. (1) $\Rightarrow (3\cos\theta)x+(2\cot\theta)y-13=0$

Eq. (2) $\Rightarrow (3\sin\theta)x+(2\tan\theta)y-13=0$ since $\theta+\phi=\dfrac{\pi}{2}$

On solving, we get

$$\frac{x}{-26\cot\theta-26\tan\theta}=\frac{y}{-39\sin\theta+39\cos\theta}=\frac{1}{6\sin\theta-6\cos\theta}$$

2	3	1	2
$2\cot\theta$	-13	$3\cos\theta$	$2\cot\theta$
$2\tan\theta$	-13	$3\sin\theta$	$2\tan\theta$

Therefore,

$$y=\frac{39(\cos\theta-\sin\theta)}{-6(\cos\theta-\sin\theta)}=\frac{-13}{2}$$

Hence, the correct answer is option (D).

4. The tangent at an extremity (in the first quadrant) of latus rectum of the hyperbola $\dfrac{x^2}{4}-\dfrac{y^2}{5}=1$, meets x-axis and y-axis at A and B respectively. Then $(OA)^2-(OB)^2$, where O is the origin, equals

(A) $-\dfrac{20}{9}$ **(B)** $\dfrac{16}{9}$ **(C)** 4 **(D)** $-\dfrac{4}{3}$

[JEE MAIN 2014 (ONLINE SET-4)]

Solution: See Fig. 15.7.

The y-coordinate of latus rectum is $\dfrac{b^2}{a}$, i.e. $\dfrac{5}{2}$.

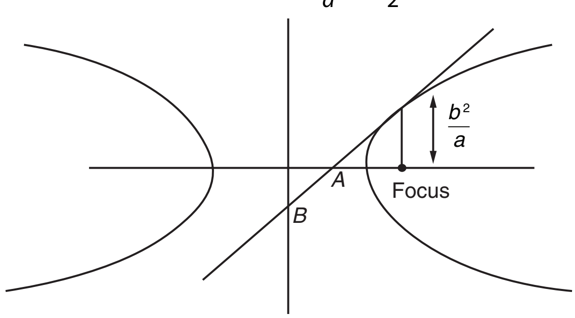


Figure 15.7

Finding x-coordinate

$$\frac{x^2}{4}-\frac{\left(\frac{5}{2}\right)^2}{5}=1\Rightarrow\frac{x^2}{4}-\frac{25^{\,5}}{4\times 5}=1\Rightarrow\frac{x^2}{4}=1+\frac{5}{4}=\frac{9}{4}\Rightarrow x=\pm 3$$

Point A is $\left(3,\dfrac{5}{2}\right)$. Equation of tangent at A is

$$\frac{x(3)}{4}-\frac{y\left(\frac{5}{2}\right)}{5}=1 \text{ or } \frac{x}{4/3}+\frac{y}{-2}=1$$

Now,

$$OA^2-OB^2=\left(\frac{4}{3}\right)^2-(-2)^2=\frac{16}{9}-4=\frac{16-36}{9}=-\frac{20}{9}$$

Hence, the correct answer is option (A).

5. If the tangent to the conic, $y-6=x^2$ at (2, 10) touches the circle, $x^2+y^2+8x-2y=k$ (for some fixed k) at a point (α, β); then (α, β) is

(A) $\left(-\dfrac{6}{17},\dfrac{10}{17}\right)$ **(B)** $\left(-\dfrac{8}{17},\dfrac{2}{17}\right)$

(C) $\left(-\dfrac{4}{17},\dfrac{1}{17}\right)$ **(D)** $\left(-\dfrac{7}{17},\dfrac{6}{17}\right)$

[JEE MAIN 2015 (ONLINE SET-1)]

Solution: See Fig. 15.8.

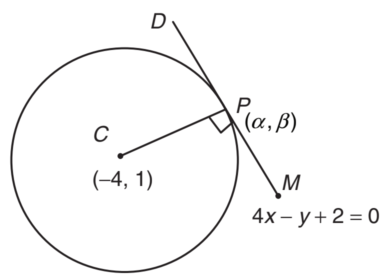


Figure 15.8

Equation tangent to $y=x^2+6$ at (2, 10) is

$$\left(\frac{y+10}{2}\right)=2x+6 \Rightarrow 4x-y+2=0 \quad (1)$$

Because Eq. (1) is tangent to circle $x^2+y^2+8x-2y=k$ and also P (α, β) must be foot of perpendicular from $C(-4, 1)$ to $4x-y+2=0$, therefore

$$\frac{x+4}{4}=\frac{y-1}{-1}=\frac{[4(-4)-1+2]}{16+1}$$

$$\Rightarrow x+4=\frac{60}{17},\ y-1=\frac{-15}{17} \Rightarrow x=\frac{-8}{17},\ y=\frac{2}{17}$$

$$\Rightarrow (\alpha,\beta)\equiv\left(\frac{-8}{17},\frac{2}{17}\right)$$

Hence, the correct answer is option (B).

6. An ellipse passes through the foci of the hyperbola, $9x^2-4y^2=36$ and its major and minor axes lie along the transverse and conjugate axes of the hyperbola respectively.

If the product of eccentricities of the two conics is $\frac{1}{2}$, then which of the following points lie on the ellipse?

(A) $(\sqrt{13},0)$ **(B)** $\left(\frac{\sqrt{39}}{2},\sqrt{3}\right)$

(C) $\left(\frac{1}{2}\sqrt{13},\frac{\sqrt{3}}{2}\right)$ **(D)** $\left(\sqrt{\frac{13}{2}},\sqrt{6}\right)$

[JEE MAIN 2015 (ONLINE SET-1)]

Solution: Given hyperbola is

$$9x^2 - 4y^2 = 36$$

$$\Rightarrow \frac{x^2}{4} - \frac{y^2}{9} = 1 \quad (1)$$

$$\Rightarrow a^2 = 4, b^2 = 9 \Rightarrow a = 2, b = 3 \Rightarrow b^2 = a^2(e^2 - 1)$$

$$\Rightarrow a^2 e^2 = a^2 + b^2 \Rightarrow a^2e^2 = 13 \Rightarrow ae = \sqrt{13}$$

Foci of Eq. (1) are $S_1(-\sqrt{13},0); S_2(\sqrt{13},0)$.

$$e_1 = \text{eccentricity of hyperbola} = \frac{\sqrt{13}}{2}$$

Given,

$$e_1.e_2 = \frac{1}{2} \Rightarrow e_2 = \frac{1}{2e_1} = \frac{1}{\sqrt{13}} = \text{eccentricity of ellipse}$$

Let the equation of ellipse be

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$$

It passes through S_1, S_2, so

$$\alpha = \sqrt{13}$$

Also,

$$\beta^2 = \alpha^2(1 - e_2^2)$$

$$\Rightarrow \beta^2 = 13\left(1 - \frac{1}{13}\right) = 12$$

Therefore, equation of ellipse is $\frac{x^2}{13} + \frac{y^2}{12} = 1$, so point $\left(\frac{\sqrt{13}}{2},\frac{\sqrt{3}}{2}\right)$ does not lie on ellipse.

Hence, the correct answer is option (C).

7. The eccentricity of the hyperbola, whose length of the latus rectum is equal to 8 and the length of its conjugate axis is equal to half of the distance between its foci is

(A) $\sqrt{3}$ **(B)** $\frac{4}{3}$ **(C)** $\frac{4}{\sqrt{3}}$ **(D)** $\frac{2}{\sqrt{3}}$

[JEE MAIN 2016 (OFFLINE)]

Solution: Length of the latus rectum of hyperbola is 8.

$$\frac{2b^2}{a} = 8 \Rightarrow b^2 = 4a$$

Length of conjugate axis $= \frac{1}{2} \times$ (Distance between the foci)

That is,

$$2b = ae$$
$$b^2 = a^2(e^2 - 1)$$

$$\frac{b^2}{a^2} = e^2 - 1 \Rightarrow \frac{e^2}{4} = e^2 - 1 \Rightarrow 1 = \frac{3e^2}{4} \Rightarrow e = \frac{2}{\sqrt{3}}$$

Hence, the correct answer is option (D).

8. Let a and b, respectively, be the semi-transverse and semi-conjugate axes of a hyperbola whose eccentricity satisfies the equation $9e^2 - 18e + 5 = 0$. If S(5, 0) is a focus and $5x = 9$ is the corresponding directrix of this hyperbola, then $a^2 - b^2$ is equal to

(A) –7 **(B)** –5 **(C)** 5 **(D)** 7

[JEE MAIN 2016 (ONLINE SET-1)]

Solution: Length of the latus rectum of hyperbola is 8. We have

$$9e^2 - 3e - 15e + 5 = 0$$
$$3e(3e - 1) - 5(3e - 1) = 0$$
$$\Rightarrow e = \frac{1}{3}, \frac{5}{3}$$

For hyperbola, $e > 1$, that is,

$$e = \frac{5}{3}$$

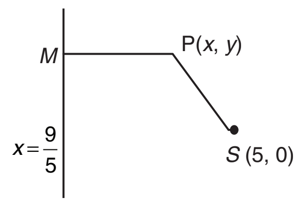


Figure 15.9

$$PS = ePM \text{ (See Fig. 15.9)}$$

$$\sqrt{(x-5)^2 + y^2} = \frac{5}{3}\left|\frac{(5x-9)}{5}\right| = \frac{1}{3}|(5x-9)|$$

$$(x-5)^2 + y^2 = \frac{1}{9}(25x^2 + 81 - 90x)$$

$$9x^2 + 225 - 90x + (y^2 \times 9) = 25x^2 - 90x + 81$$

$$16x^2 - (y^2 \times 9) = 225 - 81 = 144$$

Now,
$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

Therefore, $a^2 = 9$ and $b^2 = 16$. Hence,

$$a^2 - b^2 = -7$$

Hence, the correct answer is option (A).

9. A hyperbola whose transverse axis is along the major axis of the conic, $\frac{x^2}{3} + \frac{y^2}{4} = 4$ and has vertices the foci of this conic. If the eccentricity of the hyperbola is $\frac{3}{2}$, then which of the following points does NOT lie on it?

(A) $(\sqrt{5},2\sqrt{2})$ **(B)** (0, 2)

(C) $(5,2\sqrt{3})$ **(D)** $(\sqrt{10},2\sqrt{3})$

[JEE MAIN 2016 (ONLINE SET-2)]

Solution: See Fig. 15.10. We have

$$\frac{x^2}{12} + \frac{y^2}{16} = 1$$

Now, $a^2 = b^2(1 - e^2)$

$$12 = 16(1 - e^2)$$

$$\frac{3}{4} = 1 - e^2$$

$$\Rightarrow e^2 = \frac{1}{4}$$

$$\Rightarrow e = \frac{1}{2}$$

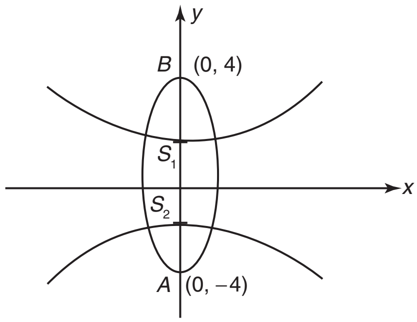


Figure 15.10

Therefore, we have the points S_1 and S_2 as follows (see Fig. 15.11):

$$S_1\left(0, 4\times\frac{1}{2}\right) = S_1(0, 2)$$

$$S_2\left(0, -4\times\frac{1}{2}\right) = S_2(0, -2)$$

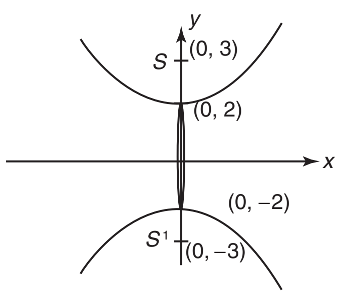


Figure 15.11

Now,

$$2b = 4 \Rightarrow b = 2$$

Therefore,

$$be = 2\times\frac{3}{2} = 3$$

Also, $a^2 = b^2(e^2 - 1)$

$$a^2 = 4\left(\frac{9}{4} - 1\right) = \frac{4.5}{4}$$

$$a^2 = 5 \Rightarrow a = \sqrt{5}$$

Therefore, the equation of hyperbola is

$$\frac{x^2}{5} - \frac{y^2}{4} = -1$$

And hence the point $(5, 2\sqrt{3})$ does not lie on the hyperbola.

Hence, the correct answer is option (C).

Previous Years' Solved JEE Advanced/ IIT-JEE Questions

1. A hyperbola, having the transverse axis of length $2\sin\theta$, is confocal with the ellipse $3x^2 + 4y^2 = 12$. Then its equation is

(A) $x^2\text{cosec}^2\theta - y^2\sec^2\theta = 1$ **(B)** $x^2\sec^2\theta - y^2\text{cosec}^2\theta = 1$

(C) $x^2\sin^2\theta - y^2\cos^2\theta = 1$ **(D)** $x^2\cos^2\theta - y^2\sin^2\theta = 1$

[IIT-JEE 2007]

Solution: The given equation of ellipse can be written as

$$\frac{x^2}{4} + \frac{y^2}{3} = 1$$

Now, $a = 2, b = \sqrt{3} \Rightarrow e = \frac{1}{2}$

The foci of ellipse is $(\pm 1, 0)$. Since the hyperbola is confocal with ellipse, the foci of hyperbola is $(\pm 1, 0)$. Now, let e' be the eccentricity of hyperbola. The transverse axis is

$$2\sin\theta = 2a'$$

Therefore, the focus is

$$(e'\sin\theta, 0) = (1, 0)$$

$$\Rightarrow e' = \text{cosec}\,\theta$$

Now,

$$b'^2 = a'^2(e'^2 - 1)$$

$$= \sin^2\theta(\text{cosec}^2\theta - 1)$$

$$= 1 - \sin^2\theta$$

$$= \cos^2\theta$$

Therefore, the equation of hyperbola is

$$\frac{x^2}{\sin^2\theta} - \frac{y^2}{\cos^2\theta} = 1$$

That is,

$$x^2\text{cosec}^2\theta - y^2\sec^2\theta = 1$$

Hence, the correct answer is option (A).

2. Let a and b be non-zero real numbers. Then, the equation $(ax^2 + by^2 + c)(x^2 - 5xy + 6y^2) = 0$ represents

(A) four straight lines, when $c = 0$ and a, b are of the same sign

(B) two straight lines and a circle, when $a = b$, and c is of sign opposite to that of a

(C) two straight lines and a hyperbola, when a and b are of the same sign and c is of sign opposite to that of a

(D) a circle and an ellipse, when a and b are of the same sign and c is of sign opposite to that of a

[IIT-JEE 2008]

Solution: We have

$$(ax^2 + by^2 + c)(x^2 - 5xy + 6y^2) = 0$$

$$\Rightarrow x^2 - 5xy + 6y^2 = 0$$

$$\Rightarrow (x - 2y)(x - 3y) = 0$$

$$\Rightarrow x - 2y = 0 \quad \text{or} \quad x - 3y = 0$$

represents two straight lines. Again

$$ax^2 + by^2 + c = 0$$

Let $a = b$. Then

$$x^2 + y^2 = \left(\frac{-c}{a}\right)$$

Therefore, if a and c are of opposite sign, then the equation represents a circle.

Hence, the correct answer is option (B).

3. Consider a branch of the hyperbola $x^2 - 2y^2 - 2\sqrt{2}x - 4\sqrt{2}y - 6 = 0$ with vertex at the point A. Let B be one of the end points of its latus rectum. If C is the focus of the hyperbola nearest to the point A, then the area of the triangle ABC is

(A) $1-\sqrt{\frac{2}{3}}$ **(B)** $\sqrt{\frac{3}{2}}-1$

(C) $1+\sqrt{\frac{2}{3}}$ **(D)** $\sqrt{\frac{3}{2}}+1$

[IIT-JEE 2008]

Solution: We have

$$x^2 - 2y^2 - 2\sqrt{2}x - 4\sqrt{2}y - 6 = 0$$

$$\frac{(x-\sqrt{2})^2}{4} - \frac{(y+\sqrt{2})^2}{2} = 1$$

$$a = 2, b = \sqrt{2} \Rightarrow c = \sqrt{\frac{3}{2}}$$

Now,

$$\text{Areas} = \frac{1}{2}(\text{Base})(\text{height})$$

$$= \frac{1}{2}(ac-a)\cdot\left(\frac{b^2}{a}\right)$$

$$= \left(\sqrt{\frac{3}{2}}-1\right)$$

Hence, the correct answer is option (B).

4. Match the conics in Column I with the statements/expressions in Column II.

Column I	Column II
(A) Circle	**(P)** The locus of the point (h, k) for which the line $hx + ky = 1$ touches the circle $x^2 + y^2 = 4$
(B) Parabola	**(Q)** Points z in the complex plane satisfying $\lvert z+2\rvert - \lvert z-2\rvert = \pm 3$
(C) Ellipse	**(R)** Points of the conic have parametric representation $x = \sqrt{3}\left(\frac{1-t^2}{1+t^2}\right), y = \frac{2t}{1+t^2}$
(D) Hyperbola	**(S)** The eccentricity of the conic lies in the interval $1 \le e \le \infty$
	(T) Points z in the complex plane satisfying $\text{Re}(z+1)^2 = \lvert z\rvert^2 + 1$

[IIT-JEE 2009]

Solution:

(P) $\frac{1}{k^2} = 4\left(1+\frac{h^2}{k^2}\right) \Rightarrow 1 = 4(k^2+h^2)$

Therefore, $h^2 + k^2 = \left(\frac{1}{2}\right)^2$, which is a circle.

(Q) If $\lvert z-z_1\rvert - \lvert z-z_2\rvert = k$, where $k < \lvert z_1 - z_2\rvert$ the locus is a hyperbola.

(R) Let $t = \tan\alpha$. Then

$$x = \sqrt{3}\cos 2\alpha \text{ and } y = \sin 2\alpha$$

or $$\cos 2\alpha = \frac{x}{\sqrt{3}} \text{ and } \sin 2\alpha = y$$

Therefore, $\frac{x^2}{3} + y^2 = \sin^2 2\alpha + \cos^2 2\alpha = 1$, which is an ellipse.

(S) If eccentricity is $[1, \infty)$, then the conic can be a parabola (if $e = 1$) and a hyperbola if $e \in (1, \infty)$.

(T) Let $z = x + iy$; $x, y \in R$. Then

$$(x+1)^2 - y^2 = x^2 + y^2 + 1 \Rightarrow y^2 = x$$

which is a parabola.

Hence, the correct matches are (A)→(P); (B)→(S, T); (C) →(R); (D)→(Q, S).

5. The locus of the orthocentre of the triangle formed by the lines $(1 + p)x - py + p(1 + p) = 0$, $(1 + q)x - qy + q(1 + q) = 0$ and $y = 0$, where $p \neq q$, is

(A) a hyperbola **(B)** a parabola

(C) an ellipse **(D)** a straight line

[IIT-JEE 2009]

Solution: Intersection point of $y = 0$ with first line is $B(-p, 0)$.
Intersection point of $y = 0$ with second line is $A(-q, 0)$.
Intersection point of the two lines is $C(pq, (p + 1)(q + 1))$.
Altitude from C to AB is $x = pq$.

Altitude from B to AC is $y = -\frac{q}{1+q}(x+p)$.

Solving these the above two equations, we get

$$x = pq \text{ and } y = -pq$$

Therefore, locus of orthocentre is $x + y = 0$.

Hence, the correct answer is option (D).

6. An ellipse intersects the hyperbola $2x^2 - 2y^2 = 1$ orthogonally. The eccentricity of the ellipse is reciprocal of that of the hyperbola. If the axes of the ellipse are along the coordinates axes, then

(A) equation of ellipse is $x^2 + 2y^2 = 2$

(B) the foci of ellipse are $(\pm 1, 0)$

(C) equation of ellipse is $x^2 + 2y^2 = 4$

(D) the foci of ellipse are $(\pm\sqrt{2}, 0)$

[IIT-JEE 2009]

Solution: Ellipse and hyperbola will be confocal. Therefore,

$$(\pm ae, 0) \equiv (\pm 1, 0)$$

$$\Rightarrow \left(\pm a \times \frac{1}{\sqrt{2}}, 0\right) \equiv (\pm 1, 0)$$

$$\Rightarrow a = \sqrt{2} \text{ and } e = \frac{1}{\sqrt{2}}$$

$$\Rightarrow b^2 = a^2(1-e^2) \Rightarrow b^2 = 1$$

Therefore, equation of ellipse is $\frac{x^2}{2} + \frac{y^2}{1} = 1$.

Hence, the correct answers are options (A) and (B).

Paragraph for Questions 7 and 8: The circle $x^2 + y^2 - 8x = 0$ and hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ intersect at the points A and B.

[IIT-JEE 2010]

7. Equation of a common tangent with positive slope to the circle as well as to the hyperbola is

(A) $2x - \sqrt{5}y - 20 = 0$ **(B)** $2x - \sqrt{5}y + 4 = 0$

(C) $3x - 4y + 8 = 0$ **(D)** $4x - 3y + 4 = 0$

Solution: Equation of a tangent to $\frac{x^2}{9}-\frac{y^2}{4}=1$ is

$$y = mx + \sqrt{9m^2-4},\ m>0$$

It is tangent to $x^2+y^2-8x=0$. Therefore,

$$\frac{4m+\sqrt{9m^2-4}}{\sqrt{1+m^2}}=4$$

$$\Rightarrow 495m^4+104m^2-400=0$$

$$\Rightarrow m^2=\frac{4}{5} \text{ or } m=\frac{2}{\sqrt{5}}$$

Therefore, the equation of the tangent is

$$y=\frac{2}{\sqrt{5}}m+\frac{4}{\sqrt{5}}$$

$$\Rightarrow 2x-\sqrt{5}y+4=0$$

Hence, the correct answer is option (B).

8. Equation of the circle with AB as its diameter is
   (A) $x^2+y^2-12x+24=0$
   (B) $x^2+y^2+12x+24=0$
   (C) $x^2+y^2+24x-12=0$
   (D) $x^2+y^2-24x-12=0$

Solution: A point on hyperbola is $(3\sec\theta, 2\tan\theta)$.

It lies on the circle, so

$$9\sec^2\theta+4\tan^2\theta-24\sec\theta=0$$

$$\Rightarrow 13\sec^2\theta-24\sec\theta-4=0$$

$$\Rightarrow \sec\theta=2,-\frac{2}{13}$$

Therefore,

$$\sec\theta=2\Rightarrow\tan\theta=\sqrt{3}$$

The points of intersection are $A(6,2\sqrt{3})$ and $B(6,-2\sqrt{3})$.

Therefore, the circle with AB as diameter is

$$(x-6)^2+y^2=(2\sqrt{3})^2\Rightarrow x^2+y^2-12x+24=0$$

Hence, the correct answer is option (A).

9. The line $2x + y = 1$ is tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If this line passes through the point of intersection of the nearest directrix and the x–axis, then the eccentricity of the hyperbola is

[IIT-JEE 2010]

Solution: Substituting $\left(\frac{a}{e},0\right)$ in $y = -2x + 1$, we have

$$0=-\frac{2a}{e}+1\Rightarrow\frac{2a}{e}=1\Rightarrow a=\frac{e}{2}$$

Also,

$$1=\sqrt{a^2m^2-b^2}$$

$$\Rightarrow 1=a^2m^2-b^2\Rightarrow 1=4a^2-b^2\Rightarrow 1=\frac{4e^2}{4}-b^2\Rightarrow b^2=e^2-1$$

Also, $b^2=a^2(e^2-1)$

Therefore, $a = 1$, $e = 2$

Hence, the correct answer is (2).

10. Let the eccentricity of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ be reciprocal to that of the ellipse $x^2+4y^2=4$. If the hyperbola passes through a focus of the ellipse, then
    (A) the equation of the hyperbola is $\frac{x^2}{3}-\frac{y^2}{2}=1$
    (B) a focus of the hyperbola is (2, 0)
    (C) the eccentricity of the hyperbola is $\sqrt{\frac{5}{3}}$
    (D) the equation of the hyperbola is $x^2-3y^2=3$

[IIT-JEE 2011]

Solution: Ellipse is

$$\frac{x^2}{2^2}+\frac{y^2}{1^2}=1$$

$$1^2=2^2(1-e^2)\Rightarrow e=\frac{\sqrt{3}}{2}$$

Therefore, eccentricity of the hyperbola is

$$\frac{2}{\sqrt{3}}\Rightarrow b^2=a^2\left(\frac{4}{3}-1\right)\Rightarrow 3b^2=a^2$$

Foci of the ellipse are $(\sqrt{3},0)$ and $(-\sqrt{3},0)$.

As hyperbola passes through $(\sqrt{3},0)$, we have

$$\frac{3}{a^2}=1\Rightarrow a^2=3 \text{ and } b^2=1$$

Therefore, equation of hyperbola is $x^2-3y^2=3$.

Now, focus of hyperbola is $(ae,0)\equiv\left(\sqrt{3}\times\frac{2}{\sqrt{3}},0\right)\equiv(2,0)$.

Hence, the correct answers are options (B) and (D).

11. Let $P(6, 3)$ be a point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If the normal at the point P intersects the x-axis at (9, 0) then the eccentricity of the hyperbola is
    (A) $\sqrt{\frac{5}{2}}$ (B) $\sqrt{\frac{3}{2}}$ (C) $\sqrt{2}$ (D) $\sqrt{3}$

[IIT-JEE 2011]

Solution: Equation of normal is

$$(y-3)=\frac{-a^2}{2b^2}(x-6)\ \Rightarrow\frac{a^2}{2b^2}=1\ \Rightarrow e=\sqrt{\frac{3}{2}}$$

Hence, the correct answer is option (B).

12. Tangents are drawn to the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$, parallel to the straight line $2x - y = 1$. The points of contact of the tangents on the hyperbola are
    (A) $\left(\frac{9}{2\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ (B) $\left(-\frac{9}{2\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$
    (C) $(3\sqrt{3},-2\sqrt{2})$ (D) $(-3\sqrt{3},2\sqrt{2})$

[IIT-JEE 2012]

Solution: Slope of tangent = 2, so equation of tangents are

$$y = 2x \pm \sqrt{9 \times 4 - 4}$$

$$\Rightarrow 2x - y = \pm 4\sqrt{2}$$

$$\Rightarrow \frac{x}{2\sqrt{2}} - \frac{y}{4\sqrt{2}} = 1 \text{ and } -\frac{x}{2\sqrt{2}} + \frac{y}{4\sqrt{2}} = 1$$

Comparing it with $\frac{xx_1}{9} - \frac{yy_1}{4} = 1$

We get point of contact as $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

Alternate Solution: Equation of tangent at $P(\theta)$ is

$$\left(\frac{\sec\theta}{3}\right)x - \left(\frac{\tan\theta}{2}\right)y = 1$$

$$\Rightarrow \text{Slope} = \frac{2\sec\theta}{3\tan\theta} = 2$$

$$\Rightarrow \sin\theta = \frac{1}{3}$$

$$\Rightarrow \text{Points are} \left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \text{ and } \left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

Hence, the correct answers are options (A) and (B).

13. Consider the hyperbola $H: x^2 - y^2 = 1$ and a circle S with centre $N(x_2, 0)$. Suppose that H and S touch each other at a point $P(x_1, y_1)$ with $x_1 > 1$ and $y_1 > 0$. The common tangent to H and S at P intersects the x-axis at point M. If (l, m) is the centroid of the triangle ΔPMN, then the correct expression(s) is(are)

(A) $\frac{dl}{dx_1} = 1 - \frac{1}{3x_1^2}$ for $x_1 > 1$

(B) $\frac{dm}{dx_1} = \frac{x_1}{3\left(\sqrt{x_1^2 - 1}\right)}$ for $x_1 > 1$

(C) $\frac{dl}{dx_1} = 1 + \frac{1}{3x_1^2}$ for $x_1 > 1$

(D) $\frac{dm}{dy_1} = \frac{1}{3}$ for $y_1 > 0$

[JEE ADVANCED 2015]

Solution: See Fig. 15.12.

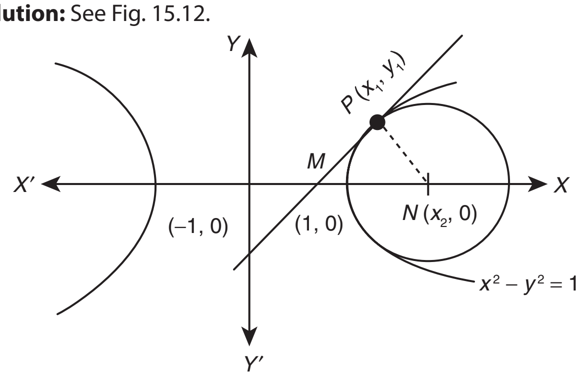


Figure 15.12

$$H: x^2 - y^2 = 1 \quad (1)$$

Let the centroid be $G \equiv (l, m)$ (2)

$$S \equiv (x - x_2)^2 + y^2 = (x_1 - x_2)^2 + y_1^2$$

or $$x^2 + y^2 - 2x_2x - x_1^2 - y_1^2 + 2x_1x_2 = 0 \quad (3)$$

Equation of tangent to circle at P is

$$xx_1 + yy_1 - x_2(x + x_1) - x_1^2 - y_1^2 + 2x_1x_2 = 0$$

or $$(x_1 - x_2)x + y_1y + x_1x_2 - x_1^2 - y_1^2 = 0$$

Also equation of tangent to hyperbola (1) at $P(x_1, y_1)$ is

$$xx_1 - yy_1 = 1$$

It meets x-axis, where $y = 0$ at M. Therefore,

$$M \equiv \left(\frac{1}{x_1}, 0\right)$$

Also slope of $PM = \frac{-1}{\text{slope of } NP}$

$$\Rightarrow \frac{y_1 - 0}{\left(x_1 - \frac{1}{x_1}\right)} = \frac{-1}{\left(\frac{y_1 - 0}{x_1 - x_2}\right)}$$

$$\Rightarrow \frac{x_1y_1}{x_1^2 - 1} = \frac{-(x_1 - x_2)}{y_1}$$

$$\Rightarrow x_1y_1^2 = -(x_1 - x_2)(x^2 - 1)$$

$$\Rightarrow x_1(x_1^2 - 1) = -(x_1 - x_2)(x_1^2 - 1) \quad (\text{Because } x_1 > 1)$$

$$\Rightarrow x_1 = -x_1 + x_2$$

$$\Rightarrow 2x_1 = x_2$$

Therefore,

$$G \equiv (l, m) \equiv \left(\frac{1}{3}\left(x_1 + x_2 + \frac{1}{x_1}\right), \frac{1}{3}(y_1)\right)$$

$$\Rightarrow l = \frac{1}{3}\left(3x_1 + \frac{1}{x_1}\right), \; m = \frac{1}{3} y_1 = \frac{1}{3} - \sqrt{x_1^2 - 1} \quad (\text{Because } y_1 > 0)$$

$$\Rightarrow \frac{dl}{dx_1} = 1 - \frac{1}{3x_1^2}, \; \frac{dm}{dx_1} = \frac{x_1}{3\sqrt{x_1^2 - 1}}$$

and $$\frac{dm}{dy_1} = \frac{1}{3}$$

Hence, the correct answers are options (A), (B) and (D).

Practice Exercise 1

1. A point on the curve $\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1$ is

(A) $(A\cos\theta, B\sin\theta)$ **(B)** $(A\sec\theta, B\tan\theta)$

(C) $(A\cos^2\theta, B\sin^2\theta)$ **(D)** None of these

2. If the eccentricities of the hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ be e and e_1, then find the value of $\frac{1}{e^2} + \frac{1}{e_1^2}$.

(A) 1 **(B)** 2

(C) 3 **(D)** None of these

3. If P is a point on the hyperbola $16x^2 - 9y^2 = 144$ whose foci are S_1 and S_2, then $PS_1 \sim PS_2$ is

(A) 4 **(B)** 6 **(C)** 8 **(D)** 12

4. If the latus rectum of a hyperbola be 8 and its eccentricity be $3/\sqrt{5}$, then the equation of the hyperbola is

(A) $4x^2 - 5y^2 = 100$ **(B)** $5x^2 - 4y^2 = 100$

(C) $4x^2 + 5y^2 = 100$ **(D)** $5x^2 + 4y^2 = 100$

5. The eccentricity of a hyperbola passing through the points (3, 0) and $(3\sqrt{2}, 2)$ is

(A) $\sqrt{13}$ **(B)** $\frac{\sqrt{13}}{3}$ **(C)** $\frac{\sqrt{13}}{4}$ **(D)** $\frac{\sqrt{13}}{2}$

6. The one which does not represent a hyperbola is

(A) $xy = 1$ **(B)** $x^2 - y^2 = 5$

(C) $(x-1)(y-3) = 3$ **(D)** $x^2 - y^2 = 0$

7. The equation of the hyperbola whose conjugate axis is 5 and the distance between the foci is 13, is

(A) $25x^2 - 144y^2 = 900$ **(B)** $144x^2 - 25y^2 = 900$

(C) $144x^2 + 25y^2 = 900$ **(D)** $25x^2 + 144y^2 = 900$

8. The length of the transverse axis of a hyperbola is 7 and it passes through the point (5, –2). The equation of the hyperbola is

(A) $\frac{4}{49}x^2 - \frac{196}{51}y^2 = 1$ **(B)** $\frac{49}{4}x^2 - \frac{51}{196}y^2 = 1$

(C) $\frac{4}{49}x^2 - \frac{51}{196}y^2 = 1$ **(D)** None of these

9. If (4, 0) and (–4, 0) be the vertices and (6, 0) and (–6, 0) be the foci of a hyperbola, then its eccentricity is

(A) 5/2 **(B)** 2 **(C)** 3/2 **(D)** $\sqrt{2}$

10. The eccentricity of the hyperbola $x^2 - y^2 = 25$ is

(A) $\sqrt{2}$ **(B)** $1/\sqrt{2}$ **(C)** 2 **(D)** $1+\sqrt{2}$

11. The equation of the transverse and conjugate axis of the hyperbola $16x^2 - y^2 + 64x + 4y + 44 = 0$ are

(A) $x = 2, y + 2 = 0$ **(B)** $x = 2, y = 2$

(C) $y = 2, x + 2 = 0$ **(D)** None of these

12. If the length of the transverse and conjugate axes of a hyperbola be 8 and 6, respectively, then the difference in the focal distances of any point of the hyperbola is

(A) 8 **(B)** 6 **(C)** 14 **(D)** 2

13. If $(0, \pm 4)$ and $(0, \pm 2)$ be the foci and vertices of a hyperbola, then its equation is

(A) $\frac{x^2}{4} - \frac{y^2}{12} = 1$ **(B)** $\frac{x^2}{12} - \frac{y^2}{4} = 1$

(C) $\frac{y^2}{4} - \frac{x^2}{12} = 1$ **(D)** $\frac{y^2}{12} - \frac{x^2}{4} = 1$

14. The locus of the point of intersection of the lines $bxt - ayt = ab$ and $bx + ay = abt$ is

(A) A parabola **(B)** An ellipse

(C) A hyperbola **(D)** None of these

15. The locus of the point of intersection of the lines $ax\sec\theta + by\tan\theta = a$ and $ax\tan\theta + by\sec\theta = b$, where θ is the parameter, is

(A) A straight line **(B)** A circle

(C) An ellipse **(D)** A hyperbola

16. If the centre, vertex and the focus of a hyperbola be (0, 0), (4, 0) and (6, 0), respectively, then the equation of the hyperbola is

(A) $4x^2 - 5y^2 = 8$ **(B)** $4x^2 - 5y^2 = 80$

(C) $5x^2 - 4y^2 = 80$ **(D)** $5x^2 - 4y^2 = 8$

17. The eccentricity of the hyperbola can never be equal to

(A) $\sqrt{9/5}$ **(B)** $2\sqrt{1/9}$ **(C)** $3\sqrt{1/8}$ **(D)** 2

18. A hyperbola passes through the points (3, 2) and (–17, 12) and has its centre at the origin and the transverse axis is along *x*-axis. The length of its transverse axis is

(A) 2 **(B)** 4 **(C)** 6 **(D)** None of these

19. The locus of the point of intersection of the lines $\sqrt{3}x - y - 4\sqrt{3}k = 0$ and $\sqrt{3}kx + ky - 4\sqrt{3} = 0$ for different value of *k* is

(A) Circle **(B)** Parabola

(C) Hyperbola **(D)** Ellipse

20. The difference of the focal distance of any point on the hyperbola $9x^2 - 16y^2 = 144$ is

(A) 8 **(B)** 7 **(C)** 6 **(D)** 4

21. The eccentricity of the hyperbola $4x^2 - 9y^2 = 16$ is

(A) 8/3 **(B)** 5/4 **(C)** $\sqrt{13}/3$ **(D)** 4/3

22. The eccentricity of the conic $x^2 - 4y^2 = 1$ is

(A) $2/\sqrt{3}$ **(B)** $\sqrt{3}/2$ **(C)** $2/\sqrt{5}$ **(D)** $\sqrt{5}/2$

23. The locus of the centre of a circle, which touches externally the given two circles, is

(A) Circle **(B)** Parabola

(C) Hyperbola **(D)** Ellipse

24. The foci of the hyperbola $2x^2 - 3y^2 = 5$ are

(A) $\left(\pm\frac{5}{\sqrt{6}}, 0\right)$ **(B)** $\left(\pm\frac{5}{6}, 0\right)$

(C) $\left(\pm\frac{\sqrt{5}}{6}, 0\right)$ **(D)** None of these

25. The latus rectum of the hyperbola $16x^2 - 9y^2 = 144$ is

(A) 16/3 **(B)** 32/3 **(C)** 8/3 **(D)** 4/3

26. The foci of the hyperbola $9x^2 - 16y^2 = 144$ are

(A) $(\pm 4, 0)$ **(B)** $(0, \pm 4)$ **(C)** $(\pm 5, 0)$ **(D)** $(0, \pm 5)$

27. The length of the transverse axis of the parabola $3x^2 - 4y^2 = 32$ is

(A) $8\sqrt{2}/\sqrt{3}$ **(B)** $16\sqrt{2}/\sqrt{3}$

(C) 3/32 **(D)** 64/3

28. The directrix of the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$ is

(A) $x=9/\sqrt{13}$ **(B)** $y=9/\sqrt{13}$
(C) $x=6/\sqrt{13}$ **(D)** $y=6/\sqrt{13}$

29. The locus of the point of intersection of straight lines $\frac{x}{a}-\frac{y}{b}=m$ and $\frac{x}{a}+\frac{y}{b}=\frac{1}{m}$ is

(A) An ellipse **(B)** A circle
(C) A hyperbola **(D)** A parabola

30. The locus of a point which moves such that the difference of its distances from two fixed points is always a constant is

(A) A straight line **(B)** A circle
(C) An ellipse **(D)** A hyperbola

31. The eccentricity of the hyperbola $2x^2-y^2=6$ is

(A) $\sqrt{2}$ **(B)** 2 **(C)** 3 **(D)** $\sqrt{3}$

32. The distance between the foci of a hyperbola is double the distance between its vertices and the length of its conjugate axis is 6. The equation of the hyperbola referred to its axes as axes of coordinates is

(A) $3x^2-y^2=3$ **(B)** $x^2-3y^2=3$
(C) $3x^2-y^2=9$ **(D)** $x^2-3y^2=9$

33. The equation $13[(x-1)^2+(y-2)^2]=3(2x+3y-2)^2$ represents

(A) A parabola **(B)** An ellipse
(C) A hyperbola **(D)** None of these

34. Find the equation of the hyperbola whose directrix is $x+2y=1$, focus is (2, 1) and eccentricity is 2.

(A) $x^2-16xy-11y^2-12x+6y+21=0$
(B) $3x^2+16xy+15y^2-4x-14y-1=0$
(C) $x^2+16xy+11y^2-12x-6y+21=0$
(D) None of these

35. The vertices of a hyperbola are at (0, 0) and (10, 0) and one of its foci is at (18, 0). The equation of the hyperbola is

(A) $\frac{x^2}{25}-\frac{y^2}{144}=1$ **(B)** $\frac{(x-5)^2}{25}-\frac{y^2}{144}=1$
(C) $\frac{x^2}{25}-\frac{(y-5)^2}{144}=1$ **(D)** $\frac{(x-5)^2}{25}-\frac{(y-5)^2}{144}=1$

36. The equation $x^2+4xy+y^2+2x+4y+2=0$ represents

(A) An ellipse **(B)** A pair of straight lines
(C) A hyperbola **(D)** None of these

37. The equation of the directrices of the conic $x^2+2x-y^2+5=0$ are

(A) $x=\pm 1$ **(B)** $y=\pm 2$
(C) $y=\pm\sqrt{2}$ **(D)** $x=\pm\sqrt{3}$

38. Foci of the hyperbola $\frac{x^2}{16}-\frac{(y-2)^2}{9}=1$ are

(A) (5, 2) (–5, 2) **(B)** (5, 2) (5, –2)
(C) (5, 2) (–5, –2) **(D)** None of these

39. The centre of the hyperbola $9x^2-16y^2+18x+32y-151=0$ is

(A) (1, –1) **(B)** (–1, 1)
(C) (–1, –1) **(D)** (1, 1)

40. Find the equation of the hyperbola whose foci are (6, 4) and (–4, 4) and eccentricity is 2.

(A) $12x^2-4y^2-24x+32y-127=0$
(B) $12x^2+4y^2+24x-32y-127=0$
(C) $12x^2-4y^2-24x-32y+127=0$
(D) $12x^2-4y^2+24x+32y+127=0$

41. The auxiliary equation of circle of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

(A) $x^2+y^2=a^2$ **(B)** $x^2+y^2=b^2$
(C) $x^2+y^2=a^2+b^2$ **(D)** $x^2+y^2=a^2-b^2$

42. The equation $x^2-16xy-11y^2-12x+6y+21=0$ represents

(A) Parabola **(B)** Ellipse
(C) Hyperbola **(D)** Two straight lines

43. The latus rectum of the hyperbola $9x^2-16y^2-18x-32y-151=0$ is

(A) 9/4 **(B)** 9 **(C)** 3/2 **(D)** 9/2

44. Find the equation of the hyperbola whose directrix is $2x+y=1$, focus is (1, 1) and eccentricity is $\sqrt{3}$.

(A) $7x^2+12xy-2y^2-2x+4y-7=0$
(B) $11x^2+12xy+2y^2-14x-14y+1=0$
(C) $11x^2+12xy+2y^2-10x-4y+1=0$
(D) None of these

45. The equation $x^2-4y^2-2x+16y-40=0$ represents

(A) A pair of straight lines **(B)** An ellipse
(C) A hyperbola **(D)** A parabola

46. The distance between the directrices of the hyperbola $x=8\sec\theta,\ y=8\tan\theta$ is

(A) $16\sqrt{2}$ **(B)** $\sqrt{2}$ **(C)** $8\sqrt{2}$ **(D)** $4\sqrt{2}$

47. The eccentricity of the hyperbola $5x^2-4y^2+20x+8y=4$ is

(A) $\sqrt{2}$ **(B)** 3/2 **(C)** 2 **(D)** 3

48. The latus rectum of the hyperbola $9x^2-16y^2-72x-32y-16=0$ is

(A) 9/2 **(B)** –9/2 **(C)** 32/2 **(D)** –32/3

49. The point of contact of the tangent $y=x+2$ to the hyperbola $5x^2-9y^2=45$ is

(A) (9/2, 5/2) **(B)** (5/2, 9/2)
(C) (–9/2, –5/2) **(D)** None of these

50. The line $lx+my+n=0$ is a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ if

(A) $a^2l^2+b^2m^2=n^2$ **(B)** $a^2l^2-b^2m^2=n^2$
(C) $am^2-b^2n^2=a^2l^2$ **(D)** None of these

51. If the line $y=2x+\lambda$ be a tangent to the hyperbola $36x^2-25y^2=3600$, then find the value of λ.

(A) 16 **(B)** −16 **(C)** ±16 **(D)** None of these

52. The line $3x-4y=5$ is a tangent to the hyperbola $x^2-4y^2=5$. The point of contact is

(A) (3, 1) **(B)** (2, 1/4) **(C)** (1, 3) **(D)** None of these

53. The equation of the tangent at the point $(a\sec\theta, b\tan\theta)$ of the conic $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

(A) $x\sec^2\theta-y\tan^2\theta=1$

(B) $\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$

(C) $\frac{x+a\sec\theta}{a^2}-\frac{y+b\tan\theta}{b^2}=1$

(D) None of these

54. The equation of the tangents to the conic $3x^2-y^2=3$ perpendicular to the line $x+3y=2$ is

(A) $y=3x\pm\sqrt{6}$ **(B)** $y=6x\pm\sqrt{3}$
(C) $y=x\pm\sqrt{6}$ **(D)** $y=3x\pm 6$

55. The equation of the tangent to the hyperbola $2x^2-3y^2=6$, which is parallel to the line $y=3x+4$, is

(A) $y=3x+5$ **(B)** $y=3x-5$
(C) $y=3x+5$ and $y=3x-5$ **(D)** None of these

56. The locus of the point of intersection of any two perpendicular tangents to the hyperbola is a circle which is called the director circle of the hyperbola, then the equation of this circle is

(A) $x^2+y^2=a^2+b^2$ **(B)** $x^2+y^2=a^2-b^2$
(C) $x^2+y^2=2ab$ **(D)** None of these

57. The equation of the tangents to the hyperbola $3x^2-4y^2=12$ which cuts the equal intercepts from the axes are

(A) $y+x=\pm 1$ **(B)** $y-x=\pm 1$
(C) $3x+4y=\pm 1$ **(D)** $3x-4y=\pm 1$

58. If m_1 and m_2 are the slopes of the tangents to the hyperbola $\frac{x^2}{25}-\frac{y^2}{16}=1$ which pass through the point (6, 2), then

(A) $m_1+m_2=\frac{24}{11}$ **(B)** $m_1m_2=\frac{20}{11}$
(C) $m_1+m_2=\frac{48}{11}$ **(D)** $m_1m_2=\frac{11}{20}$

59. The equation of the tangent to the hyperbola $4y^2=x^2-1$ at the point (1, 0) is

(A) $x=1$ **(B)** $y=1$ **(C)** $y=4$ **(D)** $x=4$

60. The value of m, for which $y=mx+6$ is a tangent to the hyperbola $\frac{x^2}{100}-\frac{y^2}{49}=1$, is

(A) $\sqrt{17/20}$ **(B)** $\sqrt{20/17}$ **(C)** $\sqrt{3/20}$ **(D)** $\sqrt{20/3}$

61. The equation of the tangent to the conic $x^2-y^2-8x+2y+11=0$ at the point (2, 1) is

(A) $x+2=0$ **(B)** $2x+1=0$
(C) $x-2=0$ **(D)** $x+y+1=0$

62. The point of contact of the line $y=x-1$ with $3x^2-4y^2=12$ is

(A) (4, 3) **(B)** (3, 4)
(C) (4, −3) **(D)** None of these

63. If the straight line $x\cos\alpha+y\sin\alpha=p$ be a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then

(A) $a^2\cos^2\alpha+b^2\sin^2\alpha=p^2$

(B) $a^2\cos^2\alpha-b^2\sin^2\alpha=p^2$

(C) $a^2\sin^2\alpha+b^2\cos^2\alpha=p^2$

(D) $a^2\sin^2\alpha-b^2\cos^2\alpha=p^2$

64. If the tangent on the point $(2\sec\phi, 3\tan\phi)$ of the hyperbola $\frac{x^2}{4}-\frac{y^2}{9}=1$ is parallel to $3x-y+4=0$, then the value of ϕ is

(A) 45° **(B)** 60° **(C)** 30° **(D)** 75°

65. The radius of the director circle of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

(A) $a-b$ **(B)** $\sqrt{a-b}$
(C) $\sqrt{a^2-b^2}$ **(D)** $\sqrt{a^2+b^2}$

66. What is the slope of the tangent line drawn to the hyperbola $xy=a(a\neq 0)$ at the point $(a, 1)$?

(A) $1/a$ **(B)** $-1/a$ **(C)** a **(D)** $-a$

67. The line $y=mx+c$ touches the curve $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ if

(A) $c^2=a^2m^2+b^2$ **(B)** $c^2=a^2m^2-b^2$
(C) $c^2=b^2m^2-a^2$ **(D)** $a^2=b^2m^2+c^2$

68. The straight line $x+y=\sqrt{2}p$ will touch the hyperbola $4x^2-9y^2=36$ if

(A) $p^2=2$ **(B)** $p^2=5$ **(C)** $5p^2=2$ **(D)** $2p^2=5$

69. The equation of the director circle of the hyperbola $\frac{x^2}{16}-\frac{y^2}{4}=1$ is given by

(A) $x^2+y^2=16$ **(B)** $x^2+y^2=4$
(C) $x^2+y^2=20$ **(D)** $x^2+y^2=12$

70. The equation of the tangent parallel to $y-x+5=0$ drawn to $\frac{x^2}{3}-\frac{y^2}{2}=1$ is

(A) $x-y-1=0$ **(B)** $x-y+2=0$
(C) $x+y-1=0$ **(D)** $x+y+2=0$

71. Let E be the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and C be the circle $x^2+y^2=9$. Let P and Q be the points (1, 2) and (2, 1), respectively. Then

(A) Q lies inside C but outside E
(B) Q lies outside both C and E
(C) P lies inside both C and E
(D) P lies inside C but outside E

72. The length of the chord of the parabola $y^2=4ax$, which passes through the vertex and makes an angle θ with the axis of the parabola, is

(A) $4a\cos\theta\,\mathrm{cosec}^2\theta$ **(B)** $4a\cos^2\theta\,\mathrm{cosec}\,\theta$
(C) $a\cos\theta\,\mathrm{cosec}^2\theta$ **(D)** $a\cos^2\theta\,\mathrm{cosec}\,\theta$

73. The equation of the normal at the point $(a\sec\theta, b\tan\theta)$ of the curve $b^2x^2-a^2y^2=a^2b^2$ is

(A) $\frac{ax}{\cos\theta}+\frac{by}{\sin\theta}=a^2+b^2$ **(B)** $\frac{ax}{\tan\theta}+\frac{by}{\sec\theta}=a^2+b^2$
(C) $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$ **(D)** $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2-b^2$

74. The condition that the straight line $lx+my=n$ may be a normal to the hyperbola $b^2x^2-a^2y^2=a^2b^2$ is given by

(A) $\frac{a^2}{l^2}-\frac{b^2}{m^2}=\frac{(a^2+b^2)^2}{n^2}$ **(B)** $\frac{l^2}{a^2}-\frac{m^2}{b^2}=\frac{(a^2+b^2)^2}{n^2}$
(C) $\frac{a^2}{l^2}+\frac{b^2}{m^2}=\frac{(a^2-b^2)^2}{n^2}$ **(D)** $\frac{l^2}{a^2}+\frac{m^2}{b^2}=\frac{(a^2-b^2)^2}{n^2}$

75. The equation of the normal to the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ at the point $(8, 3\sqrt{3})$ is

(A) $\sqrt{3}x+2y=25$ **(B)** $x+y=25$
(C) $y+2x=25$ **(D)** $2x+\sqrt{3}y=25$

76. The equation of the normal at the point (6, 4) on the hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=3$ is

(A) $3x+8y=50$ **(B)** $3x-8y=50$
(C) $8x+3y=50$ **(D)** $8x-3y=50$

77. What is the equation of the chord of hyperbola $25x^2-16y^2=400$ whose mid-point is (5, 3)?

(A) $115x-117y=17$ **(B)** $125x-48y=481$
(C) $127x+33y=341$ **(D)** $15x+121y=105$

78. The value of *m*, for which the line $y=mx+\frac{25\sqrt{3}}{3}$ is a normal to the conic $\frac{x^2}{16}-\frac{y^2}{9}=1$, is

(A) $\sqrt{3}$ **(B)** $-2/\sqrt{3}$ **(C)** $-\sqrt{3}/2$ **(D)** 1

79. The equation of the normal to the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ at $(-4, 0)$ is

(A) $y=0$ **(B)** $y=x$
(C) $x=0$ **(D)** $x=-y$

80. The eccentricity of the conjugate hyperbola of the hyperbola $x^2-3y^2=1$ is

(A) 2 **(B)** $2/\sqrt{3}$ **(C)** 4 **(D)** 4/3

81. If *e* and *e*′ are the eccentricities of a hyperbola and its conjugate, respectively, then

(A) $\left(\frac{1}{e}\right)^2+\left(\frac{1}{e'}\right)^2=1$ **(B)** $\frac{1}{e}+\frac{1}{e'}=1$
(C) $\left(\frac{1}{e}\right)^2+\left(\frac{1}{e'}\right)^2=0$ **(D)** $\frac{1}{e}+\frac{1}{e'}=2$

82. The product of the lengths of perpendiculars drawn from any point on the hyperbola $x^2-2y^2-2=0$ to its asymptotes is

(A) 1/2 **(B)** 2/3 **(C)** 3/2 **(D)** 2

83. The equation of a hyperbola, whose foci are (5, 0) and (–5, 0) and the length of whose conjugate axis is 8, is

(A) $9x^2-16y^2=144$ **(B)** $16x^2-9y^2=144$
(C) $9x^2+16y^2=144$ **(D)** $16x^2-9y^2=12$

84. The equation of the hyperbola, whose foci are the foci of the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ and the eccentricity is 2, is

(A) $\frac{x^2}{4}+\frac{y^2}{12}=1$ **(B)** $\frac{x^2}{4}-\frac{y^2}{12}=1$
(C) $\frac{x^2}{12}+\frac{y^2}{4}=1$ **(D)** $\frac{x^2}{12}-\frac{y^2}{4}=1$

85. The coordinates of the foci of the rectangular hyperbola $xy=c^2$ are

(A) $(\pm c, \pm c)$ **(B)** $(\pm c\sqrt{2}, \pm c\sqrt{2})$
(C) $\left(\pm\frac{c}{\sqrt{2}}, \pm\frac{c}{\sqrt{2}}\right)$ **(D)** None of these

86. The eccentricity of the curve $x^2-y^2=1$ is

(A) 1/2 **(B)** $1/\sqrt{2}$ **(C)** 2 **(D)** $\sqrt{2}$

87. The locus of the point of intersection of lines $(x+y)t=a$ and $x-y=at$, where *t* is the parameter, is

(A) A circle **(B)** An ellipse
(C) A rectangular hyperbola **(D)** None of these

88. The equation of the hyperbola which is referred to its axes as axes of coordinate, and whose distance between the foci is 16 and eccentricity is $\sqrt{2}$, is

(A) $x^2-y^2=16$ **(B)** $x^2-y^2=32$
(C) $x^2-2y^2=16$ **(D)** $y^2-x^2=16$

89. If the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{b^2}=1$ and the hyperbola $\frac{x^2}{144}-\frac{y^2}{81}=\frac{1}{25}$ coincide, then the value of b^2 is

(A) 1 **(B)** 5 **(C)** 7 **(D)** 9

90. A tangent to a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ intercepts a length of unity from each of the coordinate axes, then the point (*a*, *b*) lies on the rectangular hyperbola

(A) $x^2-y^2=2$ **(B)** $x^2-y^2=1$

(C) $x^2-y^2=-1$ **(D)** None of these

91. The curve $xy=c^2$ is said to be

(A) A parabola
(B) A rectangular hyperbola
(C) A hyperbola
(D) An ellipse

92. The reciprocal of the eccentricity of rectangular hyperbola is

(A) 2 **(B)** 1/2 **(C)** $\sqrt{2}$ **(D)** $1/\sqrt{2}$

93. The eccentricity of the hyperbola $\frac{\sqrt{1999}}{3}(x^2-y^2)=1$ is

(A) $\sqrt{3}$ **(B)** $\sqrt{2}$ **(C)** 2 **(D)** $2\sqrt{2}$

94. If transverse and conjugate axes of a hyperbola are equal, then its eccentricity is

(A) $\sqrt{3}$ **(B)** $\sqrt{2}$ **(C)** $1/\sqrt{2}$ **(D)** 2

95. If $5x^2+\lambda y^2=20$ represents a rectangular hyperbola, then λ equals

(A) 5 **(B)** 4 **(C)** –5 **(D)** None of these

96. If *e* and *e*′ are the eccentricities of the ellipse $5x^2+9y^2=45$ and the hyperbola $5x^2-4y^2=45$ respectively, then find *ee*′.

(A) 9 **(B)** 4 **(C)** 5 **(D)** 1

97. If the distance between the directrices of a rectangular hyperbola is 10 unit, then the distance between its foci is

(A) $10\sqrt{2}$ **(B)** 5 **(C)** $5\sqrt{2}$ **(D)** 20

98. The eccentricity of the curve $x^2-y^2=a^2$ is

(A) 2 **(B)** $\sqrt{2}$ **(C)** 4 **(D)** None of these

99. The eccentricity of a rectangular hyperbola is

(A) $1/\sqrt{2}$ **(B)** $-1/\sqrt{2}$ **(C)** $\sqrt{2}$ **(D)** >2

100. The eccentricity of the hyperbola conjugate to $x^2-3y^2=2x+8$ is

(A) $2/\sqrt{3}$ **(B)** $\sqrt{3}$ **(C)** 2 **(D)** None of these

101. The locus of a point $P(\alpha,\beta)$ which is moving under the condition that the line $y=\alpha x+\beta$ is a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

(A) A parabola **(B)** A hyperbola

(C) An ellipse **(D)** A circle

102. The eccentricity of the hyperbola $\frac{x^2}{16}-\frac{y^2}{25}=1$ is

(A) 3/4 **(B)** 3/5 **(C)** $\sqrt{41}/4$ **(D)** $\sqrt{41/5}$

103. Find the equation to the hyperbola having its eccentricity 2 and the distance between its foci is 8.

(A) $\frac{x^2}{12}-\frac{y^2}{4}=1$ **(B)** $\frac{x^2}{4}-\frac{y^2}{12}=1$

(C) $\frac{x^2}{8}-\frac{y^2}{2}=1$ **(D)** $\frac{x^2}{16}-\frac{y^2}{9}=1$

104. If θ is the acute angle of intersection at a real point of intersection of the circle $x^2+y^2=5$ and the parabola $y^2=4x$, then $\tan\theta$ is equal to

(A) 1 **(B)** $\sqrt{3}$ **(C)** 3 **(D)** $1/\sqrt{3}$

105. Find the equation of the hyperbola in the standard form (with transverse axis along the *x*-axis) which has the length of the latus rectum as 9 unit and eccentricity as 5/4.

(A) $\frac{x^2}{16}-\frac{y^2}{18}=1$ **(B)** $\frac{x^2}{36}-\frac{y^2}{27}=1$

(C) $\frac{x^2}{64}-\frac{y^2}{36}=1$ **(D)** $\frac{x^2}{36}-\frac{y^2}{64}=1$

106. If $4x^2+py^2=45$ and $x^2-4y^2=5$ cut orthogonally, then the value of *p* is

(A) 1/9 **(B)** 1/3 **(C)** 3 **(D)** 9

107. Find the equation of axis of the given hyperbola $\frac{x^2}{3}-\frac{y^2}{2}=1$ which is equally inclined to the axes

(A) $y=x+1$ **(B)** $y=x-1$

(C) $y=x+2$ **(D)** $y=x-2$

Practice Exercise 2

Single/Multiple Correct Choice Type Questions

1. If foci of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ concide with the focii of $\frac{x^2}{25}+\frac{y^2}{9}=1$ and eccentricity of the hyperbola is 2, then

(A) $a^2+b^2=16$
(B) there is no director circle to the hyperbola
(C) centre of the director circle is (0, 0)
(D) length of latus rectum of the hyperbola = 12

2. The line $y=mx\pm\sqrt{a^2m^2-b^2}$, $m>0$ touches the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point whose eccentric angle is

(A) $\sin^{-1}\left(\frac{b}{ma}\right)$ **(B)** $\pi+\sin^{-1}\left(\frac{b}{ma}\right)$

(C) $2\pi+\sin^{-1}\left(\frac{b}{ma}\right)$ **(D)** $-\sin^{-1}\left(\frac{b}{ma}\right)$

3. For the hyperbola $9x^2-16y^2-18x+32y-151=0$

(A) One of the directrix is $x=\frac{21}{5}$

(B) Length of latus rectum = $\frac{9}{2}$

(C) Foci are (6, 1) and (–4, 1)

(D) Eccentricity is $\frac{5}{4}$

4. If ($a\sec\theta$, $b\tan\theta$) and ($a\sec\phi$, $b\tan\phi$) are the ends of a focal chord of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $\tan\frac{\theta}{2}\tan\frac{\phi}{2}$ equals to

(A) $\frac{e-1}{e+1}$ **(B)** $\frac{1-e}{1+e}$ **(C)** $\frac{1+e}{1-e}$ **(D)** $\frac{e+1}{e-1}$

Comprehension Type Questions

Paragraph for Questions 5–7: If P is a variable point and F_1 and F_2 are the two fixed points such that $|PF_1 - PF_2| = 2a$. Then the locus of the point P is a hyperbola, with the points F_1 and F_2 as the two focii ($F_1F_2 > 2a$). If $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is a hyperbola, then its conjugate hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$. Let $P(x, y)$ is a variable point such that $\left|\sqrt{(x-1)^2+(y-2)^2}-\sqrt{(x-5)^2+(y-5)^2}\right|=3$.

5. If the locus of the point P represents a hyperbola of the eccentricity e, then the eccentricity e' of the corresponding conjugate hyperbola is

(A) $\frac{5}{3}$ **(B)** $\frac{4}{3}$ **(C)** $\frac{5}{4}$ **(D)** $\frac{3}{\sqrt{7}}$

6. Locus of intersection of the two perpendicular tangents to the given hyperbola is

(A) $(x-3)^2+\left(y-\frac{7}{2}\right)^2=\frac{55}{4}$

(B) $(x-3)^2+\left(y-\frac{7}{2}\right)^2=\frac{25}{4}$

(C) $(x-3)^2+\left(y-\frac{7}{2}\right)^2=\frac{7}{4}$

(D) None of these

7. If the origin is shifted to the point $\left(3,\frac{7}{2}\right)$ and the axes are rotated through an angle θ in a clockwise direction, so that the equation of the given hyperbola changes to the standard form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then θ is

(A) $\tan^{-1}\left(\frac{4}{3}\right)$ **(B)** $\tan^{-1}\left(\frac{3}{4}\right)$

(C) $\tan^{-1}\left(\frac{5}{3}\right)$ **(D)** $\tan^{-1}\left(\frac{3}{5}\right)$

Paragraph for Questions 8–10: For the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, the normal at P meets the transverse axis AA' in G and the conjugate axis BB' in g and CF be perpendicular to the normal from the centre.

8. If $PF\times PG = K\times CB^2$, then $K=$

(A) 2 **(B)** 1 **(C)** $\frac{1}{2}$ **(D)** 4

9. $PF\times Pg=$

(A) CA^2 **(B)** CF^2 **(C)** CB^2 **(D)** $CA\times CB$

10. Locus of the middle point of G and g is a hyperbola of eccentricity

(A) $\frac{1}{\sqrt{e^2-1}}$ **(B)** $\frac{e}{\sqrt{e^2-1}}$ **(C)** $2\sqrt{e^2-1}$ **(D)** $\frac{e}{2}$

Paragraph for Questions 11–13: If a circle with centre $C(\alpha, \beta)$ intersects a rectangular hyperbola with centre $L(h, k)$ at four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$, then the mean of the four points P, Q, R, S is the mean of the points C and L. In other words, the mid-point of CL coincides with the mean point of P, Q, R, S. Analytically,

$$\frac{x_1+x_2+x_3+x_4}{4}=\frac{\alpha+h}{2} \text{ and } \frac{y_1+y_2+y_3+y_4}{4}=\frac{\beta+k}{2}$$

11. If four points are taken on the circle $x^2+y^2=a^2$. A rectangular hyperbola (H) passes through these four points. If the centroid of the quadrilateral formed from these four points lies on the straight line $3x-4y+1=0$, then find the locus of the centre of the rectangular hyperbola (H).

(A) $3x-4y+2=0$ **(B)** $3x-4y+3=0$
(C) $3x-4y+4=0$ **(D)** None of these

12. A, B, C, D are the points of intersection of a circle and a rectangular hyperbola which have different centres. If AB passes through the centre of the hyperbola, then CD passes through

(A) centre of the hyperbola
(B) centre of the circle
(C) mid-point of the centres of the circle and the hyperbola
(D) None of the points mentioned in the three options

13. If the normals drawn at four concyclic points on a rectangular hyperbola $xy=c^2$ meet at point (α, β) then the centre of the circle has the coordinates

(A) (α, β) **(B)** $(2\alpha, 2\beta)$

(C) $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$ **(D)** $\left(\frac{\alpha}{4}, \frac{\beta}{4}\right)$

Matrix Match Type Questions

14. Match the following:

Column I	Column II
(A) Length of a common tangent to the hyperbola $x^2-9y^2=9$ and $y^2-9x^2=9$ is	**(p)** 9/8
(B) A line drawn through the focus of hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ and parallel to one of its asymptotes meets the curve at P, then SP is equal to	**(q)** $\sqrt{3}$

Column I	Column II
(C) Tangent to the ellipse $\frac{x^2}{12}+\frac{y^2}{324}=1$ at $(2\sqrt{3}\cos\theta, 18\sin\theta)$, $(\theta\in(0,\pi))$ is drawn. The value of $\tan\theta$ such that the sum of the length of the intercepts on axes made by this tangent is minimum, is	**(r)** 5
(D) If the line $2x+\sqrt{6}y=2$ touches the hyperbola $x^2-2y^2=4$, and slope of the line joining the origin and the point of contact is m, then $\sqrt{8}m$ is	**(s)** $-\sqrt{3}$
	(t) 1

15. Match the following:

Column I	Column II
(A) The area of the triangle that a tangent at a point of the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ makes with its asymptotes is	**(p)** 12
(B) If the line $y = 3x + \lambda$ touches the curve $9x^2 - 5y^2 = 45$, then $\|\lambda\|$ is	**(q)** 6
(C) If the chord $x \cos \alpha + y \sin \alpha = p$ of the hyperbola $\frac{x^2}{16}-\frac{y^2}{18}=1$ subtends a right angle at the centre, then the diameter of the circle, concentric with the hyperbola, to which the given chord is a tangent is	**(r)** 24
(D) If λ be the length of the latus rectum of the hyperbola $16x^2 - 9y^2 + 32x + 36y - 164 = 0$, then 3λ is equal to	**(s)** 32
	(t) 3

16. Match the following:

Column I	Column II
(A) A tangent drawn to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at $P\left(\frac{\pi}{6}\right)$ forms a triangle of area $3a^2$ square units, with coordinate axes, then the square of its eccentricity is equal to	**(p)** 17
(B) If the eccentricity of the hyperbola $x^2 - y^2 \sec^2\theta = 5\sqrt{3}$ times the eccentricity of the ellipse $x^2\sec^2\theta + y^2 = 25$, then the smallest positive value of θ is $\frac{6\pi}{p}$, value of 'p' is	**(q)** 32
(C) For the hyperbola $\frac{x^2}{3} - y^2 = 3$, acute angle between its asymptotes is $\frac{\ell\pi}{24}$, then the value of 'ℓ' is	**(r)** 16
(D) For the hyperbola $xy = 8$ any tangent of it at P meets co-ordinate axes at Q and R then the area of triangle CQR where 'C' is the centre of the hyperbola is	**(s)** 24
	(t) 8

17. Match the following:

Column I	Column II
(A) Value of c for which $3x^2 - 5xy - 2y^2 + 5x + 11y + c = 0$ is the asymptotes of the hyperbola $3x^2 - 5xy - 2y^2 + 5x + 11y - 8 = 0$	**(p)** 3
(B) If locus of a point whose chord of contact with respect to the circle $x^2 + y^2 = 4$ is a tangent to the hyperbola $xy = 1$ is $xy = c^2$, then value of c^2 is	**(q)** –4
(C) If the equation of a hyperbola whose conjugate axis is 5 and distance between its foci is 13, is $ax^2 - by^2 = c$ where a and b are co-prime natural numbers, then value of $\frac{ab}{c}$ is	**(r)** –12
(D) If the vertex of a hyperbola bisects the distance between its centre and the corresponding focus, then the ratio of square of its conjugate axis to the square of its transverse axis is	**(s)** 4
	(t) – 6

Integer Type Questions

18. If the line $3x+4y=12$ intersects the hyperbola at P & P' and its asymptotes at Q and Q' then find the value of $\frac{PQ}{P'Q'}+\frac{PQ'}{P'Q}$.

19. Chords of the circle $x^2 + y^2 = 4$ touch the hyperbola $\frac{x^2}{4}-\frac{y^2}{16}=1$. The locus of their middle points is the curve $(x^2+y^2)^2 = \lambda x^2 - 16y^2$, then find λ.

20. If a variable line has its intercepts on the co-ordinates axes e, e', where $\frac{e}{2}$, $\frac{e'}{2}$ are the eccentricities of a hyperbola and its conjugate hyperbola, then this line always touches the circle $x^2 + y^2 = r^2$, where $r =$ ______.

Answer Key

Practice Exercise 1

1. (B)	**2.** (A)	**3.** (B)	**4.** (A)	**5.** (B)	**6.** (D)
7. (A)	**8.** (C)	**9.** (C)	**10.** (A)	**11.** (C)	**12.** (A)
13. (C)	**14.** (C)	**15.** (D)	**16.** (C)	**17.** (B)	**18.** (A)
19. (C)	**20.** (A)	**21.** (C)	**22.** (D)	**23.** (C)	**24.** (A)
25. (B)	**26.** (C)	**27.** (A)	**28.** (A)	**29.** (C)	**30.** (D)
31. (D)	**32.** (C)	**33.** (C)	**34.** (A)	**35.** (B)	**36.** (C)
37. (C)	**38.** (A)	**39.** (B)	**40.** (A)	**41.** (A)	**42.** (C)
43. (D)	**44.** (A)	**45.** (C)	**46.** (C)	**47.** (B)	**48.** (A)
49. (C)	**50.** (B)	**51.** (C)	**52.** (A)	**53.** (B)	**54.** (A)
55. (C)	**56.** (B)	**57.** (B)	**58.** (A), (B)	**59.** (A)	**60.** (A)
61. (C)	**62.** (A)	**63.** (B)	**64.** (C)	**65.** (C)	**66.** (B)
67. (B)	**68.** (D)	**69.** (D)	**70.** (A)	**71.** (D)	**72.** (A)
73. (C)	**74.** (A)	**75.** (D)	**76.** (A)	**77.** (B)	**78.** (B)
79. (A)	**80.** (A)	**81.** (A)	**82.** (B)	**83.** (B)	**84.** (B)
85. (B)	**86.** (D)	**87.** (C)	**88.** (B)	**89.** (C)	**90.** (B)
91. (B)	**92.** (C)	**93.** (B)	**94.** (B)	**95.** (C)	**96.** (D)
97. (D)	**98.** (B)	**99.** (C)	**100.** (C)	**101.** (B)	**102** (C)
103. (B)	**104.** (C)	**105.** (C)	**106.** (D)	**107.** (A)	

Practice Exercise 2

1. (A), (B), (D)	**2.** (A), (B)	**3.** (A), (B), (C), (D)	**4.** (B), (C)	**5.** (C)	**6.** (D)
7. (B)	**8.** (B)	**9.** (A)	**10.** (B)	**11.** (A)	**12.** (B)
13. (C)	**14.** (A) → (r); (B) → (p); (C) → (q), (s); (D) → (s)	**15.** (A) → (p), (B) → (q), (C) → (r), (D) → (s)	**16.** (A) → (p), (B) → (s), (C) → (t), (D) → (r)	**17.** (A) → (r), (B) → (s), (C) → (s), (D) → (p)	**18.** 2
19. 4	**20.** 2				

Solutions

Practice Exercise 1

1. Parametric point on hyperbola $\frac{x^2}{A^2}-\frac{y^2}{B^2}=1$ is $(A\sec\theta, B\tan\theta)$.

2. We have

$$e=\sqrt{1+\frac{b^2}{a^2}} \Rightarrow e^2=\frac{a^2+b^2}{a^2}$$

Therefore,

$$e_1=\sqrt{1+\frac{a^2}{b^2}} \Rightarrow e_1^2=\frac{b^2+a^2}{b^2} \Rightarrow \frac{1}{e_1^2}+\frac{1}{e^2}=1$$

3. We have

$$\frac{x^2}{3^2}-\frac{y^2}{4^2}=1$$

Therefore,

$$PS_1 \sim PS_2 = 2(3) = 6$$

4. We have

$$\frac{2b^2}{a}=8 \text{ and } \frac{3}{\sqrt{5}}=\sqrt{1+\frac{b^2}{a^2}}$$

or

$$\frac{4}{5}=\frac{b^2}{a^2}$$

$$\Rightarrow a=5 \text{ and } b=2\sqrt{5}$$

Hence, the equation of the hyperbola is

$$\frac{x^2}{25}-\frac{y^2}{20}=1 \Rightarrow 4x^2-5y^2=100$$

5. We have
$$\frac{9}{a^2}=1 \Rightarrow a=3 \text{ and } \frac{18}{a^2}-\frac{4}{b^2}=1 \Rightarrow b^2=4$$
Therefore,
$$e=\sqrt{1+\frac{4}{9}}=\frac{\sqrt{13}}{3}$$

6. For the hyperbola, $\Delta \neq 0$ and $h^2 > ab$, we have $\Delta = 0$.

7. Conjugate axis is 5 and distance between foci is 13. So,
$$2b=5 \text{ and } 2ae=13$$
We know that for the hyperbola,
$$b^2=a^2(e^2-1) \Rightarrow \frac{25}{4}=\frac{(13)^2}{4e^2}(e^2-1)$$
$$\Rightarrow \frac{25}{4}=\frac{169}{4}-\frac{169}{4e^2} \text{ or } e^2=\frac{169}{144} \Rightarrow e=\frac{13}{12}$$
or $a=6$ and $b=5/2$ or the hyperbola is
$$\frac{x^2}{36}-\frac{y^2}{25/4}=1$$
$$\Rightarrow 25x^2-144y^2=900$$

8. **Trick:** $2a=7$ or $a=7/2$. Also (5, –2) satisfies it. Thus,
$$\frac{4}{49}(25)-\frac{51}{196}(4)=1$$
and
$$a^2=\frac{49}{4} \Rightarrow a=\frac{7}{2}$$

9. Vertices: $(\pm 4,0) \equiv (\pm a,0) \Rightarrow a=4$

Foci: $(\pm 6,0) \equiv (\pm ae,0) \Rightarrow e=6/4=3/2$

10. We have
$$\frac{x^2}{25}-\frac{y^2}{25}=1$$
Thus
$$\sqrt{1+\frac{b^2}{a^2}}=\sqrt{2}$$
The eccentricity is $\sqrt{2}$ since $a=b$.

11. We have
$$(4x+8)^2-(y-2)^2=-44+64-4$$
$$\Rightarrow \frac{16(x+2)^2}{16}-\frac{(y-2)^2}{16}=1$$
The transverse and conjugate axes are $y=2$ and $x=-2$.

12. We have $2a=8$ and $2b=6$. The difference of the focal distances of any point of the hyperbola is $2a=8$.

13. Foci: $(0,\pm 4) \equiv (0,\pm be) \Rightarrow be=4$

Vertices: $(0,\pm 2) \equiv (0,\pm b) \Rightarrow b=2 \Rightarrow a=2\sqrt{3}$

Hence, the equation is
$$\frac{-x^2}{(2\sqrt{3})^2}+\frac{y^2}{(2)^2}=1 \text{ or } \frac{y^2}{4}-\frac{x^2}{12}=1$$

14. Multiplying both, we get
$$(bx)^2-(ay)^2=(ab)^2 \Rightarrow \frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$
which is the standard equation of a hyperbola.

15. Squaring and subtracting, we get
$$a^2x^2-b^2y^2=a^2-b^2$$
which is the equation of the hyperbola.

16. The centre is (0, 0) and the vertex is (4, 0). Therefore, $a=4$ and the focus is (6, 0). Therefore,
$$ae=4 \Rightarrow e=\frac{3}{2}$$
Therefore, $b=2\sqrt{5}$. Hence, the equation is
$$5x^2-4y^2=80$$

17. For hyperbola, since $e>1$, we have $2/3<1$.

18. Let the equation of the hyperbola be
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$
which passes through (3, 2). Therefore,
$$\frac{9}{a^2}-\frac{4}{b^2}=1 \quad (1)$$
Also, this passes through (–17, 12). Thus,
$$\frac{(-17)^2}{a^2}-\frac{(12)^2}{b^2}=1 \quad (2)$$
On solving these, we get $a=1$ and $b=\sqrt{2}$. Hence, the length of the transverse axis is $2a=2$.

19. Multiplying both, we get
$$3x^2-y^2=48$$
or
$$\frac{x^2}{(48/3)}-\frac{y^2}{48}=1$$
which is a hyperbola.

20. The hyperbola is
$$\frac{x^2}{16}-\frac{y^2}{9}=1$$
Thus, we have the difference of focal distance as $2a=8$.

21. The given equation of hyperbola is
$$\frac{x^2}{4}-\frac{y^2}{(16/9)}=1$$
Therefore, $a=2$ and $b=4/3$. As we know,
$$b^2=a^2(e^2-1)$$
$$\Rightarrow \frac{16}{9}=4(e^2-1) \Rightarrow e^2=\frac{13}{9}$$
Therefore, $e=\sqrt{13}/3$.

22. The given conic is
$$\frac{x^2}{(1)^2}-\frac{y^2}{(1/2)^2}=1$$
Therefore,
$$b^2=a^2(e^2-1) \Rightarrow \frac{1}{4}+1=e^2 \Rightarrow e=\frac{\sqrt{5}}{2}$$

23. We know that when a circle touches the two given circles externally, then the locus of the circle is a hyperbola.

24. The given equation is

$$2x^2 - 3y^2 = 5$$

Now,

$$b^2 = a^2(e^2 - 1) \Rightarrow \frac{5}{3} = \frac{5}{2}(e^2 - 1) \Rightarrow e = \sqrt{\frac{5}{3}}$$

The foci of the hyperbola are $(\pm ae, 0)$. That is,

$$\left[\pm\sqrt{\frac{5}{2}}\left(\sqrt{\frac{5}{3}}\right), 0\right] = \left(\pm\frac{5}{\sqrt{6}}, 0\right)$$

25. The given equation of hyperbola is

$$16x^2 - 9y^2 = 144 \Rightarrow \frac{x^2}{9} - \frac{y^2}{16} = 1$$

Therefore, the L.R. is

$$\frac{2b^2}{a} = \frac{2(16)}{3} = \frac{32}{3}$$

26. The equation of the hyperbola is

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

Now, $$b^2 = a^2(e^2 - 1) \Rightarrow e = \frac{5}{4}$$

Hence, the foci are

$$(\pm ae, 0) \Rightarrow \left(\pm 4.\frac{5}{4}, 0\right)$$

That is, $(\pm 5, 0)$.

27. The given equation may be written as

$$\frac{x^2}{32/3} - \frac{y^2}{8} = 1 \text{ or } \frac{x^2}{(4\sqrt{2}/\sqrt{3})^2} - \frac{y^2}{(2\sqrt{2})^2} = 1$$

On comparing the given equation with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get

$$a^2 = \left(\frac{4\sqrt{2}}{\sqrt{3}}\right)^2 \text{ or } a = \frac{4\sqrt{2}}{\sqrt{3}}$$

Therefore, the length of the transverse axis of a hyperbola is

$$2a = 2 \times \frac{4\sqrt{2}}{\sqrt{3}} = \frac{8\sqrt{2}}{\sqrt{3}}$$

28. The directrix of hyperbola is $x = a/e$, where

$$e = \sqrt{\frac{b^2 + a^2}{a^2}} = \frac{\sqrt{b^2 + a^2}}{a}$$

The directrix is

$$x = \frac{a^2}{\sqrt{a^2 + b^2}} = \frac{9}{\sqrt{9 + 4}} \Rightarrow x = \frac{9}{\sqrt{13}}$$

29. We have

$$\frac{x}{a} - \frac{y}{b} = m \quad (1)$$

$$\frac{x}{a} + \frac{y}{b} = \frac{1}{m} \quad (2)$$

On multiplying Eqs. (1) and (2), we get

$$\left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{a} + \frac{y}{b}\right) = m\left(\frac{1}{m}\right)$$

Therefore, the equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

30. Let A and B are two fixed points and P is the moving point. According to given condition, we have

$$|PA - PB| = k, (k < AB)$$

Therefore, locus of P is hyperbola.

31. We have

$$\frac{x^2}{(6/2)} - \frac{y^2}{6} = 1$$

That is, $a^2 = 3$ and $b^2 = 6$. Therefore,

$$e = \sqrt{\frac{b^2}{a^2} + 1} \Rightarrow e = \sqrt{3}$$

32. According to given conditions,

$$2ae = 2(2a) \Rightarrow e = 2$$

and $$2b = 6 \Rightarrow b = 3$$

Hence,

$$a = \frac{3}{\sqrt{3}} = \sqrt{3}$$

Therefore, the equation is

$$\frac{x^2}{3} - \frac{y^2}{9} = 1$$

That is,

$$3x^2 - y^2 = 9$$

33. Here, the coefficient of x^2 is positive and that of y^2 is negative. That is, a hyperbola.

34. We have

$$(x-2)^2 + (y-1)^2 = 4\left[\frac{(x+2y-1)^2}{5}\right]$$

$$\Rightarrow 5[x^2 + y^2 - 4x - 2y + 5] = 4[x^2 + 4y^2 + 1 + 4xy - 2x - 4y]$$

$$\Rightarrow x^2 - 11y^2 - 16xy - 12x + 6y + 21 = 0$$

35. We have $2a = 10$. Therefore, $a = 5$.

$$ae - a = 8 \text{ or } e = 1 + \frac{8}{5} = \frac{13}{5}$$

Therefore,

$$b = 5\sqrt{\frac{13^2}{5^2} - 1} = 5 \times \frac{12}{5} = 12$$

and the centre of hyperbola is $(5, 0)$. Therefore,

$$\frac{(x-5)^2}{5^2} - \frac{(y-0)^2}{12^2} = 1$$

36. Obviously, $h^2 > ab$ and

$$\Delta = (1)(1)(2) + 2(2)(2)(2) - (1)(2)^2 - (1)(1)^2 - 2(2)^2 < 0$$

Hence, it is a hyperbola.

37. We have

$$(x+1)^2 - y^2 - 1 + 5 = 0 \Rightarrow -\frac{(x+1)^2}{4} + \frac{y^2}{4} = 1$$

The equation of the directrices of $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ are $y = \pm\frac{b}{e}$.
Here, $b = 2$ and $e = \sqrt{1+1} = \sqrt{2}$. Hence,

$$y = \pm\frac{2}{\sqrt{2}} \Rightarrow y = \pm\sqrt{2}$$

38. We have

$$a = 4, b = 3 \Rightarrow \frac{9}{16} = (e^2 - 1) \Rightarrow e = \frac{5}{4}$$

The vertex is (0, 2). Hence, the focus is $(\pm ae, 2) = (\pm 5, 2)$.

39. The centre is given by

$$\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right) = \left(\frac{+16\cdot 9}{-9\cdot 16}, \frac{-9(16)}{-9(16)}\right) = (-1, 1)$$

40. The foci are (6,4) and (–4,4), $e = 2$ and the centre is

$$\left(\frac{6-4}{2}, 4\right) = (1, 4)$$

$$\Rightarrow 6 = 1 + ae \Rightarrow ae = 5$$

$$\Rightarrow a = \frac{5}{2} \text{ and } b = \frac{5}{2}(\sqrt{3})$$

Hence, the equation is

$$\frac{(x-1)^2}{(25/4)} - \frac{(y-4)^2}{(75/4)} = 1$$

or $$12x^2 - 4y^2 - 24x + 32y - 127 = 0$$

41. By definition of auxiliary circle, equation of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $x^2 + y^2 = a^2$.

42. $\Delta = -1\cdot 11\cdot 21 + 2\cdot 8\cdot 6\cdot 3 - 1\cdot 9 + 11\cdot 36 - 21\cdot 64 \neq 0$,

$$h^2 > ab \Rightarrow 64 > -11$$

43. We have

$$9x^2 - 18x + 9 - 16y^2 - 32y - 16 = 144$$

$$\Rightarrow \frac{(x-1)^2}{16} - \frac{(y+1)^2}{9} = 1$$

Therefore, the latus rectum is

$$\frac{2b^2}{a} = \frac{2\times 9}{4} = \frac{9}{2}$$

44. We have S(1,1), the directrix is $2x + y = 1$ and $e = \sqrt{3}$. Now, let the various point be (h, k). Then, we get

$$\frac{\sqrt{(h-1)^2 + (k-1)^2}}{(2h + k - 1)/\sqrt{5}} = \sqrt{3}$$

Squaring on both the sides, we get

$$5[(h-1)^2 + (k-1)^2] = 3(2h + k - 1)^2$$

On simplification, the required locus is

$$7x^2 + 12xy - 2y^2 - 2x + 4y - 7 = 0$$

45. We have

$$x^2 - 2x - 4y^2 + 16y - 40 = 0$$

$$\Rightarrow (x^2 - 2x) - 4(y^2 - 4y) - 40 = 0$$

$$\Rightarrow (x-1)^2 - 1 - 4[(y-2)^2 - 4] - 40 = 0$$

$$\Rightarrow (x-1)^2 - 4(y-2)^2 = 25$$

$$\Rightarrow \frac{(x-1)^2}{25} - \frac{(y-2)^2}{25/4} = 1$$

which is a hyperbola.

46. Equation of hyperbola is

$$x = 8\sec\theta, y = 8\tan\theta \Rightarrow \frac{x}{8} = \sec\theta, \frac{y}{8} = \tan\theta$$

Since $\sec^2\theta - \tan^2\theta = 1$, we get

$$\frac{x^2}{8^2} - \frac{y^2}{8^2} = 1$$

Here, $a = 8$ and $b = 8$. Now,

$$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{8^2}{8^2}} = \sqrt{1+1} \Rightarrow e = \sqrt{2}$$

Therefore, the distance between the directrices is

$$\frac{2a}{e} = \frac{2\times 8}{\sqrt{2}} = 8\sqrt{2}$$

47. The given equation of hyperbola is

$$5x^2 - 4y^2 + 20x + 8y = 4$$

$$5(x+2)^2 - 4(y-1)^2 = 20 \Rightarrow \frac{(x+2)^2}{4} - \frac{(y-1)^2}{5} = 1$$

From $b^2 = a^2(e^2 - 1)$, we have $5 = 4(e^2 - 1)$. Therefore,

$$e^2 = \frac{9}{4} \Rightarrow e = \frac{3}{2}$$

48. The equation of the hyperbola is given as

$$9x^2 - 16y^2 + 72x - 32y - 16 = 0$$

$$\Rightarrow 9(x^2 + 8x) - 16(y^2 + 2y) - 16 = 0$$

$$\Rightarrow 9(x+4)^2 - 16(y+1)^2 = 144$$

$$\Rightarrow \frac{(x+4)^2}{16} - \frac{(y+1)^2}{9} = 1$$

Therefore, the latus rectum is

$$\frac{2b^2}{a} = 2\times\frac{9}{4} = \frac{9}{2}$$

49. The hyperbola is

$$\frac{x^2}{9} - \frac{y^2}{5} = 1$$

Hence, the point of contact is

$$\left[\frac{-9(1)}{\sqrt{9-5}}, \frac{-5}{\sqrt{9-5}}\right] \equiv \left[\frac{-9}{2}, \frac{-5}{2}\right]$$

Trick: The point $\left(-\frac{9}{2}, -\frac{5}{2}\right)$ satisfies both equations.

50. $lx+my+n=0 \Rightarrow y=-\frac{1}{m}x-\frac{n}{m}$ is tangent to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Apply condition of tangency,

$$c^2=a^2M^2-b^2 \Rightarrow \left(-\frac{n}{m}\right)^2=a^2\left(\frac{1}{m}\right)^2-b^2 \Rightarrow n^2=a^2l^2-b^2m^2$$

51. If $y=2x+\lambda$ is tangent to the given hyperbola, then

$$\lambda=\pm\sqrt{a^2m^2-b^2}=\pm\sqrt{(100)(4)-144}=\pm\sqrt{256}=\pm 16$$

52. If the point of contact be (h, k), then the tangent is

$$hx-4ky-5=0 \equiv 3x-4y-5=0$$

or $h=3$ and $k=1$. Hence, the point of contact is (3, 1).

53. We have

$$\frac{x(a\sec\theta)}{a^2}-\frac{y(b\tan\theta)}{b^2}=1 \Rightarrow \frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$$

54. The tangent to $\frac{x^2}{1}-\frac{y^2}{3}=1$ and perpendicular to $x+3y-2=0$ is given by

$$y=3x\pm\sqrt{9-3}=3x\pm\sqrt{6}$$

55. Let the tangent be $y=3x+c$. Then

$$c=\pm\sqrt{a^2m^2-b^2}=\pm\sqrt{3.9-2}=\pm 5 \Rightarrow y=3x\pm 5$$

56. The equation of hyperbola is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Tangents to the hyperbola are given by

$$y=mx\pm\sqrt{a^2m^2-b^2}$$

Also, the tangent perpendicular to this is

$$y=\frac{-1}{m}x\pm\sqrt{\frac{a^2}{m^2}-b^2}$$

On eliminating m, we get

$$x^2+y^2=a^2-b^2$$

57. The tangent at (h, k) is

$$\frac{x}{4/h}-\frac{y}{3/k}=1$$

Therefore,

$$\frac{4}{h}=\frac{3}{k} \Rightarrow \frac{h}{k}=\frac{4}{3} \quad (1)$$

and $$3h^2-4k^2=12 \quad (2)$$

Since the point (h, k) lies on it, on using Eqs. (1) and (2), we get the tangent as

$$y-x=\pm 1$$

58. The line through (6, 2) is

$$y-2=m(x-6) \Rightarrow y=mx+2-6m$$

Now, from the condition of tangency, we get

$$(2-6m)^2=25m^2-16$$

$$\Rightarrow 36m^2+4-24m-25m^2+16=0$$

$$\Rightarrow 11m^2-24m+20=0$$

Obviously, its roots are m_1 and m_2. Therefore,

$$m_1+m_2=\frac{24}{11} \text{ and } m_1m_2=\frac{20}{11}$$

59. The equation of the tangent to $4y^2=x^2-1$ at (1, 0) is

$$4(y\times 0)=x\times 1-1 \text{ or } x-1=0 \text{ or } x=1$$

60. If $y=mx+c$ touches

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

then we have $c^2=a^2m^2-b^2$.

Here, $c=6$, $a^2=100$ and $b^2=49$. Therefore,

$$36=100m^2-49 \Rightarrow 100m^2=85 \Rightarrow m=\sqrt{\frac{17}{20}}$$

61. The equation of the tangent to $x^2-y^2-8x+2y+11=0$ at (2, 1) is $2x-y-4(x+2)+(y+1)+11=0$ or $x=2$.

62. See Fig. 15.13.

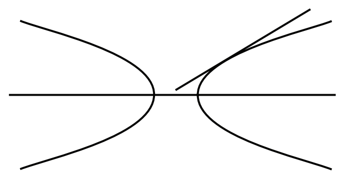


Figure 15.13

The equation of the line and hyperbola are

$$y=x-1 \quad (1)$$

$$3x^2-4y^2=12 \quad (2)$$

From Eqs. (1) and (2), we get

$$3x^2-4(x-1)^2=12$$

$$\Rightarrow 3x^2-4(x^2-2x+1)=12$$

or $$x^2-8x+16=0 \Rightarrow x=4$$

From Eq. (1), $y=3$. Therefore, the point of contact is $(4, 3)$.

63. We have

$$x\cos\alpha+y\sin\alpha=p \Rightarrow y=-\cot\alpha(x)+p\,\mathrm{cosec}\,\alpha$$

which is tangent to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Therefore,

$$p^2\mathrm{cosec}^2\alpha=a^2\cot^2\alpha-b^2$$

$$\Rightarrow a^2\cos^2\alpha-b^2\sin^2\alpha=p^2$$

64. On differentiating $x=2\sec\phi$, we get

$$\frac{dx}{d\phi}=2\sec\phi\tan\phi$$

Differentiating $y = 3\tan\phi$, we get

$$\frac{dy}{d\phi} = 3\sec^2\phi$$

Therefore, the gradient of tangent is

$$\frac{dy}{dx} = \frac{dy/d\phi}{dx/d\phi} = \frac{3\sec^2\phi}{2\sec\phi\tan\phi}$$

$$\frac{dy}{dx} = \frac{3}{2}\text{cosec}\phi \qquad (1)$$

However, the tangent is parallel to

$$3x - y + 4 = 0$$

Therefore, the gradient is

$$m = 3 \qquad (2)$$

From Eqs. (1) and (2), we get

$$\frac{3}{2}\text{cosec}\phi = 3 \Rightarrow \text{cosec}\phi = 2$$

Therefore, $\phi = 30°$.

65. The equation of director circle of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$x^2 + y^2 = a^2 - b^2$$

Therefore, the radius is $\sqrt{a^2 - b^2}$.

66. The given equation of hyperbola is $xy = a$. The slope of the tangent at point $(x_1 y_1)$ is

$$m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$$

Therefore,

$$\frac{xdy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = \frac{-y}{x}$$

At point $(a, 1)$, we get

$$m = \left(\frac{dy}{dx}\right)_{(a,1)} = -\frac{1}{a}$$

67. We know that $y = mx + c$ touches the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 - b^2$.

68. The condition for the line $y = mx + c$ to touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $c^2 = a^2m^2 - b^2$. Now, we have $m = -1$, $c = \sqrt{2}p$, $a^2 = 9$ and $b^2 = 4$. Therefore, we get $2p^2 = 5$.

69. The equation of the director circle of hyperbola is $x^2 + y^2 = a^2 - b^2$. Now, $a^2 = 16$ and $b^2 = 4$. Therefore, $x^2 + y^2$ = 12 is the required director circle.

70. The given hyperbola is

$$\frac{x^2}{3} - \frac{y^2}{2} = 1 \qquad (1)$$

The equation of tangent parallel to $y - x + 5 = 0$ is

$$y - x + \lambda = 0$$

$$\Rightarrow y = x - \lambda \qquad (2)$$

If this line [Eq. (2)] is a tangent to hyperbola [Eq. (1)], then

$$-\lambda = \pm\sqrt{3\times1-2} \qquad (\text{from } c = \pm\sqrt{a^2m^2 - b^2})$$

Therefore,

$$-\lambda = \pm1 \Rightarrow \lambda = -1, +1$$

Substituting the values of λ in Eq. (2), we get $x - y - 1 = 0$ and $x - y + 1 = 0$ which are the tangents.

71. The given ellipse is $\frac{x^2}{9} + \frac{y^2}{4} = 1$. The value of the expression $\frac{x^2}{9} + \frac{y^2}{4} - 1$ is positive for $x = 1, y = 2$ and negative for $x = 2, y = 1$. Therefore, P lies outside E and Q lies inside E. The value of the expression $x^2 + y^2 - 9$ is negative for both the points P and Q. Therefore, both P and Q lie inside C. Hence, P lies inside C, but outside E.

72. The equation of chord is $y = x\tan\theta$. The points of intersection of chord and parabola are (0, 0) and $\left(\frac{4a}{\tan^2\theta}, \frac{4a}{\tan\theta}\right)$. Hence, the length of chord is

$$4a\sqrt{\left(\frac{1}{\tan^2\theta}\right)^2 + \frac{1}{\tan^2\theta}}$$

$$= \frac{4a}{\tan\theta}\sqrt{\frac{1+\tan^2\theta}{\tan^2\theta}} = 4a\,\text{cosec}^2\theta\cos\theta$$

73. The equation of normal to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a\sec\theta, b\tan\theta)$ is

$$\frac{a^2x}{a\sec\theta} + \frac{b^2y}{b\tan\theta} = a^2 + b^2$$

74. Any normal to the hyperbola is

$$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 \qquad (1)$$

However, it is given that

$$lx + my - n = 0 \qquad (2)$$

On comparing Eqs. (1) and (2), we get

$$\sec\theta = \frac{a}{l}\left(\frac{-n}{a^2+b^2}\right) \text{ and } \tan\theta = \frac{b}{m}\left(\frac{-n}{a^2+b^2}\right)$$

Hence, on eliminating θ, we get

$$\frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2+b^2)^2}{n^2}$$

75. Applying the formula, the required normal is

$$\frac{16x}{8} + \frac{9y}{3\sqrt{3}} = 16 + 9$$

That is,

$$2x + \sqrt{3}y = 25$$

Trick: Among the given options, this is the only line (equation) on which the point $(8, 3\sqrt{3})$ is located.

76. The equation of the normal at any point (x_1, y_1) on hyperbola is

$$\frac{a^2(x-x_1)}{x_1}=\frac{b^2(y-y_1)}{-y_1}$$

Here, $a^2=267, b^2=48$ and $(x_1, y_1)=(6, 4)$. Therefore,

$$\frac{27(x-6)}{6}=-\frac{48(y-4)}{4} \Rightarrow 3(x-6)=-8(y-4)$$

$$\Rightarrow 3x+8y=50$$

77. We have

$$S \equiv 25x^2-16y^2-400=0$$

The equation of chord is

$$S_1=T \quad (1)$$

Here,

$$S_1=25(5)^2-16(3)^2-400$$

$$=625-144-400=81$$

and $$T \equiv 25xx_1-16yy_1-400$$

where $x_1=5$, $y_1=3$. Therefore

$$T \equiv 25(x)(5)-16(y)(3)-400 = 125x-48y-400$$

From Eq. (1), the required chord is

$$125x-48y-400=81$$

$$\Rightarrow 125x-48y=481$$

78. We know that the equation of the normal of the conic $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at point $(a\sec\theta, b\tan\theta)$ is

$$ax\sec\theta+by\cot\theta=a^2+b^2$$

or $$y=\frac{-a}{b}\sin\theta\, x+\frac{a^2+b^2}{b\cot\theta}$$

On comparing this equation with the equation $y=mx+\frac{25\sqrt{3}}{3}$ and taking $a=4$ and $b=3$, we get

$$\frac{a^2+b^2}{b\cot\theta}=\frac{25\sqrt{3}}{3} \Rightarrow \tan\theta=\sqrt{3} \Rightarrow \theta=60°$$

and $$m=-\frac{a}{b}\sin\theta=\frac{-4}{3}\sin 60° = \frac{-4}{3}\times\frac{\sqrt{3}}{2}=\frac{-2}{\sqrt{3}}$$

79. We have

$$\frac{x^2}{16}-\frac{y^2}{9}=1 \Rightarrow \frac{2x}{16}-\frac{2y}{9}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{2x\times 9}{16\times 2y}=\frac{9}{16}\frac{x}{y} \Rightarrow \left(\frac{-dx}{dy}\right)_{(-4,0)}=\frac{-16}{9}\frac{y}{x}=0$$

Hence, the equation of normal is

$$(y-0)=0(x+4) \Rightarrow y=0$$

80. The eccentricity of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$e=\sqrt{\frac{a^2+b^2}{a^2}}$$

The eccentricity of conjugate hyperbola is

$$e'=\sqrt{\frac{a^2+b^2}{b^2}}$$

On writing the given equation in standard form, we get

$$\frac{x^2}{1}-\frac{y^2}{1/3}=1 \Rightarrow a^2=1, b^2=\frac{1}{3}$$

Therefore,

$$e'=\sqrt{\frac{1+1/3}{1/3}}=\sqrt{4}=2$$

81. Let the hyperbola is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad (1)$$

Now, its conjugate is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1 \quad (2)$$

If e is the eccentricity of hyperbola mentioned in Eq. (1), then

$$b^2=a^2(e^2-1)$$

or $$\frac{1}{e^2}=\frac{a^2}{(a^2+b^2)} \quad (3)$$

Similarly, if e' is the eccentricity of conjugate mentioned in Eq. (2), then $a^2=b^2(e'^2-1)$ or

$$\frac{1}{e'^2}=\frac{b^2}{(a^2+b^2)} \quad (4)$$

Adding Eqs. (3) and (4), we get

$$\frac{1}{(e')^2}+\frac{1}{e^2}=\frac{a^2}{a^2+b^2}+\frac{b^2}{a^2+b^2}=1$$

82. The given equation is

$$\frac{x^2}{2}-\frac{y^2}{1}=1 \quad (1)$$

The product of the length of perpendiculars that are drawn from any point on the hyperbola mentioned in Eq. (1) to the asymptotes is

$$\frac{a^2b^2}{a^2+b^2}=\frac{2\times 1}{2+1}=\frac{2}{3}$$

83. We have

$$b=4 \Rightarrow 2ae=10 \Rightarrow 16=25-a^2 \Rightarrow a=3$$

Hence, the hyperbola is

$$16x^2-9y^2=144$$

84. Here, for the given ellipse, we have

$$a=5, b=3, b^2=a^2(1-e^2) \Rightarrow e=\frac{4}{5}$$

Therefore, the focus is (– 4, 0), (4, 0). It is given that the eccentricity of hyperbola is 2.

$$a=\frac{ae}{e}=\frac{4}{2}=2 \text{ and } b=2\sqrt{(4-1)}=2\sqrt{3}$$

Hence, the hyperbola is

$$\frac{x^2}{4}-\frac{y^2}{12}=1$$

85. We have $xy=c^2$ since $c^2=a^2/2$. Here, the coordinates of focus are

$$(ae\cos 45°,\ ae\sin 45°)\equiv(c\sqrt{2},c\sqrt{2})$$

$$(\text{Because } e=\sqrt{2},\ a=c\sqrt{2})$$

Similarly, the other focus is $(-c\sqrt{2},-c\sqrt{2})$.

Note: Students should remember this question as a fact.

86. Since it is a rectangular hyperbola, the eccentricity is $e=\sqrt{2}$.

87. On multiplying both, we get $x^2-y^2=a^2$. This is the equation of rectangular hyperbola since $a=b$.

88. We have

$$2ae=16,\ e=\sqrt{2}\ \Rightarrow\ a=4\sqrt{2}\ \text{and}\ b=4\sqrt{2}$$

Therefore, the equation is

$$\frac{x^2}{(4\sqrt{2})^2}-\frac{y^2}{(4\sqrt{2})^2}=1\Rightarrow x^2-y^2=32$$

89. The hyperbola is

$$\frac{x^2}{144}-\frac{y^2}{81}=\frac{1}{25}$$

We have

$$a=\sqrt{\frac{144}{25}},\ b=\sqrt{\frac{81}{25}},\ e_1=\sqrt{1+\frac{81}{144}}=\sqrt{\frac{225}{144}}=\frac{15}{12}=\frac{5}{4}$$

Therefore, the foci are

$$(ae_1,\ 0)=\left(\frac{12}{5}\times\frac{5}{4},\ 0\right)=(3,\ 0)$$

Hence, the focus of the ellipse is $(4e,\ 0)$, that is, $(3,\ 0)$. Therefore, $e=3/4$. Hence,

$$b^2=16\left(1-\frac{9}{16}\right)=7$$

90. The tangent at $(a\sec\theta,\ b\tan\theta)$ is

$$\frac{x}{(a/\sec\theta)}-\frac{y}{(b/\tan\theta)}=1$$

or

$$\frac{a}{\sec\theta}=1,\ \frac{b}{\tan\theta}=1$$

$$\Rightarrow a=\sec\theta,\ b=\tan\theta$$

or $(a,\ b)$ lies on $x^2-y^2=1$.

91. We have $xy=c^2$. Therefore, the rectangular hyperbola is $a^2=b^2$.

92. This is due to the reason that the eccentricity of the rectangular hyperbola is $\sqrt{2}$.

93. Here, $a=b$. Thus, it is a rectangular hyperbola. Hence, the eccentricity is $e=\sqrt{2}$.

94. The hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Here, the transverse and the conjugate axis of a hyperbola is equal. That is, $a=b$. Therefore, $x^2-y^2=a^2$, which is a rectangular hyperbola. Hence, the eccentricity is

$$e=\sqrt{1+\frac{b^2}{a^2}}=\sqrt{2}$$

95. Since the general equation of the second degree represents a rectangular hyperbola, if $\Delta\neq 0,\ h^2>ab$ and coefficient of x^2+ coefficient of $y^2=0$, the given equation represents a rectangular hyperbola if $\lambda+5=0$, that is, $\lambda=-5$.

96. Obviously, $e=2/3$ and $e'=3/2$. Therefore, $ee'=1$.

97. The distance between the directrices is $2a/e$. Since the eccentricity of rectangular hyperbola is $\sqrt{2}$, the distance between the directrices is $2a/\sqrt{2}$. It is given that

$$\frac{2a}{\sqrt{2}}=10\ \Rightarrow\ 2a=10\sqrt{2}$$

Now, the distance between the foci is

$$2ae=(10\sqrt{2})(\sqrt{2})=20$$

98. Eccentricity of hyperbola is

$$e=\sqrt{1+\frac{b^2}{a^2}}=\sqrt{1+\frac{a^2}{a^2}}$$

$$\Rightarrow e=\sqrt{2}$$

99. Eccentricity of rectangular hyperbola is $\sqrt{2}$.

100. It is given that the equation of hyperbola is

$$x^2-3y^2=2x+8$$

$$\Rightarrow\ x^2-2x-3y^2=8$$

$$\Rightarrow\ (x-1)^2-3y^2=9\ \Rightarrow\ \frac{(x-1)^2}{9}-\frac{y^2}{3}=1$$

The conjugate of this hyperbola is

$$-\frac{(x-1)^2}{9}+\frac{y^2}{3}=1$$

and its eccentricity is

$$e=\sqrt{\left(\frac{a^2+b^2}{b^2}\right)}$$

Here, we have $a^2=9,\ b^2=3$. Therefore,

$$e=\sqrt{\frac{9+3}{3}}=2$$

101. If $y=mx+c$ is the tangent to the hyperbola, then $c^2=a^2m^2-b^2$. Here, $\beta^2=a^2\alpha^2-b^2$ Hence, the locus of $P(\alpha,\beta)$ is $a^2x^2-y^2=b^2$, which is a hyperbola.

102. The equation of hyperbola is

$$\frac{x^2}{16}-\frac{y^2}{25}=1$$

The eccentricity is

$$e^2=\frac{b^2}{a^2}+1$$

That is,

$$e^2=\frac{25}{16}+1$$

$$\Rightarrow\ e^2=\frac{41}{16}\ \Rightarrow\ e=\frac{\sqrt{41}}{4}$$

103. The distance between foci is 8. Therefore, $2ae = 8$. Also

$$2a = 4$$

$$\Rightarrow a = 2 \Rightarrow a^2 = 4$$

Therefore,

$$b^2 = 4(4-1) = 12$$

Therefore, the equation of hyperbola is

$$\frac{x^2}{4} - \frac{y^2}{12} = 1$$

104. On solving equations $x^2 + y^2 = 5$ and $y^2 = 4x$, we get

$$x^2 + 4x - 5 = 0$$

That is, $x = 1, -5$

For $x = 1$, we get

$$y^2 = 4 \Rightarrow y = \pm 2$$

For $x = -5$, we get

$$y^2 = -20$$

which is of imaginary value. Thus, the points are (1, 2) and (1, –2). Now, m_1 for $x^2 + y^2 = 5$ at (1, 2) is

$$\frac{dy}{dx} = -\frac{x}{y}\bigg|_{(1,2)} = -\frac{1}{2}$$

Similarly, m_2 for $y^2 = 4x$ at (1, 2) is 1. Therefore,

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{-(1/2) - 1}{1 - (1/2)}\right| = 3$$

105. We have $2b^2/a^2 = 9$. Therefore,

$$2b^2 = 9a \quad (1)$$

Now, $$b^2 = a^2(e^2 - 1) = \frac{9}{16}a^2$$

$$\Rightarrow a = \left(\frac{4}{3}\right)b \qquad \left(\because e = \frac{5}{4}\right) \quad (2)$$

From Eqs. (1) and (2), we get $b = 6$ and $a = 8$. Hence, the equation of hyperbola is

$$\frac{x^2}{64} - \frac{y^2}{36} = 1$$

106. Slope of the first curve is

$$\left(\frac{dy}{dx}\right)_{\text{I}} = -\frac{4x}{py}$$

Slope of the second curve is

$$\left(\frac{dy}{dx}\right)_{\text{II}} = \frac{x}{4y}$$

For the orthogonal intersection, we have

$$\left(-\frac{4x}{py}\right)\left(\frac{x}{4y}\right) = -1 \Rightarrow x^2 = py^2$$

On solving the equations of the given curves, we get $x = 3$ and $y = 1$. Therefore,

$$p(1) = (3)^2 = 9 \Rightarrow p = 9$$

107. We have

$$\frac{x^2}{3} - \frac{y^2}{2} = 1$$

Since the equation of tangent are equally inclined to the axis, we get $\tan\theta = 1 = m$. Thus, the equation of tangent is

$$y = mx + \sqrt{a^2m^2 - b^2}$$

The given equation is

$$\frac{x^2}{3} - \frac{y^2}{2} = 1$$

which is an equation of hyperbola of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Now, on comparing $a^2 = 3$ and $b^2 = 2$, we get

$$y = 1(x) + \sqrt{3 \times (1)^2 - 2} \Rightarrow y = x + 1$$

Practice Exercise 2

1. For the ellipse: $e = \sqrt{\frac{25-9}{25}} = \frac{4}{5}$

Therefore, foci are (–4, 0) and (4, 0).

For the hyperbola,

$$ae = 4, e = 2$$

Therefore, $a = 2$

Now,

$$b^2 = 4\,(4 - 1) = 12$$

$$\Rightarrow b = \sqrt{12}$$

Now,

$$a^2 + b^2 = 4 + 12 = 16$$

Length of latus rectum $= \frac{2b^2}{a}$

$$= \frac{2(\sqrt{12})^2}{2}$$

$$= 12$$

2. Equation of the tangent

$$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1 \quad (1)$$

Comparing Eq. (1) with the equation of the tangent, we have

$$y = mx \pm \sqrt{a^2m^2 - b^2} \quad (2)$$

$$\Rightarrow \frac{b}{a}\,\frac{\sec\theta}{\tan\theta} = m$$

$$\Rightarrow \frac{b}{a\sin\theta} = m$$

$$\Rightarrow \sin\theta = \frac{b}{ma}$$

$$\Rightarrow \theta = \sin^{-1}\left(\frac{b}{ma}\right) \text{ and } \pi + \sin^{-1}\left(\frac{b}{ma}\right), m > 0$$

3. Given hyperbola can be written as

$$\frac{(x-1)^2}{16} - \frac{(y-1)^2}{9} = 1$$

$$\Rightarrow \frac{X^2}{16} - \frac{Y^2}{9} = 1 \qquad \{\text{where } X = x-1, Y = y-1\}$$

$$e = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{1+\frac{9}{16}} = \frac{5}{4}$$

And the directrices are

$$X = \pm\frac{a}{e}$$

$$\Rightarrow x-1 = \pm\frac{16}{5}$$

$$\Rightarrow x = \frac{21}{5} \text{ and } x = -\frac{11}{5}$$

Length of the latus rectum $= \frac{2b^2}{a} = \frac{9}{2}$

and foci are

$$X = \pm ae, Y = 0$$
$$\Rightarrow (6, 1) \text{ and } (-4, 1)$$

4. Equation of chord joining θ and ϕ

$$\frac{x}{a}\cos\frac{\theta-\phi}{2} - \frac{y}{b}\sin\frac{\theta+\phi}{2} = \cos\frac{\theta+\phi}{2}$$

It passes through $(ae, 0)$.

Therefore,

$$e\cos\frac{\theta-\phi}{2} = \cos\frac{\theta+\phi}{2}$$

$$\Rightarrow \frac{\cos\frac{\theta-\phi}{2}}{\cos\frac{\theta+\phi}{2}} = \frac{1}{e}$$

$$\Rightarrow \frac{\cos\frac{\theta-\phi}{2}-\cos\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}+\cos\frac{\theta+\phi}{2}} = \frac{1-e}{1+e}$$

$$\Rightarrow \frac{2\sin\frac{\theta}{2}\sin\frac{\phi}{2}}{2\cos\frac{\theta}{2}\cos\frac{\phi}{2}} = \frac{1-e}{1+e}$$

$$\Rightarrow \tan\frac{\theta}{2}\tan\frac{\phi}{2} = \frac{1-e}{1+e}$$

Since the chord may also pass through $(-ae, 0)$.
Similarly as above, we get

$$\tan\frac{\theta}{2}\tan\frac{\phi}{2} = \frac{1+e}{1-e}$$

5. $2a = 3$

Distance between the foci (1, 2) and (5, 5) is 5.

$$2ae = 5$$

Therefore, $$e = \frac{5}{3}$$

$$\frac{1}{e^2} + \frac{1}{e'^2} = 1 \Rightarrow e' = \frac{5}{4}$$

6. Director circle $(x-h)^2 + (y-k)^2 = a^2 - b^2$, where (h, k) is the centre. So, the centre is

$$\left(\frac{1+5}{2}, \frac{2+5}{2}\right) \Rightarrow \left(3, \frac{7}{2}\right)$$

Now,

$$b^2 = a^2(e^2-1) = \left(\frac{3}{2}\right)^2\left(\left(\frac{5}{3}\right)^2 - 1\right) = 4$$

Director circle,

$$(x-3)^2 + \left(y-\frac{7}{2}\right)^2 = \frac{9}{4} - 4$$

$$\Rightarrow (x-3)^2 + \left(y-\frac{7}{2}\right)^2 = -\frac{7}{4}$$

This does not represent any real point.

7. Slope of transverse axis is $\frac{3}{4}$.

Therefore, angle of rotation $= \theta = \tan^{-1}\frac{3}{4}$

8. Equation of normal at $P(a\sec\theta, b\tan\theta)$ is

$$ax\cos\theta + by\cot\theta = a^2 + b^2$$

Therefore,

$$G\left(\frac{a^2+b^2}{a}\sec\theta, 0\right), g\left(0, \frac{a^2+b^2}{b}\tan\theta\right)$$

Equation of CF is $bx\cot\theta - ay\cos\theta = 0$. Therefore,

$$PF = \frac{ab}{\sqrt{b^2\sec^2\phi + a^2\tan^2\phi}} \text{ and } PG^2 = \frac{b^2}{a^2}(b^2\sec^2\phi + a^2\tan^2\phi)$$

Therefore,

$$PF \times PG = b^2$$

Comparing the above equation with the given equation, we have

$$K = 1$$

9. $$Pg^2 = \frac{a^2}{b^2}(b^2\sec^2\theta + a^2\tan^2\theta)$$

$$\Rightarrow PF \times Pg = a^2 = CA^2$$

10. Locus of middle point is

$$\frac{x^2}{\frac{a^2e^4}{4}} - \frac{y^2}{\frac{a^4e^4}{4b^2}} = 1$$

Therefore,

$$e_1 = \sqrt{\frac{\frac{a^2e^4}{4}+\frac{a^4e^4}{4b^2}}{\frac{a^2e^4}{4}}} = \frac{e}{\sqrt{e^2-1}}$$

11. Let the centre of the rectangular hyperbola (H) be $P(h, k)$. Then the centroid of the quadrilateral can be given by $G\left(\frac{h+0}{2}, \frac{k+0}{2}\right)$.

{G is same as the midpoint of the centres of a circle and a rectangular hyperbola (H)}

Now, $G\left(\frac{h}{2}, \frac{k}{2}\right)$ lies on $3x - 4y + 1 = 0$. Therefore,

$$\frac{3h}{2} - \frac{4k}{2} + 1 = 0$$

$$\Rightarrow 3h - 4k + 2 = 0 \Rightarrow 3x - 4y + 2 = 0$$

12. Let the centre of the circle and the hyperbola are (α, β) and (h,k), respectively, and points are $A(x_1,y_1)$, $B(x_2,y_2)$, $C(x_3,y_3)$ and $D(x_4, y_4)$. Then

$$\frac{h+\alpha}{2} = \frac{x_1+x_2+x_3+x_4}{4} \quad (1)$$

and $$\frac{k+\beta}{2} = \frac{y_1+y_2+y_3+y_4}{4} \quad (2)$$

As any chord passing through the centre of the hyperbola is bisected at the centre, AB is bisected at (h, k).

$$\frac{x_1+x_2}{2} = h \quad (3)$$

and $$\frac{y_1+y_2}{2} = k \quad (4)$$

From Eqs. (1) and (3), we have

$$\frac{x_1+x_2}{2} + \alpha = \frac{x_1+x_2+x_3+x_4}{2} \Rightarrow \alpha = \frac{x_3+x_4}{2}$$

From Eqs. (2) and (4), we have

$$\beta = \frac{y_3+y_4}{2}$$

Now, (α, β) is the mid-point of CD and (α, β) lies on CD. Hence, centre of the circle lies on CD.

13. Let the four concyclic points at which normals to rectangular hyperbola are concurrent are $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and $D(x_4, y_4)$ and the centre of the circle be (h, k). Therefore,

$$\frac{x_1+x_2+x_3+x_4}{4} = \frac{h+0}{2}$$

and $$\frac{y_1+y_2+y_3+y_4}{4} = \frac{k+0}{2}$$

$$\Rightarrow x_1+x_2+x_3+x_4 = 2h \quad (1)$$

and $$y_1+y_2+y_3+y_4 = 2k \quad (2)$$

Normal to the rectangular hyperbola $xy = c^2$ at $\left(ct, \frac{c}{t}\right)$ is

$$ct^4 - xt^3 + yt - c = 0$$

As all normal pass through (α, β), therefore,

$$ct^4 - \alpha t^3 + \beta t - c = 0$$

$$\Rightarrow x_1+x_2+x_3+x_4 = c(t_1+t_2+t_3+t_4) = c\left(\frac{\alpha}{c}\right) = \alpha \quad (3)$$

and

$$y_1+y_2+y_3+y_4 = c\left(\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}+\frac{1}{t_4}\right) = c\left(\frac{\Sigma t_1t_2t_3}{t_1t_2t_3t_4}\right)$$

$$= c\left(\frac{-\beta | c}{-c | c}\right) = \beta \quad (4)$$

From Eqs. (1) and (3), we have

$$2h = \alpha$$

From Eqs. (2) and (4), we have

$$2k = \beta$$

Therefore,

$$(h, k) = \left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$$

14. (A) Common tangent $y = x + 2\sqrt{2}$.

Points of contact with two hyperbola are $\left(\frac{-9}{2\sqrt{2}}, \frac{-1}{2\sqrt{2}}\right)$ and $\left(\frac{1}{2\sqrt{2}}, \frac{9}{2\sqrt{2}}\right)$

Hence, length = 5.

(B) For hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, s is $(ae, 0)$ equation of line

$$y = \frac{b}{a}(x - ae)$$

and point

$$P\left(\frac{a^2(1+e^2)}{2ae}, \frac{-ab(e^2-1)}{2ae}\right)$$

Hence,

$$SP = \frac{b^2}{2a} = \frac{9}{8}$$

(C) Equation of the tangent is $\frac{x\cos\theta}{2\sqrt{3}} + \frac{y\sin\theta}{18} = 1.$

Sum of the intercept is

$$2\sqrt{3}\sec\theta + 6\operatorname{cosec}\theta = f(\theta) \text{ (say)}$$

$$\Rightarrow f'(\theta) = \frac{2\sqrt{3}\sin^3\theta - 18\cos^3\theta}{\sin^2\theta\cos^2\theta} = 0 \Rightarrow \tan^3\theta = 3\sqrt{3}$$

$$\Rightarrow \tan\theta = \sqrt{3}$$

So, by symmetry in the second quadrant, $\tan\theta = -\sqrt{3}$.

(D) Equation of the tangent at (x_1, y_1) is $xx_1 - 2yy_1 = 4$ which is same as $2x + \sqrt{6}y = 2$. Hence,

$$\frac{x_1}{2} = \frac{-2y_1}{\sqrt{6}} = \frac{4}{2} \Rightarrow x_1 = 4,\ y_1 = -\sqrt{6}$$

Therefore,

$$\text{Slope } (m) = -\frac{\sqrt{6}}{4} \Rightarrow \sqrt{8}m = -\sqrt{3}$$

15. (A) Equation of the tangent at $(a, 0)$ is $x = a$.

Equation of asymptote $y = \frac{b}{a}x$

Therefore,

$$\text{Area} = a \cdot b = ab = 4 \times 3 = 12$$

(B) $y = 3x + \lambda$ touches $9x^2 - 5y^2 = 45$

Therefore,

$$9x^2 - 5(3x+\lambda)^2 = 45$$
$$\Rightarrow -36x^2 - 30\lambda x - 5\lambda^2 - 45 = 0$$

That is, $36x^2 + 30\lambda x + 5\lambda^2 + 45 = 0$ has equal roots. So,

$$900\lambda^2 - 720\lambda^2 - 180\times 36 = 0$$
$$\Rightarrow \lambda^2 = 36$$
$$\Rightarrow \lambda = \pm 6$$
$$\Rightarrow |\lambda| = 6$$

(C) $$18x^2 - 16y^2 - 288 = 0$$
$$\Rightarrow 9x^2 - 8y^2 - 144 = 0$$
$$\Rightarrow 9x^2 - 8y^2 - 144\left(\frac{x\cos\alpha + y\sin\alpha}{p}\right)^2 = 0$$

Since, these lines are perpendicular to each other, therefore,

$$9p^2 - 8p^2 - 144(\cos^2\alpha + \sin^2\alpha) = 0$$
$$\Rightarrow p^2 = 144 \Rightarrow p = \pm 12$$

Therefore, radius of the circle = 12

And diameter of the circle = 24

(D) $16x^2 + 32x + 16 - 9(y^2 - 4y + 4) - 144 = 0$
$$\Rightarrow 16(x+1)^2 - 9(y-2)^2 = 144$$
$$\Rightarrow \frac{(x+1)^2}{9} - \frac{(y-2)^2}{16} = 1$$

Now, length of latus rectum $= \frac{2\times 16}{3} = \frac{32}{3}$

Therefore,
$$3\lambda = 32$$

16. (A) The point $P\left(\frac{\pi}{6}\right)$ is $\left(a\sec\frac{\pi}{6}, b\tan\frac{\pi}{6}\right)$

That is, $$P\left(\frac{2a}{\sqrt{3}}, \frac{b}{\sqrt{3}}\right)$$

Therefore, equation of the tangent at P is $\dfrac{x}{\frac{\sqrt{3}a}{2}} - \dfrac{y}{\sqrt{3}b} = 1$

Therefore, area of the triangle $= \frac{1}{2}\times\frac{\sqrt{3}a}{2}\times\sqrt{3}b = 3a^2$

$$\Rightarrow \frac{b}{a} = 4$$

Therefore, $e_2 = 1 + \frac{b^2}{a^2} = 17.$

(B) Eccentricity of the hyperbola $x^2 - y^2\sec^2\theta = 5$ is

$$e_1 = \sqrt{1+\cos^2\theta}$$

Eccentricity of the ellipse $x^2\sec^2\theta + y^2 = 25$ is

$$e_2 = \sqrt{\frac{\sec^2\theta - 1}{\sec^2\theta}} = |\sin\theta|$$

Now,

$$e_1 = \sqrt{3}\,e_2 \Rightarrow 1 + \cos^2\theta = 3\sin^2\theta \Rightarrow \cos\theta = \pm\frac{1}{\sqrt{2}}$$

Therefore, least positive value of θ is $\frac{\pi}{4}$.

Hence, $p = 24$.

(C) Asymptotes are $x = \pm\sqrt{3}\,y$

Therefore, angle between the asymptotes is $\frac{\pi}{3}$.

Hence, $\ell = 8$.

(D) Any point of $xy = 8$ is $P\left(\sqrt{8}t, \frac{\sqrt{8}}{t}\right)$. Therefore, equation of the tangent at P is

$$\frac{x}{\frac{16t}{\sqrt{8}}} + \frac{y}{\frac{16}{\sqrt{8}t}} = 1$$

So, area of the triangle $= \frac{1}{2}\cdot\frac{16t}{\sqrt{8}}\cdot\frac{16}{\sqrt{8}t} = 16$

17. (A) Since $3x^2 - 5xy - 2y^2 + 5x + 11y + c = 0$ are asymptotes, it represents a pair of straight lines. Therefore,

$$3(-2)c + 2\times\frac{11}{2}\left(\frac{5}{2}\right)\left(\frac{-5}{2}\right) - 3\left(\frac{11}{2}\right)^2 - (-2)\left(\frac{5}{2}\right)^2 - c\left(-\frac{5}{2}\right)^2 = 0$$

$$\Rightarrow -6c - \frac{275}{4} - \frac{363}{4} + \frac{25}{2} - \frac{25}{4}c = 0$$
$$\Rightarrow -24c - 275 - 363 + 50 - 25c = 0$$
$$\Rightarrow 49c = -588$$
$$\Rightarrow c = -12$$

(B) Let the point be (h, k). Then equation of the chord of contact is $hx + ky = 4$.

Since $hx + ky = 4$ is tangent to $xy = 1$,

$$x\left(\frac{4-hx}{k}\right) = 1 \text{ has two equal roots}$$
$$\Rightarrow hx^2 - 4x + k = 0$$
$$\Rightarrow hk = 4$$

Therefore, locus of (h, k) is $xy = 4$, that is,

$$c^2 = 4$$

(C) Equation of the hyperbola is $\dfrac{x^2}{c/a} - \dfrac{y^2}{c/b} = 1$

Eccentricity, $$e = \sqrt{\frac{a+b}{b}}$$

Therefore,

$$\sqrt{\frac{c}{b}} = \frac{5}{2} \text{ and } \frac{13}{2} = \sqrt{\frac{c}{a}}\cdot\sqrt{\frac{a+b}{b}}$$

$$\Rightarrow \frac{13}{2} = \frac{5}{2}\sqrt{1+\frac{b}{a}} \Rightarrow \frac{b}{a} = \frac{144}{25}$$

Therefore, $\frac{c}{a} = 36.$

So, the hyperbola is

$$25x^2 - 144y^2 = 900$$
$$\Rightarrow a = 25, b = 144, c = 900$$

Therefore, $\frac{ab}{c} = 4.$

(D) Let the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Then $2a = ae$, that is $e = 2$.

Therefore,

$$\frac{b^2}{a^2} = e_2 - 1 = 3$$

$$\Rightarrow \frac{(2b)^2}{(2a)^2} = 3$$

18. See Fig. 15.14. For any line in case of hyperbola, we know that $PQ = P'Q'$. Therefore,

$$\frac{PQ}{P'Q'} + \frac{PQ'}{P'Q} = 1 + \frac{PP' + P'Q'}{PP' + PQ} = 2$$

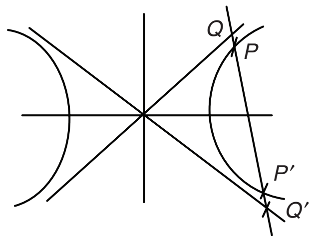


Figure 15.14

19. Let (h, k) be the mid-point of the chord of the circle $x^2 + y^2 = 4$
So, by mid-point form, equation is $T = S_1$
Now,

$$hx + ky = h^2 + k^2 \text{ or } y = -\frac{h}{k}x + \frac{h^2+k^2}{k} \Rightarrow y = mx + c$$

It will touch the hyperbola if

$$c^2 = a^2m^2 - b^2$$

$$\Rightarrow \left(\frac{h^2+k^2}{k}\right)^2 = 4\left(\frac{-h}{k}\right)^2 - 16 \Rightarrow (x^2+y^2)^2 = 4x^2 - 16y^2$$

Comparing the above equation with the given equation $(x^2 + y^2)^2 = \lambda x^2 - 16y^2$, we have

$$\lambda = 4$$

20. Since $\frac{e}{2}$ and $\frac{e'}{2}$ are eccentricities of a hyperbola and its conjugate, therefore,

$$\frac{4}{e^2} + \frac{4}{e'^2} = 1$$

$$\Rightarrow 4 = \frac{e^2e'^2}{e'^2 + e'^2}$$

Equation of variable line is

$$\frac{x}{e} + \frac{y}{e'} = 1$$

$$\Rightarrow e'x + ey - ee' = 0$$

It is tangent to the circle $x^2 + y^2 = r^2$. Therefore,

$$\frac{ee'}{\sqrt{e^2 + e'^2}} = r$$

$$\Rightarrow r^2 = \frac{e^2e'^2}{e^2 + e'^2} = 4$$

$$\Rightarrow r = 2$$

Solved JEE 2017 Questions

JEE Main 2017

1. A hyperbola passes through the point $P(\sqrt{2},\sqrt{3})$ and has foci at (±2, 0). Then the tangent to this hyperbola at P also passes through the point:

(A) $(2\sqrt{2},3\sqrt{3})$ **(B)** $(\sqrt{3},\sqrt{2})$

(C) $(-\sqrt{2},-\sqrt{3})$ **(D)** $(3\sqrt{2},2\sqrt{3})$

(OFFLINE)

Solution: Equation of hyperbola is given by

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad (1)$$

Foci is (±2, 0); hence,

$$ae = 2$$
$$a^2e^2 = 4$$
$$b^2 = a^2(e^2 - 1)$$

Therefore,

$$a^2 + b^2 = 4 \quad (2)$$

Hyperbola passes through $(\sqrt{2},\sqrt{3})$. Therefore,

$$\frac{2}{a^2}-\frac{3}{9^2}=1 \quad (3)$$

On solving Eqs. (2) and (3), we get

$a^2 = 8$ (which is rejected) and $a^2 = 1$;
$b^2 = 3$

Now, from Eq. (1), we get the equation of hyperbola as

$$\frac{x^2}{1}-\frac{y^2}{3}=1$$

Therefore, the equation of tangent is

$$\frac{\sqrt{2}x}{1}-\frac{\sqrt{3}y}{3}=1$$

That is, this tangent to the hyperbola at P passes through the point $(2\sqrt{2},3\sqrt{3})$.
Hence, the correct answer is option (A).

2. The locus of the point of intersection of the straight lines, $tx - 2y - 3t = 0$; $x - 2ty + 3 = 0$ ($t \in \mathbb{R}$), is

(A) a hyperbola with the length of conjugate axis 3.

(B) an ellipse with eccentricity $\frac{2}{\sqrt{5}}$.

(C) an ellipse with the length of major axis 6.

(D) a hyperbola with eccentricity $\sqrt{5}$.

(ONLINE)

Solution: The two given straight lines are

$$tx - 2y - 3t = 0 \quad (1)$$
$$x - 2ty + 3 = 0 \quad (2)$$

Multiplying Eq. (2) by t and subtract from Eq. (1), we get

$$(tx - 2y - 3t) - (tx - 2t^2y + 3t) = 0$$
$$\Rightarrow tx-2y-3t-tx+2t^2y-3t=0$$
$$\Rightarrow y(2t^2 - 2) - 6t = 0$$
$$y=\frac{6t}{2t^2-2}\Rightarrow\frac{6t}{t^2-1}=2y \quad (3)$$

Substituting $t = \tan\theta$, we get the locus of the point of intersection of the two given straight lines as

$$2y=\frac{3\times 2\tan\theta}{\tan^2\theta-1}\Rightarrow 2y=3(-\tan 2\theta) \quad (4)$$

$$\left(\text{since } \tan 2x=\frac{2\tan x}{1-\tan^2 x}\right)$$

Multiplying Eq. (1) by t and subtracting Eq. (2) from it, we get

$$(t^2x - 2ty - 3t^2) - (x - 2ty + 3) = 0$$
$$\Rightarrow t^2x-2ty-3t^2-x+2ty-3=0$$
$$\Rightarrow x(t^2 - 1) - 3(t^2 + 1) = 0$$
$$\Rightarrow x=\frac{3(t^2+1)}{t^2-1}$$

Substituting $t = \tan\theta$, we get

$$x=\frac{3(\tan^2\theta+1)}{\tan^2\theta-1}$$
$$\Rightarrow x=\frac{3}{-\cos 2\theta}$$
$$\Rightarrow x=-3\sec 2\theta \quad (5)$$

Using the identity $\sec^2\theta-\tan^2\theta=1$; substituting the value of $\sec 2\theta$ and $\tan 2\theta$ from Eqs. (4) and (5), we get

$$\left(\frac{-x}{3}\right)^2-\left(\frac{-2y}{3}\right)^2=1$$
$$\Rightarrow\frac{x^2}{9}-\frac{4y^2}{9}=1$$
$$\Rightarrow\frac{x^2}{9}-\frac{y^2}{9/4}=1$$

On comparing with $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we get $a = 3$, $b=\frac{3}{2}$ and the eccentricity is

$$e^2=1+\frac{b^2}{a^2}=1+\frac{9/4}{9}=1+\frac{1}{4}=\frac{5}{4}$$
$$\Rightarrow e=\frac{\sqrt{5}}{2}$$

Hence, the correct answer is option (A).

JEE Advanced 2017

1. If $2x - y + 1 = 0$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{16} = 1$, then which of the following CANNOT be sides of a right-angled triangle?

(A) a, 4, 1 **(B)** a, 4, 2
(C) $2a$, 8, 1 **(D)** $2a$, 4, 1

Solution: Line $y = mx + c$ is the tangent to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if following condition is satisfied:

$$c^2 = a^2m^2 - b^2 \quad (1)$$

It is given that the equation of line is

$$2x - y + 1 = 0$$

and hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{16} = 1$

Let us consider the equation

$$2x - y + 1 = 0$$

Therefore,

$$y = 2x + 1 \Rightarrow m = 2 \text{ and } c = 1$$

From the equation of hyperbola, we get $b^2 = 16$ and a^2 as follows:
Substituting all values in Eq. (1), we get

$$1^2 = a^2(2)^2 - 16$$

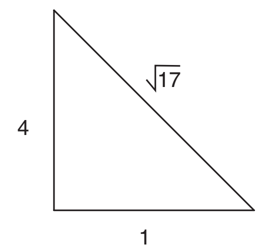
$$\Rightarrow 1 = 4a^2 - 16$$

$$\Rightarrow 4a^2 = 16 + 1 \Rightarrow 4a^2 = 17$$

$$\Rightarrow a^2 = \frac{17}{4} \Rightarrow a = \pm\sqrt{\frac{17}{4}} = \pm\frac{\sqrt{17}}{2}$$

$$\Rightarrow a = \pm\frac{\sqrt{17}}{2}$$

- **For option (A):** a, 4, 1 $\Rightarrow \pm\frac{\sqrt{17}}{2}, 4, 1$ is not a right-angled triangle $\left(\text{since } \sqrt{4^2+1^2} \neq \frac{\sqrt{17}}{2}\right)$
- **For option (B):** a, 4, 2 $\Rightarrow \pm\frac{\sqrt{17}}{2}, 4, 2$ is not a right-angled triangle $\left(\text{since } \sqrt{4^2+2^2} \neq \frac{\sqrt{17}}{2}\right)$
- **For option (C):** $2a$, 8, 1 $\Rightarrow \pm\sqrt{17}, 8, 1$ is not a right-angled triangle (since $\sqrt{8^2+1^2} \neq \sqrt{17}$)
- **For option (D):** $2a$, 4, 1 $\Rightarrow \pm\sqrt{17}, 4, 1$ is a right-angled triangle, therefore $\sqrt{4^2+1^2} = \sqrt{16+1} = \sqrt{17}$.



Hence, the correct answers are options (A), (B) and (C).

Directions for Questions 2–4: Answer the questions by appropriately matching the information given in the three columns of the following table:

Columns 1, 2 and 3 contain conics, equations of tangents to the conics and points of contact, respectively.

Column 1	Column 2	Column 3
(I) $x^2 + y^2 = a^2$	(i) $my = m^2x + a$	(P) $x\left(\frac{a}{m^2}, \frac{2a}{m}\right)$
(II) $x^2 + a^2y^2 = a^2$	(ii) $y = mx + a\sqrt{m^2+1}$	(Q) $\left(\frac{-ma}{\sqrt{m^2+1}}, \frac{a}{\sqrt{m^2+1}}\right)$
(III) $y^2 = 4ax$	(iii) $y = mx + \sqrt{a^2m^2-1}$	(R) $\left(\frac{-a^2m}{\sqrt{a^2m^2+1}}, \frac{1}{\sqrt{a^2m^2+1}}\right)$
(IV) $x^2 - a^2y^2 = a^2$	(iv) $y = mx + \sqrt{a^2m^2+1}$	(S) $\left(\frac{-a^2m}{\sqrt{a^2m^2-1}}, \frac{-1}{\sqrt{a^2m^2-1}}\right)$

2. For $a = \sqrt{2}$, if a tangent is drawn to a suitable conic (Column 1) at the point of contact (–1, 1), then which of the following options is the only CORRECT combination for obtaining its equation?

(A) (I) (i) (P) **(B)** (I) (ii) (Q)
(C) (II) (ii) (Q) **(D)** (III) (i) (P)

Solution: We discuss the combinations as follows:

- For the given equation $a = \sqrt{2}$ and the point of contact (–1, 1), the conic equation $x^2 + y^2 = a^2$ satisfies these conditions [i.e. (I) of Column 1].
- Thus, the equation of tangent is $-x + y = 2$ that is, $y = x + 2$; thus, (ii) of Column 2 satisfies this condition with $m = 1$. That is, $y = mx + a\sqrt{m^2+1}$.
- It is given that the point of constant (–1, 1); given $m = 1$ and $a = \sqrt{2}$, we see that (Q) of Column 3 $\left[\text{i.e. } \left(\frac{-ma}{\sqrt{m^2+1}}, \frac{a}{\sqrt{m^2+1}}\right)\right]$ gives the point of contact (–1, 1).

Thus, the correct combination is (I) (ii) (Q).

Hence, the correct answer is option (B).

3. If a tangent to a suitable conic (Column 1) is found to be $y = x + 8$ and its point of contact is (8, 16), then which of the following options is the only CORRECT combination?

(A) (I) (ii) (Q) **(B)** (II) (iv) (R)
(C) (III) (i) (P) **(D)** (III) (ii) (Q)

Solution: We discuss the combinations as follows:

- It is given that the tangent to the conic is $y = x + 8$ and the point of contact is (8, 16).
- On comparing equation $y = x + 8$ with the options provided in Column 2 containing equations of tangents, we get $m = 1$.
- Comparing $y = x + 8$ with option (i) of Column (2), we get $a = 8$.
- The given point of contact (8, 16) and $a = 8$, we see that (III) of Column 1 (i.e. $y^2 = 4ax$) is satisfied with these conditions.
- It is given that $a = 8$ and $m = 1$, $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ gives (8, 16) as point of contact.

Thus, the correct combination is (III) (i) (P).
Hence, the correct answer is option (C).

4. The tangent to a suitable conic (Column 1) at $\left(\sqrt{3}, \frac{1}{2}\right)$ is found to be $\sqrt{3}x + 2y = 4$, then which of the following options is the only CORRECT combination?

(A) (IV) (iii) (S) **(B)** (IV) (iv) (S)
(C) (II) (iii) (R) **(D)** (II) (iv) (R)

Solution: It is given that the tangent to conic is

$$\sqrt{3}x + 2y = 4 \tag{1}$$

and the point of contact is $\left(\sqrt{3}, \frac{1}{2}\right)$. Therefore, from Eq. (1), we get

$$2y = 4 - \sqrt{3}x$$

$$\Rightarrow y = \frac{-\sqrt{3}}{2}x + 2 \tag{2}$$

On comparing this equation with the options provided in Column 2, we get $m = \frac{-\sqrt{3}}{2}$.

Equation (1) can also be written as

$$\sqrt{3}x + 4\left(\frac{1}{2}\right)y = 4$$

On comparing with option (II), we get

$$a^2 = 4 \Rightarrow a = 2$$

Now, for $a = 2$, $m = \frac{-\sqrt{3}}{2}$, $\left(\frac{-a^2 m}{\sqrt{a^2m^2+1}}, \frac{1}{\sqrt{a^2m^2+1}}\right)$ gives $\left(\sqrt{3}, \frac{1}{2}\right)$ as the point of contact.

Thus, the correct combination is (II) (iv) (R).

Hence, the correct answer is option (D).

16 Statistics

16.1 Frequency Distribution

In statistics, **frequency distribution** is an arrangement of the values that one or more variables take in a sample. Each entry in the table contains the frequency or count of the occurrences of values within a particular group or interval, and in this way, the table summarizes the distribution of values in the sample.

Your Turn 1

1. The following data gives the number of children in 30 families in a village:
2, 3, 0, 1, 2, 4, 3, 0, 1, 2, 1, 3, 0, 2, 2, 3, 1, 1, 1, 0, 2, 4, 1, 2, 3, 2, 1, 2, 2, 1
Represent the data in the form of a frequency distribution.

Ans.

No. of children	No. of families
0	4
1	9
2	10
3	5
4	2

2. Following are the ages of 360 patients getting medical treatment in a hospital on a day:

Age (in years)	10–20	20–30	30–40	40–50	50–60	60–70
No. of patients	90	50	60	80	50	30

Construct the cumulative frequency distribution table.

Ans.

Age	No. of patients	Cumulative frequency
10–20	90	90
20–30	50	140
30–40	60	200
40–50	80	280
50–60	50	330
60–70	30	360

3. Following are marks of 75 students in a test:

Marks (up to)	30	40	50	60	70	80
No. of students	0	12	30	47	60	75

Form a frequency distribution from the data. How many students are getting more than 60 marks?

Ans.

Age	Frequency
30–40	12
40–50	18
50–60	17
60–70	13
70–80	15

There are 28 students getting more than 60 marks.

16.2 Measure of Central Tendency

Statistics is a branch of mathematics that deals with collection, grouping and analysis of the data and then deriving the results which are representative of the data collected.

Single value of a variable describing the data is called an average. Average lies generally in the middle of data so such values are called measures of central tendency. Common measures are **Mean**, **Median** and **Mode**.

The degree to which the values in a set of data are spread around an average are called dispersion of variation. The different measures of dispersion are range, quartile deviation, mean deviation, standard deviation and variance.

Statistics finds wide application in finance, economics, various surveys, planning, demography and various other analysis. All future planning of a country depends upon statistics of growth or fall in population, industrial production, crop production, etc.

16.2.1 Mean

Mean is also called arithmetic mean or simply average.

16.2.1.1 Mean of Ungrouped Data

The mean of n observations $x_1, x_2, \ldots, x_n$ is defined as

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

16.2.1.2 Mean of Grouped Data

Let $x_1, x_2, \ldots, x_n$ be n observations with corresponding frequencies as $f_1, f_2, \ldots, f_n$ respectively. Then

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + \cdots + f_nx_n}{f_1 + f_2 + \cdots + f_n} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}$$

Shortcut method: Let the assumed mean (usually the middle term of data) be A. The deviation from assumed mean, $d_i = (x_i - A)$ for each term

$$\bar{x} = A + \frac{\Sigma f_i d_i}{\Sigma f_i}$$

Step deviation method: When the data is grouped in equal class intervals with class size as h, then to simplify the calculation each $d_i = (x_i - A)$ is divided by h.

$$\bar{x} = A + \frac{\Sigma f_i d_i}{\Sigma f_i} \times h$$

Here, A = assumed mean and $d_i = \frac{x_i - A}{h}$

16.2.1.3 Weighted Mean

If $w_1, w_2, \ldots, w_n$ are the weights assigned to the values $x_1, x_2, \ldots, x_n$ respectively, then the weighted mean is given by

$$\text{Weight mean} = \frac{w_1x_1 + w_2x_2 + \cdots + w_nx_n}{w_1 + w_2 + \cdots + w_n}$$

16.2.1.4 Combined Mean

Let individual means of two sets of data are $\bar{x}_1$ and $\bar{x}_2$ with the size of data as n_1 and n_2 respectively. Then

$$\bar{x}_{12} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

Your Turn 2

1. In a factory, the workers each of age 20 years or more are grouped as follows:

Age below (in years)	25	30	35	40	45	50	55	60
No. of workers	32	60	190	236	288	350	380	400

Find the average age of the workers.

Ans. 38.3 years

2. A college sends the results of their entrance examination by post. Following is the distribution of amount spent and the number of letters dispatched:

Amount (Rs.)	**No. of letters**
2.00	2000
10.50	1500
15.00	500
30.00	300

Find the mean cost of postage per student.

Ans. Rs. 8.43 (approx).

3. The marks obtained by 20 students of a class in a test are 72, 48, 54, 65, 68, 82, 85, 47, 40, 76, 74, 73, 53, 56, 67, 65, 49, 52, 61, 36. Find the average marks of class.

Ans. 61.2

4. If mean of n observations $x_1, x_2, \ldots, x_n$ be $\bar{x}$, then find the mean of n observations

$$2x_1 + 3, 2x_2 + 3, 2x_3 + 3, \ldots, 2x_n + 3$$

Ans. $2\bar{x} + 3$

5. The mean of a set of observations is $\bar{x}$. If each observation is divided by α, $\alpha \neq 0$, and then is increased by 10, then find the mean of the new set.

Ans. $\frac{\bar{x} + 10\alpha}{\alpha}$

16.2.1.5 Geometric Mean

If $x_1, x_2, x_3, \ldots, x_n$ are n values of a variable x, none of them being zero, then the geometric mean G is defined as

$$G = (x_1, x_2, x_3, \ldots, x_n)^{1/n}$$

Geometric mean for frequency distribution: Geometric mean of n values $x_1, x_2, x_3, \ldots, x_n$ of a variable x, occurring with frequency $f_1, f_2, f_3, \ldots, f_n$ respectively is given by

$$G = \left[x_1^{f_1} \times x_2^{f_2} \times x_3^{f_3} \times \cdots \times x_n^{f_n}\right]^{1/n} \text{ or } G = \text{antilog}\left[\frac{\sum_{i=1}^{n} f_i \log x_i}{n}\right]$$

16.2.1.6 Harmonic Mean

The harmonic mean of n items $x_1, x_2, x_3, \ldots, x_n$ is defined as

$$\text{Harmonic mean} = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \cdots + \frac{1}{x_n}}$$

Harmonic mean of frequency distribution: Let $x_1, x_2, x_3, \ldots, x_n$ be n items which occur with frequencies $f_1, f_2, f_3, \ldots, f_n$, respectively. Then their harmonic mean is given by

$$\text{Harmonic mean} = \frac{f_1 + f_2 + f_3 + \cdots + f_n}{\frac{f_1}{x_1} + \frac{f_2}{x_2} + \frac{f_3}{x_3} + \cdots + \frac{f_n}{x_n}} = \frac{\sum f_i}{\sum f_i \times \frac{1}{x_i}}$$

16.2.1.7 Properties of Mean

1. If each of the n given observations are increased (decreased) by the same quantity a, then the new mean of data is also increased (decreased) by a.
2. If each of the n given observations are multiplied (divided) by the same quantity a (provided $a \neq 0$), then the mean of data is also multiplied (divided) by a.
3. The sum of the deviations of individual values from mean is always zero.

$$\sum_{i=1}^{n} f_i(x_i - \bar{x}) = 0$$

4. The sum of squares of deviation from mean is minimum, that is,

$$\sum_{i=1}^{n} f_i(x_i - \bar{x})^2 \text{ is minimum}$$

Illustration 16.1 In a class of 40 boys and 30 girls, the average age is 16 years. If the mean age of boys is 1 year more than the mean age of girls, then find the mean age of girls.

Solution: Let the mean age of 30 girls is x years so that of 40 boys is $(x + 1)$ years.

Therefore,

$$16 = \frac{40(x+1)+30x}{40+30} = \frac{4x+4+3x}{7}$$

$$\Rightarrow 7x + 4 = 112$$

$$\Rightarrow 7x = 108$$

$$x = 15\frac{3}{7} \text{ years}$$

Illustration 16.2 If n persons donate each rupees 1, 2, 4, 8, …, 2^{n-1}, respectively, then the mean donation per person is

Solution:

$$\overline{x} = \frac{1+2+2^2+2^3+\cdots+2^{n-1}}{n}$$

This is a G.P. with common ratio as 2,

$$\overline{x} = \frac{(2^n - 1)}{n\cdot(2-1)}$$

$$\overline{x} = \frac{2^n - 1}{n} \text{ rupees}$$

Illustration 16.3 The mean heights of team A and team B are 6′2″ and 5′10″, respectively. One tallest member from team A was transferred to B so that average height of team B rises by 0.5″. Then the shortest member from team B was transferred to A so that now both teams have same average height. Find the heights of members transferred respectively. Each team consists of 6 members.

Solution: Total height of 6 members of team A = 6′2″ × 6 = 37′

Total height of 6 members of team B = 5′10″ × 6 = 35′
Let one member of height h' is transferred from team A to B.
Now, total height of 5 members of team A remaining = 37′ − h'
Total height of 7 members of team B = 35′ + h'

$$\text{Mean height of team } B = \frac{35' + h'}{7} = 5'10.5''$$

$$\Rightarrow h' = 7(5'\,10.5'') - 35'$$

$$= 73.5''$$

$$= 6' \text{ and } \frac{1}{8}''$$

Let one member transferred from B to A of height H′

$$\text{So, mean height of 6 members of team } B = \frac{35 + h' - H'}{6} \text{ feet}$$

$$\text{So, mean height of 6 members of team } A = \frac{37 - h' + H'}{6} \text{ feet}$$

$$\frac{35 + h' - H'}{6} = \frac{37 - h' + H'}{6}$$

$$2h - 2H = 2$$

$$H = h - 1 = 5' \text{ and } \frac{1}{8}''$$

Therefore, the members transferred were of height 6 feet and $\frac{1}{8}$ inch, 5 feet and $\frac{1}{8}$ inch, respectively.

Your Turn 3

1. If boy goes to school from his home at a speed of x km/h and comes back at a speed of y km/h, then find the average speed of the boy.

   Ans. $\frac{2xy}{x+y}$ km/h

2. If the values 2, 8, 16, 128, 512 are given, then find the geometric mean.

   Ans. $2^{24/5}$

3. Find the harmonic mean of the following distribution:

Class	4.5–5.5	5.5–6.5	6.5–7.5	7.5–8.5	8.5–9.5
Frequency	8	10	18	6	4

Ans. 6.54

16.2.2 Median

Median is defined as the middle most or the central value of the variables in a set of observations, when the observations are arranged either in ascending or in descending order of their magnitudes. It divides the arranged series in two equal parts. Median is a position average, whereas, the arithmetic mean is the calculated average. When a series consists of an even number of terms, median is the arithmetic mean of the two central items. It is generally denoted by M.

16.2.2.1 Median of Individual Series

If there are total n observations arranged in ascending or descending order, then

1. If n is odd, then

$$\text{Median} = \text{Value of } \left(\frac{n+1}{2}\right)^{\text{th}} \text{ observation}$$

2. If n is even, then

$$\text{Median} = \text{Mean of } \left(\frac{n}{2}\right)^{\text{th}} \text{ and } \left(\frac{n}{2}+1\right)^{\text{th}} \text{ observations}$$

16.2.2.2 Median of Discrete Series

Prepare a cumulative frequency after arranging the data in ascending or descending order.

1. If n is odd, then

$$\text{Median} = \text{Size of the } \left(\frac{n+1}{2}\right)^{\text{th}} \text{ term}$$

2. If n is even, then

$$\text{Median} = \frac{\text{Size of } \left(\frac{n}{2}\right)^{\text{th}} \text{ term} + \text{Size of } \left(\frac{n}{2}+1\right)^{\text{th}} \text{ term}}{2}$$

Illustration 16.4 In a class of 20 students, the marks obtained by them in a test are grouped as follows:

Marks	No. of students
0	1
1	3
4	2
5	6
7	2
9	3
11	1
15	1
16	1
	20

Find the median marks.

Solution:

x	*f*	Σf
0	1	1
1	3	4
4	2	6
5	6	12
7	2	14
9	3	17
11	1	18
15	1	19
16	1	20

Here $n = 20$

So, $\frac{n}{2} = 10$ and $\frac{n}{2} + 1 = 11$

Size corresponding to both $\frac{n}{2}$ and $\left(\frac{n}{2}+1\right)$ is '5'.
Hence, median is 5.

16.2.2.3 Median of a Continuous Series

1. Prepare cumulative frequency table.
2. Choose the median class where $\frac{n}{2}^{\text{th}}$ observation lies.
3. Median $= l + \frac{\left(\frac{n}{2} - c_f\right)}{f} \times h$

   where l = lower limit of median class
   f = frequency of median class
   c_f = cumulative frequency of class preceding to the median class
   h = width of median class.

16.2.3 Mode

It is the value of the variate occurring most frequently or the variate having maximum frequency.

Mode of individual series is the value which is repeated maximum number of times.

Mode of discrete series is the value corresponding to maximum frequency.

Mode of continuous series: Modal class is the class having maximum frequency.

$$\text{Mode} = l + \frac{f_m - f_{m-1}}{2f_m - f_{m-1} - f_{m+1}} \times h$$

l = lower limit of modal class
h = width of modal class
f_{m-1} = frequency of class preceding modal class
f_m = frequency of modal class
f_{m+1} = frequency of class succeeding modal class.
If mode falls in other class than modal class, then

$$\text{Mode} = l + \frac{f_{m+1}}{f_{m-1} + f_{m+1}} \times h$$

When mean, median and mode coincide, such distribution is called symmetrical distribution. Otherwise for moderately skew distribution

$$\text{Mode} = 3 \text{ median} - 2 \text{ mean}$$

Illustration 16.5 The median of a set of 11 distinct numbers is 20.5. If each of the largest 5 observations is increased by 2 and each of the smallest observations is decreased by 3, then choose the correct statement.

(A) Mean remains unchanged
(B) Median remains unchanged
(C) Mean increases
(D) Median decreases

Solution: Median is the middle, that is, 6^{th} member in ascending order or descending order of given numbers. When the left or right numbers are changed the middle number remains same. Hence, median is unchanged.

Your Turn 4

1. Find the median of the data 13, 14, 16, 18, 20, 22.
   Ans. 17
2. The following data gives the distribution of heights of students.

Height (in cm)	160	150	152	161	156	154	155
No. of students	12	8	4	4	3	3	7

   Calculate the median of the distribution.
   Ans. 155
3. If the mode of a data is 18 and the mean is 24, then find median.
   Ans. 22
4. The marks obtained by 60 students in certain test are given below:

Marks	No. of students
10–20	2
20–30	3
30–40	4
40–50	5
50–60	6
60–70	12
70–80	14
80–90	10
90–100	4

   Calculate median and mode.
   Ans. Median = 68.33, Mode = 73.33

5. Find the mode of following frequency distribution:

x	40	43	46	49	52	55
f	5	8	16	9	7	3

Ans. 46

16.3 Measure of Dispersion

Dispersion means scatterness. The degree to which numerical data tend to spread about an average value is called the dispersion of the data. There are four measures of dispersion.

16.3.1 Range

$$\text{Range} = L - S$$

where L is the largest value and S is the smallest value.

$$\text{Coefficient of Range} = \frac{L-S}{L+S}$$

16.3.2 Quartile Deviation

$$\text{Quartile deviation} = \frac{Q_3 - Q_1}{2}$$

where Q_1 is first quartile and Q_3 is third quartile.

$$\text{Coefficient of Quartile deviation} = \frac{Q_3 - Q_1}{Q_3 + Q_1}$$

16.3.3 Mean Deviation

It is the average of the modulus of the deviations of the observations in a series taken from mean or median.

16.3.3.1 Methods for Calculation of Mean Deviation

Case I: For ungrouped data: In this case, the mean deviation is given by the formula

$$\text{Mean deviation} = \text{M.D.} = \frac{\Sigma|x-A|}{n} = \frac{\Sigma|d|}{n},$$

where d stands for the deviation from the mean or median and $|d|$ is always positive whether d itself is positive or negative, and n is the total number of items.

Case II: For grouped data: Let $x_1, x_2, x_3, \ldots, x_n$ occur with frequencies $f_1, f_2, f_3, \ldots, f_n$ respectively, and let $\Sigma f = n$ and M can be either mean or median, then the mean deviation is given by the formula.

$$\text{Mean Deviation} = \frac{\Sigma(f|x-M|)}{\Sigma f} = \frac{\Sigma f|d|}{n}$$

where $d = |x - M|$ and $\Sigma f = n$

$$\text{Coefficient of Mean deviation} = \frac{\text{Mean deviation}}{\text{Median}}$$

or

$$= \frac{\text{Mean deviation}}{\text{Mean}} \text{ (In case the deviations are taken from mean).}$$

Illustration 16.6 An A.P., 2, 5, 8, …, 21 terms is given. Find the mean deviation of A.P. from its mean.

Solution:

$$\sum x_i = 2 + 5 + 8 + \cdots$$

$$= \frac{21}{2}[4 + 20 \times 3]$$
$$= 21 \times 32$$
$$= 672$$

Therefore,

$$\bar{x} = \frac{672}{21} = 32$$

$$\Sigma|x_i - \bar{x}| = 30 + 27 + 24 + \cdots + 24 + 27 + 30$$
$$= 2 \times 3(10 + 9 + 8 + \cdots + 1)$$
$$= 2 \times 3\left(\frac{10 \times 11}{2}\right) = 330$$

Hence,

$$\text{Mean deviation} = \frac{330}{21} = \frac{110}{7}$$

16.3.4 Standard Deviation

The positive square root of the average of squared deviations of all observations taken from their mean is called standard deviation. It is generally denoted by the Greek alphabet σ or s.

16.3.4.1 Variance

The square of the standard deviation is called variance and is denoted by σ^2.

16.3.4.2 Coefficient of Standard Deviation

It is the ratio of the standard deviation to its A.M., that is,

$$\text{Coefficient of standard deviation} = \frac{\sigma}{\bar{x}}$$

16.3.4.3 Standard Deviation in Ungrouped Data

1. **Direct method:** In case of individual series, the standard deviation can be obtained by the formula.

$$\sigma = \sqrt{\frac{(x-\bar{x})^2}{n}} = \sqrt{\frac{\Sigma d^2}{n}} \quad \text{[First form]}$$

where $d = x - \bar{x}$ and x = value of the variable or observation, $\bar{x}$ = arithmetic mean, n = total number of observations.

2. **Shortcut method:** This method is applied to calculate standard deviation, when the mean of the data comes out to be a fraction. In that case, it is very difficult and tedious to find the deviations of all observations from the mean by the earlier method. The formula used is

$$\sigma = \sqrt{\frac{\Sigma d^2}{n} - \left(\frac{\Sigma d}{n}\right)^2} \quad \text{[Second form]}$$

where $d = x - A$, A = assumed mean, n = total number of observations.

Illustration 16.7 Find the standard deviation of first n natural numbers.

Solution:

$$\text{Sum of first } n \text{ natural numbers} = \frac{n(n+1)}{2}$$

$$\bar{x} = \frac{n+1}{2}$$

$$\sigma = \sqrt{\frac{(1-\bar{x})^2 + (2-\bar{x})^2 + \cdots + (n-\bar{x})^2}{n}}$$

$$= \sqrt{\frac{\Sigma n^2 + n\bar{x}^2 - 2\bar{x}(\Sigma n)}{n}}$$

$$= \sqrt{\frac{\frac{n(n+1)(2n+1)}{6} + \frac{n\times(n+1)^2}{4} - \frac{(n+1)\times n(n+1)}{2}}{n}}$$

$$= \sqrt{\frac{n(n+1)}{12n}(4n+2+3n+3-6n-6)}$$

$$= \sqrt{\frac{n}{12}\frac{(n+1)(n-1)}{n}}$$

$$\Rightarrow \sigma = \sqrt{\frac{(n^2-1)}{12}}$$

16.3.4.4 Standard Deviation in Grouped Data

It is calculated by the following formula.

$$\sigma = \sqrt{\frac{\Sigma f(x-\bar{x})^2}{n}} \qquad \text{[Third form]}$$

where $\bar{x}$ is A.M., x is the size of the item, and f is the corresponding frequency in the case of discrete series.

But when the mean has a fractional value, then the following formula is applied to calculate S.D.

$$\sigma = \sqrt{\frac{\Sigma fd^2}{n} - \left(\frac{\Sigma fd}{n}\right)^2} \qquad \text{[Fourth form]}$$

where $d = x - A$, A = assumed mean, $n = \Sigma f$ = total frequency.

16.3.4.5 Coefficient of Variation

Two or more series can be compared for variability by C.V. (the coefficient of variation) and it is given by

$$\text{C.V.} = \frac{\sigma}{\bar{x}} \times 100\%$$

16.3.4.6 Variance

Square of S.D. (σ^2) is called variance of distribution.

Direct method for S.D.:

$$\sigma = \sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2}$$

Shortcut method for S.D.:

(i) $\sigma = \sqrt{\frac{\Sigma d^2}{n} - \left(\frac{\Sigma d}{n}\right)^2}$ for ungrouped data, $d = (x_i - A)$. A is the assumed mean.

(ii) $\sigma = \sqrt{\frac{\Sigma fd^2}{N} - \left(\frac{\Sigma fd}{N}\right)^2}$

Step-deviation method:

$$\sigma = h\sqrt{\frac{\Sigma fd'^2}{N} - \left(\frac{\Sigma fd'}{N}\right)^2} \text{ where } d' = \frac{x_i - A}{h}$$

16.3.4.7 Combined Standard Deviation

Let two data sets having n_1 and n_2 observations have $\bar{x}_1$ and $\bar{x}_2$ as their means and σ_1 and σ_2 their S.D. Then combined S.D. is

$$\sigma_{12} = \sqrt{\frac{n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)}{n_1 + n_2}}$$

where $d_1 = (\bar{x} - \bar{x}_{12})$; $d_2 = (\bar{x}_2 - \bar{x}_{12})$

and $$\bar{x}_{12} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

16.4 Symmetric and Skew-Symmetric

In a symmetrical distribution, mean, median and mode coincide. Here, frequencies are symmetrically distributed on both sides of some central value.

A distribution which is not symmetrical is called skew-symmetrical. In a moderately skew-symmetric distribution,

Mean − Mode = 3 (Mean − Median)

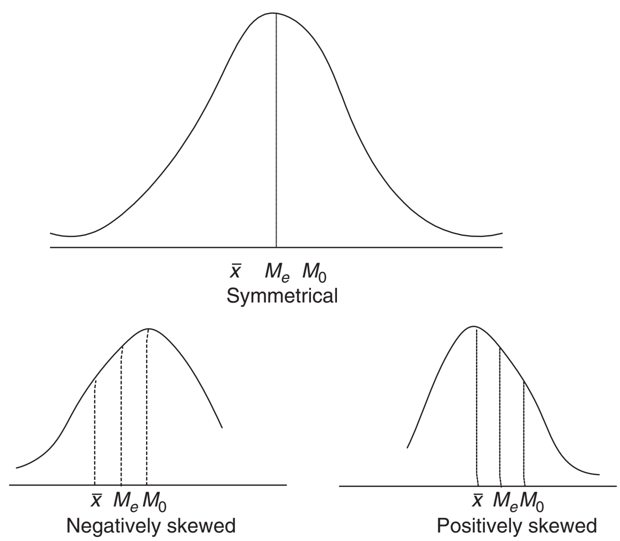


Figure 16.1

In a positively skew-symmetric distribution, the value of mean is maximum and that of mode is least, and the median lies between the two.

In a negatively skew-symmetric distribution, the value of mode is maximum and that of mean is least, and the median lies between the two. (See Fig. 16.1.)

Absolute measures of skewness are as follows:

1. $\bar{x} - M_e$
2. $\bar{x} - M_0$
3. $Q_3 + Q_1 - 2Q_2$

where, Q_1, Q_2, Q_3 are quartiles and

$$Q_1 = l + \frac{(n/4 - c_f)}{f} \times h$$

$$Q_2 = l + \frac{(n/2 - c_f)}{f} \times h$$

$$Q_3 = l + \frac{(3n/4 - c_f)}{f} \times h$$

Relative measures of skewness are as follows:

1. Karl Pearson's coefficient of skewness is equal to $\frac{\bar{x}-M_0}{\sigma}$.
   It lies between −1 and 1.
2. Bowley's coefficient of skewness is equal to $\frac{Q_3+Q_1-2Q_2}{Q_3-Q_1}$.
   It also lies between −1 and 1.
3. Kelley's coefficient of skewness $=\frac{P_{10}+P_{90}-2Q_2}{Q_{90}-P_{10}}$.

Area result:
For a symmetrical distribution (normal curve),

1. The interval $(\bar{x}-\sigma, \bar{x}+\sigma)$ contains 68.27% items.
2. The interval $(\bar{x}-2\sigma, \bar{x}+2\sigma)$ contains 95.45% items.
3. The interval $(\bar{x}-3\sigma, \bar{x}+3\sigma)$ contains 99.74% items.

Your Turn 5

1. Which of the following is not a measure of dispersion?
   (A) Mean **(B)** Variance
   (C) Mean deviation **(D)** Range

   Ans. (A)
2. Find the range of the following set of observations 2, 3, 5, 9, 8, 7, 6, 5, 7, 4, 3.

   Ans. 7
3. Find the mean and standard deviation of the following distribution:

Class interval	Frequency
31–35	2
36–40	3
41–45	8
46–50	12
51–55	16
56–60	5
61–65	2
66–70	2

   Ans. Mean = 50, Standard deviation = 7.62
4. Find the mean deviation about median for the distribution:

Marks	50	55	60	65	70
No. of students	5	8	6	4	2

   Ans. 5
5. The batting scores of two cricket players A and B in 10 innings are as follows:

Batsman A	15	17	19	27	30	36	40	90	95	110
Batsman B	10	16	21	28	37	41	36	80	82	85

   Find which of the player is more consistent.

   Ans. B
6. The weights of 9 men are 76, 74.5, 61, 64, 69, 67.5, 71, 73, 74. Find the variance and standard deviation of the weights.

   Ans. Variance = 22.94, σ = 4.79

Additional Solved Examples

1. The sum of squares of deviation for n observations taken from their mean x is y. The coefficient of variation is

   (A) $\sqrt{\frac{y}{nx}}\times 100\%$ **(B)** $\frac{1}{nx}\sqrt{y}\times 100\%$
   (C) $\frac{1}{x}\sqrt{\frac{y}{n}}\times 100\%$ **(D)** $\frac{1}{n}\sqrt{\frac{y}{n}}\times 100\%$

Solution:

$$\sigma=\sqrt{\frac{y}{n}}$$

Therefore,

$$\text{C.V.}=\frac{\sigma}{x}\times 100=\frac{1}{x}\sqrt{\frac{y}{n}}\times 100\%$$

Hence, the correct answer is option (C).

2. The mean of six observations is 7 and their variance is $\frac{25}{3}$. If four observations are 5, 6, 8, 9, then the median of all the observations are ______.

Solution:

$\bar{x}=7, N=6, \Sigma\frac{(x-\bar{x})^2}{N}=\frac{25}{3}$ and x_1 and x_2 are other two observations, then

$$x_1+x_2=42-5-6-8-9=14 \quad (1)$$

$$\sum(x-\bar{x})^2=50$$

$\Rightarrow (5-7)^2+(6-7)^2+(8-7)^2+(9-7)^2+(x_1-7)^2+(x_2-7)^2=50$
$\Rightarrow (x_1-7)^2+(x_2-7)^2=40$
$\Rightarrow (x_1-7)^2+(14-x_1-7)^2=40$ [from Eq. (1)]
$\Rightarrow (x_1-7)^2+(7-x_1)^2=40$
$\Rightarrow (x_1-7)^2=20$
$\Rightarrow x_1-7=\pm\sqrt{20}$
$\Rightarrow x_1=7\pm\sqrt{20}$
$\Rightarrow x_1=7+\sqrt{20}$
$\Rightarrow x_2=7-\sqrt{20}$

Now all the 6 observations arranged in ascending order are $7-\sqrt{20}, 5, 6, 8, 9, 7+\sqrt{20}$.

Therefore,

$$\text{Median}=\frac{6+8}{2}=7$$

3. The mean of n items is $\bar{x}$. If each of the n^{th} item is increased by n, then the new mean is ______.

Solution: Let $x_1, x_2, \ldots, x_n$ be n items. Then

$$\bar{x}=\frac{1}{n}\sum(x_1+x_2+\cdots+x_n)$$

Now, x_1 becomes x_1+1, x_2 becomes x_2+2, and so on.
So, the new mean is

$$\bar{x}'=\frac{1}{n}\sum(x_1+x_2+x_3+\cdots+x_n)+(1+2+3+\cdots+n)$$
$$=\frac{\sum(x_1+x_2+\cdots+x_n)}{n}+\frac{(1+2+3+\cdots+n)}{n}$$
$$=\bar{x}+\frac{\sum n}{n}$$
$$=\bar{x}+\frac{(n+1)}{2}$$

4. If the data is less scattered about its mean, then
   (A) its C.V. is large **(B)** its s is large
   (C) its variance is small **(D)** its mean is small

Solution: Less scattered data about mean, indicates less variation from mean, hence all measures of dispersion would be small. So variance is small is the correct statement.

Hence, the correct answer is option (C).

5. A data given below has its S.D. = 2, then find a.

$$\underbrace{-a,-a,-a,\ldots,-a}_{n \text{ times}},\ \underbrace{a,a,a,\ldots,a}_{n \text{ times}}$$

Solution:

$$\bar{x}=0$$

Therefore,

$$\sigma=2=\sqrt{\frac{n(-a-0)^2+n(a-0)^2}{2n}}$$

$$\Rightarrow 2=\sqrt{\frac{2na^2}{2n}}$$

$$\Rightarrow a^2=4$$

$$\Rightarrow a=\pm 2$$

Previous Years' Solved JEE Main/AIEEE Questions

1. The average marks of boys in a class are 52 and that of girls is 42. The average marks of boys and girls combined is 50. The percentage of boys in the class is

(A) 40 **(B)** 20
(C) 80 **(D)** 60

[AIEEE 2007]

Solution: Let number of boys in the class be x.

Let number of girls in the class be y.
Now according to question, total marks of students,

$$52x+42y=50(x+y)\Rightarrow 2x=8y$$

$$\Rightarrow \frac{x}{y}=\frac{4}{1};$$

$$\frac{x}{x+y}=\frac{4}{5}=\frac{4}{5}\times 100=80\%$$

Thus, the percentage of boys is 80%.

Hence, the correct answer is option (C).

2. The mean of the numbers a, b, 8, 5, 10 is 6 and the variance is 6.80. Then which one of the following gives possible values of a and b?

(A) $a=0, b=7$ **(B)** $a=5, b=2$
(C) $a=1, b=6$ **(C)** $a=3, b=4$

[AIEEE 2008]

Solution: Mean of a, b, 8, 5, 10 is 6. This implies that

$$\frac{a+b+8+5+10}{5}=6$$

Therefore,

$$a+b=7 \qquad (1)$$

Given that variance is 6.8. Therefore, variance is

$$\frac{\Sigma(X_i-A)^2}{n}=\frac{(a-6)^2+(b-6)^2+4+1+2}{5}=6.8$$

Therefore,

$$a^2+b^2=25\Rightarrow a^2+(7-a)^2=25\Rightarrow a^2-7a+12=0 \qquad \text{[using Eq. (1)]}$$

Therefore, $a=4, 3$ and $b=3, 4$.

Hence, the correct answer is option (D).

3. If the mean deviation of number 1, $1+d$, $1+2d$, …, $1+100d$ from their mean is 255, then d is equal to

(A) 10.0 **(B)** 20.0
(C) 10.1 **(D)** 20.2

[AIEEE 2009]

Solution:

$$\text{Mean}(\bar{x})=\frac{\text{Sum of quantities}}{n}=\frac{\frac{\not n}{2}(a_n+1)}{\not n}=\frac{1}{2}[(1+100d)+1]=1+50d$$

Mean deviation is given by

$$\frac{1}{n}\Sigma|x_i-\bar{x}|=\frac{1}{n}\left[\begin{array}{l}|1+50d-1|+|1+50d-1-d|+|1+50d-1-2d|\\+\cdots+|1+50d-1-100d|\end{array}\right]$$

$$\Rightarrow 255=\frac{1}{101}[50d+49d+48d+\cdots+d+0+d+\cdots+50d]$$

$$=\frac{\not 2 d}{101}\left[\frac{50\times 51}{\not 2}\right]$$

$$\Rightarrow d=\frac{255\times 101}{50\times 51}=10.1$$

Hence, the correct answer is option (C).

4. **Statement-1:** The variance of first n even natural numbers is $\frac{n^2-1}{4}$.

Statement-2: The sum of first n natural numbers is $\frac{n(n+1)}{2}$ and the sum of squares of first n natural numbers is $\frac{n(n+1)(2n+1)}{6}$.

(A) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(B) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(C) Statement-1 is true, Statement-2 is false
(D) Statement-1 is false, Statement-2 is true

[AIEEE 2009]

Solution: Sum of n even natural numbers = $n(n+1)$, we have

$$\text{Mean}(\bar{x})=\frac{n(n+1)}{n}=n+1$$

and

$$\text{Variance}=\left[\frac{1}{n}\sum(x_i)^2\right]-(\bar{x})^2=\frac{1}{n}[2^2+4^2+\cdots+(2n)^2]-(n+1)^2$$

$$=\frac{1}{n}2^2[1^2+2^2+\cdots+n^2]-(n+1)^2=\frac{4}{n}\frac{n(n+1)(2n+1)}{6}-(n+1)^2$$

$$=\frac{(n+1)[2(2n+1)-3(n+1)]}{3}=\frac{(n+1)[4n+2-3n-3]}{3}$$

$$=\frac{(n+1)(n-1)}{3}=\frac{n^2-1}{3}$$

Therefore, Statement-1 is false.

Also, the sum of first n natural numbers is $\frac{n(n+1)}{2}$ and the sum of squares of first n natural numbers is $\frac{n(n+1)(2n+1)}{6}$, which is true.

Hence, the correct answer is option (D).

5. For two data sets, each of size 5, the variances are given to be 4 and 5 and the corresponding means are given to be 2 and 4, respectively. The variance of the combined data set is

(A) $\frac{11}{2}$ **(B)** 6

(C) $\frac{13}{2}$ **(D)** $\frac{5}{2}$

[AIEEE 2010]

Solution: We have

$$\sigma_x{}^2 = 4; \sigma_y{}^2 = 5; \bar{x} = 2; \bar{y} = 4$$

Therefore,

$$\frac{\sum x_i}{5} = 2 \Rightarrow \sum x_i = 10; \sum y_i = 20$$

Now,

$$\sigma_x = \frac{1}{5}\sum x_i^2 - (\bar{x})^2; \sigma_y = \frac{1}{5}\sum y_i^2 - (\bar{y})^2$$

$$\Rightarrow \sum x_i{}^2 = 40; \sum y_i{}^2 = 105$$

$$\sigma_z{}^2 = \frac{1}{10}(\sum x_i{}^2 + \sum y_i{}^2) - \left(\frac{\bar{x}+\bar{y}}{2}\right)^2$$

$$= \frac{1}{10}(40+105) - 9 = \frac{145-90}{10} = \frac{55}{10} = \frac{11}{2}$$

Hence, the correct answer is option (A).

6. If the mean deviation about the median of the numbers a, $2a$, $\ldots$, $50a$ is 50, then $|a|$ equals

(A) 3 **(B)** 4

(C) 5 **(D)** 2

[AIEEE 2011]

Solution: We have

$$\frac{1}{n}\sum |x_i - A|$$

That is,

$$A = \text{Median} = \frac{25a+26a}{2} = 25.5a$$

$$\text{Mean deviation} = \frac{1}{50}\{|a-25.5a| + |2a-25.5a| + \cdots + |50a-25.5a|\}$$

$$= \frac{2}{50}\{24.5a + 23.5a + \cdots + (0.5a)\}$$

$$= \frac{2}{50}\{312.5a\} = 50$$

$$\Rightarrow 625a = 2500 \Rightarrow a = 4$$

Hence, the correct answer is option (B).

7. Let $x_1, x_2, \ldots, x_n$ be n observations, and let $\bar{x}$ be their arithmetic mean and σ^2 be their variance.

Statement-1: Variance of $2x_1, 2x_2, \ldots, 2x_n$ is $4\sigma^2$.

Statement-2: Arithmetic mean of $2x_1, 2x_2, \ldots, 2x_n$ is $4\bar{x}$.

(A) Statement-1 is false, Statement-2 is true

(B) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(C) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1

(D) Statement-1 is true, Statement-2 is false

[AIEEE 2012]

Solution: We have

$$\sigma^2 = \sum \frac{x_i^2}{n} - \left(\sum \frac{x_i}{n}\right)^2$$

Variance of $2x_1, 2x_2, \ldots, 2x_n$ is

$$\sum \frac{(2x_i)^2}{n} - \left(\sum \frac{2x_i}{n}\right)^2 = 4\left[\sum \frac{x_i^2}{n} - \left(\sum \frac{x_i}{n}\right)^2\right] = 4\sigma^2$$

from which it is clear that Statement-1 is true.

A.M. of $2x_1, 2x_2, \ldots, 2x_n$ is

$$\frac{2x_1 + 2x_2 + \cdots + 2x_n}{n} = 2\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) = 2\bar{x}$$

from which it is clear that Statement-2 is false.

Hence, the correct answer is option (D).

8. All the students of a class performed poorly in Mathematics. The teacher decided to give grace marks of 10 to each of the students. Which of the following statistical measures will not change even after the grace marks were given?

(A) Median **(B)** Mode

(C) Variance **(D)** Mean

[JEE MAIN 2013]

Solution: Before the grace marks were given, the variance of marks of the students is expressed as

$$\sigma_1^2 = \frac{\sum (x_i - \bar{x})^2}{N} \quad (1)$$

After the grace marks were given, the variance of marks of the students is expressed as

$$\sigma_2^2 = \frac{\sum [(x_i + 10) - (\bar{x} + 10)]^2}{N} == \frac{\sum (x_i - \bar{x})^2}{N} \quad (2)$$

From Eqs. (1) and (2), we get $\sigma_1^2 = \sigma_2^2$.

Hence, variance will not change even after the grace marks were given, however mean, median and mode will increase by 10.

Hence, the correct answer is option (C).

9. The variance of first 50 even natural numbers is

(A) 437 **(B)** $\frac{437}{4}$

(C) $\frac{833}{4}$ **(D)** 833

[JEE MAIN 2014 (OFFLINE)]

Solution:

$$\text{Variance, } \sigma^2 = \text{Mean of squares} - \text{Square of mean}$$

$$= \frac{\sum_{r=1}^{50}(2r)^2}{n} - \left(\frac{\sum_{r=1}^{50}(2r)}{n}\right)^2$$

$$= \frac{4(1^2 + 2^2 + 3^2 + \cdots + 50^2)}{50} - \left[\frac{2(1+2+3+\cdots+50)}{50}\right]^2$$

$$= 4 \times \frac{50(51)(101)}{6 \times 50} - 4\left(\frac{50 \times 51}{2 \times 50}\right)^2$$

$$= 3434 - 51^2 = 833$$

Hence, the correct answer is option (D).

10. In a set of $2n$ distinct observations, each of the observation below the median of all the observations is increased by 5 and each of the remaining observations is decreased by 3. Then the mean of the new set of observations:

(A) Increases by 1.
(B) Decreases by 1.
(C) Decreases by 2.
(D) Increases by 2.

[JEE MAIN 2014 (ONLINE SET 1)]

Solution: Let observation be $a_1, a_2, \ldots, a_n, a_{n+1}, a_{n+2}, \ldots, a_{2n}$. Then

$$\text{Mean} = \bar{x} = \frac{a_1 + a_2 + \cdots + a_n + a_{n+1} + a_{n+2} + \cdots + a_{2n}}{2n}$$

$$\bar{x} = \frac{(a_1+5)+(a_2+5)+\cdots+(a_n+5)+(a_{n+1}-3)+(a_{n+2}-3)+\cdots+(a_{2n}-3)}{2n}$$

$$= \frac{a_1 + a_2 + \cdots + a_n + a_{n+1} + a_{n+2} + \cdots + a_{2n}}{2n} + \frac{5n-3n}{2n}$$

$$= \bar{x} + 1$$

Hence, the correct answer is option (A).

11. Let $\bar{X}$ and M.D. be the mean and the mean deviation about $\bar{X}$ of n observations x_i, $i = 1, 2, \ldots, n$. If each of the observations is increased by 5, then the new mean and the mean deviation about the new mean, respectively, are:

(A) $\bar{X}$, M.D.
(B) $\bar{X} + 5$, M.D.
(C) $\bar{X}$, M.D. + 5
(D) $\bar{X} + 5$, M.D. + 5

[JEE MAIN 2014 (ONLINE SET 3)]

Solution:

$$\bar{X} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$$

Now, the new mean is

$$\frac{x_1 + 5 + x_2 + 5 + \cdots + x_n + 5}{n} = \frac{x_1 + x_2 + \cdots + x_n + 5n}{n} = \bar{X} + 5$$

$$\text{M.D. (New)} = \frac{\sum_{i=1}^{n} |x_i - \bar{X}|}{n} = \frac{\sum_{i=1}^{n} |(x_i + 5) - (\bar{X} + 5)|}{n}$$

$$= \frac{\sum_{i=1}^{n} |x_i - \bar{X}|}{n} = \text{M.D. (old)}$$

Hence, the correct answer is option (B).

12. Let $\bar{x}$, M and σ^2 be respectively the mean, mode and variance of n observations $x_1, x_2, \ldots, x_n$, and $d_i = -x_i - a$, $i = 1, 2, \ldots, n$, where a is any number.

Statement-1: Variance of $d_1, d_2, \ldots, d_n$ is σ^2.

Statement-2: Mean and mode of $d_1, d_2, \ldots, d_n$ are $-\bar{x} - a$ and $-M - a$, respectively.

(A) Statement-1 and Statement-2 are both false.
(B) Statement-1 and Statement-2 are both true.
(C) Statement-1 is true and Statement-2 is false.
(D) Statement-1 is false and Statement-2 is true.

[JEE MAIN 2014 (ONLINE SET 4)]

Solution:

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

M = Most frequent

$$\text{Variance of } d_1, d_2, \ldots, d_n = \frac{1}{n}\sum_{i=1}^{n}(-x_i - a - \bar{d})^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}\{-x_i - a - (-\bar{x} - a)\}^2 = \frac{1}{n}\sum_{i=1}^{n}[-x_i - \cancel{a} + \bar{x} + \cancel{a}]^2$$

$$= \frac{1}{n}\sum (x_i - \bar{x})^2 = \sigma^2$$

Therefore, Statement-1 is true.

Now,

$$\bar{d} = \text{Mean of } d_1, d_2, \ldots, d_n = \frac{d_1 + d_2 + \cdots + d_n}{n}$$

$$= \frac{(-x_1 - a) + (-x_2 - a) + \cdots (-x_n - a)}{n}$$

$$= -\bar{x} - \frac{na}{n} = -\bar{x} - a$$

Mode of $d_1, d_2, \ldots, d_n = -M - a$ (a is attached with most frequent value).

Therefore, Statement-2 is correct.

Hence, the correct answer is option (B).

13. The mean of the data set comprising 16 observations is 16. If one of the observation valued 16 is deleted and three new observations valued 3, 4 and 5 are added to the data, then the mean of the resultant data is

(A) 16.0 **(B)** 15.8
(C) 14.0 **(D)** 16.8

[JEE MAIN 2015 (OFFLINE)]

Solution: We have

$$n = 16, \ \bar{x} = 16$$

$$\Rightarrow \ \Sigma x_i = n\bar{x} = 256$$

Therefore, new mean is

$$\bar{x} = \frac{256 - 16 + 3 + 4 + 5}{18} = 14$$

Hence, the correct answer is option (C).

14. If the standard deviation of the numbers 2, 3, a and 11 is 3.5, then which of the following is true?

(A) $3a^2 - 23a + 44 = 0$
(B) $3a^2 - 26a + 55 = 0$
(C) $3a^2 - 32a + 84 = 0$
(D) $3a^2 - 34a + 91 = 0$

[JEE MAIN 2016 (OFFLINE)]

Solution: We have

$$\text{Standard deviation} = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

$$\Rightarrow 3.5 = \sqrt{\frac{4 + 9 + a^2 + 121}{4} - \frac{(16+a)^2}{4^2}}$$

$$\Rightarrow \frac{49}{4} = \frac{4(134 + a^2) - (16+a)^2}{4^2}$$

$$\Rightarrow 3a^2 - 32a + 84 = 0$$

Hence, the correct answer is option (C).

15. The mean of 5 observations is 5 and their variance is 124. If three of the observations are 1, 2 and 6; then the mean deviation from the mean of the data is

(A) 2.5 **(B)** 2.6
(C) 2.8 **(D)** 8.4

[JEE MAIN 2016 (ONLINE SET 2)]

Solution: From the given data, the mean is

$$\bar{x} = \frac{x_1 + x_2 + x_3 + x_4 + x_5}{5}$$

$$\Rightarrow \sum_{i=1}^{5} x_i = 25 \Rightarrow x_1 + x_2 = 16 \quad (1)$$

Their variance is given by

$$\sigma^2 = 124$$

That is,

$$\frac{\sum x_1^2}{5} - (\bar{x})^2 = 124$$

$$\Rightarrow \frac{\sum x_1^2}{5} = 124 + 25 = 149$$

$$\Rightarrow x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 745$$

$$\Rightarrow x_1^2 + x_2^2 = 704 \quad (2)$$

Solving Eqs. (1) and (2)

$$x_1^2 + (16 - x_1)^2 = 704$$
$$\Rightarrow 2x_1^2 - 32x_1 - 448 = 0$$
$$\Rightarrow x_1^2 - 16x_1 - 224 = 0$$
$$\Rightarrow x_1 = 8 \pm 12\sqrt{2}$$

So,

$$x_1 = 8 + 12\sqrt{2}$$
$$x_2 = 8 - 12\sqrt{2}$$

Mean deviation

$$= \frac{|1-5| + |2-5| + |6-5| + |8+12\sqrt{2}-5| + |8-12\sqrt{2}-5|}{5}$$

$$= \frac{4+3+1+3+12\sqrt{2}+12\sqrt{2}-3}{5} = \frac{8+24\sqrt{2}}{5}$$

$$= 8.39 \text{ or } \sim 8.4$$

Hence, the correct answer is option (D).

Practice Exercise 1

1. The sum of squares of deviations from mean is

(A) Least **(B)** Zero
(C) Maximum **(D)** None of these

2. The weighted mean of first n natural numbers whose weights are equal is given by

(A) $(2n + 1)/2$ **(B)** $(n + 1)/2$
(C) $(n - 1)/2$ **(D)** $(2n - 1)/n$

3. The arithmetic means of a set of observations is $\bar{x}$. If each observation is divided by β and then it is increased by 12, then the mean of the new series is

(A) $\frac{\bar{x}}{\beta} + 12$ **(B)** $(\bar{x} + 12)/\beta$
(C) $\frac{\bar{x}}{\beta} - 12$ **(D)** $\beta\bar{x} + 12$

4. The A.M. of n numbers of a series is $\bar{X}$. If the sum of first $(n - 1)$ terms is k, then the nth number is

(A) $\bar{X} - k$ **(B)** $n\bar{X} - k$
(C) $\bar{X} - nk$ **(D)** $n\bar{X} - nk$

5. The number of children in 25 families of a locality are recorded as follows:

3, 1, 4, 0, 2, 1, 1, 2, 3, 3, 2, 2, 2, 5, 0, 1, 4, 1, 2, 1, 2, 3, 0, 1, 4

Then, mean number of children per family is

(A) 2 **(B)** 3
(C) 4 **(D)** 5

6. If a variable takes the discrete value $\alpha + 4$, $\alpha - 7/2$, $\alpha - 5/2$, $\alpha - 3$, $\alpha - 2$, $\alpha + 1/2$, $\alpha - 1/2$, $\alpha + 5$, $(\alpha > 0)$, then the median is

(A) $\alpha - 5/4$ **(B)** $\alpha - 1/2$
(C) $\alpha - 2$ **(D)** $\alpha + 5/4$

7. Quartile deviation is nearly equal to

(A) σ **(B)** $3/2\sigma$
(C) $2/3\sigma$ **(D)** None of these

8. One of the methods of determining mode is

(A) Mode = 2 median − 3 mean
(B) Mode = 2 median + 3 mean
(C) Mode = 3 median − 2 mean
(D) Mode = 3 median + 2 mean

9. The best average of dealing a qualitative data is

(A) Mean **(B)** Median
(C) Mode **(D)** Harmonic mean

10. Median of 16, 10, 14, 11, 9, 8, 12, 6, 5 is

(A) 10 **(B)** 12
(C) 11 **(D)** 14

11. The points scored by a basketball team in a series of matches are as follows:

15, 3, 8, 10, 22, 5, 27, 11, 12, 19, 18, 21, 13, 14

Its median is

(A) 13 **(B)** 13.4
(C) 13.5 **(D)** 14.5

12. If a variable x takes values x_i such that $a \le x_i \le b$, for $i = 1, 2, \cdots, n$, then

(A) $a \le \text{var}(x) \le b$ **(B)** $a^2 \le \text{var}(x) \le b^2$
(C) $\frac{a^2}{4} \le \text{var}(x)$ **(D)** $(b-a)^2 \ge \text{var}(x)$

13. Mode of the data 5, 7, 2, 7, 7, 4, 7, 7, 6, 7 is

(A) 5 **(B)** 4
(C) 6 **(D)** 7

14. Geometric mean of 1, 2, 2^2, 2^3, …, 2^n is

(A) $2^{(n+2)/2}$ **(B)** $2^{n/2}$
(C) $2^{(n-1)/2}$ **(D)** $2^{(n+1)/2}$

15. If the first quartile is 104 and quartile deviation is 8, then the third quartile is

(A) 130 **(B)** 140
(C) 136 **(D)** 146

16. The variance of first n natural number is

(A) $\frac{n^2+1}{12}$ **(B)** $\frac{n^2-1}{12}$
(C) $\frac{(n+1)(2n+1)}{6}$ **(D)** None of these

17. The coefficient of variation of a series is 50. Its S.D. is 21.2. Its arithmetic mean is

(A) 36.6 **(B)** 22.6
(C) 26.6 **(D)** None of these

18. If G_1, G_2 are the geometric means of two series of observations and G is the GM of the ratios of the corresponding observations, then G is equal to

(A) (G_1/G_2) **(B)** $\log G_1 - \log G_2$
(C) $\log G_1/\log G_2$ **(D)** $\log (G_1, G_2)$

19. If a and b are two positive numbers, then

(A) AM $\leq$ GM $\geq$ HM **(B)** AM $\geq$ GM $\leq$ HM
(C) AM $\leq$ GM $\leq$ HM **(D)** AM $\geq$ GM $\geq$ HM

20. The geometric mean of 6 observations was calculated as 13. It was later observed that one of the observation was recorded as 28 instead of 36. The correct geometric mean is

(A) $\left(\frac{9}{7}\right)^{1/6}$ **(B)** $3\left(\frac{9}{7}\right)^{1/6}$
(C) $13\left(\frac{9}{7}\right)^{1/6}$ **(D)** 13

21. The algebraic sum of deviations of a set of values from the arithmetic mean is

(A) 0 **(B)** 1
(C) Dependent on mean **(D)** None of these

22. The mean of observations is 4.4 and the variance is 8.24. If three of the five observations are 1, 2 and 6, then find the other two observations.

(A) 9, 4 **(B)** 2, 11
(C) 7, 6 **(D)** 5, 8

23. The mean annual salaries paid to 1000 employees of a company was Rs. 3400. The mean annual salaries paid to male and female employees were Rs. 200 and Rs. 4200, respectively. The percentage of males and females employed by the company are

(A) 50, 50 **(B)** 40, 60
(C) 70, 30 **(D)** 20, 80

24. If G is the G.M. of the product of r sets of observations with geometric means $G_1, G_2, \ldots, G_r$ respectively, then G is equal to

(A) $\log G_1 + \log G_2 + \cdots + \log G_r$
(B) $G_1 \cdot G_2 \ldots G_r$
(C) $\log G_1 \cdot \log G_2 \ldots \log G_r$
(D) None of these

25. The mean deviation about the median of the series of batsman in ten innings 34, 38, 42, 44, 46, 48, 54, 55, 56, 76 is

(A) 8.5 **(B)** 7.6
(C) 8 **(D)** None of these

26. For a set of 100 observations, taking assumed mean as 4, the sum of the deviations is −11 cm, and the sum of the squares of these deviations is 275 cm^2. The coefficient of variation is

(A) 41.13% **(B)** 40.13%
(C) 42.13% **(D)** None of these

27. You are provided with the following raw data of two variable x and y.

$$\sum x = 235, \sum y = 250, \sum x^2 = 6750, \sum y^2 = 6840$$

Ten pairs of values are included in the survey. The standard deviation are

(A) 11.08, 7.68 **(B)** 11.02, 7.58
(C) 11.48, 7.48 **(D)** None of these

28. The mean wage of 1000 workers in a factory running in two shifts of 700 and 300 workers is Rs. 500. The mean wage of 700 workers working in day shift is Rs. 450. The mean wage of workers working in the night shift is

(A) Rs. 570 **(B)** Rs. 616.67
(C) Rs. 543.67 **(D)** None of these

29. A car owner buys petrol at Rs. 7.50, Rs. 8.00 and Rs. 8.50 per litre for 3 successive years. If he spends Rs. 4000 each year, then the average cost per litre of petrol is

(A) Rs. 8 **(B)** Rs. 8.25
(C) Rs. 7.98 **(D)** None of these

30. The mean square deviation of n observations $x_1, x_2, \ldots, x_n$ about −2 and 2 are 18 and 10 respectively. Then S.D. of the given set is

(A) 1 **(B)** 2
(C) 3 **(D)** 4

31. If mode of a data is 27 and mean is 24, then median is

(A) 24 **(B)** 25
(C) 26 **(D)** None of these

32. If a variate X is expressed as a linear function of two variates U and V in the form $X = aU + bV$, then mean $\bar{X}$ of X is

(A) $a\bar{U} + b\bar{V}$ **(B)** $\bar{U} + \bar{V}$
(C) $b\bar{U} + a\bar{U}$ **(D)** None of these

33. The mean of the set of numbers $x_1, x_2, x_3, \ldots, x_n$ is $\bar{x}$, then the mean of the numbers $x_i + 2i$, $1 \leq i \leq n$, is

(A) $\bar{x} + 2n$ **(B)** $\bar{x} + (n + 1)$
(C) $\bar{x} + 2$ **(D)** None of these

34. The mean of following frequency table is 50.

Class	Frequency
0–20	17
20–40	f_1
40–60	32
60–80	f_2
80–100	19
Total	120

The missing frequencies are

(A) 28, 24 **(B)** 24, 36
(C) 36, 28 **(D)** None of these

35. Mean of n items is $\bar{x}$. If these n items are successively increased by 2, 2^2, 2^3, …, 2^n, then the new mean is

(A) $\bar{x}+\frac{2^{n+1}}{n}$ **(B)** $\bar{x}+\frac{2^{n+1}}{n}-\frac{2}{n}$

(C) $\bar{x}+\frac{2^n}{n}$ **(D)** None of these

36. The quartile deviation of daily wages (in Rs.) of 7 persons given below is

12, 7, 15, 10, 17, 17, 25

(A) 14.5 **(B)** 2.5

(C) 9 **(D)** 4.5

37. The average weight of 25 boys was calculated to be 78.4 kg. It was later discovered that one weight was misread as 69 kg instead of 96 kg. The correct average is

(A) 79 kg **(B)** 79.48 kg

(C) 81.32 kg **(D)** None of these

38. If $\bar{x}_1$ and $\bar{x}_2$ are means to two distributions such that $\bar{x}_1<\bar{x}_2$ and $\bar{x}$ is the mean of the combined distribution, then

(A) $\bar{x}<\bar{x}_1$ **(B)** $\bar{x}>\bar{x}_2$

(C) $\bar{x}=\frac{\bar{x}_1>\bar{x}_2}{2}$ **(D)** $\bar{x}_1<\bar{x}<\bar{x}_2$

39. If a variable takes values 0, 1, 2, …, n with frequencies 1, ${}^nC_1$, ${}^nC_2$, …, ${}^nC_n$, then the A.M. is

(A) n **(B)** $(2^n/n)$

(C) $n+1$ **(D)** $(n/2)$

40. A.M. of C_0, $2\,C_1$, $3\,C_2$, …, $(n+1)\,C_n$, where $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$ is equal to

(A) $\frac{2^{n-1}(n+2)}{(n+1)}$ **(B)** $\frac{2^n(n+2)}{(n+1)}$

(C) $\frac{2^{n-1}(n+1)}{(n+2)}$ **(D)** $\frac{2^n(n+1)}{(n+2)}$

Practice Exercise 2

Single/Multiple Correct Choice Type Questions

1. The median of the items 6, 10, 4, 3, 9, 11, 22, 18 is

(A) 9 **(B)** 10

(C) 9.5 **(D)** 11

2. If the frequencies of first four numbers out of 1, 2, 4, 6, 8 are 2, 3, 3, 2, respectively, then the frequency of 8, if their AM is 5, is

(A) 4 **(B)** 5

(C) 6 **(D)** None of these

3. In a family, there are 8 men, 7 women and 5 children whose mean ages separately are respectively 24, 20 and 6 years. The mean age of the family is

(A) 17.1 years **(B)** 18.1 years

(C) 19.1 years **(D)** None of these

4. If $n = 10$, $\bar{x}=12$, $\sum x^2 = 1530$, then the coefficient of variation is

(A) 36% **(B)** 41%

(C) 25% **(D)** None of these

5. The arithmetic mean of first n odd natural numbers is

(A) n **(B)** $\frac{n+1}{2}$

(C) $n-1$ **(D)** None of these

6. The weighed mean of first n natural numbers whose weights are equal to the squares of corresponding numbers is

(A) $\frac{n+1}{2}$ **(B)** $\frac{3n(n+1)}{2(2n+1)}$

(C) $\frac{(n+1)(2n+1)}{6}$ **(D)** $\frac{n(n+1)}{2}$

7. In any discrete series (when all values are same), the relationship between MD about mean and SD is

(A) MD = SD **(B)** MD $\geq$ SD

(C) MD < SD **(D)** MD $\leq$ SD

8. A sample of 35 observations has the mean as 30 and SD as 4. A second sample of 65 observations from the same population has mean as 70 and SD as 3. The SD of the combined sample is

(A) 5.85 **(B)** 5.58

(C) 3.42 **(D)** None of these

9. The mean deviation of the numbers 3, 4, 5, 6, 7 is

(A) 0 **(B)** 1.2

(C) 5 **(D)** 25

Integer Type Questions

10. The standard deviation of a date is 6, when each observation is increased by 1, then find the S.D. of the new data.

11. If the S.D. of a set of observation is 8 and if each observation is divided by –2, then find the S.D. of the new set of observations.

12. The mean of 4, 7, 2, 8, 6 and a is 7. Find the mean deviation about median of these observations.

Answer Key

Practice Exercise 1

1. (A)	**2.** (B)	**3.** (A)	**4.** (B)	**5.** (A)	**6.** (A)
7. (C)	**8.** (C)	**9.** (B)	**10.** (A)	**11.** (C)	**12.** (D)
13. (D)	**14.** (B)	**15.** (B)	**16.** (B)	**17.** (D)	**18.** (A)
19. (D)	**20.** (C)	**21.** (A)	**22.** (A)	**23.** (D)	**24.** (B)
25. (A)	**26.** (A)	**27.** (A)	**28.** (B)	**29.** (C)	**30.** (C)
31. (B)	**32.** (A)	**33.** (B)	**34.** (A)	**35.** (B)	**36.** (B)
37. (B)	**38.** (D)	**39.** (D)	**40.** (A)		

Practice Exercise 2

1. (C)	**2.** (C)	**3.** (B)	**4.** (C)	**5.** (A)	**6.** (B)
7. (D)	**8.** (D)	**9.** (B)	**10.** 6	**11.** 4	**12.** 3

Solutions

Practice Exercise 1

1. Let sum of squares of deviation from any value A is y.

$$y=\sum_{i=1}^{n}(x_i-A)^2$$

Therefore, minimum value occurs at $A=\bar{x}$.

2. Weighted A.M. of the first n natural numbers whose weights (w) are equal is

$$\frac{w\cdot 1+w\cdot 2+\cdots+w\cdot n}{w+w+\cdots+w}=\frac{nw(n+1)}{2\cdot nw}=\frac{(n+1)}{2}$$

3. New mean $=\dfrac{\left(\frac{x_1}{\beta}+12\right)+\left(\frac{x_2}{\beta}+12\right)+\cdots+\left(\frac{x_n}{\beta}+12\right)}{n}$

$$=\frac{x_1+x_2+\cdots+x_n}{n\cdot\beta}+12=\frac{\bar{x}}{\beta}+12$$

4. $\dfrac{x_1+x_2+\cdots+x_{n-1}+x_n}{n}=\bar{X}\Rightarrow\dfrac{k+x_n}{n}=\bar{X}\Rightarrow x_n=n\bar{X}-k$

5. Mean number of children per family is

$$\frac{x_1+x_2+\cdots+x_{24}+x_{25}}{25}\Rightarrow\frac{50}{25}=2$$

6. Write down the discrete value $\alpha+4$, $\alpha-7/2$, $\alpha-5/2$, $\alpha-3$, $\alpha-2$, $\alpha+1/2$, $\alpha-1/2$, $\alpha+5$, $(\alpha>0)$, in increasing order $\alpha-7/2$, $\alpha-3$, $\alpha-5/2$, $\alpha-2$, $\alpha-1/2$, $\alpha+1/2$, $\alpha+4$, $\alpha+5$.

As the total number of terms is 8,

median = (4th term + 5th term)/2

$=(\alpha-2+\alpha-1/2)/2=\alpha-5/4$

7. Quartile deviation $=\dfrac{Q_3-Q_1}{Q_3+Q_1}$ which is approximately 2/3 of standard deviation. So, quartile deviation is nearly equal to $2/3\sigma$.

8. Median internally divides mode and mean in 2:1. Therefore,

Mode = 3 Median − 2 Mean

9. The best average of dealing a qualitative data is median.

10. Increasing order of 16, 10, 14, 11, 9, 8, 12, 6, 5 is 5, 6, 8, 9, 10, 11, 12, 14, 16. As thers are 9 terms, therefore

$$\text{Median}=\left(\frac{n+1}{2}\right)^{\text{th}}\text{ term}$$

$\Rightarrow$ Median = 5th term = 10

11. Median of 3, 5, 8, 10, 11, 12, 13, 14, 15, 18, 19, 21, 22, 27 is

(7th term + 8th term)/2 = (13 + 14)/2 = 13.5

12. $a\le x_i\le b\Rightarrow a\le x_1\le x_2\le x_3\le\cdots\le x_n\le b$ is divided into n equal parts of width $h=(b-a)/n$.

So,

$$x_i=a+i\cdot h,(i=1,2,\ldots,n)$$

Therefore,

$$\text{Var}(x)=\frac{\sum x_i^2}{n}-(\bar{x})^2=\frac{[(a+h)^2+(a+2h)^2+\cdots+(a+nh)^2]}{n}-\left(\frac{[(a+h)+(a+2h)+\cdots+(a+nh)]}{n}\right)^2$$

$$=\frac{h^2(n^2-1)}{12}=\frac{(b-a)^2(n-1)}{12(n+1)}\le(b-a)^2$$

13. '7' occurs maximum number of times, so frequency is maximum for '7'. Therefore, 7 is mode.

14. G.M. $=(1\cdot 2\cdot 2^2\cdots 2^n)^{1/(n+1)}=[2^{n(n+1)/2}]^{1/(n+1)}=2^{n/2}$

15. $Q_1=104$

and $\dfrac{Q_3-Q_1}{2}=18$

$$\Rightarrow Q_3=36+Q_1=36+104=140$$

16. $\sigma^2=\dfrac{1}{n}\displaystyle\sum_{r=1}^{n}r^2-\left(\frac{n+1}{2}\right)^2=\frac{1}{n}\cdot\frac{n(n+1)(2n+1)}{6}-\frac{(n+1)^2}{4}$

$$=(n+1)\left(\frac{2n+1}{6}-\frac{n+1}{4}\right)=\frac{(n+1)(4n+2-3n-3)}{12}=\frac{n^2-1}{12}$$

17. Coefficient of variations $=\dfrac{\sigma}{\text{Mean}}\times 100$

$$\Rightarrow 50=\frac{21.2}{\text{Mean}}\times 100$$

$\Rightarrow$ Mean = 42.4

18. Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be two series of observation.

Then $G_1^{\,n}=x_1\cdot x_2\cdot x_3\cdots x_n$, $G_2^{\,n}=y_1\cdot y_2\cdot y_3\cdots y_n$.

Now,

$$\frac{x_1}{y_1}\cdot\frac{x_2}{y_2}\cdot\frac{x_3}{y_3}\cdots\frac{x_n}{y_n}=G^n\text{ (given)}$$

$$\Rightarrow\frac{G_1^n}{G_2^n}=G^n\Rightarrow G=\frac{G_1}{G_2}$$

19. AM of two positive numbers $=\dfrac{a+b}{2}$

GM of two positive numbers $=\sqrt{ab}$

HM of two positive numbers $=\dfrac{2ab}{a+b}$

Case I: AM $\ge$ GM

Let AM < GM. Then

$$\frac{a+b}{2}<\sqrt{ab}\Rightarrow a+b-2\sqrt{ab}<0$$

$\Rightarrow (\sqrt{a}-\sqrt{b})^2 < 0$, which is not possible

$\Rightarrow$ AM $\geq$ GM

Case II: GM $\geq$ HM

Let GM < HM. Then

$$\sqrt{ab} < \frac{2ab}{a+b} \Rightarrow a+b < 2\sqrt{ab} \Rightarrow \frac{a+b}{2} < \sqrt{ab} \Rightarrow \text{AM} < \text{GM}$$

which again is not possible.

So, from case I and case II, we have

$$\text{AM} \geq \text{GM} \geq \text{HM}$$

20. Let $x_1, x_2, \ldots, x_6$ are the observations and $x_1 = 28$. Then

$$28 \cdot x_2, \ldots, x_6 = 13^6$$

$$\Rightarrow x_2, \ldots, x_6 = \frac{13^6}{28}$$

Now, correct observation is 36 instead of 28, so

$$36 \cdot x_2 \cdots x_6 = \frac{13^6}{28} \times 36$$

So, correct geometric mean = $13\left(\frac{9}{7}\right)^{1/6}$.

21. Let set of values be $x_1, x_2, \ldots, x_n$ and their mean be $\bar{x}$. Then

Deviation of values from mean = $(x_i - \bar{x})$

$$\Rightarrow \text{Sum of deviations} = \sum_{i=1}^{n}(x_i - \bar{x}) = \sum_{i=1}^{n} x_i - n\bar{x}$$

$$= \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i = 0$$

22. Let other two observation be a and b. Then

$$a + b + 1 + 2 + 6 = 5 \times 4.4$$

$$\Rightarrow a + b = 13 \quad (1)$$

Now,

$$\sigma^2 = \frac{1}{5}(a^2 + b^2 + 1 + 4 + 36) - (4.4)^2 = 8.24$$

$$\Rightarrow a^2 + b^2 = 97 \quad (2)$$

From Eqs. (1) and (2), we get

$$a = 9, b = 4$$

23. Let number of female workers be n_f and number of male workers be n_m. Then

$$\frac{200\, n_m + 4200\, n_f}{1000} = 3400$$

$$\Rightarrow 200\, n_m + 4200\,(1000 - n_m) = 34 \times 10^5$$

$\Rightarrow n_m = 200$ and $n_f = 800 \Rightarrow$ 20% male workers

Therefore, 80% are female workers.

24. Taking X as the product of variates $X_1, X_2, \ldots, X_r$ corresponding to r sets of observations, that is, $X = X_1 \cdot X_2 \cdots X_r$, we have

$$\log X = \log X_1 + \log X_2 + \cdots + \log X_r$$

$$\Rightarrow \sum \log X = \sum \log X_1 + \sum \log X_2 + \cdots + \sum \log X_r$$

$$\Rightarrow \frac{1}{n}\sum \log X = \frac{1}{n}\sum \log X_1 + \frac{1}{n}\sum \log X_2 + \cdots + \frac{1}{n}\sum \log X_r$$

$$\Rightarrow \log G = \log G_1 + \log G_2 + \cdots + \log G_r$$

$$\Rightarrow G = G_1 \cdot G_2 \ldots G_r$$

25. The data is 34, 38, 42, 44, 46, 48, 54, 55, 56, 76.

Median = Mean of 5th and 6th term = $\frac{46+48}{2} = 47 = \text{M}$

$$\text{Mean deviation} = \frac{1}{10}[|34-47| + |38-47| + |42-47| + |44-47| + |46-47| + |48-47| + |54-47| + |55-47| + |55-47| + |76-47|] = 8.5$$

26. $\bar{x} = A + \frac{\sum fd}{n} = 4 - \frac{11}{100} = 3.87$

and

$$\sigma = \sqrt{\frac{\sum d^2}{n} - \left(\frac{\sum d}{n}\right)^2} = \sqrt{\frac{257}{100} - \left(-\frac{11}{100}\right)^2} = 1.6$$

Coefficient of variation = $\frac{\sigma}{\bar{x}} \times 100 = \frac{1.6}{3.89} \times 100 = 41.13\%$

27. $\sum x = 235, \text{n} = 10, \sum x^2 = 6750$

$$\Rightarrow \sigma^2 = \frac{1}{10}\sum x_i^2 - \left(\frac{\sum x_i}{10}\right)^2 = \frac{1}{10} \cdot 6750 - \left(\frac{235}{10}\right)^2$$

$$= 675 - (23.5)^2 = 675 - 552.25 = 122.75$$

$$\Rightarrow \sigma = 11.08$$

Similarly for $\sum y = 250$, $n = 10$, $\sum y^2 = 6840$, we have

$$\sigma = 7.68$$

28. $n_1 = 700$, $n_2 = 300$, $\bar{x}_1 = 450$, $\bar{x} = 500$

It means wage of workers in the shift is $\bar{x}_2$, then

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

$$\Rightarrow 500 = \frac{700(450) + 300(\bar{x}_2)}{700 + 300}$$

$$\Rightarrow \bar{x}_2 = \frac{185000}{300} = \text{Rs. } 616.67$$

29. Petrol bought in first year = $\frac{4000}{7.50}$ litre

Petrol bought in second year = $\frac{4000}{8}$ litre

Petrol bought in third year = $\frac{4000}{8.50}$ litre

Total money = Rs. 12000

Therefore,

$$\text{Average cost per litre} = \frac{12000}{\frac{4000}{7.50} + \frac{4000}{8} + \frac{4000}{8.50}}$$

$$= \text{Rs. } 7.98$$

30. $\frac{1}{n}\sum_{i=1}^{n}(x_i+2)^2=18$ and $\frac{1}{n}\sum_{i=1}^{n}(x_i-2)^2=10$

Therefore, adding and subtracting, we get

$$\frac{2}{n}\sum_{i=1}^{n}(x_i^2+2^2)=28 \text{ and } \frac{8}{n}\sum_{i=1}^{n}x_i=8$$

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n}x_i^2=10 \text{ and } \frac{1}{n}\sum_{i=1}^{n}x_i=1$$

$$\sigma^2=\frac{1}{n}\sum_{i=1}^{n}x_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^2=10-1=9$$

$$\Rightarrow \sigma=3$$

31. Mode + 2(Mean) = 3 (Median)

$$\Rightarrow \text{Median}=\frac{27+2\times 24}{3}=25$$

32. $x_i=aU_i+bV_i$

$$\Rightarrow \frac{1}{n}\sum x_i=\frac{a\sum U_i}{n}+\frac{b\sum V_i}{n}$$

$$\Rightarrow \bar{x}=a\bar{U}+b\bar{V}$$

33. $\bar{x}=\frac{\sum_{i=1}^{n}x_i}{n}\Rightarrow \sum_{i=1}^{n}x_i=n\bar{x}$ and hence

$$\frac{1}{n}\sum_{i=1}^{n}(x_i+2i)$$

$$=\frac{1}{n}\sum_{i=1}^{n}x_i+\frac{2(1+2+3+\cdots+n)}{n}=\bar{x}+\frac{2n(n+1)}{2n}=\bar{x}+(n+1)$$

34.

Midpoint (x)	Frequency (f)	fx
10	17	170
30	f_1	$30f_1$
50	32	1600
70	f_2	$70f_2$
90	19	1710
	120	$30f_1+70f_2+3480$

Therefore,

$$\bar{x}=\frac{1}{120}\sum fx\Rightarrow 50=\frac{1}{120}\times(30f_1+70f_2+3480)$$

$$\Rightarrow 600=3f_1+7f_2+348$$

$$\Rightarrow 3f_1+7f_2=252 \quad (1)$$

Also,

$$f_1+f_2+68=120$$

$$\Rightarrow f_1+f_2=52 \quad (2)$$

Solving Eqs. (1) and (2), we get

$$f_2=24, f_1=28$$

35. New mean $=\frac{(x_1+2)+(x_2+2^2)+\cdots+(x_n+2^n)}{n}$

$$=\frac{x_1+x_2+\cdots+x_n}{n}+\frac{(2+2^2+\cdots+2^n)}{n}$$

$$=\bar{x}+\frac{2}{n}(2^n-1)=\bar{x}+\frac{2^{n+1}}{n}-\frac{2}{n}$$

36. $n=7+10+12+15+17+17+25=103$

Q_1 = Size of $\left(\frac{103+1}{4}\right)$ item = Size of 26th item = 12

Q_3 = Size of $3\left(\frac{103+1}{4}\right)$ item = Size of 78th item = 17

Therefore,

$$\frac{Q_3-Q_1}{2}=\frac{17-12}{2}=2.5$$

37. Incorrect sum of all items = 78.4 × 25 = 1960 kg

Hence,

Correct sum = 1960 − 69 + 96 = 1987 kg

and

$$\text{Correct average}=\frac{1987}{25}=79.48 \text{ kg}$$

38. If n_1 and n_2 are the number of items in the two distributions having means $\bar{x}_1$ and $\bar{x}_2$.

$$\bar{x}=\frac{n_1\bar{x}_1+n_2\bar{x}_2}{n_1+n_2}$$

Therefore,

$$\bar{x}-\bar{x}_1=\frac{n_1\bar{x}_1+n_2\bar{x}_2}{n_1+n_2}-\bar{x}_1=\frac{n_2(\bar{x}_2-\bar{x}_1)}{n_1+n_2}>0 \quad (\because \bar{x}_2>\bar{x}_1)$$

$$\Rightarrow \bar{x}>\bar{x}_1$$

Similarly,

$$\bar{x}-\bar{x}_2=\frac{n(\bar{x}_1-\bar{x}_2)}{n_1+n_2}<0$$

$$\Rightarrow \bar{x}<\bar{x}_2$$

Hence,

$$\bar{x}_1<\bar{x}<\bar{x}_2$$

39. $\bar{X}=\frac{0\cdot 1+1\cdot{}^nC_1+2\cdot{}^nC_2+3\cdot{}^nC_3+\cdots+n\cdot{}^nC_n}{1+{}^nC_1+{}^nC_2+\cdots+{}^nC_n}$

$$=\frac{\sum_{r=0}^{n}r\cdot{}^nC_r}{\sum_{r=0}^{n}{}^nC_r}=\frac{\sum_{r=1}^{n}r\cdot\frac{n}{r}\,{}^{n-1}C_{r-1}}{\sum_{r=0}^{n}{}^nC_r}=\frac{n\sum_{r=1}^{n}{}^{n-1}C_{r-1}}{\sum_{r=0}^{n}{}^nC_r}=\frac{n2^{n-1}}{2^n}=\frac{n}{2}$$

40. $(1+x)^n=C_0+C_1x+C_2x^2+\cdots+C_nx^n$

Multiplying with x, we get

$$x(1+x)^n=C_0x+C_1x^2+C_2x^3+\cdots+C_nx^{n+1}$$

Differentiating with respect to x, we get

$$nx(1+x)^{n-1} + (1+x)^n = C_0 + 2C_1x + 3C_2x^2 + \cdots + (n+1)C_nx^n$$

Putting $x = 1$, we get

$$n(2^{n-1}) + 2^n = C_0 + 2C_1 + 3C_2 + \cdots + (n+1)C_n$$

So,

$$\text{A.M.} = \frac{C_0+2C_1+3C_2+\cdots+(n+1)C_n}{(n+1)} = \frac{2^{n-1}(n+2)}{(n+1)}$$

Practice Exercise 2

1. Let us arrange the items in ascending order 3, 4, 6, 9, 10, 11, 18, 22.

In this data, the number of items is $n = 8$, which is even. Therefore,

$$\text{Median} = M = \text{Average of } \left(\frac{n}{2}\right)^{\text{th}} \text{ and } \left(\frac{n}{2}+1\right)^{\text{th}} \text{ terms}$$

$$= \text{Average of } \left(\frac{8}{2}\right)^{\text{th}} \text{ and } \left(\frac{8}{2}+1\right)^{\text{th}} \text{ terms}$$

$$= \text{Average of } 4^{\text{th}} \text{ and } 5^{\text{th}} \text{ terms}$$

$$= \frac{9+10}{2} = \frac{19}{2} = 9.5$$

2. Here, mean $A = 5$

Let the frequency of 8 be x. Then, by the formula, we have

$$A = \frac{\sum xf}{\sum f}$$

$$5 = \frac{1\cdot 2 + 2\cdot 3 + 4\cdot 3 + 6\cdot 2 + 8\cdot x}{2+3+3+2+x} = \frac{32+8x}{10+x}$$

$\Rightarrow 18 = 3x \Rightarrow x = 6$

3. Here, we have three collections for which $A_1 = 24$, $n_1 = 8$, $A_2 = 20$, $n_2 = 7$ and $A_3 = 6$, $n_3 = 5$. Their combined mean is the required mean.

By the formula,

$$A = \frac{n_1A_1 + n_2A_2 + n_3A_3}{n_1+n_2+n_3}$$

Hence,

$$A = \frac{8\times 24 + 7\times 20 + 5\times 6}{8+7+3}$$

$$= \frac{192+140+30}{20} = \frac{362}{20} = 18.1$$

4. We know that

$$\sigma = \sqrt{\frac{\sum x^2}{n} - (\bar{x})^2}$$

$$= \sqrt{\frac{1530}{10} - (12)^2} = \sqrt{9} = 3$$

Therefore, coefficient of variation $= \dfrac{\sigma}{\bar{x}} \times 100 = \dfrac{3\times 100}{12} = 25\%$

5. The first n odd natural number are 1, 3, 5, …, $(2n - 1)$
Sum of these numbers

$$\sum x = 1 + 3 + 5 + \ldots + (2n-1)$$

$$= \frac{n}{2}[1 + (2n-1)] = \frac{n}{2}2n = n^2$$

Thus,

$$\bar{x} = \frac{1}{n}\sum x = \frac{n^2}{n} = n$$

6. Weighted mean $= \dfrac{1\cdot 1^2 + 2\cdot 2^2 + 3\cdot 3^2 + \cdots + n\cdot n^2}{1^2+2^2+3^2+\cdots+n^2}$

$$= \frac{\sum n^3}{\sum n^2} = \frac{\frac{[n(n+1)]}{2}\frac{n(n+1)}{2}}{\frac{n(n+1)(2n+1)}{6}} = \frac{3n(n+1)}{2(2n+1)}$$

7. Let x_i/f_i; $i = 1, 2, \ldots, n$ be a frequency distribution. Then

$$\text{SD} = \sqrt{\frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^2}$$

and

$$\text{MD} = \frac{1}{N}\sum_{i=1}^{n} f_i\,|x_i - \bar{x}|$$

Let $|x_i - \bar{x}| = z_i;\ i = i, 2, \ldots, n$. Then

$$(\text{SD})^2 - (\text{MD})^2 = \frac{1}{N}\sum_{i=1}^{n} f_i z_i^2 - \left[\frac{1}{N}\sum_{i=1}^{n} f_i z_i\right]^2 = \sigma^2 \geq 0$$

$$\Rightarrow \text{SD} \geq \text{MD}$$

8. Here,

$$n_1 = 35,\ \bar{x}_1 = 30,\ \sigma_1 = 4;\ n_2 = 65,\ \bar{x}_2 = 70,\ \sigma_2 = 3$$

Therefore,

$$\bar{x}_{12} = \frac{35\times 30 + 65\times 70}{35+65} = 56$$

$$\sigma_{12} = \sqrt{\frac{35\times 16 + 65\times 9 + 35\times 26^2 + 65\times 14^2}{35+65}}$$

$$= \sqrt{\frac{35(16+676) + 65(9+196)}{100}}$$

$$= \sqrt{375.45} = 19.39$$

9. Mean $\bar{x} = \dfrac{3+4+5+6+7}{5} = 5$

$N = 5$

x	$\lvert x - \bar{x} \rvert$
3	2
4	1
5	0
6	1
7	2
	$\sum \lvert x - \bar{x} \rvert = 6$

Therefore, mean deviation from the mean $= \dfrac{\sum |x - \bar{x}|}{N} = \dfrac{6}{5} = 1.2.$

10. We know that X and Y are two variables such that $Y = X + b$, where b is a constant.

Then,

$$\text{var}(X) = \text{var}(Y) \Rightarrow \sigma_X = \sigma_Y$$

Thus, the standard deviation of the new data is also 6.

11. We know that if X and Y are two variables such that

$$Y = \frac{X}{a},\ a \neq 0$$

Then,

$$\sigma_Y = \frac{1}{|a|}\sigma_X, \text{ S.D. of the new observations}$$

$$= \frac{8}{|-2|} = 4$$

12. Here, number of observations, $n = 6$

Given

$$\text{Mean} = 7$$

$$\Rightarrow \frac{4+7+2+8+6+a}{6} = 7 \Rightarrow 27 + a = 42 \Rightarrow a = 15$$

Therefore, arranging the observations in ascending order, we get 2, 4, 6, 7, 8, 15.

Hence, Median is

$$k = \text{Mean of } \left(\frac{n}{2}\right)^{\text{th}} \text{ observation and } \left(\frac{n}{2}+1\right)^{\text{th}} \text{ observation}$$

$$= \frac{3^{\text{rd}} \text{ observations} + 4^{\text{th}} \text{ observation}}{2} = \frac{6+7}{2} = 6.5$$

Calculation of mean deviation:

x_i	$\lvert x_i - k \rvert$
2	4.5
4	2.5
6	0.5
7	0.5
8	1.5
15	8.5
Total	18

Therefore, mean deviation about median is

$$\frac{\sum |x_i - k|}{n} = \frac{18}{6} = 3$$

Solved JEE 2017 Questions

JEE Main 2017

1. A box contains 15 green and 10 yellow balls. If 10 balls are randomly drawn, one-by-one, with replacement, then the variance of the number of green balls drawn is:

(A) 6 **(B)** 4

(C) $\frac{6}{25}$ **(D)** $\frac{12}{5}$

(OFFLINE)

Solution: Total number of balls = 25 (15 balls are in box + 10 yellow balls).

The variance is

$$\sigma^2 = npq$$

Let n be the total number of trials, p be the probability of happening and q be the probability of not happening:

$$n = 10;\ p = \frac{15}{25} = \frac{3}{5};\ q = \frac{10}{25} = \frac{2}{5}$$

Thus, the variance of the number of green balls drawn is obtained as follows:

$$\sigma^2 = 10 \times \frac{3}{5} \times \frac{2}{5} = \frac{60}{25} = \frac{12}{5}$$

Hence, the correct answer is option (D).

2. The mean age of 25 teachers in a school is 40 years. A teacher retires at the age of 60 years and a new teacher is appointed in his place. If now the mean age of the teachers in this school is 39 years, then the age (in years) of the newly appointed teacher is

(A) 25 **(B)** 35

(C) 30 **(D)** 40

(ONLINE)

Solution: The mean age of 25 teachers is 40.

The sum of ages of 25 teachers is 40 × 25 = 1000.
A teacher retires at the age of 60 and a new teacher is appointed.
Let the age of newly appointed teacher be x. Then it is given that new mean age is 39. Therefore,

$$\frac{1000 - 60 + x}{25} = 39$$

$$\Rightarrow 1000 - 60 + x = 39 \times 25$$

$$\Rightarrow 940 + x = 975$$

$$\Rightarrow x = 975 - 940$$

$$\Rightarrow x = 35$$

Thus, the age of newly appointed teacher is 35 years.
Hence, the correct answer is option (B).

3. The sum of 100 observations and the sum of their squares are 400 and 2474, respectively. Later on, three observations, 3, 4 and 5, were found to be incorrect. If the incorrect observations are omitted, then the variance of the remaining observations is

(A) 8.00 **(B)** 8.25

(C) 9.00 **(D)** 8.50

(ONLINE)

Solution: We have

$$\sum_{i=1}^{100} x_i = 400$$

$$\sum_{i=1}^{100} x_i^2 = 2425$$

The variance of the remaining observations is

$$\sigma^2 = \frac{\sum x_i^2}{N} - \left(\frac{\sum x_i}{N}\right)^2$$

$$\Rightarrow \frac{2425}{97} - \left(\frac{388}{97}\right)^2$$

$$\Rightarrow \frac{2425}{97} - 16 \Rightarrow \frac{2425 - 1552}{97} = \frac{873}{97} = 9$$

Hence, the correct answer is option (C).

Appendix

Chapterwise Solved JEE 2018 Questions

Chapter 1: Sets, Relations and Functions

JEE Main 2018

1. Two sets A and B are as under:

$A = \{(a, b) \in \mathbb{R} \times \mathbb{R} : |a - 5| < 1 \text{ and } |b - 5| < 1\}$

$B = \{(a, b) \in \mathbb{R} \times \mathbb{R} : 4(a - 6)^2 + 9(b - 5)^2 \leq 36\}$

Then

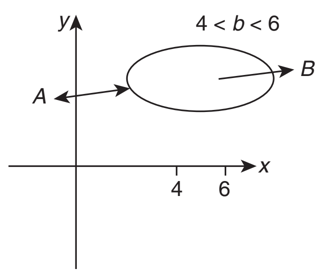
(A) $A \subset B$
(B) $A \cap B = \phi$ (an empty set)
(C) neither $A \subset B$ nor $B \subset A$
(D) $B \subset A$

(Offline)

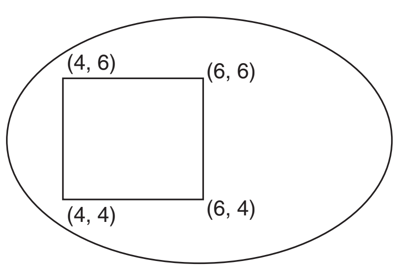
Solution

(A) We have

$$-1 < a - 5 < 1$$
$$4 < a < 6$$
$$4 < b < 6$$



$$\frac{(a-6)^2}{3^2} + \frac{(b-5)^2}{2^2} = 1$$



The ellipse passes through (4, 6); therefore,

$$\frac{16+9-36}{36} = \frac{25-36}{36} < 0$$

That is, $A \subset B$.

2. Let $S = \{x \in \mathbb{R} : x \geq 0 \text{ and } 2|\sqrt{x} - 3| + \sqrt{x}(\sqrt{x} - 6) + 6 = 0\}$. Then, S

(A) contains exactly one element.
(B) contains exactly two elements.
(C) contains exactly four elements.
(D) is an empty set.

(Offline)

Solution

(B) Given: $2\left|\sqrt{x} - 3\right| + \sqrt{x}(\sqrt{x} - 6) + 6 = 0$

- **Case 1:** $\sqrt{x} \geq 3 \Rightarrow x \geq 9$

$$2\sqrt{x} - 6 + x - 6\sqrt{x} + 6 = 0$$
$$x - 4\sqrt{x} = 0$$
$$\sqrt{x}(\sqrt{x} - 4) = 0$$
$$\sqrt{x} = 0, 4 \;\Rightarrow\; x = 0, 16$$

So, $x = 16$ is accepted.

- **Case 2:** $\sqrt{x} < 3 \Rightarrow x < 9$

$$-2\sqrt{x} + 6 + x - 6\sqrt{x} + 6 = 0$$
$$x - 8\sqrt{x} + 12 = 0$$
$$(\sqrt{x} - 6)(\sqrt{x} - 2) = 0$$
$$\sqrt{x} = 2, 6 \Rightarrow x = 4, 36$$

So, $x = 4$ is accepted.

Therefore, S has two elements: $x = 4$ and $x = 16$.

3. Let N denote the set of all natural numbers. Define two binary relations on N as $R_1 = \{(x, y) \in N \times N : 2x + y = 10\}$ and $R_2 = \{(x, y) \in N \times N : x + 2y = 10\}$. Then

(A) range of R_1 is {2, 4, 8}.
(B) range of R_2 is {1, 2, 3, 4}.
(C) both R_1 and R_2 are symmetric relations.
(D) both R_1 and R_2 are transitive relations.

(Online)

Solution

(B) Given: N is set of all natural numbers

$$R_1 = \{(x, y) \in N \times N : 2x + y = 10\}$$

and $$R_2 = \{(x, y) \in N \times N : x + 2y = 10\}$$

So, $R_1 = \{(1, 8), (2, 6), (3, 4), (4, 2)\}$ and
$R_2 = \{(8, 1), (6, 2), (4, 3), (2, 4)\}$

Therefore, range of $R_2 = \{1, 2, 3, 4\}$

4. Consider the following two binary relations on the set $A = \{a, b, c\}$:

$R_1 = \{(c, a), (b, b), (a, c), (c, c), (b, c), (a, a)\}$ and
$R_2 = \{(a, b), (b, a), (c, c), (c, a), (a, a), (b, b), (a, c)\}$.

Then:

(A) both R_1 and R_2 are not symmetric.
(B) R_1 is not symmetric but it is transitive.
(C) R_2 is symmetric but it is not transitive.
(D) both R_1 and R_2 are transitive. **(Online)**

Solution

(C) Given: $R_1 = \{(c, a), (b, b), (a, c), (c, c), (b, c), (a, a)\}$
$R_2 = \{(a, b), (b, a), (c, c), (c, a), (a, a), (b, b), (a, c)\}$

Here, R_2 is symmetric as for any $(a_1, a_2) \in R_2$, we have $(a_2, a_1) \in R_2$. However, R_1 is not symmetric.

For checking transitivity, we observe for R_2 that $(b, a) \in R_2$, $(a, c) \in R_2$ but $(b, c) \notin R_2$.

Similarly, for R_1 $(b, c) \in R_1$, $(c, a) \in R_1$ but $(b, a) \notin R_1$. Therefore, neither R_1 nor R_2 is transitive.

5. Let $f : A \to B$ be a function defined as $f(x) = \frac{x-1}{x-2}$, where $A = R - \{2\}$ and $B = R - \{1\}$. Then, f is

(A) invertible and $f^{-1}(y) = \frac{3y-1}{y-1}$

(B) invertible and $f^{-1}(y) = \frac{2y-1}{y-1}$

(C) invertible and $f^{-1}(y) = \frac{2y+1}{y-1}$

(D) not invertible **(Online)**

Solution

(B) Given: $f : A \to B$ be a function defined as $f(x) = \frac{x-1}{x-2}$, where $A = R - \{2\}$ and $B = R - \{1\}$

So, $f(x)$ is bijective function or invertible function.

$$y = \frac{x-1}{x-2} \Rightarrow xy - 2y = x - 1$$
$$\Rightarrow x(y-1) = 2y - 1$$
$$\Rightarrow x = \frac{2y-1}{y-1}$$
$$\Rightarrow f-1(y) = \frac{2y-1}{y-1}$$

JEE Advanced 2018

6. For every twice differentiable function $f: \mathbb{R} \to [-2, 2]$ with $(f(0))^2 + (f'(0))^2 = 85$, which of the following statement(s) is(are) TRUE?

(A) There exist $r, s \in \mathbb{R}$, where $r < s$, such that f is one–one on the open interval (r, s).
(B) There exists $x_0 \in (-4, 0)$ such that $|f'(x_0)| \leq 1$.
(C) $\lim_{x\to\infty} f(x) = 1$.
(D) There exists $\alpha \in (-4, 4)$ such that $f(\alpha) + f''(\alpha) = 0$ and $f'(\alpha) \neq 0$.

(Paper-1)

Solution

(A), (B), (D) Let us check all four options as follows:

- **Option (A):** It is given that

$$f: \mathbb{R} \to [-2, 2] \text{ with } (f(0))^2 + (f'(0))^2 = 85$$

Suppose $f(x)$ is constant, then

$$f'(x) = 0$$

Therefore, $(f(0))^2 = 85$

$$\Rightarrow f(0) = \pm\sqrt{85}$$

However, $f: \mathbb{R} \to [-2, 2]$

Therefore, $f(0) \neq \pm\sqrt{85}$

Thus, $f(x)$ cannot be constant throughout the domain and it is possible to find (r, s), where $f(x)$ is one–one.

Hence, option (A) is true.

- **Option (B):** Using Lagrange's mean value theorem for $x_0 \in (-4, 0)$, we get

$$|f'(x_0)| = \left|\frac{f(-4) - f(0)}{-4}\right| \leq 1$$

Hence, option (B) is true.

- **Option (C):** Let is consider

$$f(x) = a\sin bx$$

Therefore, $f'(x) = ab\cos bx$

Since $f(x) \in [-2, 2]$, we can write as $a \in [-2, 2]$.

Also, it is given that

$$(f(0))^2 + (f'(0))^2 = 85$$

That is, $(0)^2 + (ab)^2 = 85 \Rightarrow (ab)^2 = 85$

Therefore, we have $\lim_{x\to\infty} f(x) \neq 1$.

Hence, option (C) is incorrect.

- **Option (D):** Now, for $\alpha \in (-4, 4)$, we get

$$f(\alpha) + f''(\alpha) = 0$$

and $f'(\alpha) \neq 0$

Now, $g(x) = f^2(x) + (f'(x))^2$

Using Lagrange's mean value theorem, we have

$$|f'(x)| \leq 1$$

Also, $|f(x)| \leq 2$

Therefore,

$$g(x_1) \leq 2^2 + 1^2$$
$$\Rightarrow g(x_1) \leq 5 \ \forall x_1 \in (-4, 0)$$

Similarly, for $x_2 \in (0, 4)$:

$$g(x_2) \leq 5$$

and $g(0) = 85$

Thus $g(x)$ has maxima in (x_1, x_2) at α. Therefore,

$$g'(\alpha) = 0 \text{ and } g(\alpha) \geq 85$$

Therefore, $2f'(\alpha)(f(\alpha) + f''(\alpha)) = 0$

Now, suppose $f'(\alpha) = 0$, then

$$g(\alpha) = f^2(\alpha) \ngeq 85$$

That is, $f'(\alpha) \neq 0$

Hence, option (D) is correct.

7. The number of 5 digit numbers which are divisible by 4, with digits from the set {1, 2, 3, 4, 5} and the repetition of digits is allowed, is ___________.

(Paper-1)

Solution

(625) The given set is {1, 2, 3, 4, 5}

For numbers to be divisible by 4, the last two digits of this 5 digit number can be 12, 24, 32, 44 and 52.

Hence, the total number of 5 digit numbers can be

$$5 \times 5^3 = 625$$

8. Let X be a set with exactly 5 elements and Y be a set with exactly 7 elements. If α is the number of one–one functions from X to Y and β is the number of onto functions from Y to X, then the value of $\frac{1}{5!}(\beta - \alpha)$ is___________. **(Paper-2)**

Solution

(119) Given:

- X is a set with 5 elements.
- Y is set with exactly 7 elements.
- The number of one–one functions from X to Y is α.
- The number of onto functions from Y to X is β.

Now, we need to find the value of $\frac{1}{5!}(\beta - \alpha)$:

Let us make 5 groups out of 7 elements of Y and doing permutation up to 5 elements of X. Therefore,

$$\beta - \alpha = \frac{7!}{3! \times 4!} \times 5! + \frac{7!}{(2!)^3 \times 3!} \times 5!$$
$$= ({}^7C_3 + 3\,{}^7C_3)5!$$
$$= 4 \times {}^7C_3 \times 5!$$

Therefore,

$$\frac{(\beta - \alpha)}{5!} = 4 \times {}^7C_3 - {}^7C_5 = 4 \times \frac{7!}{4!3!} - \frac{7!}{5!2!}$$
$$= \frac{4 \times 7 \times 6 \times 5}{3 \times 2} - \frac{7 \times 6}{2}$$
$$= 4 \times 35 - 2!$$
$$= 119$$

9. Let $E_1 = \left\{x \in \mathbb{R} : x \neq 1 \text{ and } \frac{x}{x-1} > 0\right\}$ and

$$E_2 = \left\{x \in E_1 : \sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right) \text{is a real number}\right\}$$

(Here, the inverse trigonometric function $\sin^{-1}x$ assumes values in $\left[\frac{\pi}{2}, \frac{\pi}{2}\right]$.)

Let $f : E_1 \to \mathbb{R}$ be the function defined by $f(x) = \log_e\left(\frac{x}{x-1}\right)$ and $g: E_2 \to \mathbb{R}$ be the function defined by $g(x) = \sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right)$.

LIST-I	LIST-II
P. The range of f is	**1.** $\left(-\infty \frac{1}{1-e}\right] \cup \left[\frac{e}{e-1}, \infty\right)$
Q. The range of g contains	**2.** (0, 1)
R. The domain of f contains	**3.** $\left[-\frac{1}{2}, \frac{1}{2}\right]$
S. The domain of g is	**4.** $(-\infty, 0) \cup (0, \infty)$
	5. $\left(-\infty, \frac{e}{e-1}\right]$
	6. $(-\infty, 0) \cup \left(\frac{1}{2}, \frac{e}{e-1}\right]$

The correct option is:

(A) P→4; Q→2; R→1; S→1

(B) P→3; Q→3; R→6; S→5

(C) P→4; Q→2; R→1; S→6

(D) P→4; Q→3; R→6; S→5 **(Paper-2)**

Solution

(A) We have

$$E_1 = \left\{x \in \mathbb{R} : x \neq 1 \text{ and } \frac{x}{x-1} > 0\right\}$$
$$f : E_1 \to \mathbb{R}; f(x) = \log_e\left(\frac{x}{x-1}\right)$$
$$E_2 = x \in E_1 : \sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right) \text{ is the real number.}$$
$$g : E_2 \to \mathbb{R};\ g(x) = \sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right)$$

- Domain of $f(x) : (-\infty, 0) \cup (1, \infty)$ and this is contained in
$$\left(-\infty, \frac{1}{1-e}\right] \cup \left[\frac{e}{e-1}, \infty\right)$$
- Range of $f(x) : (-\infty, 0) \cup (0, \infty)$
$$\text{Domain of } g(x) = \left(-\infty, \frac{1}{1-e}\right] \cup \left[\frac{e}{e-1}, \infty\right)$$
- Range of $g(x) = (0, 1)$
Therefore, the correct mapping is P→4; Q→2; R→1; S→1.
Hence, option (A) is correct.

Chapter 2: Trigonometric Ratios and Identities

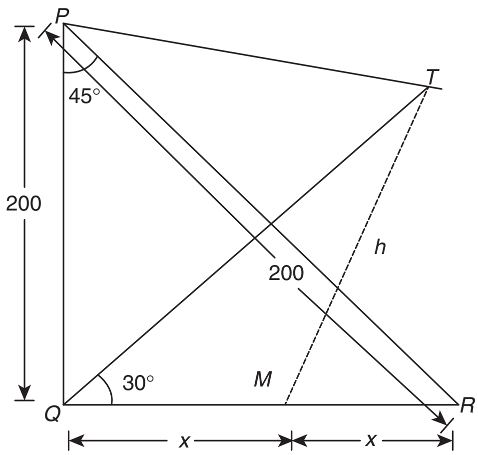
JEE Main 2018

10. *PQR* is a triangular park with *PQ* = *PR* = 200 m. A TV tower stands at the mid-point of *QR*. If the angles of elevation of the top of the tower at *P*, *Q* and *R* are, respectively, 45°, 30° and 30°, then the height of the tower (in m) is

(A) 50 **(B)** $100\sqrt{3}$
(C) $50\sqrt{3}$ **(D)** 100 **(Offline)**

Solution

(D) Let height of the tower be $TM = h$ and $QM = MR = x$.



From this figure, we can say that

$$PM = \sqrt{40000 - x^2}$$

$$\tan 45° = \frac{TM}{PM} = \frac{h}{\sqrt{40000 - x^2}}$$

$$h^2 = 40000 - x^2$$

$$h^2 + x^2 = 40000 \quad (1)$$

Therefore, $\tan 30° = \frac{TM}{QM}$

$$x = \sqrt{3}h \quad (2)$$

From Eqs. (1) and (2), we have

$$4h^2 = 40000$$

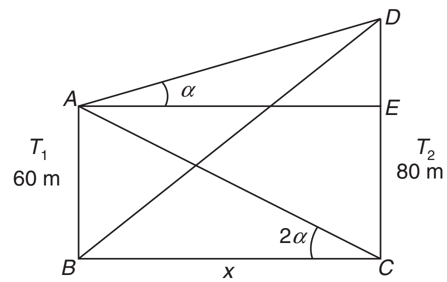
$$h = 100 \text{ m}$$

11. A tower T_1 of height 60 m is located exactly opposite to a tower T_2 of height 80 m on a straight road. From the top of T_1, if the angle of depression of the foot of T_2 is twice the angle of elevation of the top of T_2, then the width (in m) of the road between the feet of the towers T_1 and T_2 is

(A) $10\sqrt{2}$ **(B)** $10\sqrt{3}$
(C) $20\sqrt{3}$ **(D)** $20\sqrt{2}$ **(Online)**

Solution

(C) From given data, we draw the following figure:



Now, ΔADE $\tan\alpha = \frac{DE}{AE} = \frac{20}{x}$, $\tan 2\alpha = \frac{60}{x}$

$$\Rightarrow \frac{2\tan\alpha}{(1-\tan^2\alpha)\tan\alpha} = \frac{60}{20} = 3$$

$$\Rightarrow \frac{2}{3} = 1 - \tan^2\alpha \Rightarrow \tan^2\alpha = \frac{1}{3} \Rightarrow \tan\alpha = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \alpha = 30°$$

Therefore, $\tan\alpha = \frac{20}{x} \Rightarrow x = 20\cot\alpha = 20\sqrt{3}$

12. If an angle *A* of a ΔABC satisfies $5\cos A + 3 = 0$, then the roots of the quadratic equation, $9x^2 + 27x + 20 = 0$ are

(A) sec*A*, cot*A* **(B)** sin*A*, sec*A*
(C) sec*A*, tan*A* **(D)** tan*A*, cos*A* **(Online)**

Solution

(C) Given:

$$5\cos A + 3 = 0$$

$$\Rightarrow \cos A = \frac{-3}{5}$$

So, it is clear that $A \in (90°, 180°)$.

Now, the roots of equation $9x^2 + 27x + 20 = 0$ are $-5/3$ and $-4/3$.

Therefore, the roots of the quadratic equation, $9x^2 + 27x + 20 = 0$, are sec*A* and tan*A*.

Chapter 3: Trigonometric Equation and Inequation

JEE Main 2018

13. If sum of all the solutions of the equation $8\cos x \left[\cos\left(\frac{\pi}{6}+x\right)\cdot\cos\left(\frac{\pi}{6}-x\right)-\frac{1}{2}\right] = 1$ in $[0, \pi]$ is $k\pi$, then k is equal to

(A) $\frac{13}{9}$ **(B)** $\frac{8}{9}$
(C) $\frac{20}{9}$ **(D)** $\frac{2}{3}$ **(Offline)**

Solution

(A) Given:

$$8\cos x\left[\cos\left(\frac{\pi}{6}+x\right)\cdot\cos\left(\frac{\pi}{6}-x\right)-\frac{1}{2}\right]=1$$

That is, $$8\cos x\left(\left(\cos^2\frac{\pi}{6}-\sin^2 x\right)-\frac{1}{2}\right)=1$$

$$8\cos x\left(\left(\frac{3}{4}-\sin^2 x\right)-\frac{1}{2}\right)=1$$

$6\cos x - 8\cos x(\sin^2 x) - 4\cos x = 1$

$6\cos x - 8\cos x(1 - \cos^2 x) - 4\cos x - 1 = 0$

$8\cos^3 x - 6\cos x - 1 = 0$

$2(4\cos^3 x - 3\cos x) = 1$

$2(\cos 3x) = 1$

That is, $\cos 3x = \frac{1}{2}$

Now, $3x = 2n\pi \pm \frac{\pi}{3}$

$$x=(6n\pm 1)\frac{\pi}{9}$$

Now, for $n = 0$: $x=\frac{\pi}{9}$

and for $n = 1$: $x=\frac{7\pi}{9}, \frac{5\pi}{9}$

Now, the sum is

$$S=\frac{13\pi}{9}=k\pi \Rightarrow k=\frac{13}{9}$$

14. If $\tan A$ and $\tan B$ are the roots of the quadratic equation, $3x^2 - 10x - 25 = 0$, then the value of $3\sin^2(A+B)-10\sin(A+B)\cos(A+B)-25\cos^2(A+B)$ is

(A) −10 **(B)** 10
(C) −25 **(D)** 25 **(Online)**

Solution

(C) Given: The roots of equation $2x^2 - 10x - 25 = 0$ are $\tan A$ and $\tan B$.

So, $\tan A+\tan B=\frac{10}{3}$, $\tan A\tan B=\frac{-25}{3}$

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}=\frac{(10/3)}{1+\frac{25}{3}}=\frac{10}{28}=\frac{5}{14}$$

Then, $\sin(A+B)=\frac{5}{\sqrt{221}}$, $\cos(A+B)=\frac{14}{\sqrt{221}}$

Therefore,

$$3\sin^2(A+B)-10\sin(A+B)\cos(A+B)-25\cos^2(A+B)$$

$$=\frac{75}{221}-\frac{10\times 70}{221}-\frac{25\times 196}{221}=-25$$

15. The number of solutions of $\sin 3x = \cos 2x$, in the interval $\left(\frac{\pi}{2},\pi\right)$, is

(A) 1 **(B)** 2
(C) 3 **(D)** 4 **(Online)**

Solution

(A) Given: $\sin 3x = \cos 2x$

That is, $3\sin x - 4\sin^3 x = 1 - 2\sin^2 x$

$\Rightarrow 4\sin^3 x - 2\sin^2 x - 3\sin x + 1 = 0$

$\Rightarrow 4\sin^2 x(\sin x - 1) + 2\sin x(\sin x - 1) - (\sin x - 1) = 0$

$\Rightarrow (\sin x - 1)(4\sin^2 x + 2\sin x - 1)$

$\Rightarrow \sin x = 1$ or $4\sin^2 x + 2\sin x - 1 = 0$

$4\sin^2 x + 2\sin x - 1$ $\sin x=\frac{-2\pm\sqrt{4+16}}{2.4}$

$4\sin^2 x + 2\sin x - 1$ $\sin x=\frac{-1\pm\sqrt{5}}{4}$

We know that $\sin x>0\,\forall x\in(\pi/2,\pi)$ and $\sin x \neq 1$,

$$\sin x=\frac{\sqrt{5}-1}{4}$$

Therefore, the number of solution is only one.

16. The number of values of k for which the system of linear equations,

$(k + 2)x + 10y = k$
$kx + (k + 3)y = k - 1$

has **no solution,** is

(1) 1 **(2)** 2
(3) 3 **(4)** infinitely many

(JEE Main 2018 Online Paper-3)

Solution

(1) Given:

$(k + 2)x + 10y = k$
$kx + (k + 3)y = k - 1$

For no solution

$$\frac{k+2}{k}=\frac{10}{k+3}\neq\frac{k}{k-1}$$

$\Rightarrow (k + 2)(k + 3) = 10k$

$\Rightarrow k^2 - 5k + 6 = 0$

$\Rightarrow k = 2, 3$

But for $k \neq 2$ for $k = 2$ both lines identical

So, $k = 3$ only

Therefore, the number of values of k is 1.

Chapter 4: Properties of Triangle

JEE Main 2018

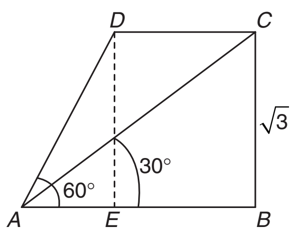
17. An aeroplane flying at a constant speed, parallel to the horizontal ground, $\sqrt{3}$ km above it, is observed at an elevation of 60° from a point on the ground. If, after five seconds, its elevation from the same point, is 30°, then the speed (in km h^{-1}) of the aeroplane, is

(A) 1500 **(B)** 1440

(C) 750 **(D)** 720 **(Online)**

Solution

(B) From given data, we draw the diagram as shown in the following figure:



The height of the plane is

$$DE = CD = \sqrt{3}$$

$$\tan 60° = \frac{\sqrt{3}}{AE} \Rightarrow AE = 1$$

Now, In ΔACB: $\tan 30° = \frac{\sqrt{3}}{AB} \Rightarrow AB = 3$

The distance travelled by plane in 5 s is

$$CD = CE = 3 - 1 = 2 \text{ km}$$

Therefore, $2 = 5 \times v \Rightarrow v = \frac{2}{5} \text{ km s}^{-1}$

$$= \frac{2}{5} \times 3650 \text{ km h}^{-1}$$

$$= 2 \times 720 = 1440 \text{ km h}^{-1}$$

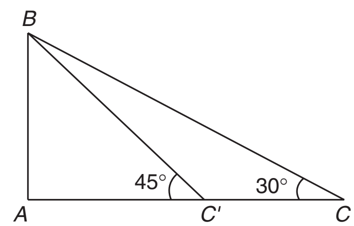
18. A man on the top of a vertical tower observes a car moving at a uniform speed towards the tower on a horizontal road. If it takes 18 min for the angle of depression of the car to change from 30° to 45°; then after this, the time taken (in min) by the car to reach the foot of the tower, is

(A) $9(1+\sqrt{3})$ **(B)** $18(1+\sqrt{3})$

(C) $18(\sqrt{3}-1)$ **(D)** $\frac{9}{2}(\sqrt{3}-1)$ **(Online)**

Solution

(A) Let the length of the tower be h.



$$AC' = AB = h$$

$$AC = AB\cot 30° = \sqrt{3}h$$

$$CC' = (\sqrt{3}-1)h$$

The time taken by the car from C to C' is 18 min.

Therefore, the time taken by the car to reach the foot of the tower is

$$\frac{18}{\sqrt{3}-1} \text{ min} = 9(\sqrt{3}+1) \text{ min}$$

Chapter 5: Complex Number

JEE Main 2018

19. The set of all $\alpha \in \mathbb{R}$, for which $\omega = \frac{1+(1-8\alpha)z}{1-z}$ is a purely imaginary number, for all $z \in \mathbb{C}$ satisfying $|z| = 1$ and Re $z \neq 1$, is

(A) an empty set **(B)** $\{0\}$

(C) $\left\{0, \frac{1}{4}, -\frac{1}{4}\right\}$ **(D)** equal to $\mathbb{R}$

(Online)

Solution

(B) Given: $\omega = \frac{1+(1-8\alpha)z}{1-z}$ is a purely imaginary number.

Therefore, $\omega + \overline{\omega} = 0$

$$\Rightarrow \frac{1+(1-8\alpha)z}{1-z} + \frac{1+(1-8\alpha)\overline{z}}{1-\overline{z}} = 0$$

$$\Rightarrow (1-\overline{z}) + (1-8\alpha)(z - z\overline{z}) + (1-z) + (1-8\alpha)(\overline{z} - z\overline{z}) = 0$$

$$\Rightarrow 2 - (\overline{z} + z) + (1-8\alpha)(z + \overline{z} = 2|z|^2) = 0$$

$$\Rightarrow 2 - (z + \overline{z}) + (1-8\alpha)(z + \overline{z} - 2) = 0$$

Since $z = x + iy$, we get

$$\overline{z} = x - iy \Rightarrow z + \overline{z} = 2x$$

So, $2 - 2x + (1 - 8\alpha)(2x - 2) = 0$

$$\Rightarrow (1 - x) + (8\alpha - 1)(1 - x) = 0$$

$$\Rightarrow (1 - x)(1 + 8\alpha - 1) = 0$$

$$\Rightarrow (1 - x)8\alpha = 0$$

$$\Rightarrow x = 1 \text{ or } \alpha = 0$$

According to the given data, since $x \neq 1$, we get $\alpha = 0$.

20. If $|z - 3 + 2i| \le 4$ then the difference between the greatest value and the least value of $|z|$ is

(A) $2\sqrt{13}$ **(B)** 8

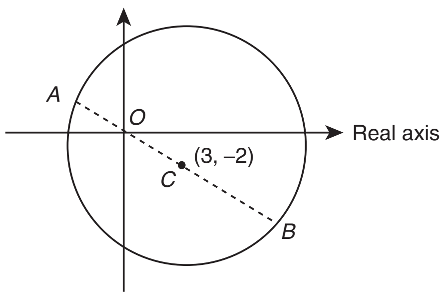
(C) $4+\sqrt{13}$ **(D)** $\sqrt{13}$

(Online)

Solution

(A) Given:

$$|z - 3 + 2i| \le 4$$
$$\Rightarrow |z - (3 - 2i)| \le 4$$



Now, $|z|_{max} = OC + CB = OB = \sqrt{13}+4$

Now, $|z|_{min} = AC - OC = OA = 4-\sqrt{13}$

Therefore, $|z|_{max} - |z|_{min} = (4+\sqrt{13})-(4-\sqrt{13}) = 2\sqrt{13}$

21. The least positive integer n for which $\left(\frac{1+i\sqrt{3}}{1-i\sqrt{3}}\right)^n = 1$, is

(A) 2 **(B)** 3

(C) 5 **(D)** 6 **(Online)**

Solution

(B) Given: $\left(\frac{1+i\sqrt{3}}{1-i\sqrt{3}}\right)^n = 1.$

We know that $1+i\sqrt{3} = -2\omega^2$ and $1-i\sqrt{3} = -2\omega$.

$$\left(\frac{-2\omega^2}{-2\omega}\right)^n = 1$$

$$\omega^n = 1$$

$$\omega^3 = 1 \Rightarrow n = 3$$

Therefore, the least positive integer value of n is 3.

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22. For a non-zero complex number z, let arg(z) denote the principal argument with $-\pi < \arg(z) \le \pi$. Then, which of the following statement(s) is(are) **FALSE**?

(A) $\arg(-1 - i) = \frac{\pi}{4}$, where $i = \sqrt{-1}$.

(B) The function $f: \mathbb{R} \to (-\pi, \pi]$, defined by $f(t) = \arg(-1 + it)$ for all $t \in \mathbb{R}$, is continuous at all points of $\mathbb{R}$, where $i = \sqrt{-1}$.

(C) For any two non-zero complex numbers z_1 and z_2, $\arg\left(\frac{z_1}{z_2}\right) - \arg(z_1) + \arg(z_2)$ is an integer multiple of 2π.

(D) For any three given distinct complex numbers z_1, z_2 and z_3, the locus of the point z satisfying the condition $\arg\left(\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}\right) = \pi$, lies on a straight line.

(Paper-1)

Solution

(A), (B), (D)

Let us check all four options as follows:

- **Option (A):** We know that

$$\arg(-1-i) = -\pi + \tan^{-1}(1) = -\pi + \frac{\pi}{4} = \frac{-3\pi}{4}$$

Hence, option (A) is false.

- **Option (B):** It is given that

$$f(t) = \arg(-1 + it)\ \forall\, t \in \mathbb{R}$$
$$f(t) = -\pi - \tan^{-1}|t|$$

Therefore, $f(t) = -\pi - \tan^{-1} t$ for $t < 0$

$= \pi - \tan^{-1} t$ for $t \ge 0$

Thus, the function is discontinuous at $t = 0$.

Hence, option (B) is false.

- **Option (C):** It is given that

$$\arg\left(\frac{z_1}{z_2}\right) - \arg(z_1) + \arg(z_2)$$

which can be shown as follows:

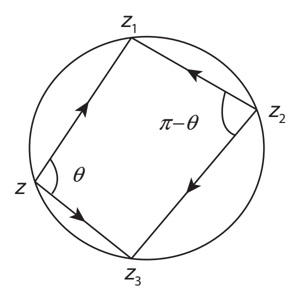
$$\arg\left(\frac{z_1}{z_2}\right) - \arg(z_1) + \arg(z_2) = \arg\left(\frac{z_1}{z_2}\cdot\frac{z_2}{z_1}\right) + 2\pi n$$
$$= \arg(1) + 2\pi n$$
$$= 2\pi n$$

Therefore, for any two non-zero complex numbers z_1 and z_2, $\arg\left(\frac{z_1}{z_2}\right) - \arg(z_1) + \arg(z_2)$ is an integer multiple of 2π.

Hence, option (C) is true.

- **Option (D):** It is given that

$$\arg\left(\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}\right) = \pi$$



Now, $\arg\left(\frac{z_1 - z}{z_3 - z}\right) = \theta$

and $\arg\left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \pi - \theta$

Therefore, $\arg\left(\frac{z_1 - z}{z_3 - z}\right) + \arg\left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \theta + \pi - \theta = \pi$

That is, $\arg\left(\frac{(z_1 - z)}{(z_3 - z)} \cdot \frac{(z_3 - z_2)}{(z_1 - z_2)}\right) = \arg\left(\frac{(z_1 - z)(z_3 - z_2)}{(z_3 - z)(z_1 - z_2)}\right) = \pi$

This implies that for any three given distinct complex numbers z_1, z_2 and z_3, the locus of the point z satisfying the condition $\arg\left(\frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}\right) = \pi$, lies on a circle.

Thus, option (D) is false.

23. Let s, t, r be non-zero complex numbers and L be the set of solutions $z = x + iy$ $\{x, y \in \mathbb{R}, i = \sqrt{-1}\}$ of the equation $sz + t\bar{z} + r = 0$, where $\bar{z} = x - iy$. Then, which of the following statement(s) is (are) TRUE?

(A) If L has exactly one element, then $|s| \neq |t|$.

(B) If $|s| = |t|$, then L has infinitely many elements.

(C) The number of elements in $L \cap \{z : |z - 1 + i| = 5\}$ is at most 2.

(D) If L has more than one element, then L has infinitely many elements.

(Paper-2)

Solution

(A), (C), (D)

We have the equation as

$$sz + t\bar{z} + r = 0 \quad (1)$$

Taking conjugate, we get

$$\bar{s}\bar{z} + \bar{t}z + \bar{r} = 0 \quad (2)$$

Now, eliminating $\bar{z}$ from Eqs. (1) and (2), we get

$$(s\bar{s}z + t\bar{s}\bar{z} + \bar{s}r) - (t\bar{s}\bar{z} + t\bar{t}z + t\bar{r}) = 0$$

$$|s|^2 z + t\bar{s}\bar{z} + \bar{s}r - t\bar{s}\bar{z} - |t|^2 z - t\bar{r} = 0$$

$$\Rightarrow z(|s|^2 - |t|^2) = t\bar{r} - \bar{s}r$$

- **Option (A):** If $|s| \neq |t|$, then z has unique value. Therefore, if L has exactly one element, then $|s| \neq |t|$.
  Thus, option (A) is true.
- **Option (B):** If $|s| = |t|$ and $\bar{r}t - r\bar{s} = 0$, then z has infinitely many values. If $|s| = |t|$ and $\bar{r}t - r\bar{s} \neq 0$, then, z has no values thus, L may be empty set or infinite. Therefore, if $|s| = |t|$, then L has infinitely many elements or no elements.
  Thus, option (B) is false.
- **Option (C):** Locus of z is null or singleton set or a line in all cases. It will intersect the given circle at most two points. The number of elements in $L \cap \{z : |z - 1 + i| = 5\}$ is at most two.
  Hence, option (C) is true.
- **Option (D):** If L has more than one element, then L has infinitely many elements.
  Hence, option (D) is true.

Chapter 6: Quadratic Equations

JEE Main 2018

24. If $\alpha, \beta \in \mathbb{C}$ are the distinct roots of the equation $x^2 - x + 1 = 0$, then $\alpha^{101} + \beta^{107}$ is equal to

(A) 0 **(B)** 1

(C) 2 **(D)** –1 **(Offline)**

Solution

(B) Given: $x^2 - x + 1 = 0$

The roots of the equation are

$$x = \frac{1 \pm \sqrt{1^2 - 4 \times 1 \times 1}}{2 \times 1}$$

$$x = \frac{1 \pm i\sqrt{3}}{2}$$

Thus, we get the following values:

$$\alpha = \frac{1 + i\sqrt{3}}{2} = -\omega$$

$$\beta = \frac{1 - i\sqrt{3}}{2} = -\omega^2$$

Therefore, $\alpha^{101} + \beta^{107} = (-\omega)^{101} + (-\omega^2)^{107}$

$$= -\omega^{101} - \omega^{214}$$

$$= -\omega^{99+2} - \omega^{213+1} = -\omega^{3\times 11}\omega^2 - \omega^{3\times 71}\omega$$

Since $\omega^3 = 1$, $1 + \omega + \omega^2 = 0$, we get

$$= -\omega^2 - \omega = 1$$

25. If $\lambda \in \mathbb{R}$ is such that the sum of the cubes of the roots of the equation, $x^2 + (2 - \lambda)x + (10 - \lambda) = 0$ is minimum, then the magnitude of the difference of the roots of this equation is

(A) $4\sqrt{2}$ **(B)** $2\sqrt{5}$

(C) $2\sqrt{7}$ **(D)** 20 **(Online)**

Solution

(B) Let the root of the equation $x^2 + (2 - \lambda)x + (10 - \lambda) = 0$ be α and β.

Now, $\alpha + \beta = \lambda - 2$, $\alpha\beta = 10 - \lambda$

$$\alpha^3 + \beta^3 = (\alpha + \beta)((\alpha + \beta)^2 - 3\alpha\beta)$$

$$= (\lambda - 2)((\lambda - 2)^2 + 3(\lambda - 10))$$

$$= (\lambda - 2)(\lambda^2 - \lambda - 26) = f(\lambda)$$

Now, $f'(\lambda) = \lambda^2 - \lambda - 26 + (\lambda - 2)(2\lambda - 1)$

$$\Rightarrow f'(\lambda) = 3\lambda^2 - 6\lambda - 24$$

$$\Rightarrow f'(\lambda) = 3(\lambda^2 - 2\lambda - 8)$$

$$\Rightarrow f'(\lambda) = 3(\lambda - 4)(\lambda + 2) = 0$$

and $f''(\lambda) = 6(\lambda - 1),\ f'(\lambda) = 0 \Rightarrow \lambda = -2, 4$

$f''(-2) < 0$ and $f''(4) > 0$

Thus, the minimum value occurs at $\lambda = 4$.

$$x^2 - 2x + 6 = 0$$

$$\Rightarrow \alpha, \beta = \frac{2 \pm \sqrt{4 - 24}}{2} = \frac{2 \pm i\sqrt{20}}{2} = 1 \pm i\sqrt{5}$$

Therefore, $|\alpha - \beta| = 2\sqrt{5}$

26. If $f(x)$ is a quadratic expression such that $f(1) + f(2) = 0$, and -1 is a root of $f(x) = 0$, then the other root of $f(x) = 0$ is

(A) $-\frac{5}{8}$ **(B)** $-\frac{8}{5}$

(C) $\frac{5}{8}$ **(D)** $\frac{8}{5}$ **(Online)**

Solution

(D) Given: One root of $f(x)$ is -1. Then, $f(x) = a(x + 1)(x - \alpha)$, where α is other root of quadratic equation. Therefore,

$$f(1) + f(2) = 0$$

$$\Rightarrow 2a(1 - \alpha) + 3a(2 - \alpha) = 0$$

$$\Rightarrow 2 - 2\alpha + 6 - 3\alpha = 0$$

$$\Rightarrow \alpha = \frac{8}{5}$$

27. Let p, q and r be real numbers ($p \neq q$, $r \neq 0$), such that the roots of the equation $\frac{1}{x+p} + \frac{1}{x+q} = \frac{1}{r}$ are equal in magnitude but opposite in sign, then the sum of squares of these roots is equal to

(A) $\frac{p^2 + q^2}{2}$ **(B)** $p^2 + q^2$

(C) $2(p^2 + q^2)$ **(D)** $p^2 + q^2 + r^2$ **(Online)**

Solution

(B) Given: Let p, q and r be real the numbers ($p \neq q$, $r \neq 0$) and

$$\frac{1}{x+p} + \frac{1}{x+q} = \frac{1}{r}$$

$$(2x + p + q)r = (x + p)(x + q)$$

$$x^2 + (p + q - 2r)x + pq - pr - qr = 0$$

The general equation is

$$x^2 + (\alpha + \beta)x + \alpha\beta = 0$$

Given: $\alpha = -\beta \Rightarrow \alpha + \beta = 0$

Therefore, $p + q = 2r$ (1)

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$= 0 - 2(pq - pr - qr)$$

$$= -2pq + 2r(p + q)$$

$$\Rightarrow -2pq + (p + q)^2 = 0$$

$$\Rightarrow p^2 + q^2$$

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28. Let a, b, c be three non-zero real numbers such that the equation $\sqrt{3}\, a \cos x + 2\, b \sin x = c$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, has two distinct real roots α and β with $\alpha + \beta = \frac{\pi}{3}$. Then, the value of $\frac{b}{a}$ is ____________. **(Paper-1)**

Solution

(0.5) The given equation is

$$\sqrt{3}a \cos x + 2b \sin x = c$$

Dividing by a, we get

$$\sqrt{3} \cos x + \frac{2b}{a} \sin x = \frac{c}{a}$$

Now, $$\sqrt{3} \cos \alpha + \frac{2b}{a} \sin \alpha = \frac{c}{a}$$

and $$\sqrt{3} \cos \beta + \frac{2b}{a} \sin \beta = \frac{c}{a}$$

Subtracting the above two equations, we get

$$\sqrt{3}(\cos \alpha - \cos \beta) + \frac{2b}{a}(\sin \alpha - \sin \beta) = 0$$

Using $\cos A - \cos B = -2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$, we get

$$\sin A - \sin B = 2 \cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$$

$$\sqrt{3}\left[-2 \sin\left(\frac{\alpha+\beta}{2}\right) \sin\left(\frac{\alpha-\beta}{2}\right)\right] + \frac{2b}{a}\left[2 \cos\left(\frac{\alpha+\beta}{2}\right) \sin\left(\frac{\alpha-\beta}{2}\right)\right] = 0$$

Given: $\alpha + \beta = \frac{\pi}{3}$. Therefore,

$$\sqrt{3}\left[-2 \sin\frac{\pi}{6} \sin\left(\frac{\alpha-\beta}{2}\right)\right] + \frac{2b}{a}\left[2 \cos\frac{\pi}{6} \sin\left(\frac{\alpha-\beta}{2}\right)\right] = 0$$

$$\sqrt{3}\left(-2 \times \frac{1}{2}\right) + \frac{2b}{a}\left(2 \times \frac{\sqrt{3}}{2}\right) = 0$$

$$-\sqrt{3} + 2\sqrt{3}\frac{b}{a} = 0$$

or $$2\sqrt{3}\frac{b}{a} = \sqrt{3}$$

Therefore, we get the value of b/a as follows:

$$\frac{b}{a} = \frac{1}{2} = 0.5$$

Chapter 7: Permutation and Combination

JEE Main 2018

29. From 6 different novels and 3 different dictionaries, 4 novels and 1 dictionary are to be selected and arranged in a row on a shelf so that the dictionary is always in the middle. The number of such arrangements is

(A) less than 500.

(B) at least 500 but less than 750.

(C) at least 750 but less than 1000.

(D) at least 1000. **(Offline)**

Solution

(D) The number of ways, the novels are arranged, is

$$x = {}^6C_4 \times {}^3C_1 \times 4!$$
$$= 15 \times 3 \times 24 = 1080$$

30. n-digit numbers are formed using only three digits 2, 5 and 7. The smallest value of n for which 900 such distinct numbers can be formed, is

(A) 6 **(B)** 7

(C) 8 **(D)** 9 **(Online)**

Solution

(B) Let us consider the following empty box to be filled up with n digit number.

Every cell can be filled by any one of 2, 5 and 17 or by 3 ways.

Thus, the number of n digit numbers is 3^n.

Therefore, $3^n > 900 \Rightarrow n \geq 7$

$$\Rightarrow n = 7$$

31. The number of four letter words that can be formed using the letters of the word **BARRACK** is

(A) 120 **(B)** 144

(C) 264 **(D)** 270 **(Online)**

Solution

(D) Given word: BARRACK

B(1), A(2), R(2), C(1), K(1)

For four letter words:

(i) All distinct letters can be formed in the following ways:

$${}^5C_4 \cdot 4! = 120$$

(ii) Two distinct letters and two alike letters can be formed in the following ways:

$${}^2C_1 \cdot {}^4C_2 \cdot \frac{4!}{2!} = 144$$

(iii) Two alike first type letter, two alike second type letter can be formed in the following way:

$${}^2C_2 \cdot \frac{4!}{2!2!} = 6$$

Therefore, the total number of four letter words that can be formed using the letters of the word BARRACK is

$$120 + 144 + 6 = 270$$

32. The number of numbers between 2000 and 5000 that can be formed with the digits 0, 1, 2, 3, 4 (repetition of digits is not allowed) and are multiple of 3 is

(A) 24 **(B)** 30

(C) 36 **(D)** 48 **(Online)**

Solution

(B) The numbers between 2000 and 5000 are of 4-digit numbers.

We should use 0, 1, 2, 3, 4 (repetition of digits is not allowed) and are multiple of 3.

For the multiple of 3, the sum of all digits should be divisible of 3.

Thus, the number can be formed 0, 1, 2, 3 (sum is 6 which is divisible of 3) or 0, 2, 3, 4 (sum is 9 which is divisible of 3). 0 and 1 cannot be on highest digit in the number.

Therefore, the number of numbers between 2000 and 5000 that can be formed with the digits 0, 1, 2, 3, 4 (repetition of digits is not allowed) and are multiple of 3 is

$$2 \times 3! + 3 \times 3! = 30$$

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33. In a high school, a committee has to be formed from a group of 6 boys $M_1, M_2, M_3, M_4, M_5, M_6$ and 5 girls G_1, G_2, G_3, G_4, G_5.

(i) Let α_1 be the total number of ways in which the committee can be formed such that the committee has 5 members, having exactly 3 boys and 2 girls.

(ii) Let α_2 be the total number of ways in which the committee can be formed such that the committee has at least 2 members, and having an equal number of boys and girls.

(iii) Let α_3 be the total number of ways in which the committee can be formed such that the committee has 5 members, at least 2 of them being girls.

(iv) Let α_4 be the total number of ways in which the committee can be formed such that the committee has 4 members, having at least 2 girls and such that both M_1 and G_1 are **NOT** in the committee together.

LIST–I	LIST–II
P. The value of α_1 is	**1.** 136
Q. The value of α_2 is	**2.** 189
R. The value of α_3 is	**3.** 192
S. The value of α_4 is	**4.** 200
	5. 381
	6. 461

The correct option is:

(A) P→4; Q→6; R→2; S→1

(B) P→1; Q→4; R→2; S→3

(C) P→4; Q→6; R→5; S→2

(D) P→4; Q→2; R→3; S→1 **(Paper-2)**

Solution

(C) Let us check for each given item:

(i) The total number of ways in which the committee can be formed such that the committee has 5 members, having exactly 3 boys and 2 girls, is

$$\alpha_1 = {}^6C_3 \times {}^5C_2 = \frac{6!}{3!3!}\times\frac{5!}{3!2!} = \frac{6\times5\times4}{3\times2}\times\frac{5\times4}{2} = 20\times10 = 200$$

(ii) The total number of ways in which the committee can be formed such that the committee has at least 2 members, and having an equal number of boys and girls, is

$$\alpha_2 = {}^6C_1\times{}^5C_1 + {}^6C_2\times{}^5C_2 + {}^6C_3\times{}^5C_3 + {}^6C_4\times{}^5C_4 + {}^6C_5\times{}^5C_5$$

$$= 6\times5 + \frac{6\times5}{2}\times\frac{5\times4}{2} + \frac{6\times5\times4}{3\times2}\times\frac{5\times4}{2} + \frac{6\times5}{2}\times5 + 6\times1$$

$$= 30+150+200+75+6 = 461$$

(iii) The total number of ways in which the committee can be formed such that the committee has 5 members, at least 2 of them being girls, is

$$\alpha_3 = {}^5C_2\times{}^6C_3 + {}^5C_3\times{}^6C_2 + {}^5C_4\times{}^6C_1 + {}^5C_5\times{}^6C_0$$

$$= \frac{5\times4}{2}\times\frac{6\times5\times4}{3\times2} + \frac{5\times4}{2}\times\frac{6\times5}{2} + 5\times6 + 1$$

$$= 200+150+30+1 = 381$$

(iv) The total number of ways in which the committee can be formed such that the committee has 4 members, having at least 2 girls and such that both M_1 and G_1 are **NOT** in the committee together, is

$$\alpha_4 = ({}^5C_2\times{}^6C_3 - {}^4C_1\times{}^5C_1) + ({}^5C_3\times{}^6C_1 - {}^4C_2\times{}^5C_0) + {}^5C_4$$

$$= \left(\frac{5\times4}{2}\times\frac{6\times5\times4}{3\times2} - 4\times5\right) + \left(\frac{5\times4}{2}\times6 - \frac{4\times3}{2}\times1\right) + 5$$

$$= (150-20)+(60-6)+5 = 189$$

Therefore, the correct mapping is P → 4; Q → 6; R → 5; S → 2. Hence, option (C) is correct.

Chapter 8: Binomial Theorem

JEE Main 2018

34. The sum of the coefficients of all odd degree terms in the expansion of $(x+\sqrt{x^3-1})^5 + (x-\sqrt{x^3-1})^5, (x>1)$ is

(A) 0 **(B)** 1

(C) 2 **(D)** –1 **(Offline)**

Solution

(C) Given: $(x+\sqrt{x^3-1})^5 + (x-\sqrt{x^3-1})^5$

Let us expand the given identity as follows:

$$(T_1+T_2+T_3+T_4+T_5+T_6) + (T_1-T_2+T_3-T_4+T_5-T_6)$$

$$= 2(T_1+T_3+T_5)$$

$$= 2\left({}^5C_0(x)^5 + {}^5C_2(x)^3\left(\sqrt{x^3-1}\right)^2 + {}^5C_4(x)^1\left(\sqrt{x^3-1}\right)^4\right.$$

$$= 2(x^5 + 10x^3(x^3-1) + 5x(x^6+1-2x^3))$$

$$= 2(x^5 + 10x^6 - 10x^3 + 5x^7 + 5x - 10x^4)$$

$$= 2(5x^7 + 10x^6 + x^5 - 10x^4 - 10x^3 + 5x)$$

$$= (10x^7 + 20x^6 + 2x^5 - 10x^4 - 20x^3 + 10x)$$

Therefore, the sum of the coefficients of all odd degree terms in the expansion of $(x+\sqrt{x^3-1})^5 + (x-\sqrt{x^3-1})^5, (x>1)$ is

$$10 + 2 - 20 + 10 = 2$$

35. If n is the degree of the polynomial, $\left[\frac{2}{\sqrt{5x^3+1}-\sqrt{5x^3-1}}\right]^8 + \left[\frac{2}{\sqrt{5x^3+1}+\sqrt{5x^3-1}}\right]^8$ and m is the coefficient of x^n in it, then the ordered pair (n, m) is equal to

(A) $(24, (10)^8)$ **(B)** $(8, 5(10)^4)$

(C) $(12, (20)^4)$ **(D)** $(12, 8(10)^4)$ **(Online)**

Solution

(D) Let us consider that

$$\left[\frac{2(\sqrt{5x^3+1}+\sqrt{5x^3-1})}{2}\right]^8 + \left[\frac{2(\sqrt{5x^3+1}-\sqrt{5x^3-1})}{2}\right]^8$$

Let $a=\sqrt{5x^3+1}$ and $b=\sqrt{5x^3-1}$. Then,

$$(a+b)^8 + (a-b)^8$$

$$= ({}^8C_0a^8 + {}^8C_2a^6b^2 + {}^8C_4a^4b^4 + {}^8C_6a^2b^6 + {}^8C_8b^8)$$

$$= {}^8C_0(5x^3+1)^4 + {}^8C_2(5x^3+1)^3(5x^3-1) + {}^8C_4(5x^3+1)^2(5x^3-1)^2 + {}^8C_6(5x^3+1)(5x^3-1)^3 + {}^8C_8(5x^3-1)^4$$

Thus, the degree of polynomial is $n = 12$ and coefficient of $x^n = x^{12}$ is

$${}^8C_05^4 + {}^8C_25^4 + {}^8C_55^4 + {}^8C_65^4 + {}^8C_85^4$$

$$= 5^4\times2^{8-1} = 5^4\times2^7 = 8\times10^4$$

Therefore, the ordered pair (n, m) is given by

$$(n, m) = (12, 8(10^4))$$

36. The coefficient of x^{10} in the expansion of $(1+x)^2(1+x^2)^3(1+x^3)^4$ is equal to

(A) 52 **(B)** 56

(C) 50 **(D)** 44 **(Online)**

Solution

(A) Given: $(1-x)^2(1+x^2)^3(1+x^3)^4$

Coefficient of x^{10} in $(1 + x^2 + 2x)(1 + 3x^2 + 3x^4 + x^6)(1 + 4x^3 + 6x^6 + 4x^9 + x^{12})$ is obtained as

$$18 + 18 + 8 + 8 = 52$$

37. The coefficient of x^2 in the expansion of the product $(2 - x^2)\cdot((1 + 2x + 3x^2)^6 + (1 - 4x^2)^6)$ is

(A) 107 **(B)** 106

(C) 108 **(D)** 155 **(Online)**

Solution

(B) Given: $(2 - x^2)\cdot((1 + 2x + 3x^2)^6 + (1 - 4x^2)^6)$

Coefficient of x^2 = 2Coefficient of x^2 in $((1 + 2x + 3x^2)^6 + (1 - 4x^2)^6)$ – Constant Term

$$(1 + 2x + 3x^2)^6 = \sum_{r=0}^{6} {}^6C_r(2x+3x^2)^r$$

$$= {}^6C_0 + {}^6C_1(2x+3x^2) + {}^6C_2(2x+3x^2)^2$$

Now, coefficient of x^2 in the expansion of the product $(2 - x^2)\cdot((1 + 2x + 3x^2)^6 + (1 - 4x^2)^6)$ is

$$2(18 + 60 - 24) - 2 = 108 - 2 = 106$$

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38. Let $X = ({}^{10}C_1)^2 + 2({}^{10}C_2)^2 + 3({}^{10}C_3)^2 + \ldots + 10({}^{10}C_{10})^2$, where ${}^{10}C_r$, $r \in \{1, 2, \ldots, 10\}$ denote binomial coefficients. Then, the value of $\frac{1}{1430}X$ is _______. **(Paper-2)**

Solution

(646) It is given that

$$X = ({}^{10}C_1)^2 + 2({}^{10}C_2)^2 + 3({}^{10}C_3)^2 + \cdots + 10({}^{10}C_{10})^2$$

Using binomial expansion, we get

$$(1+x)^{10} = {}^{10}C_0 + x\,{}^{10}C_1 + x^2\,{}^{10}C_2 + \cdots + x^{10}\,{}^{10}C_{10}$$

Differentiating it w.r.t. x, we get

$$10(1+x)^9 = 0 + {}^{10}C_1 + x\cdot 2\,{}^{10}C_2 + x^2\cdot 3\,{}^{10}C_3 + \cdots + x^9\cdot 10\,{}^{10}C_{10}$$

Also, $(1+x)^{10} = {}^{10}C_0x^{10} + {}^{10}C_1x^9 + {}^{10}C_2x^8 + \cdots + {}^{10}C_{10}x^0$

Now, the coefficient of x^9 in $10(1+x)^9(1+x)^{10}$ is

$$({}^{10}C_1)^2 + 2({}^{10}C_2)^2 + 3({}^{10}C_3)^2 + \cdots + 10({}^{10}C_{10})^2$$

Therefore, the coefficient of x^9 in

$$10(1+x)^9(1+x)^{10} = X$$

The coefficient of x^9 in $10(1+x)^{19} = 10\cdot{}^{19}C_9$ is

$$10\,{}^{19}C_9 = X$$

Therefore,

$$\frac{1}{1430}X = \frac{1}{1430}\times 10\,{}^{19}C_9$$

$$= \frac{1}{1430}\times 10 \times \frac{19!}{9!\times 10!}$$

$$\Rightarrow \frac{X}{1430} = 646$$

Chapter 9: Sequence and Series

JEE Main 2018

39. Let $a_1, a_2, a_3, \ldots, a_{49}$ be in AP such that $\sum_{k=0}^{12} a_{4k+1} = 416$ and $a_9 + a_{43} = 66$. If $a_1^2 + a_2^2 + \cdots + a_{17}^2 = 140m$, then m is equal to

(A) 68 **(B)** 34

(C) 33 **(D)** 66 **(Offline)**

Solution

(B) Given:

$$a_1 + a_5 + a_9 + a_{13} + \ldots + a_{49} = 416$$

That is, $a_1 + (a_1 + 4d) + (a_1 + 8d) + (a_1 + 12d) + \ldots + (a_1 + 48d) = 416$

$$13a_1 + 4d(1 + 2 + 3 + \ldots + 12) = 416$$

$$13a_1 + 4d \times \frac{12\times 13}{2} = 416$$

$$13a_1 + 24 \times 13d = 416$$

$$a_1 + 24d = 32 \quad (1)$$

We have $a_9 + a_{43} = 66$

$$a_1 + 8d + a_1 + 42d = 66$$

$$2a_1 + 50d = 66$$

$$a_1 + 25d = 33 \quad (2)$$

Solving Eqs. (1) and (2), we get $d = 1$. Therefore, $a_1 = 8$.

$$a_1^2 + (a_1+d)^2 + (a_1+2d)^2 + \cdots + (a_1+16d)^2 = 140m$$

$$17a_1^2 + d^2(1^2 + 2^2 + \cdots + 16)^2 + 2a_1d$$

$$(1 + 2 + 3 + \cdots + 16) = 140m$$

$$17\times 64 + \frac{16\times 17\times 33}{6} + 2\times 8\times 1\times \frac{16\times 17}{2} = 140m$$

$$17 \times 64 + 8 \times 11 \times 17 + 8 \times 16 \times 17 = 140m$$

$$17 \times 16 + 22 \times 17 + 2 \times 16 \times 17 = 35m$$

$$272 + 374 + 544 = 35m$$

$$1190 = 35m$$

Therefore, $m = 34$.

40. Let A be the sum of the first 20 terms and B be the sum of the first 40 terms of the series $1^2 + 2\cdot 2^2 + 3^2 + 2\cdot 4^2 + 5^2 + 2\cdot 6^2 + \ldots$ If $B - 2A = 100\lambda$, then λ is equal to

(A) 248 **(B)** 464

(C) 496 **(D)** 232 **(Offline)**

Solution

(A) The given series is

$$1^2 + 2\cdot 2^2 + 3^2 + 2\cdot 4^2 + 5^2 + 2\cdot 6^2 + \ldots$$

The sum of the first 20 terms is

$$A = 1^2 + 2.2^2 + \ldots + 2.20^2$$

$= (1^2 + 2^2 + \ldots + 20^2) + (2^2 + 4^2 + \ldots + 20^2)$

$= \dfrac{20\times 21\times 41}{6} + \dfrac{4\times 10\times 11\times 21}{6}$

$\Rightarrow \dfrac{20\times 21}{6}(41+22) = 70\times 63 = 4410$

The sum of the first 40 terms is

$B = 1^2 + 2\cdot 2^2 + \ldots + 2\cdot 40^2$

$= (1^2 + 2^2 + \ldots + 40^2) + (2^2 + 4^2 + \ldots + 40^2)$

$= \dfrac{40\times 41\times 81}{6} + \dfrac{4\times 20\times 21\times 41}{6}$

$= \dfrac{40\times 41}{6}(81+42)$

$= \dfrac{40\times 41}{6}\times(123)$

$\Rightarrow 20(41)^2 = 33620$

Therefore,

$$B - 2A = 100\lambda \Rightarrow \lambda = \frac{33620 - 8820}{100} = 248$$

41. If $x_1, x_2, \ldots, x_n$ and $\dfrac{1}{h_1}, \dfrac{1}{h_2}, \cdots, \dfrac{1}{h_n}$ are two APs such that $x_3 = h_2 = 8$ and $x_8 = h_7 = 20$, then $x_5 h_{10}$ equals

(A) 2560 **(B)** 2650

(C) 3200 **(D)** 1600 **(Online)**

Solution

(A) Given: There are two APs $x_1, x_2, \ldots, x_n$ and $\dfrac{1}{h_1}, \dfrac{1}{h_2}, \cdots, \dfrac{1}{h_n}$.

Now, $x_3 = x_1 + 2d_1$

where d_1 is common difference of first AP and

$$\frac{1}{h_2} = \frac{1}{h_1} + d_2$$

where d_2 is common difference of second AP. We know that

$x_3 = h_2$

Therefore, $8 = x_1 + 2d_1$ (1)

$\dfrac{1}{h_2} = \dfrac{1}{h_1} + d_2$ (2)

and $x_8 = h_7 = 20$

$x_1 + 7d_1 = 20$ (3)

$\dfrac{1}{20} = \dfrac{1}{h_1} + 6d_2$ (4)

From Eqs. (1) and (3), we get

$$d_1 = \frac{12}{5} \text{ and } x_1 = \frac{16}{5}$$

From Eqs. (2) and (4), we get

$$\frac{1}{20} - \frac{1}{8} = 5d \Rightarrow \frac{-12}{20\times 8} = 5d_2 \Rightarrow d_2 = \frac{-3}{200}$$

and $\dfrac{1}{h_1} = \dfrac{1}{20} + \dfrac{3\times 6}{200} \Rightarrow \dfrac{1}{h_1} = \dfrac{10+18}{200} \Rightarrow \dfrac{1}{h_1} = \dfrac{7}{50}$

Thus,

$x_5 \cdot h_{10} = (x_1 + 4d_1)\dfrac{1}{h_1 + 9d_2}$

$= \left(\dfrac{16}{5} + 4\cdot\dfrac{12}{5}\right)\cdot\dfrac{1}{\dfrac{28}{200} - \dfrac{27}{200}}$

$= \dfrac{200}{5}(48+16) = \dfrac{200\times 64}{5} = 40\times 64 = 2560$

42. If b is the first term of an infinite GP whose sum is five, then b lies in the interval:

(A) $(-\infty, -10]$ **(B)** $(-10, 0)$

(C) $(0, 10)$ **(D)** $[10, \infty)$ **(Online)**

Solution

(C) Given: $b = 1^{st}$ term of GP

and $5 = b + br + br^2 + \ldots + \infty$

$5 = \dfrac{b}{1-r}$

$1 - r = \dfrac{b}{5}$

$-1 < r < 1$

$\Rightarrow -1 - 1 < r - 1 < 1 - 1$

$\Rightarrow 0 < 1 - r < 2$

$\Rightarrow 0 < \dfrac{b}{5} < 2$

$\Rightarrow 0 < b < 10$

Therefore, b lies in the interval (0, 10).

43. If a, b, c are in AP and a^2, b^2, c^2 are in GP such that $a < b < c$ and $a+b+c = \dfrac{3}{4}$, then the value of a is

(A) $\dfrac{1}{4} - \dfrac{1}{4\sqrt{2}}$ **(B)** $\dfrac{1}{4} - \dfrac{1}{3\sqrt{2}}$

(C) $\dfrac{1}{4} - \dfrac{1}{2\sqrt{2}}$ **(D)** $\dfrac{1}{4} - \dfrac{1}{\sqrt{2}}$ **(Online)**

Solution

(C) Given a, b, c in AP.

$\Rightarrow 2b = a + c$ (1)

and a^2, b^2, c^2 are in GP.

$\Rightarrow b^4 = a^2c^2$

$\Rightarrow b^2 = \pm ac$ (2)

Taking $b^2 = -ac$, we get

$\left(\dfrac{a+c}{2}\right)^2 = -ac$ [from Eq. (1)]

$a^2 + c^2 + 2ac = -4ac$

$a^2 + c^2 + 6ac = 0$ (3)

$$a+b+c=\frac{3}{4} \quad \text{(given)}$$

$$\Rightarrow \frac{a+c}{2}+(a+c)=\frac{3}{4} \quad \text{[from Eq. (1)]}$$

$$\Rightarrow a+c=\frac{1}{2} \quad (4)$$

$$\Rightarrow a^2+c^2=\frac{1}{4}-2ac \quad \text{[squaring Eq. (4)]}$$

$$\frac{1}{4}-2ac+6ac=0 \quad \text{[from Eq. (3)]}$$

$$\Rightarrow ac=\frac{-1}{16} \quad (5)$$

$$\Rightarrow b^2=\frac{1}{16}\Rightarrow b=\frac{1}{4},\frac{-1}{4}$$

$$\Rightarrow a\left(\frac{1}{2}-a\right)=\frac{-1}{16} \quad \text{[from Eqs. (1) and (5)]}$$

$$\Rightarrow a^2-\frac{a}{2}-\frac{1}{16}=0$$

$$\Rightarrow a=\frac{\frac{1}{2}\pm\sqrt{\frac{1}{4}+\frac{1}{4}}}{2}=\frac{1}{4}\pm\frac{1}{2\sqrt{2}}$$

$$b>a\Rightarrow a=\frac{1}{4}-\frac{1}{2\sqrt{2}}$$

44. Let $A_n=\left(\frac{3}{4}\right)-\left(\frac{3}{4}\right)^2+\left(\frac{3}{4}\right)^3-\cdots+(-1)^{n-1}\left(\frac{3}{4}\right)^n$ and $B_n=1-A_n$. Then, the least odd natural number p, so that $B_n>A_n$, for all $n\ge p$, is

(A) 9 **(B)** 7

(C) 11 **(D)** 5 **(Online)**

Solution

(B) Given:

$$A_n=\left(\frac{3}{4}\right)-\left(\frac{3}{4}\right)^2+\left(\frac{3}{4}\right)^3-\cdots+(-1)^{n-1}\left(\frac{3}{4}\right)^n$$

$$A_n=(3/4)\frac{(1-(-3/4)^n))}{(1+3/4)}=\frac{3}{7}\left(1-\left(\frac{-3}{4}\right)^n\right)$$

$$B_n>A_n\Rightarrow 1-A_n>A_n\Rightarrow A_n<\frac{1}{2}$$

$$\Rightarrow \frac{3}{7}\left(1-\left(\frac{-3}{4}\right)^n\right)<\frac{1}{2}$$

$$\Rightarrow 1-\left(\frac{-3}{4}\right)^n<\frac{7}{6}\Rightarrow\left(\frac{-3}{4}\right)^n>\frac{-1}{6}$$

$$\Rightarrow n>6.228\Rightarrow n=7$$

45. Let $\frac{1}{x_1},\frac{1}{x_2},\cdots\frac{1}{x_n}$ ($x_i\ne 0$ for $i=1, 2,\ldots, n$) be in AP such that $x_1=4$ and $x_{21}=20$. If n is the least positive integer for which $x_n>50$, then $\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)$ is equal to

(A) $\frac{1}{8}$ **(B)** 3

(C) $\frac{13}{8}$ **(D)** $\frac{13}{4}$ **(Online)**

Solution

(D) Given:

$$\text{AP: } \frac{1}{x_1},\frac{1}{x_2},\cdots\frac{1}{x_n}$$

and $x_1=4, x_{21}=20$

Therefore, $\frac{1}{4}+20\cdot d=\frac{1}{20}$

$$20d=\frac{1}{20}-\frac{1}{4}=\frac{1-5}{20}$$

$$\Rightarrow 20d=\frac{-4}{20}$$

$$\Rightarrow d=\frac{-4}{20\times 2}\Rightarrow d=\frac{-1}{100}$$

Now, $\frac{1}{x_n}<\frac{1}{50}$

$$\frac{1}{4}-\frac{n-1}{100}<\frac{1}{50}\Rightarrow n>24$$

That is, $n=25$.

Therefore,

$$\sum_{i=1}^{25}\left(\frac{1}{x_i}\right)=\frac{25}{2}\left(2\times\frac{1}{4}-\frac{1}{100}\times 24\right)=\frac{13}{4}$$

46. The sum of the first 20 terms of the series $1+\frac{3}{2}+\frac{7}{4}+\frac{15}{8}+\frac{31}{16}+\cdots+$, is

(A) $38+\frac{2}{2^{19}}$ **(B)** $38+\frac{2}{2^{20}}$

(C) $39+\frac{1}{2^{20}}$ **(D)** $39+\frac{1}{2^{19}}$ **(Online)**

Solution

(A) Given: $1+\frac{3}{2}+\frac{7}{4}+\frac{15}{8}+\frac{31}{6}+\cdots$

$$=(2-1)+\left(2-\frac{1}{2}\right)+\left(2-\frac{1}{4}\right)+\left(2-\frac{1}{8}\right)+\left(2-\frac{1}{16}\right)+\cdots$$

up to 20 terms

$$=40-\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\text{up to 20 terms}\right)$$

$$=40-\left(\frac{1-\left(\frac{1}{2}\right)^{20}}{1-\frac{1}{2}}\right)$$

$$=40-2+\frac{1}{2^{19}}=38+\frac{1}{2^{19}}$$

JEE Advanced 2018

47. Let X be the set consisting of the first 2018 terms of the arithmetic progression 1, 6, 11, ..., and Y be the set consisting of the first 2018 terms of the arithmetic progression 9, 16, 23, ... Then, the number of elements in the set $X \cup Y$ is___________.

(Paper-1)

Solution

(3748) Given: $X = \{1, 6, 11, \ldots, 10086\}$

$Y = \{9, 16, 23, \ldots, 14128\}$

Then, $X \cap Y = \{16, 51, 86, \ldots\}$

Thus, by the definition of arithmetic progression, we get

$a = 16$ and $d = 35$

Therefore, $s_n = a + (n-1)d$

$16 + (n-1)35 \le 10086$

Therefore, $n \le 288.71 \sim 288$

Thus, $n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$

$= 2018 + 2018 - 288$

Therefore, the number of elements in set $X \cup Y$ is

$n(X \cup Y) = 3748$

Chapter 10: Cartesian Coordinates and Straight Lines*

Chapter 11: Pair of Straight Lines*

Chapter 12: Circle

JEE Main 2018

48. Let the orthocentre and centroid of a triangle be $A(-3, 5)$ and $B(3, 3)$, respectively. If C is the circumcentre of this triangle, then the radius of the circle having line segment AC as diameter, is

(A) $2\sqrt{10}$ **(B)** $3\sqrt{\frac{5}{2}}$

(C) $\frac{3\sqrt{5}}{2}$ **(D)** $\sqrt{10}$ **(Offline)**

Solution

(B) From the given data, we can draw as shown in the following figure:

We know that centroid divides orthocentre and circumcentre in ratio 2 : 1. Therefore,

$$3 = \frac{2x - 3}{3}$$

$$9 = 2x - 3$$

$$12 = 2x \Rightarrow x = 6$$

Also, $$3 = \frac{2y + 5}{3}$$

$$9 = 2y + 5$$

$$2y = 4 \Rightarrow y = 2$$

Therefore, coordinates of the centroid is (6, 2).

Now, the radius of the circle is

$$r = \frac{AC}{2} = \frac{1}{2}\sqrt{(6-(-3))^2 + (2-5)^2} = \frac{1}{2}\sqrt{81 + 9}$$

$$= \frac{1}{2}\sqrt{90} = \frac{3}{2}\sqrt{10}$$

$$r = 3\sqrt{\frac{5}{2}}$$

49. A circle passes through the points (2, 3) and (4, 5). If the centre lies on the line, $y - 4x + 3 = 0$, then its radius is equal to

(A) 2 **(B)** $\sqrt{5}$

(C) $\sqrt{2}$ **(D)** 1 **(Online)**

Solution

(A) Centre (α, β) lies on line $y - 4x + 3 = 0$.

Then, $\beta = 4\alpha - 3$

and

$$\text{Radius} = \sqrt{(\alpha-2)^2 + (\beta-3)^2} = \sqrt{(\alpha-4)^2 + (\beta-5)^2}$$

$$\alpha^2 + \beta^2 + 13 - 4\alpha - 6\beta = \alpha^2 + \beta^2 + 41 - 8\alpha - 10\beta$$

$$4\alpha + 4\beta = 28 \Rightarrow \alpha + \beta = 7$$

$$\Rightarrow \alpha + 4\alpha - 3 = 7$$

$$\Rightarrow \alpha = 2, \beta = 5$$

Therefore, $\text{Radius} = \sqrt{(2-2)^2 + (5-3)^2} = 2$

50. The tangent to the circle $C_1 : x^2 + y^2 - 2x - 1 = 0$ at the point (2, 1) cuts off a chord of length 4 from a circle C_2 whose centre is (3, −2). The radius of C_2 is

**No question from this chapter appeared in 2018 JEE papers.*

(A) 2 **(B)** $\sqrt{2}$

(C) 3 **(D)** $\sqrt{6}$ **(Online)**

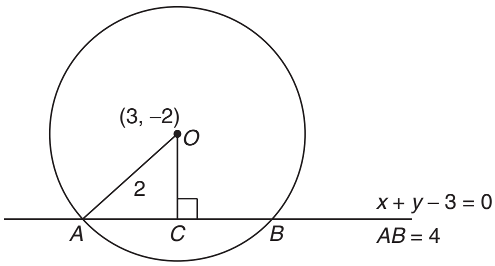
Solution

(D) Equation of tangent to circle $x^2 + y^2 - 2x - 1 = 0$ at point (2, 1) is

$$2x + y - (x + 2) - 1 = 0$$
$$x + y - 3 = 0$$

This line is chord of circle C_2. Therefore,

$$OC = \frac{|3 - 2 - 3|}{\sqrt{2}} = \sqrt{2}$$



Therefore, the radius of circle is

$$\sqrt{CA^2 + OC^2} = \sqrt{4 + 2} = \sqrt{6}$$

51. If a circle C, whose radius is 3, touches externally the circle, $x^2 + y^2 + 2x - 4y - 4 = 0$ at the point (2, 2), then the length of the intercept cut by this circle C, on the x-axis is equal to

(A) $2\sqrt{5}$ **(B)** $3\sqrt{5}$

(C) $\sqrt{5}$ **(D)** $2\sqrt{3}$ **(Online)**

Solution

(A) The given circle equation is

$$x^2 + y^2 + 2x - 4y - 4 = 0$$

The centre of the given circle is (–1, 2). Thus, the radius of the given circle is

$$r = \sqrt{1 + 4 + 4} = 3$$

The centre of the required circle is (5, 2) or (–4, 2).

The length of intercept on x-axis is square in both circles. Therefore, one required circle is

$$(x - 5)^2 + (y - 2)^2 = 3^2$$
$$x^2 + y^2 - 10x - 4y + 20 = 0$$

The length of x intercept is

$$2\sqrt{g^2 - c} = 2\sqrt{25 - 20} = 2\sqrt{5}$$

JEE Advanced 2018

Paragraph X for Questions 51 and 52: Let S be the circle in the xy-plane defined by the equation $x^2 + y^2 = 4$.

52. Let E_1E_2 and F_1F_2 be the chords of S passing through the point $P_0(1, 1)$ and parallel to the x-axis and the y-axis, respectively. Let G_1G_2 be the chord of S passing through P_0 and having slope –1. Let the tangents to S at E_1 and E_2 meet at E_3, the tangents to S at F_1 and F_2 meet at F_3, and the tangents to S at G_1 and G_2 meet at G_3. Then, the points E_3, F_3, and G_3 lie on the curve

(A) $x + y = 4$ **(B)** $(x - 4)^2 + (y - 4)^2 = 16$

(C) $(x - 4)(y - 4) = 4$ **(D)** $xy = 4$

(Paper-1)

Solution

(A) Given: The slope of the chord G_1G_2. Then, the equation of chord G_1G_2 is

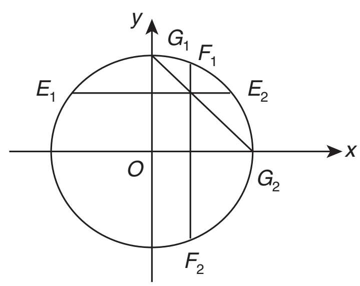
$$(y - 1) = (-1)(x - 1)$$
$$\Rightarrow y - 1 = 1 - x$$
$$\Rightarrow x + y = 2$$

Therefore, we have the following coordinates:

$G_1(0, 2)$ and $G_2(2, 0)$

$E_1(-\sqrt{3}, 1)$ and $E_2(\sqrt{3}, 1)$

$F_1(1, \sqrt{3})$ and $F_2(1, -\sqrt{3})$



Now, the intersection of F_1F_2 on x-axis lies on (4, 0) the intersection of E_1 and E_2 on y-axis lies on (0, 4); intersection of G_1 and G_2 is (2, 2).

Thus, the equation of the curve passing through (4, 0), (0, 4) and (2, 2) is

$$x + y = 4$$

53. Let P be a point on the circle S with both coordinates being positive. Let the tangent to S at P intersect the coordinate axes at the points M and N. Then, the mid-point of the line segment MN must lie on the curve

(A) $(x + y)^2 = 3xy$ **(B)** $x^{2/3} + y^{2/3} = 2^{4/3}$

(C) $x^2 + y^2 = 2xy$ **(D)** $x^2 + y^2 = x^2y^2$

(Paper-1)

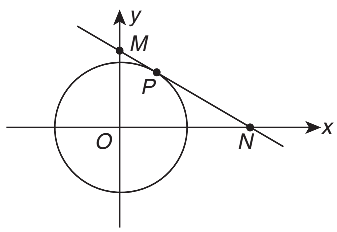
Solution

(D) Suppose point P has coordinates $(2\cos\theta, 2\sin\theta)$. Then, the equation of tangent is

$$x\cos\theta + y\sin\theta = 2$$

Therefore,

$$M = \left(0, \frac{2}{\sin\theta}\right) \text{ and } N = \left(\frac{2}{\cos\theta}, 0\right)$$



Suppose the coordinates of midpoint are (h, k), then

$$h=\frac{1}{\cos\theta}=\sec\theta \text{ and } k=\frac{1}{\sin\theta}=\operatorname{cosec}\theta$$

We know that

$$\sin^2\theta+\cos^2\theta=1$$

$$\Rightarrow \frac{1}{h^2}+\frac{1}{k^2}=1$$

Therefore, the locus of the midpoint is

$$\frac{1}{x^2}+\frac{1}{y^2}=1 \Rightarrow x^2+y^2=x^2y^2$$

54. Let T be the line passing through the points $P(-2,7)$ and $Q(2, -5)$. Let F_1 be the set of all pairs of circles (S_1, S_2) such that T is tangent to S_1 at P and tangent to S_2 at Q, and also such that S_l and S_2 touch each other at a point, say, M. Let E_1 be the set representing the locus of M as the pair (S_1, S_2) varies in F_1. Let the set of all straight line segments joining a pair of distinct points of E_1 and passing through the point $P(1, 1)$ be F_2. Let E_2 be the set of the mid-points of the line segments in the set F_2. Then, which of the following statement(s) is(are) TRUE?

(A) The point $(-2, 7)$ lies in E_1.

(B) The point $\left(\frac{4}{5},\frac{7}{5}\right)$ does **NOT** lie in E_2.

(C) The point $\left(\frac{1}{2},1\right)$ lies in E_2.

(D) The point $\left(0,\frac{3}{2}\right)$ does **NOT** lie in E_1. **(Paper-2)**

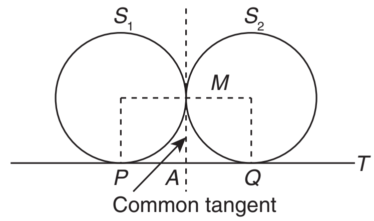
Solution

(B), (D) Let us depict the given geometrical situation in a diagram as shown in the following figure:

Here, we see that

$$AP = AQ = AM$$

The locus of M is a circle with PQ as its diameter

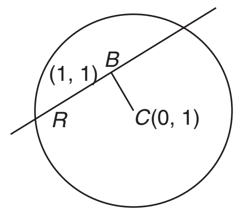


Here, the common tangent is the radial axis. Now, we have

$$E_1:\ (x-2)(x+2)+(y-7)(y+5)=0;\ x\neq\pm2$$

and its centre is given by $(0, 1)$.

The locus of point B is a circle with diameter RC as shown in the following figure:



We have

$$E_2:\ x(x-1)+y(y-1)^2=0$$

Now, let us relate this with the given four options:

- **Option (A):** Despite this option satisfies E_1, the line T generally touches on two points in circle S_1.
- **Option (B):** Despite the he point $\left(\frac{4}{5},\frac{7}{5}\right)$ satisfies E_2, one end of the chord is $(-2, 7)$ which is not a part of E_1. Hence, the point $\left(\frac{4}{5},\frac{7}{5}\right)$ does **NOT** lie in E_2.
- **Option (C):** The point $\left(\frac{1}{2},1\right)$ does not satisfy E_2 and thus, it does not lie on E_2.
- **Option (D):** The point $\left(0,\frac{3}{2}\right)$ does **NOT** satisfy E_1 and thus it does not lie on E_1.

Chapter 13: Parabola

JEE Main 2018

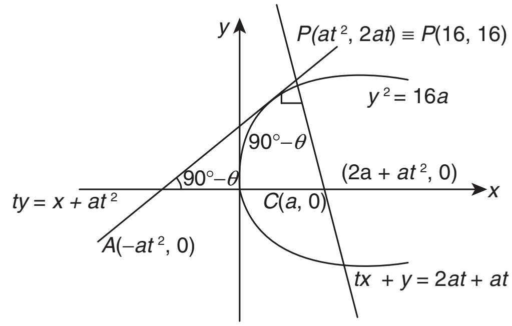
55. Tangent and normal are drawn at $P(16, 16)$ on the parabola $y^2 = 16x$, which intersect the axis of the parabola at A and B, respectively. If C is the centre of the circle through the points P, A and B and $\angle CPB=\theta$, then a value of $\tan\theta$ is

(A) 2 **(B)** 3

(C) $\frac{4}{3}$ **(D)** $\frac{1}{2}$ **(Offline)**

Solution

(A) From the given data, we plot the graph as shown in the following figure:



From above graph we can say

$$\angle CPB = \theta$$
$$\angle APC = 90 - \theta$$
$$\Rightarrow \angle PAC = 90 - \theta$$

We know that the general equation of the tangent is

$$y^2 = 4ax$$

So, $16x = 4ax \Rightarrow a = 4$

and $2at = 16 \Rightarrow 2 \times 4 \times t = 16 \Rightarrow t = 2$

Now, the slope of the tangent is

$$\tan(90° - \theta) = \frac{1}{t} = \frac{1}{2}$$
$$\cot\theta = \frac{1}{2}$$

Therefore, $\tan\theta = 2$.

56. If the tangent at (1, 7) to the curve $x^2 = y - 6$ touches the circle $x^2 + y^2 + 16x + 12y + c = 0$, then the value of c is

(A) 185 **(B)** 85
(C) 95 **(D)** 195 **(Offline)**

Solution

(C) The equation of tangent at (1, 7) for $x^2 = y - 6$ is

$$x = \frac{1}{2}(y + 7) - 6$$
$$\Rightarrow 2x = y + 7 - 12$$
$$\Rightarrow 2x - y + 5 = 0$$

The centre of the circle $x^2 + y^2 + 16x + 12y + c = 0$ is (–8, –6). Perpendicular from centre (–8, –6) to the tangent $2x - y + 5 = 0$ should be equal to radius of the circle.

Therefore,

$$\left|\frac{-16 + 6 + 5}{\sqrt{5}}\right| = \sqrt{64 + 36 - c}$$
$$\sqrt{5} = \sqrt{100 - c} \Rightarrow c = 95$$

57. Two parabolas with a common vertex and with axes along x-axis and y-axis, respectively, intersect each other in the first quadrant. If the length of the latus rectum of each parabola is 3, then the equation of the common tangent to the two parabolas is

(A) $4(x + y) + 3 = 0$ **(B)** $3(x + y) + 4 = 0$
(C) $8(2x + y) + 3 = 0$ **(D)** $x + 2y + 3 = 0$
(Online)

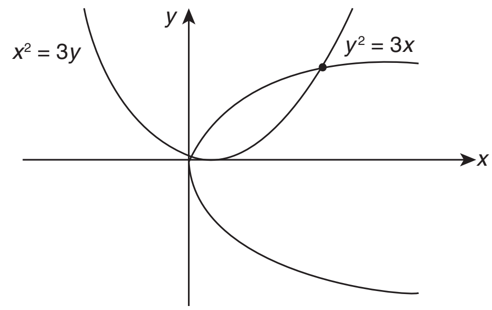
Solution

(A) Equation of tangent of the parabola $y^2 = 3x$ is

$$y = mx + \frac{a}{m}$$

Here, $4a = 3$. So, $y = mx + \dfrac{3}{4m}$

is also the tangent of parabola $x^2 = 3y$



So, $x^2 = 3\left(mx + \dfrac{3}{4m}\right) \Rightarrow x^2 = 3mx + \dfrac{9}{4m}$

$$\Rightarrow x^2 - 3mx - \frac{9}{4m} = 0$$
$$D = 0$$
$$9m^2 + 4 \times \frac{9}{4m} = 0$$
$$\Rightarrow \frac{m^3 + 1}{m} = 0 \Rightarrow m = -1$$

Equation of tangent is

$$y = -x - \frac{3}{4} \Rightarrow 4(x + y) + 3 = 0$$

57. Tangents drawn from the point (–8, 0) to the parabola $y^2 = 8x$ touch the parabola at P and Q. If F is the focus of the parabola, then the area of the triangle PFQ (in sq. units) is equal to

(A) 24 **(B)** 32
(C) 48 **(D)** 64 **(Online)**

Solution

(C) Given: $y^2 = 8x$

Equation of tangents to parabola at point $P(t_1)$ and $Q(t_2)$ is are

$$P(t_1) = P(at_1^2, 2at_1) = P(2t_1^2, 4t_1)$$

That is $4a = 8 \Rightarrow a = 2$

Now, $Q(t_2) = Q(2t_1^2, 4t_2)$

$$y \cdot 4t_1 = 4(x + 2t_1^2)$$

We have $t_1 y = x + 2t_1^2$, $t_2 y = x + 2t_1^2$ intersection point of tangents is

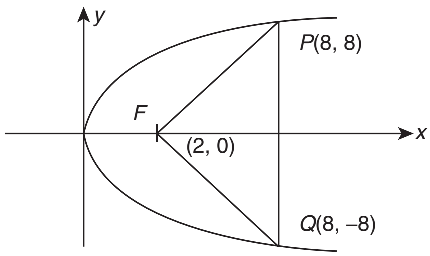
$$(2t_1t_2, 2(t_1 + t_2)) = (-8, 0)$$

So, $t_1 + t_2 = 0 \Rightarrow t_2 = -t_1$

and $t_1t_2 = -4 \Rightarrow t_1^2 = 4$

$$\Rightarrow t_1 = 2 \text{ or } -2$$
$$\Rightarrow t_2 = -2 \text{ or } 2$$

Therefore, $P(8, 8)$, $Q(8, -8)$ and focus (2, 0)



Then, area of ΔFPQ is $= \frac{1}{2} \times 16 \times 6 = 8 \times 6 = 48$ sq. unit

59. Let P be a point on the parabola, $x^2 = 4y$. If the distance of P from the centre of the circle, $x^2 + y^2 + 6x + 8 = 0$ is minimum, then the equation of the tangent to the parabola at P, is

(A) $x + 4y - 2 = 0$ **(B)** $x - y + 3 = 0$

(C) $x + y + 1 = 0$ **(D)** $x + 2y = 0$

(Online)

Solution

(C) Let $P(2t, t^2)$, equation normal at P to $x^2 = 4y$ be

$$y - t^2 = \frac{-1}{t}(x - 2t)$$

It passes through (–3, 0)

$$0 - t^2 = \frac{-1}{t}(-3 - 2t)$$

$$t^3 + 2t + 3 = 0$$

$$(t + 1)(t^2 - t + 3) = 0$$

$$t = -1$$

So, point P is (–2, 1).

Therefore, equation of tangent to $x^2 = 4y$ at (–2, 1) is

$$x(-2) = 2(y + 1)$$

$$x + y + 1 = 0$$

Chapter 14: Ellipse

JEE Main 2018

60. If the curves $y^2 = 6x$, $9x^2 + by^2 = 16$ intersect each other at right angles, then the value of b is

(A) $\frac{7}{2}$ **(B)** 4

(C) $\frac{9}{2}$ **(D)** 6 **(Offline)**

Solution

(C) Given: $y^2 = 6x$ and $9x^2 + by^2 = 16$.

Slopes of the given curves is given by

$$2y\frac{dy}{dx} = 6$$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{y} \quad (1)$$

and

$$18x + 2by\frac{dy}{dx} = 0$$

$$9x + by\frac{dy}{dx} = 0$$

$$by\frac{dy}{dx} = -9x$$

$$\Rightarrow \frac{dy}{dx} = \frac{-9x}{by} \quad (2)$$

Both curves intersect each other at right angle, that is, slopes of both curves are perpendicular to each other. Therefore,

$$\frac{3}{y} \times \frac{-9x}{by} = -1$$

$$-27x = -by^2$$

$$-27x = -b \times 6x$$

Therefore,

$$b = \frac{27}{6} = \frac{9}{2}$$

61. If β is one of the angles between the normal to the ellipse, $x^2 + 3y^2 = 9$ at the points $(3\cos\theta, \sqrt{3}\sin\theta)$ and $(-3\sin\theta, \sqrt{3}\cos\theta)$; $\theta \in \left(0, \frac{\pi}{2}\right)$; then $\frac{2\cot\beta}{\sin 2\theta}$ is equal to

(A) $\frac{2}{\sqrt{3}}$ **(B)** $\frac{1}{\sqrt{3}}$

(C) $\sqrt{2}$ **(D)** $\frac{\sqrt{3}}{4}$ **(Online)**

Solution

(A) Given

$$x^2 + 3y^2 = 9$$

$$\frac{x^2}{9} + \frac{y^2}{3} = 1$$

Equation of tangent at point $\left(3\cos\theta, \sqrt{3}\sin\theta\right)$

$$\frac{x\cos\theta}{3} + \frac{y\sin\theta}{\sqrt{3}} = 1$$

Thus, the slope is

$$m_1 = \frac{\cos\theta}{3} \cdot \frac{\sqrt{3}}{\sin\theta} = \frac{-\cot\theta}{\sqrt{3}}$$

Equation of tangent a point $\left(-3\sin\theta, \sqrt{3}\cos\theta\right)$ is

$$\frac{-x\sin\theta}{3} + \frac{y\cos\theta}{\sqrt{3}} = 1$$

Therefore, the slope is

$$m_2 = \frac{\sin\theta}{3} \cdot \frac{\sqrt{3}}{\cos\theta} = \frac{\tan\theta}{\sqrt{3}}$$

Slopes of the normal are $\frac{\sqrt{3}}{\cot\theta}$ and $-\frac{\sqrt{3}}{\tan\theta}$. Therefore, the angle between the normal to the ellipse is given by

$$\tan\beta = \left|\frac{\sqrt{3}\tan\theta + \frac{\sqrt{3}}{\tan\theta}}{1 - \frac{\sqrt{3}}{\cot\theta}\cdot\frac{\sqrt{3}}{\tan\theta}}\right| = \left|\frac{\sqrt{3}(\tan^2\theta + 1)}{-2\tan\theta}\right|$$

$$\Rightarrow \tan\beta = \frac{\sqrt{3}}{\sin 2\theta} \Rightarrow \frac{2\cot\beta}{\sin 2\theta} = \frac{2}{\sqrt{3}}$$

62. If the length of the latus rectum of an ellipse is 4 units and the distance between a focus and its nearest vertex on the major axis is $\frac{3}{2}$ units, then its eccentricity is

(A) $\frac{1}{2}$ **(B)** $\frac{1}{3}$

(C) $\frac{2}{3}$ **(D)** $\frac{1}{9}$ **(Online)**

Solution

(B) Given:

$$\frac{2b^2}{a} = 4$$

$$\Rightarrow b^2 = 2a \quad (1)$$

$$b^2 = a^2(1 - e^2) \quad (2)$$

$$a(1 - e) = \frac{3}{2} \quad (3)$$

So, $$2 = a(1 - e)(1 + e)$$

$$\Rightarrow 2 = \frac{3}{2}(1 + e) \Rightarrow 4 = 3 + 3e$$

$$\Rightarrow e = \frac{1}{3}$$

JEE Advanced 2018

63. Consider two straight lines, each of which is tangent to both the circle $x^2 + y^2 = \frac{1}{2}$ and the parabola $y^2 = 4x$. Let these lines intersect at the point Q. Consider the ellipse whose centre is at the origin $O(0, 0)$ and whose semi-major axis is OQ. If the length of the minor axis of this ellipse is $\sqrt{2}$, then which of the following statement(s) is (are) TRUE?

(A) For the ellipse, the eccentricity is $\frac{1}{\sqrt{2}}$ and the length of the latus rectum is 1.

(B) For the ellipse, the eccentricity is $\frac{1}{2}$ and the length of the latus rectum is $\frac{1}{2}$.

(C) The area of the region bounded by the ellipse between the lines $x = \frac{1}{\sqrt{2}}$ and $x = 1$ is $\frac{1}{4\sqrt{2}}(\pi - 2)$.

(D) The area of the region bounded by the ellipse between the lines $x = \frac{1}{\sqrt{2}}$ and $x = 1$ is $\frac{1}{16}(\pi - 2)$.

(Paper-1)

Solution

(A), (C) The given circle is

$$x^2 + y^2 = \frac{1}{2}$$

and the given parabola is

$$y^2 = 4x$$

Let the equation of common tangent be

$$y = mx + \frac{1}{m}$$

$$\Rightarrow \left|\frac{0 + 0 + \frac{1}{m}}{\sqrt{1 + m^2}}\right| = \frac{1}{\sqrt{2}}$$

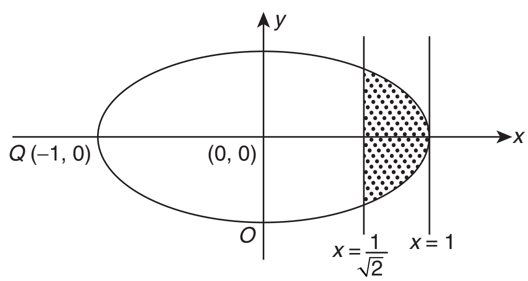
$$\Rightarrow \frac{\left|\frac{1}{m}\right|}{\sqrt{1 + m^2}} = \frac{1}{\sqrt{2}} \Rightarrow m^4 + m^2 - 2 = 0$$

$$\Rightarrow m = \pm 1$$

Thus, the equations of common tangents are as follows:

$$y = x + 1 \text{ and } y = -x - 1$$

Thus, $Q = (-1, 0)$.



Now, the equation of ellipse is

$$\frac{x^2}{1^2} + \frac{y^2}{1/\sqrt{2}} = 1$$

Now, let us find the value of eccentricity (e) as follows:

$$b^2 = a^2(1 - e^2)$$

$$\Rightarrow \frac{1}{2} = 1(1 - e^2)$$

$$\Rightarrow e = \frac{1}{\sqrt{2}}$$

The length of the latus rectum is

$$\frac{2b^2}{a} = \frac{\left(2 \times \frac{1}{2}\right)}{1} = 1$$

Hence, option (A) is correct and option (B) is incorrect.

Therefore, $$y = \pm\frac{1}{\sqrt{2}}\sqrt{1-x^2}$$

Now, the area of the region bounded by the ellipse between the lines $x = \frac{1}{\sqrt{2}}$ and $x = 1$ is expressed as follows:

$$A = 2\times\frac{1}{\sqrt{2}}\int_{1/\sqrt{2}}^{1}\sqrt{1-x^2}\,dx$$

$$= \sqrt{2}\left(\frac{1}{2}x\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x\Big|_{1/\sqrt{2}}^{1}\right) = \frac{(\pi-2)}{4\sqrt{2}}$$

Hence, option (C) is correct and option (D) is incorrect.

Chapter 15: Hyperbola

JEE Main 2018

64. Tangents are drawn to the hyperbola $4x^2 - y^2 = 36$ at the points P and Q. If these tangents intersect at the point $T(0, 3)$ then the area (in sq. units) of ΔPTQ is

(A) $54\sqrt{3}$ **(B)** $60\sqrt{3}$

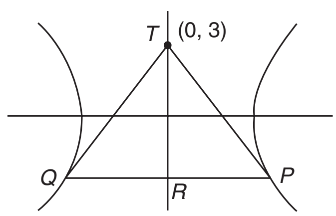
(C) $36\sqrt{5}$ **(D)** $45\sqrt{5}$ **(Offline)**

Solution

(D) The given equation can be rewrite as

$$\frac{x^2}{9} - \frac{y^2}{36} = 1$$

Now, plotting the graph according to given data, we have



$$\frac{xx_1}{9} - \frac{yy_1}{36} = 1$$

Now, $x_1 = 0$ and $y_1 = 3$.

That is, $$\frac{0\times x}{9} - \frac{3y}{36} = 1$$

$$\frac{-y}{12} = 1 \Rightarrow y = -12$$

Also, $$\frac{x^2}{9} - \frac{144}{36} = 1$$

$$\frac{x^2}{9} = \frac{180}{36} \Rightarrow x = \pm 3\sqrt{5}$$

Therefore, $P = \left(3\sqrt{5}, -12\right)$ and $Q = \left(-3\sqrt{5}, -12\right)$.

Therefore, the area of ΔPTQ is

$$A = \frac{1}{2}\begin{vmatrix} 0 & 3 & 1 \\ 3\sqrt{5} & -12 & 1 \\ -3\sqrt{5} & -12 & 1 \end{vmatrix}$$

$$= \frac{1}{2}(-3(6\sqrt{5}) - 36\sqrt{5} - 36\sqrt{5})$$

$$= \frac{1}{2}(-18\sqrt{5} - 36\sqrt{5} - 36\sqrt{5}) \Rightarrow 45\sqrt{5} \text{ sq. unit}$$

65. If the tangents drawn to the hyperbola $4y^2 = x^2 + 1$ intersect the co-ordinate axes at the distinct points A and B, then the locus of the midpoint of AB is

(A) $x^2 - 4y^2 + 16x^2y^2 = 0$ **(B)** $x^2 - 4y^2 - 16x^2y^2 = 0$

(C) $4x^2 - y^2 + 16x^2y^2 = 0$ **(D)** $4x^2 - y^2 - 16x^2y^2 = 0$

(Online)

Solution

(B) Given $$4y^2 = x^2 + 1$$

$$x^2 - 4y^2 = -1 \Rightarrow \frac{y^2}{(1/4)} - x^2 = 1$$

So, $\left(\tan\theta, \frac{1}{2}\sec\theta\right)$ lies on hyperbola. Then, equation of the tangent is

$$x\tan\theta - 4\times\frac{1}{2}y\sec\theta = -1$$

$$\Rightarrow x\tan\theta - 2y\sec\theta = -1$$

$$\frac{x}{(-\cot\theta)} + \frac{y}{\left(\frac{1}{2}\cos\theta\right)} = 1$$

Points on axis are $A(-\cot\theta, 0)$, $B(0, \frac{1}{2}\cos\theta)$.

Let the mid-point of AB be (h, k).

$$2h = -\cot\theta,\ 2k = \frac{1}{2}\cos\theta$$

$$\tan\theta = -\frac{1}{2h},\ \sec\theta = \frac{1}{4k}$$

$$\sec^2\theta - \tan^2\theta = 1 \Rightarrow \frac{1}{16k^2} - \frac{1}{4h^2} = 1$$

The locus of the midpoint of AB is

$$\frac{1}{4y^2} - \frac{1}{2x^2} = 4$$

$$\Rightarrow x^2 - 4y^2 = 16x^2y^2$$

$$\Rightarrow x^2 - 4y^2 - 16x^2y^2 = 0$$

66. A normal to the hyperbola, $4x^2 - 9y^2 = 36$ meets the coordinate axes x and y at A and B, respectively. If the parallelogram $OABP$ (O being the origin) is formed, then the locus of P is:

(A) $4x^2 + 9y^2 = 121$ **(B)** $9x^2 + 4y^2 = 169$

(C) $4x^2 - 9y^2 = 121$ **(D)** $9x^2 - 4y^2 = 169$

(Online)

Solution

(D) Given: $4x^2 - 9y^2 = 36$

Equation of hyperbola: $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$

Now, point $Q(3\sec\theta, 2\tan\theta)$ is on hyperbola.

Thus, equation of tangent is

$$\frac{x\sec\theta}{3} - \frac{y\tan\theta}{2} = 1,\ m = \frac{2\sec\theta}{3\tan\theta}$$

and equation of normal is

$$(y - 2\tan\theta) = \frac{-3\tan\theta}{2\sec\theta}(x - 3\sec\theta)$$

$$\Rightarrow \frac{y}{\tan\theta} - 2 = \frac{-3}{2}\left(\frac{x}{\sec\theta} - 3\right)$$

$$\frac{x}{(2/3\sec\theta)} + \frac{y}{\tan\theta} = \frac{9}{2} + 2 = \frac{13}{2}$$

Point $A\left(\dfrac{13}{3}\sec\theta, 0\right)$ and $B\left(0, \dfrac{13}{2}\tan\theta\right)$

Now, $OABP$ is parallelogram; let the coordinate of P be (h, k).

Then, Mid-point of AB = Mid-point of OP

$$\frac{\frac{13}{3}\sec\theta + 0}{2} = \frac{h}{2} \Rightarrow \sec\theta = \frac{3}{13}h$$

$$\frac{0 + \frac{13}{2}\tan\theta}{2} = \frac{k}{2} \Rightarrow \tan\theta = \frac{2k}{13}$$

$$\sec^2\theta - \tan^2\theta = 1$$

$$9h^2 - 4k^2 = 169$$

Therefore, the locus of P is

$$9x^2 - 4y^2 = 169$$

67. The locus of the point of intersection of the lines, $\sqrt{2}x - y + 4\sqrt{2}k = 0$ and $\sqrt{2}kx + ky - 4\sqrt{2} = 0$ (k is any non-zero real parameter), is

(A) an ellipse whose eccentricity is $\dfrac{1}{\sqrt{3}}$.

(B) an ellipse with length of its major axis $8\sqrt{2}$.

(C) a hyperbola whose eccentricity is $\sqrt{3}$.

(D) a hyperbola with length of its transverse axis $8\sqrt{2}$.

(Online)

Solution

(D) Given:

$$\sqrt{2}x - y + 4\sqrt{2}k = 0 \quad (1)$$

$$\sqrt{2}kx + ky - 4\sqrt{2} = 0 \quad (2)$$

Now, eliminating k from Eq. (2) by putting value of k from Eq. (1), we get

$$(\sqrt{2}x + y)\left(\frac{\sqrt{2x} - y}{-4\sqrt{2}}\right) = 4\sqrt{2}$$

$$2x^2 - y^2 = -32$$

$$\frac{y^2}{32} - \frac{x^2}{16} = 1$$

The above equation represents the hyperbola.

Therefore, the eccentricity e of this hyperbola is

$$e = \sqrt{1 + \frac{16}{32}} = \sqrt{\frac{3}{2}}$$

Length of transverse axis $= 8\sqrt{2}$.

JEE Advanced 2018

68. Let $H: \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, where $a > b > 0$, be a hyperbola in the xy-plane whose conjugate axis LM subtends an angle of 60° at one of its vertices N. Let the area of the triangle LMN be $4\sqrt{3}$. The correct option is:

LIST–I	LIST–II
P. The length of the conjugate axis of H is	**1.** 8
Q. The eccentricity of H is	**2.** $\dfrac{4}{\sqrt{3}}$
R. The distance between the foci of H is	**3.** $\dfrac{2}{\sqrt{3}}$
S. The length of the latus rectum of H is	**4.** 4

(A) P→4; Q→2; R→1; S→3

(B) P→4; Q→3; R→1; S→2

(C) P→4; Q→1; R→3; S→2

(D) P→3; Q→4; R→2; S→1

(Paper-2)

Solution

(B) It is given that

$$H: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (1)$$

The area of ΔLMN is

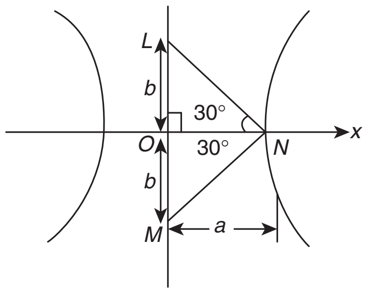
$$\text{Area}\big|_{\Delta LMN} = 4\sqrt{3} \quad (2)$$

Now, we have

$$\tan 30° = \frac{b}{a}$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{b}{a}$$

$$\Rightarrow b = \frac{a}{\sqrt{3}} \quad (3)$$



The area of ΔOLN is

$$\text{Area}\big|_{\Delta OLN} = \frac{1}{2}a\times b$$

Using Eq. (2), the area of ΔOLN is obtained as

$$\text{Area}\big|_{\Delta OLN} = \frac{4\sqrt{3}}{2}$$

$$\Rightarrow \frac{4\sqrt{3}}{2} = \frac{1}{2}a\times b \Rightarrow ab = 4\sqrt{3}$$

Using Eq. (3), we get

$$a\times\frac{a}{\sqrt{3}} = 4\sqrt{3} \Rightarrow a^2 = 4\times 3 = 12$$

$$\Rightarrow a = 2\sqrt{3} \text{ and } b = 2$$

Now, the eccentricity is obtained as

$$e = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{1+\frac{4}{12}} = \sqrt{1+\frac{1}{3}} = \frac{2}{\sqrt{3}}$$

Now, the distance between foci is obtained as

$$2ae = 2\times 2\sqrt{3}\times\frac{2}{\sqrt{3}} = 8$$

Also, the length of the latus rectum is

$$\frac{2b^2}{a} = \frac{2\times 4}{2\sqrt{3}} = \frac{4}{\sqrt{3}}$$

Therefore, the correct mapping is P→4; Q→3; R→1; S→2. Hence, option (B) is correct.

Chapter 16: Statistics

JEE Main 2018

69. If $\sum_{i=1}^{9}(x_i-5)=9$ and $\sum_{i=1}^{9}(x_i-5)^2=45$, then the standard deviation of the nine items $x_1, x_2, ..., x_9$ is

(A) 4 **(B)** 2
(C) 3 **(D)** 9

(Offline)

Solution

(B) Standard deviation (SD) is independent of shifting of origin. Therefore, the standard deviation of the nine items is

$$SD = +\sqrt{\text{var}(x_i-5)}$$

Given: $\sum_{i=1}^{9}(x_i-5)=9$ and $\sum_{i=1}^{9}(x_i-5)^2=45$

$$\sum_{i=1}^{9}x_i - 45 = 9 \Rightarrow \sum_{i=1}^{9}x_i = 54$$

$$\sum_{i=1}^{9}(x_i^2+25-10x_1) = 45 = \sum_{i=1}^{9}x_i^2 - 10\sum_{i=1}^{9}x_1 + 225 = 45$$

$$\Rightarrow \sum_{i=1}^{9}x_i^2 = 360$$

Variation is given by

$$\text{var}(x) = \frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2$$

Therefore, standard deviation is given as

$$SD = \sqrt{\frac{360}{9}-\left(\frac{54}{9}\right)^2} \Rightarrow \sqrt{40-36} \Rightarrow \sqrt{4} = 2$$

70. The mean of a set of 30 observations is 75. If each observation is multiplied by a non-zero number λ and then each of them is decreased by 25, their mean remains the same. Then λ is equal to

(A) $\frac{1}{3}$ **(B)** $\frac{2}{3}$
(C) $\frac{4}{3}$ **(D)** $\frac{10}{3}$ **(Online)**

Solution

(C) Let observations be $x_1, x_2 \cdots, x_{30}$

So, $$\frac{x_1+x_2+\cdots+x_{30}}{30} = 75 \Rightarrow x_1+\cdots+x_{30} = 30\times 75$$

and $$\frac{(\lambda x_1-25)+\cdots+(\lambda x_{30}-25)}{30} = 75$$

$$\Rightarrow \lambda(x_1+\cdots+x_{30}) - 25\times 30 = 75\times 30$$

$$\Rightarrow \lambda\,(30\times 75) = 100\times 30 \Rightarrow \lambda = \frac{100}{75} = \frac{4}{3}$$

71. If the mean of the data: 7, 8, 9, 7, 8, 7, λ, 8 is 8, then the variance of this data is

(A) $\frac{7}{8}$ **(B)** 1
(C) $\frac{9}{8}$ **(D)** 2 **(Online)**

Solution

(B) Given:

$$\text{Mean }(\bar{x}) = 8 = \frac{7+8+9+7+8+7+\lambda+8}{8}$$

$$\Rightarrow 64 = 54+\lambda \Rightarrow \lambda = 10$$

Therefore, Variance $(\sigma^2) = \dfrac{\Sigma(x_i - \bar{x})^2}{8}$

$$= \frac{1}{8}((7-8)^2 + (8-8)^2 + (9-8)^2 + (9-8)^2 + (7-8)^2$$
$$+ (8-8)^2 + (7-8)^2 + (10-8)^2 + (8-8)^2$$
$$= \frac{1}{8}(8) = 1$$

72. The mean and the standard deviation (SD) of five observations are 9 and 0, respectively. If one of the observations is changed such that the mean of the new set of five observations becomes 10, then their SD is

(A) 0 **(B)** 1
(C) 2 **(D)** 4 **(Online)**

Solution

(C) In the first case:

Mean of 5 observation = 9

Standard Deviation = 0

Since standard deviation is 0, all the five observations remain the same value of mean 9.

So, 5 observations are as follows: 9, 9, 9, 9, 9

In the second case, one of the observations is changed so that mean becomes as follows:

$$\frac{9+9+9+9+x}{5} = 10$$
$$36 + x = 50$$
$$x = 14$$

Now,

Observations (x)	**Mean ($\bar{x}$)**	**($x - \bar{x}$)**	**($x - \bar{x})^2$**
9	10	–1	1
9	10	–1	1
9	10	–1	1
9	10	–1	1
14	10	4	16

Therefore, $\text{S.D.} = \sqrt{\dfrac{(x-\bar{x})^2}{n}} = \sqrt{\dfrac{20}{5}} = \sqrt{4} = 2$

About the Book

Wiley Mathematics Problem Book covers the complete mathematics course for JEE. It is focused on the development of problem-solving skills in JEE aspirants. The chapter flow of the book is closely aligned with the JEE (Main) syllabus and its coverage in the classroom. However, the topics required for JEE (Advanced) are also covered. The problems presented systematically cover all important concepts pertaining to the topic and the possible questions that can be framed on them.

Key Features

- Focused mainly on the preparation of JEE (Main) but also suitable for JEE (Advanced)
- **All concepts and formulae** as per JEE (Main) and JEE (Advanced) syllabus.
- **Topic-wise Illustrations** and **Additional Solved Problems** for grasping the concepts.
- Topic-wise questions provided in **Your Turn** sections for practicing the concepts.
- Solved last **10-year JEE-Main/AIEEE** questions and **10-year JEE-Advanced/IIT-JEE** questions provided (2007-2016).
- Practice Questions at two levels, i.e. **Practice Exercise 1** (JEE Main) and **Practice Exercise 2** (JEE Advanced).
- All **question types** as per JEE (Main and Advanced) covered:
  - Single Correct Choice Type
  - Multiple Correct Choice Type
  - Passage Type
  - Matrix-Match Type
  - Integer Type
- **Answer Key and Solutions** for all Practice Exercises provided.
- Covers Solved JEE 2017 (Main and Advanced) Questions Chapterwise.
- **Appendix:** Chapterwise Solved JEE 2018 (Main and Advanced) Questions.

About Maestro Series

- **Idea:** Best-suited content designed as per the needs of JEE aspirants.
- **Process:** Developed in consultation with experienced teachers after in-depth study of syllabus and relative weightage of topics.
- **Two-fold advantage:**
  - **Conceptual strength** provided by authoritative yet precise content as per syllabus requirement.
  - **Assessment as per JEE** through thought provoking end-of-chapter exercises as per JEE question pattern.

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ISBN 978-81-265-7629-6